Supplement: Technical details and Monte Carlo Evidence

Appendix EC.1: Proofs of theorems

EC.1.1. Proof of Theorem 1

Consider the function $\Lambda : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined by $\Lambda(x) = x^{-1}$. Clearly, $\Lambda(\cdot)$ is monotonic, invertible, and differentiable with nonzero derivatives on $\mathbb{R}_{++}$. Hence, Theorem 3.2 of KSW2021 still holds true if $\Lambda(\cdot)$ substitutes the function $\Gamma : \mathbb{R}_{+} \to \mathbb{R}_{+}$ defined by $\Gamma(x) = x^{0.5}$ in the theorem. Similarly, the proof of Theorem 3.4 of KSW2021 is still valid if $\Lambda(\cdot)$ replaces the function $\Gamma : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined by $\Gamma(x) = \log x^{0.5}$ there. Combining these facts with Lemmas 3.1 and 3.2 of KSW2021, we can obtain asymptotic properties of the conical Farrell-type output-oriented efficiency estimators, which is analogous to Theorem 3.4 of KSW2021. Formally, there exist constants $\overline{C}_{st} \in (0, \infty)$ such that as $n \to \infty$,

$$E\left(\lambda_C(Z_i^s|\mathcal{X}_n^t) - \lambda_C(Z_i^s|\Psi^t)\right) = \overline{C}_{st} n^{-\kappa} + O(v_{n,\kappa}), \tag{EC.1}$$

$$E\left(\left[\lambda_C(Z_i^s|\mathcal{X}_n^t) - \lambda_C(Z_i^s|\Psi^t)\right]^2\right) = o(n^{-\kappa}), \tag{EC.2}$$

$$\left|E\left([\lambda_C(Z_i^s|\mathcal{X}_n^t) - E(\lambda_C(Z_i^s|\mathcal{X}_n^t))][\lambda_C(Z_j^{s^*}|\mathcal{X}_n^{t^*}) - E(\lambda_C(Z_j^{s^*}|\mathcal{X}_n^{t^*}))]\right)\right| = o(n^{-1}), \tag{EC.3}$$

for all $i, j \in \{1, \ldots, n\}, i \neq j;\ s, t, s^*, t^* \in \{1, 2\}$.

For each $i = 1, \ldots, n$, since the first two moments of $w^2 Y_i^2$ are finite and

$$\widehat{U}_{1,i} - U_{1,i} = \left(\lambda_C(Z_i^2|\mathcal{X}_n^1) - \lambda_C(Z_i^2|\Psi^1)\right) w^2 Y_i^2, \tag{EC.4}$$

the order of the first two moments of $\widehat{U}_{1,i} - U_{1,i}$ inherits those from $\lambda_C(Z_i^2|\mathcal{X}_n^1) - \lambda_C(Z_i^2|\Psi^1)$ and similarly, we have the same conclusions for $\widehat{U}_{s,i} - U_{s,i}$ $(s = 2, 3, 4)$. Also, the order of $Cov(\widehat{U}_{1,i}, \widehat{U}_{2,j})$ is the same as that of $Cov(\lambda_C(Z_i^2|\mathcal{X}_n^1), \lambda_C(Z_j^2|\mathcal{X}_n^2))$ $(j = 1, \ldots, n; j \neq i)$, and similar conclusions can be drawn for the other analogous covariances. These facts imply that Theorem 1 holds true for $s, t \in \{1, 2, 3, 4\}$.

EC.1.2. Proof of Theorem 2

(i) This part is a direct consequence of (21).

