E-Companion

EC.1. Monte Carlo volume estimation of W_{x^p}

As the mapping μ^p transforming V_{x^p} into W_{x^p} is non-linear, in contrast to V_{x^p}, the set W_{x^p} has non-linear bounds. To efficiently compute the volume of W_{x^p} we apply a numerical estimation approach to determine the size $\lambda(W_{x^p})$. We use the simple and widely applied hit-or-miss method for Monte Carlo volume estimation as presented in Fok and Crevier (1989). Just as V_{x^p}, the set W_{x^p} is a subset of a $(|S_\mathrm{r}^{-p}|-1)$-dimensional simplex. For W_{x^p}, we can define this simplex Δ as:

$$\Delta = \left\{ r^p \in \mathbb{R}^{|S_\mathrm{r}^{-p}|} \middle| r^p_{x^{-p}} \geq 0, \sum_{x^{-p} \in S_\mathrm{r}^{-p}} r^p_{x^{-p}} = 1 \right\} \tag{EC.1}$$

We sample s points $\tilde{r}^p \sim \mathrm{Dir}(\boldsymbol{\alpha})$ from a Dirichlet distribution, where the parameter vector $\boldsymbol{\alpha}$ consists of all ones, i.e., a uniform distribution over this simplex. Let $s_{W_{x^p}}$ be the number of sampled points that lie within W_{x^p} (i.e., $\tilde{r}^p \in W_{x^p}$). Given a large enough sample size s, the size $\lambda(W_{x^p})$ can be approximated as:

$$\lambda(W_{x^p}) \approx \frac{s_{W_{x^p}}}{s} \lambda(\Delta) \tag{EC.2}$$

Note that although the size $\lambda(\Delta)$ of the simplex Δ can be easily calculated, we do not require any information about $\lambda(\Delta)$ as we determine a player's probability of a strategy profile based on the relative incentive of the strategy profile over all alternatives (37). In our implementation we use a sample size of $n = 1 \times 10^5$, yielding consistent results for the dimensionality encountered in the examined numerical studies.

EC.2. n-player probability measurement

The dimension of W_{x^p} is driven by the size $|S_\mathrm{r}^{-p}|$ of the competitor strategy set in the reduced sampled game. This dimension significantly increases with the number of involved players. Imagine a 3-player game where, in the reduced sampled game, each player can only choose from 3 strategies. To determine the stability set for the first player, we would have to solve best-response conditions (30) for $3 \times 3 = 9$ possible competitor strategy combinations. The volume of the resulting

8-dimensional image set W_{x^p} can easily be approximated using the Monte Carlo approach described in EC.1. For a higher number of players or strategies ($|S_{\mathrm{r}}^{-p}|$), the dimensionality of W_{x^p} – and thereby the number of Monte Carlo samples required for a sufficiently accurate result – increases exponentially, limiting practical applicability. We therefore suggest an alternative approach for n-player games: Instead of determining the probability for player p directly by taking all possible competitor strategy combinations X^{-p} into account at the same time, we gradually reduce dimensionality by assigning one competitor at a time to a single strategy. We calculate the (conditional) probability $\mathrm{Prob}(\Theta^p = x^p | x^{\tilde{p}})$ under the condition that a competing player $\tilde{p} \neq p$ selects the strategy $x^{\tilde{p}}$. We repeat the process for all strategies $x^{\tilde{p}} \in X^{\tilde{p}}$ and arrive at an average probability $\mathrm{Prob}(\Theta^p = x^p | \tilde{p})$ across the action set of player $\tilde{p}$. The application of the same procedure for all $n-1$ competitors yields the probability for x^p:

$$\mathrm{Prob}(\Theta^p = x^p) = \frac{\sum_{\tilde{p} \neq p} \mathrm{Prob}(\Theta^p = x^p | \tilde{p})}{n-1} \tag{EC.3}$$

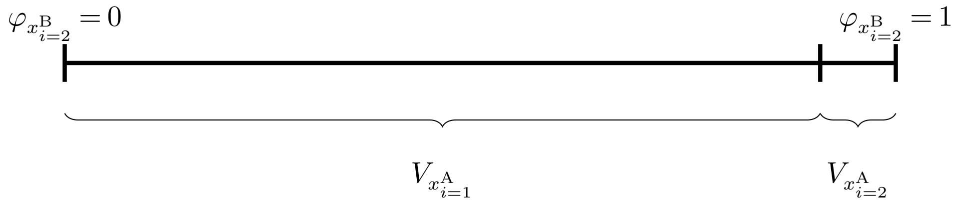
EC.3. Equilibrium selection in the knapsack game

