Online Supplement for "Lagrangian Dual Decision Rules for Multistage Stochastic Mixed-Integer Programming"

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This supplement paper provides all appendices referenced in the body of the paper, including proofs and further details on the results.

Appendix A: Proof of Lemma 1

Expanding the objective function in (5a) and rearranging the terms yields

$$\sum_{t\in[T]}\left(\mathbb{E}_{\boldsymbol{\xi}^T}\left[c_t(\boldsymbol{\xi}^t)^\top y_t(\boldsymbol{\xi}^T)\right]+\mathbb{E}_{\boldsymbol{\xi}^T}\left[\gamma_t(\boldsymbol{\xi}^T)^\top y_t(\boldsymbol{\xi}^T)\right]-\mathbb{E}_{\boldsymbol{\xi}^T}\left[\gamma_t(\boldsymbol{\xi}^T)^\top\mathbb{E}_{\boldsymbol{\xi}'^T}[y_t(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t]\right]\right). \tag{1}$$

For the last term in (1), the following equalities hold

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\xi}^T}\left[\gamma_t(\boldsymbol{\xi}^T)^\top\mathbb{E}_{\boldsymbol{\xi}'^T}[y_t(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t]\right] &= \mathbb{E}_{\boldsymbol{\xi}^T}\left[\mathbb{E}_{\boldsymbol{\xi}'^T}[\gamma_t(\boldsymbol{\xi}^T)^\top y_t(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t]\right] && \text{(2a)}\\
&= \mathbb{E}_{\boldsymbol{\xi}'^T}\left[\mathbb{E}_{\boldsymbol{\xi}^T}[\gamma_t(\boldsymbol{\xi}^T)^\top y_t(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t]\right] && \text{(2b)}\\
&= \mathbb{E}_{\boldsymbol{\xi}'^T}\left[\mathbb{E}_{\boldsymbol{\xi}^T}[\gamma_t(\boldsymbol{\xi}^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t]\, y_t(\boldsymbol{\xi}'^T)\right] && \text{(2c)}\\
&= \mathbb{E}_{\boldsymbol{\xi}^T}\left[\mathbb{E}_{\boldsymbol{\xi}'^T}[\gamma_t(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t] y_t(\boldsymbol{\xi}^T)\right]. && \text{(2d)}
\end{aligned}$$

First note that, in $\mathbb{E}_{\boldsymbol{\xi}^T}\left[\gamma_t(\boldsymbol{\xi}^T)^\top\mathbb{E}_{\boldsymbol{\xi}'^T}[y_t(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}'^t=\boldsymbol{\xi}^t]\right]$, since the dual function $\gamma_t(\boldsymbol{\xi}^T)$ is fixed, $\gamma_t(\boldsymbol{\xi}^T)$ is a vector of numbers inside the first expectation, hence it can be pushed inside the second expectation (Equation (2a)). As we have assumed that $\gamma_t(\boldsymbol{\xi}^T)$ is a member of the set Γ_t with its expectation being bounded, and $\mathbb{E}[\text{dia}(Y_t(\boldsymbol{\xi}'^t))]$, the expectation of the diameter of the set $Y_t(\boldsymbol{\xi}'^t)$

to which $y_t(\boldsymbol{\xi}'^t)$ belongs, is finite, using Fubini-Tonelli Theorem (Knapp 2005) the order of the two expectations can be exchanged (Equation (2b)). Then we can take $y_t(\boldsymbol{\xi}'^T)$ out from the inside expectation, as it is just a vector of numbers inside (Equation (2c)). Since $\boldsymbol{\xi}^T$ and $\boldsymbol{\xi}'^T$ have the same support and distribution, the last equality (Equation (2d)) is satisfied. Equality (6) can then be proven by substitution.

