EC.1. Parameter Estimation

We propose some heuristic methods for estimating the unknown parameters. To begin with, if the transformed input data $\{D_t\}$ and the simulation outputs $\{X_i\}$ are uniformly bounded, then ν_s and $\bar{\sigma}_{ij}$ can be computed using explicit bounds of their support. For example, if $X_i(\theta), X_j(\theta) \in [a,b]$ for all $\theta \in \Theta$, then $(b-a)$ would be a valid value of $\bar{\sigma}_{ij}$, but it tends to be conservative. An alternative is to approximate ν and $\bar{\sigma}_{ij}$ using the sample standard deviation of D_t and $X_i - X_j$. For $\bar{L}_{ij}$, assuming that $\delta_{ij}(\cdot)$ is sufficiently smooth, by the mean value theorem and Hölder's inequality,

$$|\delta_{ij}(\theta_1) - \delta_{ij}(\theta_2)| = \nabla\delta_{ij}(\tilde{\theta})^{\intercal}(\theta_1 - \theta_2) \le \sup_{\theta\in\Theta} \|\nabla\delta_{ij}(\theta)\|_\infty \|\theta_1 - \theta_2\|_1,$$

where ∇ denotes the gradient operator and $\tilde{\theta}$ is a convex combination of θ_1 and θ_2. The issue is that $\sup_{\theta\in\Theta}\|\nabla\delta_{ij}(\theta)\|_\infty$ is both conservative and difficult to estimate. Empirically, we find that approximating $\bar{L}_{ij}$ with $\|\nabla\delta_{ij}(\hat{\theta})\|_\infty$ usually preserves statistical validity for the procedure.In addition, $\nabla\mu_i(\theta)$ can be estimated via the likelihood ratio estimator

$$\widehat{\nabla\mu}_i(\theta) = \frac{1}{n_0}\sum_{r=1}^{n_0} X_{i,r}(\theta)\frac{\nabla_\theta f(\xi_{i,r}(\theta)\mid\theta)}{f(\xi_{i,r}(\theta)\mid\theta)}, \quad j = 1,\dots,d_s, \tag{EC.1}$$

where n_0 is an initial sample size, $f(\cdot|\theta)$ is the probability density (or the probability mass function) of X under parameter θ, and $\xi(\theta)$ denotes simulation sample generated from P_θ. Then $\widehat{\nabla\delta}_{ij}(\theta) = \widehat{\nabla\mu}_i(\theta) - \widehat{\nabla\mu}_j(\theta)$Other methods for estimating the gradient can be found in Fu et al. (2015).

The impact of estimation errors on the procedures is discussed as follows. To begin with, underestimating ν_s, σ_{ij} or $\bar{L}_{ij}$ can cause premature stopping and undershooting the target PCS since the resulting confidence bands may fail to achieve desired coverage. However, as we shall see in Section 7, this risk is mitigated by the conservatism of Bonferroni inequality. On the other hand, overestimating the parameters can lead to longer runtime and overshooting the target PCS, and thus we do not recommend estimators that are typically positively biased (e.g., $\sup_{\theta\in\Theta}\|\nabla\hat{\delta}_{ij}(\theta)\|_\infty$).

EC.2. Boosting Through η: More Analysis about Impact of Drop Rate

To minimize the impact caused by IU and SU, the analysis becomes difficult since we want to minimize a matrix-valued function in some sense. However, since our goal is to reduce the running time, which is often dominated by the difference of expected performance between the best and the second best design, we can then minimize the variance $\sigma^2_{bb'}$ to boost the procedure. Here b and b' denote the true best and the second best design, respectively. For this purpose, let $\sigma^2_{ij,\infty}$ denote the asymptotic variance of $\hat{\delta}_{ij,t}$. Then by applying Theorem 3 with only two designs i and j, we have

$$\sigma^2_{ij,\infty} := \lambda_{I,\eta} \nabla\delta_{ij}(\theta^c)^\intercal \bar{\Sigma}_D \nabla\delta_{ij}(\theta^c) + \lambda_{S,\eta}\bar{m}^{-1}\sigma^2_{ij}(\theta^c).$$

Then, for any η between $(0,1)$, $\sigma^2_{ij,\infty}$, the limiting variance for design i and j, satisfies

$$\sigma^2_{ij,\infty} \propto \left\{\frac{\kappa}{1-\eta} + \frac{2\eta\ln\eta}{(1-\eta)^2}\right\},$$

where $\propto$ denotes "is proportional to", and

$$\kappa := \frac{2\sigma^2_{ij,IU} + \sigma^2_{ij,SU}}{2\sigma^2_{ij,IU}},$$

in which $\sigma^2_{ij,IU} := \nabla\delta_{ij}(\theta^c)^\intercal \bar{\Sigma}_D \nabla\delta_{ij}(\theta^c)$ and $\sigma^2_{ij,SU} := \bar{m}^{-1}\sigma^2_{ij}(\theta^c)$. Notice that for two specific designs i and j, the larger $\sigma^2_{ij,\infty}$, the harder for us to distinguish between them since we will get a larger confidence band $w_{ij,t}$. Hence, if we want elimination between i and j happens earlier, we should minimize $\sigma^2_{ij,\infty}$. Minimizing $\sigma^2_{ij,\infty}$ is equivalent to solving the following optimization problem,

$$\inf_{\eta\in(0,1)} \psi_\kappa(\eta) := \frac{\kappa}{1-\eta} + \frac{\eta\ln\eta}{(1-\eta)^2}, \quad \kappa \geq 1, \tag{EC.2}$$

which does not have a closed-form solution. However, the following result characterizes some important properties of the optimal solution.

Proposition EC.1. *For any fixed $\kappa > 1$, ψ_κ is strictly convex in $(0,1)$ and always has a unique minimizer in $(0,1)$, denoted by $\eta^*(\kappa)$, which satisfies*

$$\eta^*(\kappa) \leq \kappa^{-1}, \quad \lim_{\kappa\to 1}\eta^*(\kappa) = 1. \tag{EC.3}$$

Furthermore,

$$\inf_{\kappa>1} \frac{\psi_\kappa(\eta^*(\kappa))}{\psi_\kappa(0)} = \frac{1}{2}. \tag{EC.4}$$

Proposition EC.1 guarantees the existence and uniqueness of a minimizer, and the convexity of ψ_κ makes it numerically easy to solve (EC.2). Moreover, connecting back to the definition of κ yields the following interpretation.

(i) If $\sigma^2_{ij,SU} \gg \sigma^2_{ij,IU}$, i.e., SU dominates IU, then κ is large and the optimal η is near 0. In other words, we should discard only a tiny portion of "stale" simulation outputs because averaging more outputs is the most effective way to reduce SU.

(ii) If $\sigma^2_{ij,SU} \ll \sigma^2_{ij,IU}$, i.e., IU overshadows SU, then $\kappa \approx 1$ and the optimal η would be near 1, meaning that it is more advantageous to discard most of the previous simulation output. Indeed, in the extreme case where there is no SU (i.e., the simulation output is $\mu_i(\hat{\theta}_t)$), there would be no need to keep anything but the latest output.

The lower bound (EC.4) indicates that compared with blindly setting $\eta = 0$, choosing the optimal η can reduce the asymptotic variance of $\hat{\delta}_{ij,t}$ by at most a factor of 2. Again, this happens when there is no SU, i.e., $\kappa = 1$, since we have seen in Section 5.2 that retaining only the latest output halves the asymptotic variance resulted from averaging all outputs.