Let us assume a setting for the knapsack game (Section 3.1), where two players $N = \{\mathrm{A}, \mathrm{B}\}$ select one out of $|I| = 2$ technologies. Imagine that a given technology $i \in I$ will only pay off if both competitors select it, meaning that $v_i^p = 0$. In this case, if both competitors prefer the same technology, equilibrium selection would be trivial. In the following, we assume A strongly prefers $i = 1$, and B strongly prefers $i = 2$:

In a symmetric case where $c_1^{\mathrm{AB}} = 1$, $c_2^{\mathrm{AB}} = 10$ and $c_2^{\mathrm{BA}} = 10$, $c_2^{\mathrm{BA}} = 1$, we have two pure Nash equilibria (with both players successfully jointly selecting either technology) and a mixed equilibrium with combinations of the two technologies, yielding the set of all equilibria: $\Phi = \left\{((1,0),(1,0)), ((0,1),(0,1)), \left(\left(\frac{1}{11}, \frac{10}{11}\right), \left(\frac{10}{11}, \frac{1}{11}\right)\right)\right\}$.

Note as the players only have two pure strategies each, the stability set $V_{x^{\mathrm{A}}}$ of player A using strategy x^{A} is a line. Let $x_{i=1}^{\mathrm{A}} = (1,0)$ be a pure strategy where A selects technology 1 and $x_{i=2}^{\mathrm{A}} = (0,1)$ the alternate pure strategy where A selects 2. Figure EC.1 shows the stability sets $V_{x_{i=1}^{\mathrm{A}}}$ and $V_{x_{i=2}^{\mathrm{A}}}$ for player A.

Figure EC.1 Stability sets $V_{x^{\mathrm{A}}_{i=1}}$ and $V_{x^{\mathrm{A}}_{i=2}}$ for the coordination knapsack game



Using the function μ^p, we determine the respective sets $W_{x^{\mathrm{A}}}$ and calculate the incentive $\lambda(W_{x^{\mathrm{A}}})$ for A to play a strategy x^{A}. Following the same process for B, we arrive at the following incentives for pure strategies:

$$\lambda(W_{x^{\mathrm{A}}_{i=1}}) = \lambda(W_{x^{\mathrm{B}}_{i=2}}) = \frac{1}{11}$$
$$\lambda(W_{x^{\mathrm{A}}_{i=2}}) = \lambda(W_{x^{\mathrm{B}}_{i=1}}) = \frac{10}{11}$$

For the mixed equilibrium $\hat{\varphi} = \left(\left(\frac{1}{11}, \frac{10}{11}\right), \left(\frac{10}{11}, \frac{1}{11}\right)\right)$ we get:

$$\Psi^{\mathrm{A}}_{\hat{\varphi}^{\mathrm{A}}} = \frac{1}{11}\lambda(W_{x^{\mathrm{A}}_{i=1}}) + \frac{10}{11}\lambda(W_{x^{\mathrm{A}}_{i=2}}) = \frac{101}{121}$$
$$\Psi^{\mathrm{B}}_{\hat{\varphi}^{\mathrm{B}}} = \frac{10}{11}\lambda(W_{x^{\mathrm{B}}_{i=1}}) + \frac{1}{11}\lambda(W_{x^{\mathrm{B}}_{i=2}}) = \frac{101}{121}$$

Scaling Ψ to ensure a cumulative probability of one per player, yields the probabilities:

$$\mathrm{Prob}(\Theta^{\mathrm{A}} = x^{\mathrm{A}}_{i=1}) = \mathrm{Prob}(\Theta^{\mathrm{B}} = x^{\mathrm{B}}_{i=2}) = 5\%$$

$$\mathrm{Prob}(\Theta^{\mathrm{A}} = x^{\mathrm{A}}_{i=2}) = \mathrm{Prob}(\Theta^{\mathrm{B}} = x^{\mathrm{B}}_{i=1}) = 50\%$$

$$\mathrm{Prob}(\Theta^{\mathrm{A}} = \hat{\varphi}^{\mathrm{A}}) = \mathrm{Prob}(\Theta^{\mathrm{B}} = \hat{\varphi}^{\mathrm{B}}) = 45\%$$

This suggests that successful coordination to either of the two pure equilibria is equally unlikely with $\mathrm{Prob}(\Theta^{\mathrm{A}} = x^{\mathrm{A}}_{i=1}) \cdot \mathrm{Prob}(\Theta^{\mathrm{B}} = x^{\mathrm{B}}_{i=1}) = \mathrm{Prob}(\Theta^{\mathrm{A}} = x^{\mathrm{A}}_{i=2}) \cdot \mathrm{Prob}(\Theta^{\mathrm{B}} = x^{\mathrm{B}}_{i=2}) = 2.25\%$, whereas the mixed equilibrium $\hat{\varphi}$ is the expected outcome of the game with $\mathrm{Prob}(\Theta^{\mathrm{A}} = \hat{\varphi}^{\mathrm{A}}) \cdot \mathrm{Prob}(\Theta^{\mathrm{B}} = \hat{\varphi}^{\mathrm{B}}) =$ 24.55%.