Appendix B: Proof of Lemma 2

First observe that since the objective in (12) is linear, the restricted Lagrangian dual problem (14) can be written as

$$\begin{aligned} z^{\text{RLD}} = \max_{\lambda,\alpha} \ \min_{x} \ & \{c^\top x + \lambda^\top (d - Dx) : x \in \text{conv}(X)\} \\ \text{s.t.} \ & \lambda - G\alpha = 0 \end{aligned}$$

Let $\{x^i\}_{i\in[M]}$ and $\{r^k\}_{k\in[K]}$ be the complete set of extreme points and extreme rays of $\text{conv}(X)$, respectively. Then, for any fixed λ, we have

$$z(\lambda) = \begin{cases} -\infty, & \exists r^k : (c^\top - \lambda^\top D) r^k < 0 \\ \min\limits_{i\in[M]} \{c^\top x^i + \lambda^\top (d - Dx^i)\}, & \text{otherwise.} \end{cases}$$

Therefore we can reformulate the Lagrangian dual problem as

$$\begin{aligned} z^{\text{LD}} = \max_{\lambda,\alpha} \ \min_{i\in[M]} \ & \{c^\top x^i + \lambda^\top (d - Dx^i)\} \\ \text{s.t.} \ & \lambda - G\alpha = 0 \\ & (c^\top - \lambda^\top D) r^k \geq 0, \quad k \in [K] \end{aligned}$$

which is equivalent to

$$\begin{aligned} z^{\text{LD}} = \max_{\lambda,\alpha,\eta} \ & \eta && && \text{(3a)} \\ \text{s.t.} \ & \lambda - G\alpha = 0 && && (\theta) \quad \text{(3b)} \\ & \lambda^\top D r^k \leq c^\top r^k && k \in [K] && (\beta_k) \quad \text{(3c)} \\ & \eta + \lambda^\top (Dx^i - d) \leq c^\top x^i && i \in [M] && (\gamma_i) \quad \text{(3d)} \end{aligned}$$

where θ, β_k and γ_i are the dual variables associated with constraints (3b), (3c) and (3d) respectively. Now take the dual of the above problem

$$\begin{aligned} \min_{\theta,\beta,\gamma} \ & c^\top \left(\sum_{i\in[M]} \gamma_i x^i + \sum_{k\in[K]} \beta_k r^k \right) \\ \text{s.t.} \ & \sum_{i\in[M]} \gamma_i = 1 \end{aligned}$$

$$
\begin{aligned}
& D\left(\sum_{i\in[M]}\gamma_i x^I + \sum_{k\in[K]}\beta_k r^k\right) + \theta = d \\
& -G^\top \theta = 0 \\
& \gamma, \beta \geq 0
\end{aligned}
$$

As we know that $\text{conv}(X) = \left\{ \sum_{i\in[M]}\gamma_i x^i + \sum_{k\in[K]}\beta_k r^k : \sum_{i\in[M]}\gamma_i = 1, \gamma_i, \beta_k \geq 0,\ i \in [M], k \in [K] \right\}$, we have

$$
\begin{aligned}
\min_{x,\theta} \quad & c^\top x \\
\text{s.t.} \quad & x \in \text{conv}(X) \\
& Dx + \theta = d \\
& -G^\top \theta = 0.
\end{aligned} \tag{4}
$$

Eliminating the θ variables using (4) yields the result.

Note that if $\theta = \mathbf{0}$, then the original MIP solution, say x^*, is feasible. So, this gives a relaxation of the original MIP.

Appendix C: The Lot-sizing Problem

C.1. Stagewise Lagrangian Dual

Relax state equations (27b), except for $t = 1$. Then, we have

$$
\begin{aligned}
\min \quad & \mathbb{E}\left[L(\lambda)\right] && \text{(5a)} \\
\text{s.t.} \quad & i_{1j}^-(\boldsymbol{\xi}^1) - i_{1j}^+(\boldsymbol{\xi}^1) = d_{1j}(\boldsymbol{\xi}^1), \quad j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi && \text{(5b)} \\
& (27c) - (27i) && \text{(5c)}
\end{aligned}
$$