EC.3. Supplement Material for Numerical Experiment

EC.3.1. Quadratic problem

The first problem is to minimize the expected value of a quadratic function $X_i = (i-\xi)^2$, where the true distribution of ξ is Poisson with mean θ^c. Let $\mathcal{I}$ be all the designs to be evaluated. Also let $\theta^c = 1 \in \mathcal{I}$ so that the best design is $b = \theta^c = 1$.

EC.3.2. Production-inventory problem

We also test the procedures on a production-inventory problem, where the objective function does not have a closed form but needs to be evaluated via simulation.

Suppose that we are running a capacitated production system and we want to minimize the expected total cost over a finite number of stages. The decision variable is the order-up-to level, i.e., the quantity we should fill up to once the inventory falls below that level. Please note this is an offline planning problem, where we select the best inventory policy after finishing the simulation. Meanwhile, the production amount in each stage is capped. At the beginning of a stage, the amount produced in the previous stage arrives. Then, the demand is revealed and we fulfill the total demand (both backlog and current demand) to the best allowed by on-hand inventory, after which unfulfilled demand becomes the new backlog. Decision of the production amount is carried out at the end of the stage.

The variables are listed as follows: i is the order-up-to level, I_s is the inventory level at the end of the sth stage, ξ_s is the demand at the sth stage, and R_s is the production amount at the sth stage. Let $I_0 = i$ and $R_0 = 0$. Starting from $s = 1$, the system dynamics evolve according to the following equations,

$$\begin{aligned} I_s &= I_{s-1} + R_{s-1} - \xi_s, \\ R_s &= \min\{R^*, (i - I_s)^+\}, \end{aligned}$$

where $a^+ := \max\{0, a\}$ and R^* is the maximum production amount. Assume that the demand quantities are independent random variables, where each ξ_s follows an exponential distribution with mean θ_s^c. Let c_H be the holding cost per unit and c_B be the backlog cost per unit. Then, the cost at the sth stage is

$$c_s := c_H(R_{s-1} + I_s^+) + c_B I_s^-,$$

where $a^- := -\min\{a, 0\}$. The expected total cost over S stages is

$$\mu_i(\theta^c) = \mathbb{E}\left(\sum_{s=1}^{S} c_s\right).$$

The goal is to select the optimal order-up-to quantity among candidate set $\mathcal{I}$.

EC.3.3. Experiment result for optimizing η

For generality, we use random batch sizes in both the quadratic example in Section 7.2.1 and inventory-production example in Secton 7.2.2. We optimize η to minimize the termination stage.

Table EC.1 Performance of SEIU-MCB: $\eta = 0.1$ vs. optimized η.

	Quadratic example		Production-inventory example	
$\bar{k}$	$\eta = 0.1$	Optimized η	$\eta = 0.1$	Optimized η
10	94% ± 3%	91% ± 4%	93% ± 3%	89% ± 4%
	47 ± 6.4	35 ± 4.8	55 ± 4.7	37 ± 3.8
20	95% ± 4%	92% ± 6%	96% ± 3%	94% ± 3%
	25 ± 2.7	20 ± 3.7	28 ± 2.4	22 ± 2.1
30	98% ± 3%	96% ± 4%	98% ± 1%	98% ± 1%
	20 ± 2.6	16 ± 2.5	21 ± 1.5	17 ± 1.3

Table EC.1 summarizes the average termination stage and the corresponding empirical PCS along with their 95% confidence intervals, based on 100 independent runs of each procedure.

From Table EC.1, we can conclude that the drop rate η balances the trade off between PCS and run time. In almost all settings, the optimal drop rate results in less run time of the procedure while achieving a relatively lower PCS. This is consistent with our intuition that less total amount of data (corresponding to earlier termination stage) increases input uncertainty and hence results in a lower PCS. It is worth noting that although the optimal drop rate results in a lower empirical PCS, it still achieves the target PCS. From a practical viewpoint, a good selection of the drop rate will boost the efficiency by reducing the run time while still satisfying the probabilistic guarantee.

Technical Proofs

EC.4. Proofs for SEIU

Proof of Proposition 1 Suppress i, j from the index for convenience. Letting

$$\bar{\delta}_t := \mathbb{E}[\hat{\delta}_t \mid \hat{\theta}_1, \ldots, \hat{\theta}_t] = [M(t) - M(t_\eta)]^{-1} \sum_{\ell = t_\eta + 1}^{t} m(\ell)\delta(\hat{\theta}_\ell),$$

$$Y_r = Y_{ij,r} := X_{i,r} - X_{j,r},$$

we have

$$\mathbb{P}\left\{|\hat{\delta}_t - \delta(\theta^c)| > x + y\right\} \leq \mathbb{P}\left\{|\hat{\delta}_t - \bar{\delta}_t| > x\right\} + \mathbb{P}\left\{|\bar{\delta}_t - \delta(\theta^c)| > y\right\} \tag{EC.5}$$

For the first term on the RHS of (EC.5),

$$\begin{aligned}\mathbb{P}\left\{|\hat{\delta}_t - \bar{\delta}_t| > x\right\} &= \mathbb{E}\left[\mathbb{P}\left\{\left|[M(t) - M(t_\eta)]^{-1}\sum_{\ell=t_\eta+1}^{t}\sum_{r=1}^{m(\ell)}[Y_r(\hat{\theta}_\ell) - \delta(\hat{\theta}_\ell)]\right| > x \,\middle|\, \hat{\theta}_1, \ldots, \hat{\theta}_t\right\}\right] \\ &\leq 2\exp\left\{-\frac{[M(t) - M(t_\eta)]x^2}{2\bar{\sigma}^2}\right\},\end{aligned}$$

where the inequality holds since conditioned on $\hat{\theta}_1, \ldots, \hat{\theta}_t$, $\{Y_r(\hat{\theta}_\ell) - \delta(\hat{\theta}_\ell)\}_r$ are independent zero-mean $subG(\bar{\sigma}^2)$ random variables. For the second term on the RHS of (EC.5),

$$\begin{aligned}\mathbb{P}\left\{|\bar{\delta}_t - \delta(\theta^c)| > y\right\} &\leq \mathbb{P}\left\{\left|[M(t) - M(t_\eta)]^{-1}\sum_{\ell=t_\eta+1}^{t} m(\ell)[\delta(\hat{\theta}_\ell) - \delta(\theta^c)]\right| > y\right\} \\ &\leq \mathbb{P}\left\{\sum_{\ell=t_\eta+1}^{t} m(\ell)|\delta(\hat{\theta}_\ell) - \delta(\theta^c)| > [M(t) - M(t_\eta)]y\right\} \\ &\leq \mathbb{P}\left\{\sum_{\ell=t_\eta+1}^{t} m(\ell)\|\hat{\theta}_\ell - \theta^c\|_1 > [M(t) - M(t_\eta)]\frac{y}{\bar{L}}\right\} \\ &= \mathbb{P}\left\{\sum_{\ell=t_\eta+1}^{t} m(\ell)\sum_{s=1}^{S}\sum_{j=1}^{d_s}|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > [M(t) - M(t_\eta)]\frac{y}{\bar{L}}\right\} \\ &\leq \sum_{s=1}^{S}\sum_{j=1}^{d_s}\mathbb{P}\left\{\sum_{\ell=t_\eta+1}^{t} m(\ell)|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > [M(t) - M(t_\eta)]\frac{y}{d\bar{L}}\right\} \\ &:= \sum_{s=1}^{S}\sum_{j=1}^{d_s} p_{s,j},\end{aligned}$$