EC.4. Knapsack Game: 3-player results

Table EC.1 Computational results for the coordination Knapsack game with $n = 3$

Instances		SGM			eSGM			
$\|I\|$	m	$\|\Phi\|$	time (s)	Prob*(%)	$\|\Phi\|$	$\|\Phi_{\text{mixed}}\|$	time (s)	Prob*(%)
10	0	1	0	100	1	0	0.1	100
	1	1	0	100	1	0	0.1	100
	2	1	0	100	1	0	0.1	100
	3	1	0	100	1	0	0.1	100
	4	1	0	100	1	0	0.2	100
	5	1	0.5	71	2	2	15.1	**71**
	6	1	0.1	100	1	0	0.3	100
	7	1	1.4	29	2	1	13.9	71
	8	1	0.1	100	1	0	0.2	100
	9	1	0.1	100	1	0	0.2	100
20	0	1	0	100	1	0	0.1	100
	1	1	0.1	100	1	0	0.2	100
	2	1	0.1	100	1	0	0.3	100
	3	1	0	100	1	0	0.5	100
	4	1	0.1	100	1	0	1.0	100
	5	1	0.1	100	1	0	1.6	100
	6	1	0.1	100	1	0	1.1	100
	7	1	0.1	100	1	0	0.5	100
	8	1	0.1	100	1	0	0.5	100
	9	1	0.1	100	1	0	0.5	100
40	0	1	0	100	1	0	0.2	100
	1	1	1.3	50	2	2	10.3	**50**
	2	1	7.6	21	2	2	43.3	**79**
	3	1	0.1	100	1	0	10.5	100
	4	1	0.2	99	2	0	38.6	99
	5	1	0.2	2	2	0	52.3	98
	6	1	0.1	100	1	0	53.7	100
	7	1	0.1	100	1	0	55.1	100
	8	1	0.1	100	1	0	48.9	100
	9	1	0.1	100	1	0	55.5	100

⋆ indicates the probability of the selected equilibrium among all equilibria. **Bold** probabilities indicate that the equilibrium selected through the eSGM (highest probability) is a mixed equilibrium.

EC.5. Combinatorial competitive uncapacitated lot-sizing game

EC.5.1. Problem description

Carvalho et al. (2018b) present a combinatorial variant of the competitive uncapacitated lot-sizing problem. In this combinatorial problem, n players produce a (single) homogeneous good in a finite planning horizon $|T|$. The price of the good $p_t(q_t)$ is determined by the market based on the total quantity $q_t = \sum_{p \in N} q_t^p$ of offered units across all players: $p_t(q_t) = \max\{a_t - b_t q_t, 0\}$, with $a_t, b_t \geq 0$ as market characteristics and q_t^p as the quantity offered by player p in period t. z_t^p denotes production quantities, the binary variable $y_t^p \in \{0,1\}$ denotes production periods. In each period of active production ($y_t^p = 1$), fixed setup costs f^p are incurred. Goods that are produced but not sold $z_t^p > q_t^p$ increase the inventory h_t^p. As Carvalho et al. (2018b), we assume no holding costs and focus on the combinatorial decision problem of finding optimal periods of active production y_t^p. Decisions on sales and production quantities (q_t^p and z_t^p, respectively) follow from the zero-inventory property (Wagner and Whitin 1958). Players optimize individual profits Π^p as the difference between operational margin and fixed costs, with $\bar{u}_t^p$ defining the last period of active production prior to and including t and $c^p_{\bar{u}_t^p}$ as the production cost in this period (Carvalho et al. 2018b):

$$\max_{y_t^p, y_t^{-p}} \Pi^p(y_t^p, y_t^{-p}) = \sum_{t \in T | \sum_{\tau \in \{0,\ldots,t\}} y_\tau^p \geq 1} \bar{q}_t^p (p_t(q_t) - c^p_{\bar{u}_t^p}) - \sum_{t \in T} f^p y_t^p \tag{EC.4}$$