where

$$
\begin{aligned}
L(\lambda) := \quad & \sum_{t\in[T]}\left(\sum_{j\in[J]}\left(C_{tj}^{i^+}(\boldsymbol{\xi}^t) i_{tj}^+(\boldsymbol{\xi}^t) + C_{tj}^{i^-}(\boldsymbol{\xi}^t) i_{tj}^-(\boldsymbol{\xi}^t) + C_{tj}^{y}(\boldsymbol{\xi}^t) y_{tj}(\boldsymbol{\xi}^t)\right) + C_t^o(\boldsymbol{\xi}^t) o_t(\boldsymbol{\xi}^t)\right) \\
& + \sum_{t\in[T]}\sum_{j\in[J]} \lambda_{tj}(\boldsymbol{\xi}^t)\left(i_{tj}^-(\boldsymbol{\xi}^t) - i_{tj}^+(\boldsymbol{\xi}^t) + i_{t-1,j}^+(\boldsymbol{\xi}^{t-1}) - i_{t-1,j}^-(\boldsymbol{\xi}^{t-1}) + x_{t-1,j}(\boldsymbol{\xi}^{t-1}) - D_{tj}(\boldsymbol{\xi}^t)\right).
\end{aligned}
$$

Restrict $\lambda_t(\boldsymbol{\xi}^t)$ to follow LDDR

$$
\lambda_{tj}(\boldsymbol{\xi}^t) = \sum_{k\in[K_t]} \Phi_{tjk}(\boldsymbol{\xi}^t)\alpha_{tjk}.
$$

Then, for fixed $\hat{\alpha}$, the objective function (5a) is equivalent to

$$
\sum_{t\in[T]} \mathbb{E}\left[L_t(\hat{\alpha})\right] \tag{6}
$$

where

$$
\begin{aligned}
&L_t(\hat{\alpha}, \boldsymbol{\xi}^t) := \\
&\sum_{j\in[J]} \Bigg[\Bigg(\sum_{k\in[K_t]} \Phi_{tjk}(\boldsymbol{\xi}^t)\alpha_{tjk} \Bigg) \big(- D_{tj}(\boldsymbol{\xi}^t) \big) \\
&+ \Bigg(\mathbb{E}\Bigg[\sum_{k\in[K_{t+1}]} \Phi_{t+1,j,k}(\boldsymbol{\xi}^{t+1})\alpha_{t+1,jk} \Bigg| \boldsymbol{\xi}^t \Bigg] \Bigg) x_{tj} \\
&+ \Bigg(C_{tj}^{i^+}(\boldsymbol{\xi}^t) - \sum_{k\in[K_t]} \Phi_{tjk}(\boldsymbol{\xi}^t)\alpha_{tjk} + \mathbb{E}\Bigg[\sum_{k\in[K_{t+1}]} \Phi_{t+1,j,k}(\boldsymbol{\xi}^{t+1})\alpha_{t+1,jk} \Bigg| \boldsymbol{\xi}^t \Bigg] \Bigg) i_{tj}^+ \\
&+ \Bigg(C_{tj}^{i^-}(\boldsymbol{\xi}^t) + \sum_{k\in[K_t]} \Phi_{tjk}(\boldsymbol{\xi}^t)\alpha_{tjk} - \mathbb{E}\Bigg[\sum_{k\in[K_{t+1}]} \Phi_{t+1,j,k}(\boldsymbol{\xi}^{t+1})\alpha_{t+1,jk} \Bigg| \boldsymbol{\xi}^t \Bigg] \Bigg) i_{tj}^- \\
&+ C_{tj}^{y}(\boldsymbol{\xi}^t) y_{tj} \Bigg] + C_t^o(\boldsymbol{\xi}^t) o_t.
\end{aligned}
$$

For fixed $\hat{\alpha}$, $\boldsymbol{\xi}^t$ and $t > 1$, define $\mathcal{L}_t(\hat{\alpha}, \boldsymbol{\xi}^t)$ as follows

$$
\begin{aligned}
\mathcal{L}_t(\hat{\alpha}, \boldsymbol{\xi}^t) := \min\ & L_t(\hat{\alpha}, \boldsymbol{\xi}^t) \\
\text{s.t.}\ & \sum_{j\in[J]} \big(TS_j y_{tj} + TB_j x_{tj} \big) - o_t \le C_t \\
& M_{tj}\ y_{tj} - x_{tj} \ge 0, && j \in [J] \\
& i_{tj}^+ \le I_{tj}, && j \in [J] \\
& i_{tj}^+ + x_{tj} \le I_{t+1,j}, && j \in [J] \\
& 0 \le o_t \le O_t \\
& x_{tj}, i_{tj}^+, i_{tj}^- \ge 0, && j \in [J] \\
& y_{tj} \in \{0,1\}, && j \in [J].
\end{aligned}
$$