where $\hat{\theta}_{s,j,\ell}$ denotes the jth coordinate of $\hat{\theta}_{s,\ell}$. Note that for any nonnegative sequence $\{\omega_{s,\ell}\}_{\ell=t_\eta+1}^{t}$ that satisfies $\sum_{\ell=t_\eta+1}^{t}\omega_{s,\ell} = 1$, we have

$$\begin{aligned}p_{s,j} &\leq \mathbb{P}\left\{\bigcup_{\ell=t_\eta+1}^{t}\left\{m(\ell)|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > \omega_{s,\ell}[M(t) - M(t_\eta)]\frac{y}{d\bar{L}}\right\}\right\} \\ &\leq \sum_{\ell=t_\eta+1}^{t}\mathbb{P}\left\{m(\ell)|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > \omega_{s,\ell}[M(t) - M(t_\eta)]\frac{y}{d\bar{L}}\right\} \qquad \text{(EC.6)} \\ &\leq \sum_{\ell=t_\eta+1}^{t} 2\exp\left\{-\frac{N_s(\ell)\omega_\ell^2[M(t) - M(t_\eta)]^2 y^2}{2d^2\bar{L}^2\nu_s^2[m(\ell)]^2}\right\},\end{aligned}$$

where the second inequality follows from the fact that $\hat{\theta}_{s,j,\ell} \sim subG\left(\nu_s^2/N_s(\ell)\right)$. Take

$$\omega_{s,\ell} = \frac{m(\ell)/\sqrt{N_s(\ell)}}{\gamma_{s,\eta}(t)}$$

and we further have

$$p_{s,j} \leq 2(t-t_\eta)\exp\left\{-\frac{[M(t)-M(t_\eta)]^2 y^2}{2d^2\bar{L}^2\nu_s^2\gamma_{s,\eta}^2(t)}\right\}. \tag{EC.7}$$

Plugging (EC.7) back into (EC.5) completes the proof. ■

Proof of Lemma 1 Similar to the proof of Proposition 1, let

$$\bar{\delta}_{i,t} := \mathbb{E}[\hat{\delta}_{i,t} \mid \hat{\theta}_1, \ldots, \hat{\theta}_t] = [M_i(t) - M_i(t_\eta)]^{-1} \sum_{\ell=t_\eta+1}^{t} m_i(\ell)\delta_i(\hat{\theta}_\ell)$$

and we have

$$\mathbb{P}\left\{\bigcup_{i<j}\left\{|\hat{\delta}_{ij,t} - \delta_{ij}(\theta^c)| > x_{ij} + y_{ij}\right\}\right\} \leq \sum_{i<j}\mathbb{P}\left\{|\hat{\delta}_{i,t} - \bar{\delta}_{i,t}| > x_{ij}\right\} + \mathbb{P}\left\{\cup_{i<j}\mathcal{E}_{ij,t}\right\},$$

where $\mathcal{E}_{ij,t} := \left\{|\bar{\delta}_{ij,t} - \delta_{ij}(\theta^c)| > y_{ij}\right\}$. By the same conditioning argument in the proof of Proposition 1 and a sub-Gaussian tail bound, we have

$$\sum_{i<j}\mathbb{P}\left\{|\hat{\delta}_{ij,t} - \bar{\delta}_{ij,t}| > x_{ij}\right\} \leq \sum_{i<j} 2\exp\left\{-\frac{[M(t)-M(t_\eta)]x_{ij}^2}{2\bar{\sigma_{ij}}^2}\right\}, \tag{EC.8}$$

which gives the first part of the desired bound. For $\mathcal{E}_{ij,t}$, in light of the Lipschitz continuity of δ_{ij},

$$\begin{aligned}
\mathcal{E}_{ij,t} &\subseteq \left\{\sum_{\ell=t_\eta+1}^{t} m(\ell)\|\hat{\theta}_\ell - \theta^c\|_1 > [M(t)-M(t_\eta)]\frac{y_{ij}}{\bar{L}_{ij}}\right\} \\
&= \left\{\sum_{j=1}^{d}\sum_{\ell=t_\eta+1}^{t} m(\ell)|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > [M(t)-M(t_\eta)]\frac{y_{ij}}{\bar{L}_{ij}}\right\} \\
&\subseteq \bigcup_{s=1}^{S}\bigcup_{j=1}^{d_s}\left\{\sum_{\ell=t_\eta+1}^{t} m(\ell)|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > [M(t)-M(t_\eta)]\frac{y_{ij}}{d\bar{L}_{ij}}\right\} \\
&\subseteq \bigcup_{s=1}^{S}\bigcup_{j=1}^{d_s}\bigcup_{\ell=t_\eta+1}^{t}\left\{|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > \frac{\omega_{s,\ell}[M(t)-M(t_\eta)]y_{ij}}{m(\ell)d\bar{L}_{ij}}\right\},
\end{aligned}$$

where $\omega_{s,\ell} = m(\ell)/\left[\sqrt{N_s(\ell)}\gamma_{s,\eta}(t)\right]$ satisfies $\sum_{\ell=t_\eta+1}^{t}\omega_{s,\ell} = 1$. Plugging in the value of $\omega_{s,\ell}$ and we further have

$$\cup_{i<j}\mathcal{E}_{ij,t} \subseteq \bigcup_{s=1}^{S}\bigcup_{j=1}^{d_s}\bigcup_{\ell=t_\eta+1}^{t}\left\{|\hat{\theta}_{s,j,\ell} - \theta^c_{s,j}| > \min_{i<j}\left\{\frac{[M(t)-M(t_\eta)]y_{ij}}{d\bar{L}_{ij}\sqrt{N_s(\ell)}\gamma_\eta(t)}\right\}\right\},$$

and it follows from the fact $\hat{\theta}_{s,j,\ell} \sim subG(\bar{\nu}^2/N_s(\ell))$ that

$$\mathbb{P}(\cup_{i<j}\mathcal{E}_{ij,t}) \leq 2(t-t_\eta)\sum_{s=1}^{S} d_s \max_{i<j}\left\{\exp\left\{-\frac{[M(t)-M(t_\eta)]^2 y_{ij}^2}{2\nu_s^2 d^2 \bar{L}_{ij}^2 \gamma_{s,\eta}^2(t)}\right\}\right\}. \tag{EC.9}$$