In case a period is not preceded by a period of active production, $\bar{u}_t^p$ and by consequence $c^p_{\bar{u}_t^p}$ are undefined. Therefore, in (EC.4), we limit the sum of the operational margin to periods with active production or periods that are preceded by a period of active production.

$$\bar{u}_t^p = \max\{u \mid u \in T, u \leq t, y_u^p = 1\} \qquad \forall t \in T,\ \forall p \in N \tag{EC.5}$$

In (EC.5), $\bar{u}_t^p$ is formally defined as the period $u \in T$ with active production ($y_u^p = 1$) and the highest cardinality ($\max u$) smaller or equal to t.

Note that (EC.4) expects a decision only on y_t^p, y^{-p}. To translate the decision on the set of production periods into optimal production quantities $\bar{q}_t^p$, Carvalho et al. (2018b) derive an optimality condition by setting the partial derivative of Π^p with regards to $\bar{q}_t^p$ to zero:

$$\frac{\partial \Pi^p}{\partial \bar{q}_t^p} = 0 \rightarrow \bar{q}_t^p = \frac{\max\left\{a_t - b_t \sum_{\tilde{p} \neq p} q_t^{\tilde{p}} - c^p_{\bar{u}_t^p}, 0\right\}}{2b_t} \qquad \forall p \in N \tag{EC.6}$$

Reformulation yields:

$$\bar{q}_t^p = \begin{cases} \frac{\max\left\{p_t - c^p_{\bar{u}_t^p}, 0\right\}}{b_t} & , \text{if } \sum_{\tau \in \{0,\ldots,t\}} y_\tau^p \geq 1 \\ 0 & , \text{else} \end{cases} \qquad \forall t \in T,\ \forall p \in N \tag{EC.7}$$

$$p_t = \frac{a_t + \sum_{p \in N | y_t^p = 1} c^p_{\bar{u}_t^p}}{\sum_{p \in N} y_t^p + 1} \qquad \forall t \in T \tag{EC.8}$$

This combinatorial model features no independent continuous decision variables. We solve the model using eSGM with the following implementation of (EC.4) and (EC.5): First, we replace $\max\{p_t - c^p_{\bar{u}^p_t}, 0\}$ with a new auxiliary variable k^p_t.

$$\Pi^p(y^p_t, y^{-p}_t) = \sum_{t\in T} \frac{(k^p_t)^2}{b_t} - \sum_{t\in T} f^p y^p_t \tag{EC.9}$$

We introduce $o^p_t \in \{0,1\}$ as an auxiliary binary variable and ensure $k^p_t = \max\{p_t - C^p_{\bar{u}^p_t}, 0\}$ as follows:

$$k^p_t \leq \sum_{\tau\in\{0,\ldots,t\}} M y^p_\tau \qquad \forall t\in T,\ \forall p\in N \tag{EC.10}$$

$$k^p_t \leq p_t - c^p_{\bar{u}^p_t} + o^p_t M \qquad \forall t\in T,\ \forall p\in N \tag{EC.11}$$

$$k^p_t \geq M(o^p_t - 1) \qquad \forall t\in T,\ \forall p\in N \tag{EC.12}$$

$$M(1-o^p_t) \geq p_t - c^p_{\bar{u}^p_t} \qquad \forall t\in T,\ \forall p\in N \tag{EC.13}$$

Whenever $p_t - c^p_{\bar{u}^p_t} \leq 0$, (EC.13) sets $o^p_t = 1$ and by consequence $k^p_t = 0$ (EC.12). In all other cases k^p_t is maximized according to the objective function (EC.9) and hence equal to $k^t_p = p_t - c^p_{\bar{u}^p_t}$ according to (EC.11). Furthermore, to calculate $c^p_{\bar{u}^p_t}$, we introduce the binary auxiliary variable $u^p_{t\tau} \in \{0,1\}$. We ensure $u^p_{t\tau} = 1$, if and only if τ was the last period of active production for p prior to t (i.e., $\bar{u}^p_t = \tau$):

$$u^p_{t\tau} = 0 \qquad \forall t\in T,\ \tau\in\{t+1,\ldots,|T|\},\ \forall p\in N \tag{EC.14}$$

$$u^p_{t\tau} \leq y^p_\tau \qquad \forall t,\tau\in T,\ \forall p\in N \tag{EC.15}$$

$$\sum_{\tau\in T} u^p_{t\tau} \leq 1 \qquad \forall t\in T,\ \forall p\in N \tag{EC.16}$$