(ii) To begin with, we decompose $Cov(\widehat{U}_{t,i}, \widehat{U}_{t^*,i})$ as

$$\begin{aligned} Cov(\widehat{U}_{t,i}, \widehat{U}_{t^*,i}) &= E\left(\left[\widehat{U}_{t,i} - E(\widehat{U}_{t,i})\right]\left[\widehat{U}_{t^*,i} - E(\widehat{U}_{t^*,i})\right]\right) \\ &= E\left(\left[\widehat{U}_{t,i} - U_{t,i} + U_{t,i} - E(\widehat{U}_{t,i})\right]\left[\widehat{U}_{t^*,i} - U_{t^*,i} + U_{t^*,i} - E(\widehat{U}_{t^*,i})\right]\right) \\ &= E\left([\widehat{U}_{t,i} - U_{t,i}][\widehat{U}_{t^*,i} - U_{t^*,i}]\right) + E\left([U_{t,i} - E(\widehat{U}_{t,i})][U_{t^*,i} - E(\widehat{U}_{t^*,i})]\right) \\ &\quad + E\left([\widehat{U}_{t,i} - U_{t,i}][U_{t^*,i} - E(\widehat{U}_{t^*,i})]\right) + E\left([U_{t,i} - E(\widehat{U}_{t,i})][\widehat{U}_{t^*,i} - U_{t^*,i}]\right). \end{aligned} \tag{EC.5}$$

Now we evaluate the rates of convergence of each term in the above decomposition of $Cov(\widehat{U}_{t,i}, \widehat{U}_{t^*,i})$.

First, by the Cauchy-Schwartz inequality and (22), we have

$$\left(E([\widehat{U}_{t,i} - U_{t,i}][\widehat{U}_{t^*,i} - U_{t^*,i}])\right)^2 \le E\left([\widehat{U}_{t,i} - U_{t,i}]^2\right) E\left([\widehat{U}_{t^*,i} - U_{t^*,i}]^2\right) = o(n^{-\kappa})o(n^{-\kappa}), \tag{EC.6}$$

and hence,

$$E\left([\widehat{U}_{t,i} - U_{t,i}][\widehat{U}_{t^*,i} - U_{t^*,i}]\right) = o(n^{-\kappa}). \tag{EC.7}$$

Next, the second term in (EC.5) can be expressed as

$$\begin{aligned} &E\left([U_{t,i} - E(\widehat{U}_{t,i})][U_{t^*,i} - E(\widehat{U}_{t^*,i})]\right) \\ &= E\left([U_{t,i} - E(U_{t,i}) + E(U_{t,i}) - E(\widehat{U}_{t,i})][U_{t^*,i} - E(U_{t^*,i}) + E(U_{t^*,i}) - E(\widehat{U}_{t^*,i})]\right) \\ &= E\left([U_{t,i} - E(U_{t,i})][U_{t^*,i} - E(U_{t^*,i})]\right) + E\left([U_{t,i} - E(U_{t,i})][E(U_{t^*,i}) - E(\widehat{U}_{t^*,i})]\right) \\ &\quad + E\left([E(U_{t,i}) - E(\widehat{U}_{t,i})][U_{t^*,i} - E(U_{t^*,i})]\right) + E\left([E(U_{t,i}) - E(\widehat{U}_{t,i})][E(U_{t^*,i}) - E(\widehat{U}_{t^*,i})]\right) \\ &= \sigma_{tt^*} + 0 + 0 + E(U_{t,i} - \widehat{U}_{t,i})E(U_{t^*,i} - \widehat{U}_{t^*,i}) \\ &= \sigma_{tt^*} + E(U_{t,i} - \widehat{U}_{t,i})E(U_{t^*,i} - \widehat{U}_{t^*,i}). \end{aligned} \tag{EC.8}$$

Thus, by (21), we have

$$E\left([U_{t,i} - E(\widehat{U}_{t,i})][U_{t^*,i} - E(\widehat{U}_{t^*,i})]\right) = \sigma_{tt^*} + (C_t n^{-\kappa} + o(n^{-\kappa}))(C_{t^*} n^{-\kappa} + o(n^{-\kappa}))$$

$$= \sigma_{tt^*} + o(n^{-\kappa}). \tag{EC.9}$$

Third, by the Cauchy-Schwartz inequality, (22) and applying (EC.9) with $t = t^*$, we obtain

$$\begin{aligned}\left(E([\widehat{U}_{t,i} - U_{t,i}][U_{t^*,i} - E(\widehat{U}_{t^*,i})])\right)^2 &\leq E\left([\widehat{U}_{t,i} - U_{t,i}]^2\right) E\left([U_{t^*,i} - E(\widehat{U}_{t^*,i})]^2\right) \\ &= o(n^{-\kappa})\left(\sigma_{t^*t^*} + o(n^{-\kappa})\right) = o(n^{-\kappa}). \end{aligned} \tag{EC.10}$$