For $t = 1$ we have

$$
\begin{aligned}
\mathcal{L}_1(\hat{\alpha}, \boldsymbol{\xi}^1) := \min\ & L_1(\hat{\alpha}, \boldsymbol{\xi}^1) \\
\text{s.t.}\ & i_{1j}^- - i_{1j}^+ = d_{1j}, && j \in [J] \\
& \sum_{j\in[J]} \big(TS_j y_{1j} + TB_j x_{1j} \big) - o_1 \le C_1 \\
& M_{1j}\ y_{1j} - x_{1j} \ge 0, && j \in [J] \\
& i_{1j}^+ \le I_{1j}, && j \in [J] \\
& i_{1j}^+ + x_{1j} \le I_{2,j}, && j \in [J] \\
& 0 \le o_1 \le O_1
\end{aligned}
$$

$$
\begin{aligned}
& x_{1j}, i^+_{1j}, i^-_{1j} \geq 0, && j \in [J] \\
& y_{1j} \in \{0,1\}, && j \in [J].
\end{aligned}
$$

Finally, the LDDR-restricted stagewise Lagrangian dual problem is defined as

$$
\begin{aligned}
\max \ & \sum_{t\in[T]} \mathbb{E}\left[\mathcal{L}_t(\alpha, \boldsymbol{\xi}^t)\right] \\
\text{s.t. } & \alpha_{tk} \in \mathbb{R}^J, \qquad t \in [T], k \in [K_t].
\end{aligned}
$$

C.2. Nonanticipative Lagrangian Dual

Reformulate the MSLot problem as follows

$$
\begin{aligned}
\min \ & \mathbb{E}\Bigg[\sum_{t\in[T]}\Bigg[\sum_{j\in[J]}\Big[C^{i^+}_{tj}(\boldsymbol{\xi}^t) i^{+,na}_{tj}(\boldsymbol{\xi}^T) + C^{i^-}_{tj}(\boldsymbol{\xi}^t) i^{-,na}_{tj}(\boldsymbol{\xi}^T) + C^y_{tj}(\boldsymbol{\xi}^t) y^{na}_{tj}(\boldsymbol{\xi}^T)\Big] + C^o_t(\boldsymbol{\xi}^t) o^{na}_t(\boldsymbol{\xi}^T)\Bigg]\Bigg] \\
\text{s.t. } & i^{-,na}_{tj}(\boldsymbol{\xi}^T) - i^{+,na}_{tj}(\boldsymbol{\xi}^T) + i^{+,na}_{t-1,j}(\boldsymbol{\xi}^T) - i^{-,na}_{t-1,j}(\boldsymbol{\xi}^T) + x^{na}_{t-1,j}(\boldsymbol{\xi}^T) = D_{tj}(\boldsymbol{\xi}^t), \\
& && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& \sum_{j\in[J]} (TS_j y^{na}_{tj}(\boldsymbol{\xi}^T) + TB_j x^{na}_{tj}(\boldsymbol{\xi}^T)) - o^{na}_t(\boldsymbol{\xi}^T) \leq C_t, && t \in [T], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& M_{tj} \ y^{na}_{tj}(\boldsymbol{\xi}^T) - x^{na}_{tj}(\boldsymbol{\xi}^T) \geq 0, && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& i^{+,na}_{tj}(\boldsymbol{\xi}^T) \leq I_{tj}, && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& i^{+,na}_{tj}(\boldsymbol{\xi}^T) + x^{+,na}_{tj}(\boldsymbol{\xi}^T) \leq I_{t+1,j}, && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& 0 \leq o^{na}_t(\boldsymbol{\xi}^T) \leq O_t, && t \in [T], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& x^{na}_{tj}(\boldsymbol{\xi}^T) = \mathbb{E}_{\boldsymbol{\xi}'^T}[x^{na}_{tj}(\boldsymbol{\xi}'^T) \,|\, \boldsymbol{\xi}^t], && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& x^{na}_{tj}(\boldsymbol{\xi}^T), i^{+,na}_{tj}(\boldsymbol{\xi}^T), i^{-,na}_{tj}(\boldsymbol{\xi}^T) \geq 0, && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi \\
& y^{na}_{tj}(\boldsymbol{\xi}^T) \in \{0,1\}, && t \in [T], j \in [J], \mathbb{P}\text{-a.e. } \boldsymbol{\xi}^T \in \Xi
\end{aligned}
$$