Finally, combining (EC.8) and (EC.9) yields the result. ■

Proof of Theorem 1 According to (5), it suffices to show that

$$\mathbb{P}\left\{\bigcup_{i<j}\left\{|\hat{\delta}_{ij,t} - \delta_{ij}(\theta^c)| > w_{ij,t}\right\}\right\} \leq \frac{6\alpha}{\pi^2 t^2}, \tag{EC.10}$$

which would imply that the confidence bands $\{w_{ij,t}\}$ jointly achieve at least $1-\alpha$ coverage probability. Apply Lemma 1 to obtain

$$\begin{aligned}
&\mathbb{P}\left\{\bigcup_{i<j}\left\{|\hat{\delta}_{ij,t} - \delta_{ij}(\theta^c)| > w_{ij,t}\right\}\right\} = \mathbb{P}\left\{\bigcup_{i<j}\left\{|\hat{\delta}_{ij,t} - \mu_{ij}(\theta^c)| > u_{ij,t} + v_{ij,t}\right\}\right\} \\
&\leq \sum_{i<j} 2\exp\left\{-\frac{[M(t)-M(t_\eta)]u_{ij,t}^2}{2\bar{\sigma_{ij}}^2}\right\} && (\dagger) \\
&\quad + 2(t-t_\eta)d\max_{i<j}\left\{\exp\left\{-\frac{[M(t)-M(t_\eta)]^2 v_{ij,t}^2}{2\nu_s^2 d^2 \bar{L}_{ij}^2 \gamma_{s,\eta}^2(t)}\right\}\right\} && (\dagger\dagger)
\end{aligned}$$

We check (†) and (††) as follows.

$$\begin{aligned}
(\dagger) &= \sum_{i<j} 2\exp\left\{-\frac{M(t)-M(t_\eta)}{2\bar{\sigma}_{ij}^2}\cdot\frac{4\bar{\sigma}_{ij}^2\ln\left(\sqrt{\frac{K(K-1)\pi^2}{3\alpha}}t\right)}{M(t)-M(t_\eta)}\right\} \\
&= \sum_{i<j} 2\exp\left\{-\ln\left(\frac{K(K-1)\pi^2 t^2}{3\alpha}\right)\right\} \\
&= K(K-1)\cdot\frac{3\alpha}{K(K-1)\pi^2 t^2} = \frac{3\alpha}{\pi^2 t^2}.
\end{aligned}$$

For (††), first note that

$$\exp\left\{-\frac{[M(t)-M(t_\eta)]^2 v_{ij,t}^2}{2\nu_s^2 d^2 \bar{L}_{ij}^2 \gamma_{s,\eta}^2(t)}\right\} \leq \exp\left\{-\ln\left(\frac{2Sd_s\pi^2(1-\eta)t^3}{3\alpha}\right)\right\} = \frac{3\alpha}{2Sd_s\pi^2(1-\eta)t^3},$$

and it follows that

$$\begin{aligned}
(\dagger\dagger) &= 2(t-t_\eta)\sum_{s=1}^{S} d_s\frac{3\alpha}{2Sd_s\pi^2(1-\eta)t^3} \\
&= \frac{3\alpha}{\pi^2 t^2}.
\end{aligned}$$

Combining (†) and (††) gives (EC.10). It remains to show the upper bound on $\mathbb{E}[\tilde{\tau}]$. We claim that if all $\delta_{ij}(\theta^c)$ fall within their corresponding confidence bands $[\hat{\delta}_{ij,t} - w_{ij,t}, \hat{\delta}_{ij,t} + w_{ij,t}]$, then any suboptimal design $i \neq b$ must be eliminated prior to stage $\min_{j\neq i}\tilde{\tau}_{ij}$. To see this, note that $\tilde{\tau}_{ij}$ is how long design i can last before getting eliminated by design j. Thus, $\min_{j\neq i}\tilde{\tau}_{ij}$ is the longest survival time of design i. Furthermore, the procedure must terminate before stage $\tilde{\tau}_* := \max_{i\neq b}\min_{j\neq i}\tilde{\tau}_{ij}$. In other words, if the procedure has not terminated by some stage $t > \tilde{\tau}_*$, then at least one design's confidence band has failed to cover its true mean at stage t, i.e.,

$$\{\tilde{\tau} > t\} \subseteq \bigcup_{i<j}\left\{|\hat{\delta}_{i,t} - \delta_{ij}(\theta^c)| > w_{ij,t}\right\}, \quad \forall t > \tilde{\tau}_*.$$

Let $\mathcal{E}_t := \bigcup\limits_{i<j}\left\{|\hat{\delta}_{ij,t} - \delta_{ij}(\theta^c)| > w_{ij,t}\right\}$,

$$\begin{aligned}\mathbb{E}[\tilde{\tau}] &= \sum_{t=0}^{\infty}\mathbb{P}(\tilde{\tau} > t) = \sum_{t=0}^{\tilde{\tau}_*-1}\mathbb{P}(\tilde{\tau} > t) + \sum_{t=\tilde{\tau}_*}^{\infty}\mathbb{P}(\tilde{\tau} > t)\\ &\leq \tilde{\tau}_* + \sum_{t=\tilde{\tau}_*}^{\infty}\mathbb{P}(\mathcal{E}_t) \leq \tilde{\tau}_* + \sum_{t=1}^{\infty}\mathbb{P}(\mathcal{E}_t) \leq \tilde{\tau}_* + \alpha,\end{aligned}$$

where the last inequality follows from (EC.10). ■

Proof of Corollary 1 According to our SE framework in Section 3, we know that with probability as least $1-\alpha$, the confidence bands achieve perfect coverage. Hence at any stage t, the best design b remains in the remaining set $\mathcal{S}$.

Now, with probability at least $1-\alpha$, we have

$$\mu_{i^*}(\theta^c) - \mu_b(\theta^c) \geq \hat{\delta}_{i^*b,t} - w_{i^*b,t} \geq -w_{i^*b,t} \geq -\max_{j\in\mathcal{S},j\neq i*} w_{i*j,t}.$$

Hence the current design with the largest sample mean is ϵ_t optimal, where $\epsilon_t = \max_{j\in\mathcal{S},j\neq i*} w_{i*j,t}$.

Proof of Corollary 2 The validity of the SEIU(IZ) is the result of Theorem 1 and Corollary 1. For the termination time, notice that for $t > \tau$ and any $j \in \mathcal{S}, j \neq i^*$

$$w_{i*j,t} \leq \max_{i\neq j} w_{ij,t} < \epsilon$$

Hence, $\max\limits_{j\in\mathcal{S},j\neq i^*} w_{i^*j,t} < \epsilon$. The procedure must terminate before τ.

EC.5. Proofs for SEIU-MCB

We establish a series of Lemmas for proving Theorem 3.

LEMMA EC.1 **(The Lindeberg-Feller Theorem (Proposition 2.2.7 in Van der Vaart (2000))**. *For each n let $Y_{n,1},\ldots, Y_{n,k_n}$ be independent random vectors with finite variances such that*

(i) $\sum_{i=1}^{k_n} \operatorname{Cov} Y_{n,i} \to \Sigma$,

(ii) $\sum_{i=1}^{k_n} \mathrm{E}\left\|Y_{n,i}\right\|^2 1\left\{\left\|Y_{n,i}\right\|>\varepsilon\right\} \to 0, \quad$ *every* $\varepsilon>0$.