Consequently,

$$E\left([\widehat{U}_{t,i} - U_{t,i}][U_{t^*,i} - E(\widehat{U}_{t^*,i})]\right) = o(n^{-\kappa/2}). \tag{EC.11}$$

By swapping the positions of t and t^* in (EC.11), we obtain the result for the last term in (EC.5) below.

$$E\left([U_{t,i} - E(\widehat{U}_{t,i})][\widehat{U}_{t^*,i} - U_{t^*,i}]\right) = o(n^{-\kappa/2}). \tag{EC.12}$$

Combining (EC.5), (EC.7), (EC.9), (EC.11), and (EC.12), we have part (ii) proved.

(iii) It can be seen that

$$\begin{aligned} Cov(\widehat{U}_{s,i}, U_{r,i}) &= E\left([\widehat{U}_{s,i} - E(\widehat{U}_{s,i})][U_{r,i} - \mu_r]\right) \\ &= E\left([\widehat{U}_{s,i} - U_{s,i} + U_{s,i} - \mu_s + \mu_s - E(\widehat{U}_{s,i})][U_{r,i} - \mu_r]\right) \\ &= E\left([\widehat{U}_{s,i} - U_{s,i}][U_{r,i} - \mu_r]\right) + \sigma_{sr} + E\left([\mu_s - E(\widehat{U}_{s,i})][U_{r,i} - \mu_r]\right) \\ &= E\left([\widehat{U}_{s,i} - U_{s,i}][U_{r,i} - \mu_r]\right) + \sigma_{sr}. \end{aligned} \tag{EC.13}$$

By the Cauchy-Schwartz inequality and (22),

$$\left(E([\widehat{U}_{s,i} - U_{s,i}][U_{r,i} - \mu_r])\right)^2 \leq E\left([\widehat{U}_{s,i} - U_{s,i}]^2\right) E\left([U_{r,i} - \mu_r]^2\right) = o(n^{-\kappa})\sigma_{rr} = o(n^{-\kappa}).$$

Thus, $E\left([\widehat{U}_{s,i} - U_{s,i}][U_{r,i} - \mu_r]\right) = o(n^{-\kappa/2})$ and part (iii) follows directly from (EC.13).

EC.1.3. Proof of Theorem 3

Consider the random variables $\chi_{s,n} = n^{-1}\sum_{i=1}^{n}(\widehat{U}_{s,i} - U_{s,i}) = \widehat{\widehat{\mu}}_{s,n} - \widehat{\mu}_{s,n}$ $(s = 1,\dots,4)$. Part (i) follows directly from Theorem 2(i) as follows.

$$\widetilde{\mu}_{s,n} - \mu_s = E(\chi_{s,n}) = n^{-1}\sum_{i=1}^{n} E(\widehat{U}_{s,i} - U_{s,i}) = C_s n^{-\kappa} + o(n^{-\kappa}). \tag{EC.14}$$

Now from Theorem 1 we have

$$\begin{aligned} Var(\chi_{s,n}) &= n^{-2}\sum_{i=1}^{n} Var(\widehat{U}_{s,i} - U_{s,i}) = n^{-2}\sum_{i=1}^{n}\left(E\left([\widehat{U}_{s,i} - U_{s,i}]^2\right) - \left(E(\widehat{U}_{s,i} - U_{s,i})\right)^2\right) \\ &= n^{-1}\left(o(n^{-\kappa}) - (C_s n^{-\kappa} + o(n^{-\kappa}))^2\right) = n^{-1}o(n^{-\kappa}). \end{aligned} \tag{EC.15}$$

By Markov's inequality (see, e.g., Van der Vaart 2000, page 10), we have that for any $\epsilon > 0$,

$$Pr(\sqrt{n}|\chi_{s,n} - E(\chi_{s,n})| > \epsilon) \leq \frac{E\left(n(\chi_{s,n} - E(\chi_{s,n}))^2\right)}{\epsilon^2} = \frac{nVar(\chi_{s,n})}{\epsilon^2} = \frac{o(n^{-\kappa})}{\epsilon^2}. \tag{EC.16}$$