where for any variable $a^{na}(\boldsymbol{\xi}^T)$ the superscript na indicates the anticipative copy variable corresponding to original variable $a(\boldsymbol{\xi}^t)$. Relaxing the nonanticipativity constraints using dual variables $\gamma_t(\boldsymbol{\xi}^T)$ and enforcing LDDR on these duals as

$$
\gamma_{tj}(\boldsymbol{\xi}^T) = \sum_{k'\in[K'_t]} \Psi_{tjk'}(\boldsymbol{\xi}^T)\alpha_{tjk'},
$$

the LDDR-restricted nonanticipative Lagrangian dual problem is obtained as follows

$$
\begin{aligned}
\max \ & \mathbb{E}\left[\mathcal{L}(\alpha, \boldsymbol{\xi}^T)\right] \\
\text{s.t. } & \alpha_{tk'} \in \mathbb{R}^J, \qquad t \in [T], k' \in [K'_t]
\end{aligned}
$$

where for fixed $\hat{\alpha}$

$$
\begin{aligned}
\mathcal{L}(\hat{\alpha},\boldsymbol{\xi}^T) = \min \quad & \sum_{t\in[T]}\left[\sum_{j\in[J]}\left[C_{tj}^{i^+}(\boldsymbol{\xi}^t)i_{tj}^{+,na} + C_{tj}^{i^-}(\boldsymbol{\xi}^t)i_{tj}^{-,na} + C_{tj}^{y}(\boldsymbol{\xi}^t)y_{tj}^{na}\right] + C_t^o(\boldsymbol{\xi}^t)o_t^{na}\right] \\
& + \sum_{t\in[T]}\sum_{j\in[J]}\sum_{k'\in[K_t']}\left(\Psi_{tjk'}(\boldsymbol{\xi}^T) - \mathbb{E}_{\boldsymbol{\xi}'^T}[\Psi_{tjk'}(\boldsymbol{\xi}'^T)|\boldsymbol{\xi}^t]\right)\hat{\alpha}_{tjk'}x_{tj}^{na} \\
\text{s.t.} \quad & i_{tj}^{-,na} - i_{tj}^{+,na} + i_{t-1,j}^{+,na} - i_{t-1,j}^{-,na} + x_{t-1,j}^{na} = D_{tj}(\boldsymbol{\xi}^t), && t\in[T], j\in[J] \\
& \sum_{j\in[J]}\left(TS_j y_{tj}^{na} + TB_j x_{tj}^{na}\right) - o_t^{na} \le C_t, && t\in[T] \\
& M_{tj}\ y_{tj}^{na} - x_{tj}^{na} \ge 0, && t\in[T], j\in[J] \\
& i_{tj}^{+,na} \le I_{tj}, && t\in[T], j\in[J] \\
& i_{tj}^{+,na} + x_{tj}^{+,na} \le I_{t+1,j}, && t\in[T], j\in[J] \\
& 0 \le o_t^{na} \le O_t, && t\in[T] \\
& x_{tj}^{na}, i_{tj}^{+,na}, i_{tj}^{-,na} \ge 0, && t\in[T], j\in[J] \\
& y_{tj}^{na} \in \{0,1\}, && t\in[T], j\in[J].
\end{aligned}
$$

C.3. Parameters of the MSLot Instances

Our instances are generated using parameters that are loosely based on the work of Helber et al. (2013). Section 5.3.1 explains the demand model. In this section, we specify the demand generation procedure and the rest of the parameters. Using consecutive substitutions, we can deduce that the following holds for the conditional expectation

$$
\mathbb{E}[D_{t+h,j}|\epsilon_{tj},\delta_{tj}] = \mu_{t+h,j}\left(\rho^Y\rho^h\left(\frac{D_{tj} - (1-\rho^Y)\,\delta_{tj}}{\rho^Y\,\mu_{tj}} - 1\right) + 1\right).
$$