Then the sequence $\sum_{i=1}^{k_n}\left(Y_{n,i}-\mathrm{E} Y_{n,i}\right)$ *converges in distribution to a normal* $N_s(0, \Sigma)$ *distribution.*

LEMMA EC.2. *Let* $\{X_n\}$ *be independent random vectors with* $\mathbb{E}X_n = 0$. *If* $X_n \Rightarrow X$ *and* $\operatorname{Cov} X_n \to \operatorname{Cov} X$ *as* $n\to\infty$, *then for any* $\eta \in [0,1)$,

$$(n-n_\eta)^{-1/2}\sum_{i=n_\eta+1}^{n} X_i \Rightarrow \mathcal{N}(0, \operatorname{Cov} X) \quad \text{as } n\to\infty.$$

Proof of Lemma EC.2. Let $Y_{n,i} := 0$ if $i \le n_\eta$ and $X_i/\sqrt{n-n_\eta}$ otherwise. We apply Lemma EC.1 to $Y_{n,i}$. Condition (i) is satisfied since

$$\sum_{i=1}^{n} \operatorname{Cov} Y_{n,i} = \sum_{i=n_\eta+1}^{n} \operatorname{Cov} X_i/(n-n_\eta) \to \operatorname{Cov} X.$$

In addition, by a generalized Dominated Convergence Theorem (note that the integrand is dominated by $||X_n||^2$ and $\operatorname{Cov} X_n \to \operatorname{Cov} X$),

$$\mathbb{E}\left[||X_n||^2 \mathbb{1}_{\{||X_n||>\epsilon\sqrt{n-n_\eta}\}}\right] \to 0, \quad \text{as } n\to\infty$$

Thus, as $n\to\infty$,

$$\sum_{i=1}^{n} \mathbb{E}[||Y_{n,i}||^2 \mathbb{1}_{\{||Y_{n,i}||>\epsilon\}}] = \frac{1}{n-n_\eta}\sum_{i=n_\eta+1}^{n} \mathbb{E}[||X_i||^2 \mathbb{1}_{\{||X_i||>\epsilon\sqrt{n-n_\eta}\}}] \to 0,$$

which verifies condition (ii), and the result follows. ■

The next few lemmas revolve around the convergence properties of

$$\phi_{s,t}(k) := \sum_{\ell=k}^{t} m(\ell)/N_s(\ell), \quad k \le t, \tag{EC.11}$$

which plays a crucial role in the proof of Theorem 3.

Lemma EC.3. *Under Assumption 4, $\phi_t(t_\eta) \to -\bar{m}\ln\eta/\bar{n}$ as $t \to \infty$.*

Lemma EC.3 We drop the index s for simplicity. Note that

$$\left|\phi_t(t_\eta) - \sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{\bar{n}\ell}\right| \le \left|\phi_t(t_\eta) - \sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{N(\ell)}\right| + \left|\sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{N(\ell)} - \sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{\bar{n}\ell}\right|.$$

For the first term in the RHS,

$$\begin{aligned}\left|\phi_t(t_\eta) - \sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{N_s(\ell)}\right| &= \left|\sum_{\ell=t_\eta}^{t} \frac{m(\ell)-\bar{m}}{N(\ell)}\right| \le \frac{1}{N(t_\eta)}\left|\sum_{\ell=t_\eta}^{t} m(\ell) - (t-t_\eta+1)\bar{m}\right| \\ &= \frac{t-t_\eta+1}{N(t_\eta)}\frac{1}{t-t_\eta+1}\left|\sum_{\ell=t_\eta}^{t} m(\ell) - (t-t_\eta+1)\bar{m}\right| \to 0, \quad \text{as } t\to\infty.\end{aligned}$$

For the second term,

$$\left|\sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{N(\ell)} - \sum_{\ell=t_\eta}^{t} \frac{\bar{m}}{\bar{n}\ell}\right| = \frac{\bar{m}}{\bar{n}}\left|\sum_{\ell=t_\eta}^{t} \frac{\bar{n}\ell - N(\ell)}{N(\ell)\ell}\right| \le \frac{\bar{m}}{\bar{n}}\frac{1}{N(t_\eta)}\left|\sum_{\ell=t_\eta}^{t}\left(\bar{n} - \frac{N(\ell)}{\ell}\right)\right| \to 0, \quad \text{as } t\to\infty.$$

Therefore, we have

$$\lim_{t\to\infty}\phi_t(t_\eta) = \frac{\bar{m}}{\bar{n}}\lim_{t\to\infty}\sum_{\ell=t_\eta}^{t}\frac{1}{\ell} = \frac{\bar{m}}{\bar{n}}\lim_{t\to\infty}\int_{t_\eta}^{t}\frac{1}{x}dx = -\frac{\bar{m}\ln\eta}{\bar{n}},$$

which shows the desired limit. ■

Lemma EC.4. *For $k \le t$,*

$$\sum_{\ell=1}^{k}\phi_t^2(\ell)n(\ell) = \sum_{\ell=1}^{k-1}\frac{m^2(\ell)}{N(\ell)} + 2\sum_{\ell=1}^{k-2}\frac{M(\ell)m(\ell+1)}{N(\ell+1)} + 2M(k-1)\phi_t(k) + N(k)\phi_t^2(k).$$

Proof of Lemma EC.4 By expanding $\phi_t(\ell)$ into $m(\ell)/N(\ell) + \phi_t(\ell+1)$ recursively, we have

$$\begin{aligned}\sum_{\ell=1}^{k}\phi_t^2(\ell)n(\ell) &= \phi_t^2(1)n(1) + \sum_{\ell=2}^{k}\phi_t^2(\ell)n(\ell) \\ &= n(1)\left[\frac{m(1)}{N(1)} + \phi_t(2)\right]^2 + \sum_{\ell=2}^{k}\phi_t^2(\ell)n(\ell) \\ &= \frac{m^2(1)}{N(1)} + 2m(1)\phi_t(2) + N(2)\phi_t^2(2) + \sum_{\ell=3}^{k}\phi_t^2(\ell)n(\ell) \\ &= \frac{m^2(1)}{N(1)} + 2m(1)\phi_t(2) + N(2)\left[\frac{m(2)}{N(2)} + \phi_t(3)\right]^2 + \sum_{\ell=3}^{k}\phi_t^2(\ell)n(\ell) \\ &= \cdots \\ &= \sum_{\ell=1}^{k-1}\frac{m^2(\ell)}{N(\ell)} + 2\sum_{\ell=1}^{k-1}m(\ell)\phi_t(\ell+1) + N(k)\phi_t^2(k). \qquad (*)\end{aligned}$$

Similarly, expanding the second term in (*) gives

$$\begin{aligned}\sum_{\ell=1}^{k-1} m(\ell)\phi_t(\ell+1) &= m(1)\left[\frac{m(2)}{N(2)}+\phi_t(3)\right]+\sum_{\ell=2}^{k-1} m(\ell)\phi_t(\ell+1)\\ &= \frac{m(1)m(2)}{N(2)}+M(2)\phi_t(3)+\sum_{\ell=3}^{k-1} m(\ell)\phi_t(\ell+1)\\ &= \frac{M(1)m(2)}{N(2)}+\frac{M(2)m(3)}{N(3)}+M(3)\phi_t(4)+\sum_{\ell=4}^{k-1} m(\ell)\phi_t(\ell+1)\\ &= \cdots\\ &= \sum_{\ell=1}^{k-2}\frac{M(\ell)m(\ell+1)}{N(\ell+1)}+M(\ell-1)\phi_t(k). \end{aligned} \tag{**}$$

Plugging (**) into (*) yields the final result. ■

Lemma EC.5. *Let Assumption 4 hold. Then,*

$$[M(t)-M(t_\eta)]^{-1}\,[\phi_t^2(t_\eta+1)N_s(t_\eta)+\sum_{\ell=t_\eta+1}^{t}\phi_t^2(\ell)n_s(\ell)]\to\beta_{s,\eta},\quad \text{as } t\to\infty, \tag{EC.12}$$

where

$$\beta_{s,\eta}=\frac{\bar m}{\bar n_s}\left(2+\frac{2\eta\ln\eta}{1-\eta}\right). \tag{EC.13}$$