Therefore, $\sqrt{n}(\chi_{s,n} - E(\chi_{s,n})) = o_p(1)$ or equivalently, $\chi_{s,n} - E(\chi_{s,n}) = o_p(n^{-1/2})$. Consequently,

$$(\widehat{\widehat{\mu}}_{s,n} - \widetilde{\mu}_{s,n}) - (\widehat{\mu}_{s,n} - \mu_s) = (\widehat{\widehat{\mu}}_{s,n} - \widehat{\mu}_{s,n}) - (\widetilde{\mu}_{s,n} - \mu_s) = \chi_{s,n} - E(\chi_{s,n}) = o_p(n^{-1/2}), \tag{EC.17}$$

implying (ii).

To prove part (iii), we recall from (75) that

$$\sqrt{n}(\widehat{\mu}_{s,n} - \mu_s) \xrightarrow{d} \mathcal{N}(0, \sigma_{ss}), \quad s = 1,\dots,6. \tag{EC.18}$$

Combining this with (ii), we have

$$\sqrt{n}(\widehat{\widehat{\mu}}_{s,n} - \widetilde{\mu}_{s,n}) = \sqrt{n}(\widehat{\mu}_{s,n} - \mu_s + o_p(n^{-1/2})) = \sqrt{n}(\widehat{\mu}_{s,n} - \mu_s) + o_p(1) \xrightarrow{d} \mathcal{N}(0, \sigma_{ss}), \tag{EC.19}$$

and hence, (iii) is proved.

Now it follows from parts (i) and (iii) that $\widehat{\widehat{\mu}}_{s,n} \xrightarrow{p} \mu_s$ for $s = 1,\dots,4$. Combining this with Theorem 2 and using the standard central limit theorem, we have that as $n \to \infty$,

$$\widehat{\widehat{\sigma}}_{st,n} = n^{-1}\sum_{i=1}^{n}(\widehat{U}_{s,i} - \widehat{\widehat{\mu}}_{s,n})(\widehat{U}_{t,i} - \widehat{\widehat{\mu}}_{t,n}) = n^{-1}\sum_{i=1}^{n}\widehat{U}_{s,i}\widehat{U}_{t,i} - \widehat{\widehat{\mu}}_{s,n}\widehat{\widehat{\mu}}_{t,n}$$

$$\xrightarrow{p} E(\widehat{U}_{s,1}\widehat{U}_{t,1}) - \mu_s\mu_t = Cov(\widehat{U}_{s,1}, \widehat{U}_{t,1}) + E(\widehat{U}_{s,1})E(\widehat{U}_{t,1}) - \mu_s\mu_t$$

$$= (\sigma_{st} + o(n^{-\kappa/2})) + (\mu_s + C_s n^{-\kappa} + O(v_{n,\kappa}))(\mu_t + C_t n^{-\kappa} + O(v_{n,\kappa})) - \mu_s\mu_t$$

$$\longrightarrow \sigma_{st}, \tag{EC.20}$$

which implies part (iv). Part (v) can be proved by using similar arguments as those used for part (iv); meanwhile part (vi) is a well-known result as mentioned in (81).

Appendix EC.2: A uniform delta method (Van der Vaart 2000)

We reproduce Theorem 3.8 of Van der Vaart (2000) here with a small change of notation and presentation for consistency throughout this paper and for convenience in the application in the proofs of our theorems.

LEMMA EC.1. *Let $\Theta : \mathbb{R}^k \to \mathbb{R}^m$ be a continuously differentiable function in a neighborhood of $h \in \mathbb{R}^k$. Let H_n be random vectors taking their values in the domain of Θ. If $\vartheta_n(H_n - h_n) \xrightarrow{d} H$ for vectors $h_n \to h$ and numbers $\vartheta_n \to \infty$, then we have*

$$\text{(i) } \Theta(H_n) - \Theta(h_n) = \nabla\Theta(h)(H_n - h_n) + o_p(\vartheta_n^{-1}), \tag{EC.21}$$

$$\text{(ii) } \vartheta_n\left(\Theta(H_n) - \Theta(h_n)\right) \xrightarrow{d} \nabla\Theta(h)H, \tag{EC.22}$$

where $\nabla\Theta(h)$ denotes the ($m \times k$ matrix) gradients of $\Theta(\cdot)$ evaluated at h.