We have generated instances with $T = 2,\ldots,10$ stages, and $J = 3, 6$ product types. There are two sets of data instances for $J = 3$ with $\rho = 0.2, \rho^Y = 0.6$ and $\rho = 0.6, \rho^Y = 0.2$ leading to different levels of correlation and variation among demands. The combination $\rho = 0.6, \rho^Y = 0.2$ is additionally used in the generation of instances with $J = 6$. For solving the SW and NA dual problems, $\lceil \frac{50}{T}\rceil|\beta|_\#$ and $\lceil \frac{100}{T}\rceil|\alpha|_\#$ scenarios are generated respectively, and the solutions are evaluated using $\lceil \frac{250}{T}\rceil|\alpha|_\#$ scenarios, where $|\cdot|_\#$ operator denotes the total number of variables of a given type (which depends on the choice of basis functions, discussed in Section 5.3.2). We do not consider a production cost. The overtime cost is 100 per unit of overtime. Holding cost is 15 per unit, while the backlog cost is $(\delta_{i^-})(c_{tj}^{i^+})$, where $\delta_{i^-} = 2$. For the last stage though, we have an end of horizon effect and $c_{Tj}^{i^-} = 150$. TB_j is set to 1, and $TS_j = ts^{rel}\ \overline{\mathbb{E}[D_j]}\ TB_j$, where $\overline{\mathbb{E}[D_j]} = \frac{\sum_{t\in[T]}\mathbb{E}[D_{tj}(\boldsymbol{\xi}^t)]}{T}$ is the average expected demand of product j, and ts^{rel} is 0.25. The setup cost is $c_{tj}^{y} = \delta_y\ \overline{\mathbb{E}[D_j]}\ TBO^2\ c_{ij}^{i^+}$, where TBO is the processing time between orders (set to 2), and δ_y is 1.2. Production capacity is

$C_t = 0.9 \frac{\sum_{j\in[J]} \mathbb{E}[D_{tj}(\boldsymbol{\xi}^t)]}{Util}$, where $Util$ is 0.6. Inventory capacity is $I_{tj} = \delta_I \ \overline{\mathbb{E}[D_j]}$, with $\delta_I = 10$. The bound on overtime is $O_t = \delta_O C_t$, where $\delta_O = 0.25$. For big-M values in the MSLot formulation, we have $M_j = 6 \ \overline{\mathbb{E}[D_j]}$.

C.4. NA Variable Selection

In NA problem reformulation, another possibility to reduce the restricted dual problem size is to ignore altogether the NA constraints (4c) on some sets of variables. For example, in order to obtain a valid formulation it is sufficient to enforce (4c) only on *state variables*, i.e., variables that appear in more than one decision stage, and thus one may choose to only penalize the NA constraints associated with those variables. On the other hand, in the MSLot problem, once the recourse variables $x_t(\boldsymbol{\xi}^t)$ are determined, the optimal values of the state variables $i^-(\boldsymbol{\xi}^t)$ and $i^+(\boldsymbol{\xi}^t)$ are immediate from the equations (27b), suggesting it may be beneficial to penalize violation of the NA constraints on these variables. (A similar approach is used by Lulli and Sen (2004).) A natural question for our LDDR approach is, can enforcing nonanticipativity additionally on the other variables (which is redundant before the relaxation) improve the bound?

In Table 5, we consider three options for instances with $J = 3, \rho = 0.6, \rho^Y = 0.2$: having either x or state variables i^+, i^- as the NA variables, or considering a combination of them. Results of Table 5 show that the combination of x, i^+, i^- obtains the best bounds in a longer running time. The difference between the means of the options was tested using a t-test. We find that the bounds obtained by penalizing the NA constraints on only the x variables are not significantly worse than those obtained by penalizing the NA constraints on all variables. In addition, including a penalty of the NA constraints x variables leads to an improvement over just including a penalty of the NA constraints on the state variables i^+ and i^-. Since penalizing only the NA constraints on the x variables yields lower bounds that are indistinguishable from the best, in less time, we use this option in the remaining experiments.

C.5. Detailed Results

In the next pages, the actual numbers used in Section 5 are provided, without any scaling and normalization. Also, the confidence intervals are given for the bounds, in the form of (mean ± width).