Proof of Lemma EC.5 Fix s so that it can be suppressed from the indices. Define

$$\tilde\beta_\eta(t):=[M(t)-M(t_\eta)]^{-1}\,[\phi_t^2(t_\eta+1)N(t_\eta)+\sum_{\ell=t_\eta+1}^{t}\phi_t^2(\ell)n(\ell)]. \tag{EC.14}$$

The goal is to show that $\tilde\beta_\eta(t)\to\beta_\eta$ as $t\to\infty$. In light of Lemma EC.4,

$$\begin{aligned}\sum_{\ell=t_\eta+1}^{t}\phi_t^2(\ell)n(\ell)=\sum_{\ell=t_\eta}^{t}\frac{m^2(\ell)}{N(\ell)}+2\sum_{\ell=t_\eta+1}^{t-2}\frac{M(\ell)m(\ell+1)}{N(\ell)}\\ +2M(t-1)\phi_t(t)-2M(t_\eta-1)\phi_t(t_\eta)\\ +N(t)\phi_t^2(t)-N(t_\eta)\phi_t^2(t_\eta)\end{aligned}$$

Plugging it into $\tilde\beta_\eta(t)$ and we have

$$\begin{aligned}\tilde\beta_\eta(t)=\frac{t-t_\eta}{M(t)-M(t_\eta)}\cdot\frac{1}{t-t_\eta}\cdot\Bigg\{\phi_t^2(t_\eta+1)N(t_\eta)+\sum_{\ell=t_\eta}^{t}\frac{m^2(\ell)}{N(\ell)}\\ +2\sum_{\ell=t_\eta+1}^{t-2}\frac{M(\ell)m(\ell+1)}{N(\ell)}+2M(t-1)\phi_t(t)-2M(t_\eta-1)\phi_t(t_\eta)\\ +N(t)\phi_t^2(t)-N(t_\eta)\phi_t^2(t_\eta)\ \Bigg\}\end{aligned}$$

We will compute the limit of each term. To begin with,

$$\lim_{t\to\infty}\frac{t-t_\eta}{M(t)-M(t_\eta)}=\frac{1}{\bar{m}}.$$

Next, since $\lim_{t\to\infty}\phi_t(t_\eta)$ exists by Lemma EC.3, the first and last terms in the curly bracket cancel out in the limit:

$$\lim_{t\to\infty}\frac{1}{t-t_\eta}\left[\phi_t^2(t_\eta+1)N(t_\eta)-N(t_\eta)\phi_t^2(t_\eta)\right]=0.$$

Meanwhile, since $m(t)$ is bounded and $N(t)\to\infty$ as $t\to\infty$,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t-t_\eta}\sum_{\ell=t_\eta}^{t}\frac{m^2(\ell)}{N(\ell)}=0,\\
&\lim_{t\to\infty}\frac{1}{t-t_\eta}2M(t-1)\phi_t(t)=\frac{2\bar{m}}{1-\eta}\lim_{t\to\infty}\frac{m(t)}{N(t)}=0\\
&\lim_{t\to\infty}\frac{1}{t-t_\eta}N(t)\phi_t^2(t)=\frac{\bar{n}}{1-\eta}\lim_{t\to\infty}\frac{m(t)}{N(t)}=0.
\end{aligned}$$

For the remaining terms, we further have

$$\lim_{t\to\infty}\frac{1}{t-t_\eta}2M(t_\eta-1)\phi_t(t_\eta)=\frac{\bar{m}^2}{\bar{n}}\frac{2\eta\ln\eta}{1-\eta}.$$

Also,

$$\lim_{t\to\infty}\frac{1}{t-t_\eta}\cdot 2\sum_{\ell=t_\eta+1}^{t-2}\frac{M(\ell)m(\ell+1)}{N(\ell)}=\frac{2\bar{m}^2}{\bar{n}},$$

which can be seen as follows. Since $M(t)/N(t)\to\bar{m}/\bar{n}$ as $t\to\infty$, for any $\epsilon>0$ there exists $T>0$ such that $|M(t)/N(t)-\bar{m}/\bar{n}|<\epsilon$ for all $t>T$. Hence, for all $t>T$,

$$\begin{aligned}
\left(\frac{\bar{m}}{\bar{n}}-\epsilon\right)\frac{1}{t-t_\eta}\sum_{\ell=t_\eta+1}^{t-2}m(\ell+1)&\le\frac{1}{t-t_\eta}\sum_{\ell=t_\eta+1}^{t-2}\frac{M(\ell)m(\ell+1)}{N(\ell)}\\
&\le\left(\frac{\bar{m}}{\bar{n}}+\epsilon\right)\frac{1}{t-t_\eta}\sum_{\ell=t_\eta+1}^{t-2}m(\ell+1),
\end{aligned}$$

where taking $t\to\infty$ and $\epsilon\to 0$ confirms the limit. Putting everything together,

$$\lim_{t\to\infty}\tilde{\beta}_\eta(t)=\frac{1}{\bar{m}}\left(\frac{2\bar{m}^2}{\bar{n}}+\frac{\bar{m}^2}{\bar{n}}\frac{2\eta\ln\eta}{1-\eta}\right)=\frac{\bar{m}}{\bar{n}}\left(2+\frac{2\eta\ln\eta}{1-\eta}\right)=\beta_\eta.$$

■

Proof of Theorem 3 It suffices to show

$$\sqrt{M(t)-M(t_\eta)}\Delta_{i,t} \Rightarrow \mathcal{N}(0,\tilde{\Sigma}_{i,\infty}), \qquad \text{as } t\to\infty,$$

where

$$\tilde{\Sigma}_{i,\infty}(j,j') := \nabla\delta_{ij}(\theta^c)^\intercal \tilde{\Sigma}_D \nabla\delta_{ij'}(\theta^c) + \mathbf{Cov}\left(X_i(\theta^c)-X_j(\theta^c), X_i(\theta^c)-X_{j'}(\theta^c)\right),$$

in which $\tilde{\Sigma}_D := diag(\beta_{i,\eta}\Sigma_{D,1},\ldots,\beta_{S,\eta}\Sigma_{D,S})$ and β is defined in (EC.13). Let

$$\Delta_i(\theta) = \left(\delta_{i1}(\theta),\cdots,\delta_{ii-1}(\theta),\delta_{ii+1}(\theta),\cdots,\delta_{iK}(\theta)\right),$$

$$\hat{\Delta}_{i,t} = \left(\hat{\delta}_{i1,t},\cdots,\hat{\delta}_{ii-1,t},\hat{\delta}_{ii+1,t},\cdots,\hat{\delta}_{iK,t}\right),$$

$$\bar{\Delta}_{i,t} = \left(\bar{\delta}_{i1,t},\cdots,\bar{\delta}_{ii-1,t},\bar{\delta}_{ii+1,t},\cdots,\bar{\delta}_{iK,t}\right),$$

where

$$\bar{\delta}_{ij,t} := [M(t)-M(t_\eta)]^{-1}\sum_{\ell=t_\eta+1}^{t} m(\ell)[\mu_i(\hat{\theta}_\ell)-\mu_j(\hat{\theta}_\ell)].$$