Proof. See Theorem 3.8 of Van der Vaart (2000).

Appendix EC.3: Monte Carlo Experiments

We have conducted Monte Carlo experiments to investigate the finite-sample performance of the statistical inferences proposed in this paper, focusing on confidence intervals and hypothesis testing. Regarding the confidence intervals, we analyze their estimated coverages which are calculated as the percentage of times an estimated confidence interval covers the true value of the parameter of interest, given the pre-determined level of significance α. For hypothesis testing, we examine the size and power of tests of productivity change from period 1 to period 2 via rates of rejection of the null hypothesis. In particular, the null hypothesis of no productivity change corresponds to $\xi = 0$, depending on which estimator is used to conduct the test. Meanwhile, the alternative hypothesis that the productivity has changed from period 1 to period 2 corresponds to $\xi \neq 0$. The rejection rates are computed as the percentage of times an estimated confidence interval does not cover zero.

EC.3.1. A data generating process

In this paper, we employ a new data generating process that possesses the essential properties needed for this type of experiment, typically: (i) the technologies in both periods exhibit VRS; (ii) the productivity change between two periods can be controlled via a pre-determined parameter, say δ; (iii) the technical efficiency scores of any particular DMU in two time periods are correlated and can be controlled via a pre-determined parameter, namely ρ_λ; (iv) the sizes (i.e., inputs amounts) of any particular DMU in two time periods are correlated and can be controlled via a pre-determined parameter, say ρ.

In particular, we assume that the true technology in the period t is characterized by

$$\Psi^t = \{(x,y) \in \mathbb{R}^p_+ \times \mathbb{R}^1_+ : y \leq \psi^t(x)\}, \quad t = 1, 2, \tag{EC.23}$$

where $\psi^t(x) = \Upsilon^t (x_1 - b^t_1)^{\beta^t_1} \ldots (x_p - b^t_p)^{\beta^t_p}$, $\beta^t_j, b^t_j > 0$, $\sum_{j=1}^p \beta^t_j < 1$. Note that $\psi^t(x)$ resembles the Stone-Geary utility function (Geary 1950, Stone 1954) and was used in a production context by Beattie and Aradhyula (2015).

In order to control the degree to which the production frontier in period 2 is different from that in period 1, we set $\Upsilon^2 = \Upsilon^1 + \delta$ and $b_j^1 = b_j^2$, $\beta_j^2 = \beta_j^1 + \delta$ $(j = 1, \ldots, p)$, where δ is the predetermined parameter mentioned above. In order to guarantee that the function $\psi^t(.)$ is well-defined, we also need $x_j \geq b_j^t$ for $j = 1, \ldots, p$, $t = 1, 2$. This constraint is reasonable from the economic viewpoint since it illustrates that firms incur fixed costs or some threshold expenses (i.e., b_i^t) in order to start producing positive outputs. Our data generating process includes three steps, as described below.

1. For each $t = 1, 2$, generate n points of the form $(x_i^t, \psi^t(x_i^t))$ $(i = 1, \ldots, n)$. Here $x_i^t = (x_{i_1}^t, \ldots, x_{i_p}^t)$ is the realization of random vectors $X_i^t = (X_{i_1}^t, \ldots, X_{i_p}^t)$ where $X_{i_j}^t \sim \text{Uniform}(1, 10)$ such that $corr(X_{i_j}^1, X_{i_j}^2) = \rho$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, p\}$. The points $(x_i^t, \psi^t(x_i^t))$ generated in this step serve as optimal points, i.e., hypothetically fully efficient DMUs. The parameter ρ predetermines the correlation of inputs of each DMU in two different time periods. The details on how to generate correlated random numbers are given in e-companion EC.3.
2. Generate $\lambda(x_i^t, y_i^t | \Psi^t)$ for DMUs such that the efficiencies of the same DMU in different time periods are correlated according to ρ_λ. Formally, for each $i \in \{1, \ldots, n\}$,

$$\lambda(X_i^1, Y_i^1 | \Psi^1),\ \lambda(X_i^2, Y_i^2 | \Psi^2) \sim 1 + |\mathcal{N}(0, 0.25^2)|, \tag{EC.24}$$