Option	# Basis functions			Time (s)			Bound		
	$T=4$	$T=6$	$T=8$	$T=4$	$T=6$	$T=8$	$T=4$	$T=6$	$T=8$
1	315	750	1365	353.8	1412.5	6273.9	32474.6 ± 448.6	41291.3 ± 1194.2	69849.2 ± 5151.9
2	180	300	420	91.7	243.7	632.5	30947.9 ± 363.9	39835.2 ± 496.4	66388.5 ± 5159.7
3	135	300	525	109	288.7	795.9	31053.7± 371.9	40195.4 ± 671.9	66727.2 ± 2203.9
4	90	150	210	157.2	795.9	323.6	29245.5 ± 396.9	38958.3 ± 1086.0	56882.1 ± 4201.2

Table 1: Basis function selection analysis for the SW dual

Option	# Basis functions			Time (s)			Bound		
	$T=4$	$T=6$	$T=8$	$T=4$	$T=6$	$T=8$	$T=4$	$T=6$	$T=8$
1	756	1800	3276	76.9	403.2	1800.8	53352.5 ± 1261.5	71767.2 ± 2515.2	123173.0 ± 5101.1
2	432	720	1008	44.9	232.3	770.5	49940.7 ± 403.5	67696.9 ± 501.2	114913.0 ± 1834.5
3	324	720	1260	61.1	250	776.8	53328.5 ± 1186.6	71352.1 ± 1539.8	122755.0 ± 3085.8
4	216	360	504	34.2	219.5	735.3	49902.7 ± 407.0	67247.6 ± 442.2	114829.0 ± 1715.5

Table 2: Basis function selection analysis for the NA dual

Option	Time (s)			Bound		
	$T=4$	$T=6$	$T=8$	$T=4$	$T=6$	$T=8$
x	61.1	68.1	175.4	53328.5 ± 1186.6	71352.1 ± 1539.8	122755.0 ± 3085.8
i^+, i^-	250.0	188.0	849.9	49456.0 ± 403.9	65348.7 ± 392.2	111605.0 ± 1778.4
x, i^+, i^-	776.8	653.1	1887.3	53459.1 ± 1012.2	71844.8 ± 939.4	123403.0 ± 2400.2

Table 3: NA variable selection analysis

λ	Bound		
	$T=4$	$T=6$	$T=8$
0.00	58292.3 ± 481.1	81073.9 ± 672.5	143503.0 ± 1912.6
0.25	56624.5 ± 503.2	80815.3 ± 689.7	143503.2 ± 1928.4
0.50	56925.1 ± 499.4	81138.1 ± 680.5	144290.8 ± 1891.7
1.00	118093.6 ± 1357.2	175553.3 ± 1921.6	282653.2 ± 2651.1

Table 4: Analysis of different values for λ in SW UB for the lot-sizing problem

Instance	T	LB PI	LB SW	LB NA	UB Cond Exp	UB SW	UB NA
$\rho = 0.2, \rho^Y = 0.6, J = 3$	2	46584.6	33327.5	47189.4	49406.4	49406.4	49001.1
	3	45487.7	29308.6	46735.3	51076.7	50660.4	50953.9
	4	48868.7	29962.1	50537.6	56731.2	54878.6	53614.3
	5	55920.5	31638.4	56965.3	64759.2	64062.4	61419.5
	6	64844.7	38036.7	67777.5	75596.2	75515.2	75384.4
	7	72980.9	42580.4	75263.3	84230.4	83838.4	>6h
	8	104296.0	67443.0	112093.0	129166.0	127026.0	>6h
	9	102604.0	60101.1	104760.0	121131.0	117865.0	>6h
	10	114903.0	70844.6	122374.0	135756.0	134728.0	>6h
$\rho = 0.6, \rho^Y = 0.2, J = 3$	2	46587.0	35050.1	46876.1	48484.6	48484.6	48277.4
	3	45547.9	29282.9	46585.9	50607.9	50109.8	49404.2
	4	49070.8	32474.3	53328.5	58292.3	56624.5	54892.8
	5	55927.6	35362.3	59325.2	67492.5	66928.6	63767.0
	6	65167.3	41291.3	71352.1	81073.9	80815.3	82093.8
	7	73532.0	46230.9	80838.4	91635.6	90969.8	>6h
	8	110344.0	69849.2	122755.0	145747.0	143503.0	>6h
	9	105611.0	63981.7	115548.0	135670.0	130932.0	>6h
	10	120331.0	67950.0	134594.0	155783.0	154407.0	>6h
$\rho = 0.6, \rho^Y = 0.2, J = 6$	2	97422.8	72505.3	98890.1	103233.0	103233.0	102963.5
	3	90470.1	56340.1	92717.4	104545.0	104218.0	103844.1
	4	94000.1	53199.8	100338	113226.0	111675.8	111718.4
	5	107992.0	62849.1	114973.0	134017.0	132604.4	132614.5
	6	121141.0	67740.8	131226.0	153875.0	153151.2	>6h
	7	118421.0	65444.7	130542.0	175972.0	174715.2	>6h
	8	139644.0	106919.0	157804.0	240137.0	238465.8	>6h
	9	127301.0	107217.7	142417.0	235657.0	233470.2	>6h
	10	135019.0	118333.6	156096.0	273716.0	270377.2	>6h