We have the decomposition,

$$\begin{aligned}&[M(t)-M(t_\eta)]^{1/2}[\hat{\Delta}_{i,t}-\Delta_i(\theta^c)]\\ =&\underbrace{[M(t)-M(t_\eta)]^{1/2}(\hat{\Delta}_{i,t}-\bar{\Delta}_{i,t})}_{Y_t}+\underbrace{[M(t)-M(t_\eta)]^{1/2}[\bar{\Delta}_{i,t}-\Delta_i(\theta^c)]}_{Z_t}\end{aligned}$$

It suffices to show the respective convergence of Z_t and $Y_t \mid \mathcal{F}_t$, where $F_t := \sigma(\hat{\theta}_1,\cdots,\hat{\theta}_t)$, the filtration generated by input parameter estimates. We will show convergence of Y_t+Z_t through its characteristic function

$$\Phi_t(x) = \mathbb{E}[e^{\mathbf{i}x^T(Y_t+Z_t)}] = \mathbb{E}\left[e^{\mathbf{i}x^T Z_t}\mathbb{E}[e^{\mathbf{i}x^T Y_t} \mid \mathcal{F}_t]\right].$$

If we can show the respective convergence of Z_t and $Y_t \mid \mathcal{F}_t$, then the result follows from the Dominated Convergence Theorem for complex-valued random variables.

(1) Convergence of $Y_t \mid \mathcal{F}_t$: Expand Y_t to get

$$\begin{aligned}
Y_t =& [M(t)-M(t_\eta)]^{1/2}(\hat{\Delta}_{i,t}-\bar{\Delta}_{i,t}) \\
=& [M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t}\sum_{r=1}^{m(t)}\begin{pmatrix} X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{1,r}(\hat{\theta}_\ell)-\mu_1(\hat{\theta}_\ell)) \\ \vdots \\ X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{i-1,r}(\hat{\theta}_\ell)-\mu_{i-1}(\hat{\theta}_\ell)) \\ X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{i+1,r}(\hat{\theta}_\ell)-\mu_{i+1}(\hat{\theta}_\ell)) \\ \vdots \\ X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{K,r}(\hat{\theta}_\ell)-\mu_K(\hat{\theta}_\ell)) \end{pmatrix}
\end{aligned}$$

where, conditioned on $\mathcal{F}_t$, $(X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell-(X_{j,r}(\hat{\theta}_\ell)-\mu_j(\hat{\theta}_\ell))_{j\neq i}$ are independent random vectors with mean 0 and covariance matrix $\Sigma_i(\hat{\theta}_\ell)$. Since $\Sigma_{X,i}(\hat{\theta}_\ell)\to\Sigma_{X,i}(\theta^c)$ almost surely by the continuity of $\Sigma(\cdot)$. we can apply Lemma EC.2 to get (conditioned on $\mathcal{F}_t$)

$$[M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t}\sum_{r=1}^{m(t)}\begin{pmatrix} X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{1,r}(\hat{\theta}_\ell)-\mu_1(\hat{\theta}_\ell)) \\ \vdots \\ X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{i-1,r}(\hat{\theta}_\ell)-\mu_{i-1}(\hat{\theta}_\ell)) \\ X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{i+1,r}(\hat{\theta}_\ell)-\mu_{i+1}(\hat{\theta}_\ell)) \\ \vdots \\ X_{i,r}(\hat{\theta}_\ell)-\mu_i(\hat{\theta}_\ell)-(X_{K,r}(\hat{\theta}_\ell)-\mu_K(\hat{\theta}_\ell)) \end{pmatrix}\Rightarrow\mathcal{N}(0,\Sigma_{X,i}(\theta^c)) \text{ as } t\to\infty$$

(2) Convergence of Z_t:

Since $\mu_j(\theta)$ is twice continuously differentiable for each j, so is $\Delta_i(\theta)$. By Taylor expansion,

$$\begin{aligned}
Z_t =& [M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t} m(\ell)[\Delta_i(\hat{\theta}_\ell)-\Delta_i(\theta^c)] \\
=& [M(t)-M(t_\eta)]^{-1/2}\nabla(\Delta_i(\theta^c))\sum_{\ell=t_\eta+1}^{t} m(\ell)(\hat{\theta}_\ell-\theta^c) \qquad (Z_{t,1}) \\
&+[M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t} m(\ell)^\intercal\mathcal{O}(||\hat{\theta}_\ell-\theta^c||_2)(\hat{\theta}_\ell-\theta^c). \qquad (Z_{t,2})
\end{aligned}$$

We show the convergence of $Z_{t,1}$ and $Z_{t,2}$, respectively.

(2.1) Convergence of $Z_{t,1}$

Our goal is to show that

$$[M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t} m(\ell)(\hat{\theta}_\ell-\theta^c)\Rightarrow\mathcal{N}(0,\tilde{\Sigma}),\quad \text{as } t\to\infty.$$

Since the input distributions are mutually independent, it suffices to show that for any s,

$$[M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t} m(\ell)(\hat{\theta}_{s,\ell}-\theta^c_s)\Rightarrow\mathcal{N}(0,\beta_{s,\eta}\Sigma_{D,s}),\quad \text{as } t\to\infty. \tag{EC.15}$$

Note that $\{\hat{\theta}_{s,\ell}\}_{\ell=1}^{t}$ are linear combinations of $\{D_{s,j}\}_{j=1}^{N_s(t)}$, so we can rearrange the terms to obtain

$$\sum_{\ell=t_\eta+1}^{t} m(\ell)(\hat{\theta}_{s,\ell}-\theta^c_s)=\phi_{i,s,t}(t_\eta+1)\sum_{j=1}^{N_s(t_\eta)} D_{s,j}+\sum_{\ell=t_\eta+1}^{t}\phi_{i,s,t}(\ell)\sum_{j=N_s(\ell-1)+1}^{N_s(\ell)} D_{s,j},$$

where ϕ is defined in (EC.11). Since $\{D_{s,j}\}$ are i.i.d. and EC.12 holds by assumption, (EC.15) can be verified by invoking Lemma EC.1 (details omitted).

Hence,

$$Z_{t,1}\Rightarrow\mathcal{N}\left(0,\nabla(\Delta_{i,t}(\theta^c))^\intercal\beta_{s,\eta}\Sigma_{D,s}\nabla(\Delta_{i,t}(\theta^c))\right)\ \text{as } t\to\infty.$$

(2.2) Convergence of $Z_{t,2}$

It is left to show that the residual term $Z_{t,2}$ vanishes in probability. Since a.s.,

$$\|\hat{\theta}_\ell-\theta^c\|_2^2=\sum_{s=1}^{S}\|\hat{\theta}_{s,\ell}-\theta^c_s\|_2^2\le M_2\sum_{s=1}^{S}\frac{\log\log N_s(\ell)}{N_s(\ell)}\le M_2 S\log(U\ell)/\ell,$$

for some $M_2>0, U\ge\sup_\ell\{\sup_s n_s(\ell), m(\ell)\}$ by the Law of the Iterated Logarithm. Then we have a.s. $\exists M_1>0$,

$$\begin{aligned}
||Z_{t,2}||_2&\le[M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t} m(\ell)M_1||\hat{\theta}_\ell-\theta^c||_2^2\\
&\le[M(t)-M(t_\eta)]^{-1/2}\sum_{\ell=t_\eta+1}^{t} m(\ell)M_1M_2S\log(U\ell)/\ell\\
&\le(t-t_\eta)^{-1/2}UM_1M_2S\log(Ut)\sum_{\ell=t_\eta+1}^{t}1/\ell\\
&\le(t-t_\eta)^{-1/2}UM_1M_2S\log(Ut)\log(t),
\end{aligned}$$

which converges to 0 as $t \to \infty$. Hence we get

$$Z_t \Rightarrow \mathcal{N}\left(0, \nabla(\Delta_{i,t}(\theta^c))^\intercal \beta_{s,\eta} \Sigma_{D,s} \nabla(\Delta_{i,t}(\theta^c))\right) \text{ as } t \to \infty.$$

Together with the convergence of $Y_t | \mathcal{F}_t$, we complete the proof.