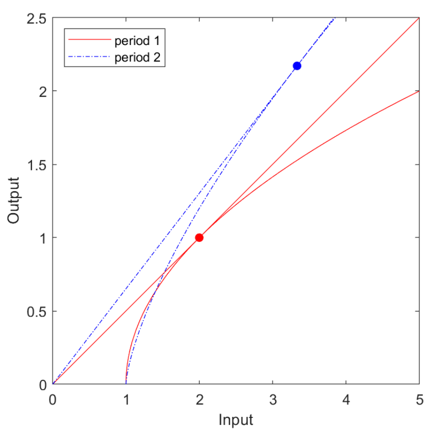
   where $corr\left(\lambda(X_i^1, Y_i^1 | \Psi^t), \lambda(X_i^2, Y_i^2 | \Psi^t)\right) \neq 0$ and depends on the pre-determined parameter ρ_λ mentioned above (see e-companion EC.3 for more details).
3. Use the generated efficiency scores to project the optimal points away from the corresponding frontiers to obtain the set $\{(x_i^t, y_i^t)\}_{i \in \{1, \ldots, n\}, t \in \{1, 2\}}$ where $y_i^t = \psi^t(x_i^t) / \lambda(x_i^t, y_i^t | \Psi^t)$, newly generated in each Monte Carlo trial. Note that we project the optimal points from the frontiers of Ψ^t, not the conical hull $\mathcal{C}(\Psi^t)$.

It is important to highlight that as $q = 1$, the vectors of output prices are scalars and they cancel out in calculations. As such, we set output prices to be unity in our Monte Carlo experiments, i.e., $w^1 = w^2 = 1$. The summary of parameters used in our Monte Carlo experiments is presented in Table EC.1 below. In addition, we provide graphical illustrations of production frontiers in the two and three dimensional spaces in Figure EC.1.

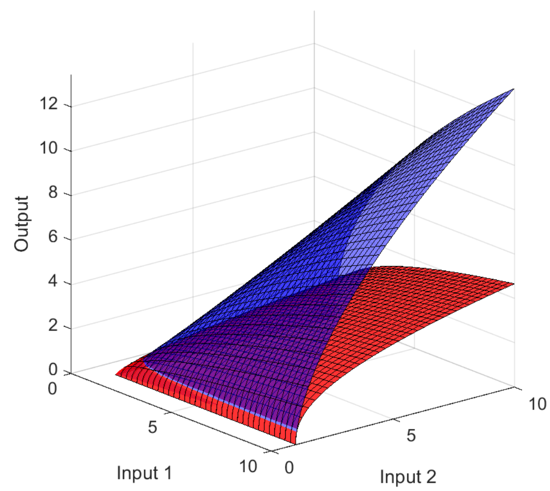
Table EC.1 Summary of experiment parameters.

Parameters	Value
Number of Monte Carlo trials in each experiment (MC)	1000
Number of iterations in estimating the bias (L)	10
Level of significance (α)	$0.01, 0.05, 0.10$
Sample size (n)	10, 20, 50, 100, 500, 1000
Number of inputs (p)	$1, 2, 3, 4$
Parameter controlling deviation from the null hypothesis (δ)	$0.00, 0.01, 0.02, 0.03, 0.04$
Parameter controlling correlation of inputs in periods 1 and 2 (ρ)	0.5
Parameter controlling correlation of efficiencies in periods 1 and 2 (ρ_λ)	0.5
Distribution of the input variables	Uniform(1,10)
Distribution of efficiency	$1 + \lvert\mathcal{N}(0, 0.25^2)\rvert$
Exponents of the production frontier function in period 1 ($\beta_1^1, \ldots, \beta_p^1$)	
$p = 1$	0.5
$p = 2$	$(0.3, 0.4)$
$p = 3$	$(0.1, 0.2, 0.3)$
$p = 4$	$(0.1, 0.15, 0.2, 0.25)$

$\Upsilon^1 = 1$, $\Upsilon^2 = \Upsilon^1 + \delta$; $b_i^t = 1$ $(i = 1, \ldots, p;\ t = 1, 2)$; $\beta_i^2 = \beta_i^1 + \delta$ $(i = 1, \ldots, p)$; number of output $q = 1$.