Table 5: Bound comparison for various instances

Instance	T	SW Time (s) LB	SW Time (s) UB (avg./sample path)	NA Time (s) LB	NA Time (s) UB (avg./sample path)
$\rho = 0.2, \rho^Y = 0.6, J = 3$	2	3.4	<0.1s	3.9	<0.1s
	3	18.3	<0.1s	24.4	0.2
	4	161.4	<0.1s	59.7	0.8
	5	338.3	<0.1s	115.9	1.7
	6	750.7	0.1	339.0	3.7
	7	960.0	0.2	437.4	7.7
	8	2184.1	0.3	605.3	9.5
	9	2460.0	0.4	714.0	12.4
	10	2711.4	0.5	1975.4	17.6
$\rho = 0.6, \rho^Y = 0.2, J = 3$	2	3.5	<0.1s	3.9	<0.1s
	3	17.6	<0.1s	25.2	0.2
	4	67.6	<0.1s	58.2	0.7
	5	115.1	<0.1s	72.5	1.8
	6	675.2	0.1	374.6	4.0
	7	883.8	0.1	413.5	7.4
	8	1499.0	0.3	645.0	9.6
	9	1725.6	0.5	696.5	12.2
	10	1919.7	0.4	1741.7	16.5
$\rho = 0.6, \rho^Y = 0.2, J = 6$	2	20.2	<0.1s	19.0	0.1
	3	288.6	<0.1s	119.8	0.6
	4	512.7	<0.1s	679.9	2.0
	5	4747.9	<0.1s	1334.4	5.0
	6	6154.1	0.1	2290.8	8.2
	7	10963.4	0.2	3749.9	14.2
	8	12859.1	0.3	7805.3	25.6
	9	13946.1	0.5	8912.0	29.9
	10	15524.3	0.5	11228.1	42.0

Table 6: Solution time of bounding techniques in seconds for the MSLot problem. For UB techniques, the average time of one sample path is given. For NA upper bounds with $T > 6$, the reported value is the average time over an evaluation sample of size 10.

Appendix D: Solution Times for MSRWA Instances

		Basis Function Option 1 Time (s)		Basis Function Option 2 Time (s)	
$(\lvert\Omega\rvert, \bar{R}, \gamma_0, \delta_1)$	T	NA UB	SW LB (avg./sample path)	NA UB	SW LB (avg./sample path)
$(1, 10, 3, 0.2)$	3	122.2	0.2	76.0	0.2
	4	417.0	0.4	200.2	0.4
	5	1852.0	0.8	1193.2	0.9
$(1, 10, 4.5, 0.05)$	3	113.5	0.2	93.1	0.2
	4	420.4	0.4	277.6	0.5
	5	1325.7	0.8	951.0	0.9
$(3, 30, 9, 0.2)$	3	772.2	0.2	302.3	0.3
	4	1533.4	0.5	787.0	0.5
	5	5240.1	0.9	3339.9	0.9
$(3, 30, 13.5, 0.05)$	3	945.4	0.2	494.6	0.2
	4	2800.8	0.5	1045.2	0.5
	5	7117.1	0.9	3061.1	0.9

Table 7: Solution time of bounding techniques in seconds for the MSRWA problem.

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