Proof of Proposition EC.1 The proof is developed in three steps.

1. Strict convexity of ψ_κ. The first-order and second-order derivatives of ψ_κ are given by

$$\begin{aligned} \psi_\kappa'(\eta) &= \frac{(1+\kappa)(1-\eta) + (1+\eta)\ln\eta}{(1-\eta)^3}, \\ \psi_\kappa''(\eta) &= \frac{2(1+\kappa)(1-\eta) + (4+2\eta)\ln\eta - \eta + \eta^{-1}}{(1-\eta)^4}. \end{aligned} \tag{EC.16}$$

We claim that $\psi_\kappa''(\eta) > 0$ on $(0,1)$. To see this, denote the numerator of $\psi_\kappa''(\eta)$ as

$$g_\kappa(\eta) := 2(1+\kappa)(1-\eta) + (4+2\eta)\ln\eta - \eta + \eta^{-1}.$$

It can be verified that

$$g_\kappa'(\eta) = -(1+2\kappa) + 4\eta^{-1} - \eta^{-2} + 2\ln\eta, \quad g_\kappa''(\eta) = 2\eta^{-3}(\eta-1)^2.$$

We have $g_\kappa'(1) = 2(1-\kappa) < 0$ and $g_\kappa''(\eta) > 0$ for $\eta \in (0,1)$, which implies that $g_\kappa'(\eta) < 0$ for $\eta \in (0,1)$. Furthermore, since $g_\kappa(1) = 0$, we know that $g_\kappa(\eta) > 0$ for $\eta \in (0,1)$ and therefore ψ_κ is strictly convex in $(0,1)$. Furthermore, since $\lim_{\eta \to 0} \psi_\kappa'(\eta) = -\infty$ and $\lim_{\eta \to 1} \psi_\kappa'(\eta) = 1+\kappa$, ψ_κ must always have a unique minimizer in $(0,1)$, which we shall denote by $\eta^*(\kappa)$.

2. Limiting behavior of $\eta^*(\kappa)$ as κ approaches 0 or 1. To begin with, according to (EC.16), $\eta^*(\kappa)$ is the solution to the following equation.

$$\varphi_\kappa(\eta) := (1+\kappa)(1-\eta) + (\eta+1)\ln\eta = 0.$$

(i) $\eta^*(\kappa) \le \kappa^{-1}$.

Note that φ_κ is concave since $\varphi_\kappa''(\eta) = \eta^{-1} - \eta^{-2} < 0$ for $\eta \in (0,1)$. Together with the fact that $\lim_{\eta \to 0} \varphi_\kappa(\eta) = -\infty$ and $\lim_{\eta \to 1} \varphi_\kappa(\eta) = 1+\kappa$, φ_κ can cross 0 only once. Thus, for any $\eta' \in (0,1)$ such that $\varphi_\kappa(\eta') \ge 0$, we must have $\eta^*(\kappa) \le \eta'$. Finally,

$$\begin{aligned} \varphi_\kappa(\kappa^{-1}) &= \kappa - \kappa^{-1} - (\kappa^{-1}+1)\ln\kappa \\ &\ge \kappa - \kappa^{-1} - (\kappa^{-1}+1)(\kappa-1) = 0, \end{aligned}$$

where the last inequality follows from the fact that $\ln\eta \le \eta - 1$ for all $\eta > 0$. This implies that $\eta^*(\kappa) \le \kappa^{-1}$.

(ii) $\lim_{\kappa\to 1}\eta^*(\kappa) = 1$.

By a similar argument, for any $\epsilon \in (0,1)$ such that $\varphi_\kappa(1-\epsilon) \le 0$, we would have $\eta^*(\kappa) \ge 1-\epsilon$. Therefore, it suffices to show that

$$\liminf_{\kappa\to 1}\varphi_\kappa(1-\epsilon) \le 0, \quad \forall \epsilon \in (0,1). \tag{EC.17}$$

as this would imply that $\liminf_{\kappa\to 1}\eta^*(\kappa) \ge 1-\epsilon$ for any fixed $\epsilon \in (0,1)$. Note that for any fixed ϵ,

$$\liminf_{\kappa\to 1}\varphi_\kappa(1-\epsilon) = 2\epsilon + (2-\epsilon)\ln(1-\epsilon) := h(\epsilon),$$

and note that $h(0) = 0$ and $h(\epsilon) \to -\infty$ as $\epsilon \to 1$. Furthermore,

$$\begin{aligned} h'(\epsilon) &= 2 - \ln(1-\epsilon) - \frac{2-\epsilon}{1-\epsilon} \\ &\le 2 - \left(1 - \frac{1}{1-\epsilon}\right) - \frac{2-\epsilon}{1-\epsilon} = 0, \end{aligned}$$

where the last inequality follows from the fact that $\ln x \ge 1 - x^{-1}$ for all $x > 0$. We conclude that $h(\epsilon) \le 0$ for any $\epsilon \in (0,1)$.

Step 3. Maximum variance reduction rate. We first show that

$$\inf_{\kappa>1}\frac{\psi_\kappa(\eta^*(\kappa))}{\psi_\kappa(0)} \le \frac{1}{2}. \tag{EC.18}$$

Note that $\psi_\kappa(0) = \kappa$ for any $\kappa > 1$. Consider $\psi_\kappa(\kappa^{-k}), k \in \mathbb{Z}^+$, for which we have

$$\psi_\kappa(\kappa^{-k}) = \frac{\kappa^k[\kappa(\kappa^k - 1) - k\ln\kappa]}{(\kappa^k - 1)^2},$$

where sending κ to 1 and applying L'Hospital's Rule gives

$$\inf_{\kappa>1}\frac{\psi_\kappa(\eta^*(\kappa))}{\psi_\kappa(0)} \le \lim_{\kappa\to 1}\frac{\psi_\kappa(\kappa^{-k})}{\psi_\kappa(0)} = \frac{1}{2} + \frac{1}{k}.$$

This implies (EC.18) as k can be arbitrarily large. To prove the opposite direction of (EC.18), note that for any fixed $\eta \in (0,1)$, $\psi_\kappa(\eta)/\psi_\kappa(0)$ is a decreasing function in κ. Therefore,

$$\inf_{\kappa>1}\frac{\psi_\kappa(\eta^*(\kappa))}{\psi_\kappa(0)} \ge \inf_{\kappa>1}\frac{\psi_1(\eta^*(\kappa))}{\psi_1(0)} \ge \inf_{\eta\in(0,1)}\psi_1(\eta) = \inf_{\eta\in(0,1)}\left\{\frac{1}{1-\eta} + \frac{\eta\ln\eta}{(1-\eta)^2}\right\} = \frac{1}{2},$$

which completes the proof of (EC.4). ■