$$u^p_{t\tau} + (1-y^p_\tau)M \geq u^p_{t\hat{t}} \qquad \forall t\in T,\ \forall\tau\in\{0,\ldots,t\},\ \forall\hat{t}\in\{0,\ldots,\tau\},\ \forall p\in N \tag{EC.17}$$

$$u^p_{tt} = y^p_t \qquad \forall t\in T,\ \forall p\in N \tag{EC.18}$$

(EC.14) sets $u_{t\tau}$ to zero for all periods $\tau > t$. (EC.15) ensures that $u_{t\tau}$ can only be one, if τ is a period of active production. As we determine the (single) last period of active production, the sum of $u_{t\tau}$ over $t \in T$ needs to add up to one (EC.16), and a more recent active period always needs to supersede a later period of active production (EC.17). Using $u^p_{t\tau}$, we can calculate $c^p_{\bar{u}^p_t}$ as:

$$c^p_{\bar{u}^p_t} = c^p_t \sum_{\tau \in T} u^p_{t\tau} \qquad \forall t \in T,\ \forall p \in N \qquad \text{(EC.19)}$$

The combinatorial problem (EC.4)-(EC.19) builds on the property that players decide on the quantity $\bar{q}^p_t$ according to optimality condition (EC.6). If only a single player was to unilaterally change their production periods (and by consequence quantity), all other players are expected to implicitly adapt their production $\bar{q}^p_t$ to their respective new optimum. Carvalho et al. (2018b) show that this implies that the set of equilibria of the combinatorial problem is a subset of the equilibria of an alternative mixed integer formulation, as equilibria in the combinatorial problem are not only stable against unilateral changes but also against (implicit) reactions in production quantities from competitors.

EC.5.2. Computational results for the competitive uncapacitated lot-sizing game

We draw the parameters a_t and b_t independently from discrete uniform distributions for three different time horizons $|T| \in \{10, 20, 50\}$: $a_t \sim \mathcal{U}\{20, 30\},\ b_t \sim \mathcal{U}\{3, 5\}$. For each time horizon, we generate 10 instances for $n = 2$ and $n = 3$ players. We assume setup costs are constant $f^p = 1$, whereas variable costs start with $\bar{c}^p \sim \mathcal{U}\{29, 31\}$ for $t = 0$ and decrease gradually over the time horizon T: $c^p_t = \bar{c}^p \left(1 - 0.35 \frac{t}{|T|}\right)$. The motivation for this type of cost function is two-fold, as it allows to model learning behavior or continuous performance improvements and provides an indirect way to account for holdings costs: Producing goods in period $t-1$ to sell in period t leads to an increase in cost by factor $\frac{0.35}{|T|}$.

Table EC.2 shows aggregate (min, mean, max) results for the runtime (RT) and the identified number of equilibria $|\Phi|$ for each combination $(n, |T|)$ using SGM and eSGM. Using the eSGM, we show that for the majority of examined instances there is no unique equilibrium, and some instances show up to 70 equilibria. More than a third of all equilibria are mixed, but only in 7% of instances the selected equilibrium is a mixed one. While on average, the conditional probability of the equilibrium selected by the eSGM only marginally outperforms the unique equilibrium identified using SGM, there are some cases where the probability of the latter is almost equal to zero whereas the eSGM selects a highly probable ($> 50\%$) equilibrium.

Table EC.2 Computational results for the competitive uncapacitated lot-sizing game (RT=runtime in s)

Problem		SGM				eSGM						
n	$\|T\|$	RT_{min}	RT_{mean}	RT_{max}	Prob*(%)	RT_{min}	RT_{mean}	RT_{max}	$\|\Phi\|_{\text{av}}$	$\|\Phi_{\text{mixed}}\|_{\text{av}}$	$\|\Phi\|_{\text{max}}$	Prob*(%)
2	10	0.0	0.1	0.2	93	0.1	0.3	0.7	1.8	0.4	7	93
	20	0.3	0.5	0.8	82	0.6	3.6	15.1	3.8	1.8	17	84
	50	3.5	3.9	4.6	76	7.7	172.2	1533.9	4	1.8	17	83
3	10	0.1	0.1	0.3	91	0.2	0.7	2.2	2.1	0.7	10	93
	20	0.4	0.6	1.2	93	2.4	46.6	372.8	2	0.5	5	93
	50	2.3	5.1	8.0	62	20.7	1772.5	15968.2	9.7	7.4	70	77

$\star$ indicates the probability of the selected equilibrium among all equilibria.