(a) One input one output



(b) Two inputs one output

Figure EC.1 Production frontiers used in Monte Carlo experiments. Periods 1 and 2 are illustrated by red and blue curves/surfaces, respectively. The parameter δ representing the difference between two time periods is set to be 0.2.

EC.3.2. Simulation results

As can be seen in Table EC.1, we conduct 120 Monte Carlo experiments, each including 1000 trials ($MC = 1000$) and corresponding to a combination of n, p and δ. (To sense the intensity of these simulations, we note that on a machine with 3.3GHz Intel Core i5-6600 CPU, 8GB RAM and MATLAB®R2018b, the runtime of one trial corresponding to the scenario where $p=4$, $n=1000$, $\delta=0.04$ was about 7 minutes and 58 minutes for $L=10$ and $L=100$, respectively.) The rejection rates for testing for productivity change and the estimated coverage of confidence intervals corresponding to $p=1,2,3,4$ are reported in Sections EC.3.3, EC.3.4, EC.3.5, EC.3.6, respectively. In order to obtain the true values of ξ for evaluating the coverages of confidence intervals, we conduct prior Monte Carlo simulations of $n=10\times10^6$ observations without DEA estimation (see e-companion EC.3.8 for more details).

We focus on analyzing the performance of statistical inferences based on the following types of confidence intervals:

(i) Using the "naive" standard central limit theorems, i.e., using the confidence interval (83) with unobserved elements being replaced by their respective DEA estimates.

(ii) Using Theorem 8 for $\kappa\geq 1/2$ $(p+q=2,3)$, i.e., confidence interval (64).

(iii) Using Theorem 8 for $\kappa<1/2$ $(p+q=4,5,6\ldots)$, i.e., confidence interval (65).

(iv) Using confidence interval (68), the re-centered version of (65) as discussed in Remark 4.

For convenience, we will refer to types (i)–(iv) as "ST", "CL1", "CL2", "RC2", respectively.

In general, the simulation results support our newly developed theory. Regarding the hypothesis testing, given a sample size, the rejection rates increase as the parameter δ, representing departure from the null hypothesis, increases. Moreover, for $\delta=0$, the rejection rates (now representing the estimated size of the test) generally tend to approximate the nominal size (i.e., the respective value of α) as the sample size increases. Especially, the rejection rates for $\delta=0$ corresponding to the re-centered confidence intervals RC2 converge to zero as n increases regardless of the values of the nominal size of the test α, confirming our Remark 3 in Section 7. Meanwhile, for $\delta\neq0$, the rejection rates (now representing the estimated power of the test) increase toward unity as the sample size increases, and especially, the bigger values of n show faster convergence to unity.

With regard to the performance of confidence intervals, given a value of δ, when the sample size increases, the estimated coverage shows an upward trend to approximate the respective nominal coverage $1-\alpha$. Similar to KSW2021, the re-centered confidence intervals RC2 show greater coverage, which also converges to 1 rather than the nominal $1-\alpha$, although they have the same width as the confidence intervals CL2. This verifies our Remark 3 in Section 7.

Furthermore, it can be seen that the performance of developed statistical inferences is quite impressive even with small sample sizes such as $n = 10$, e.g., the estimated coverage for ξ using confidence intervals CL1 when $p = 2, \alpha = 0.05, \delta = 0$ is 0.794.

The Monte Carlo evidence here also confirms our Remark 4 mentioned in Section 7. Indeed, when $\kappa = 2/5$ (i.e., $p = 3, q = 1$), the confidence intervals CL2 generally outperform the corresponding CL1 in terms of the size of the test and coverages (see Tables EC.7). Note that, on the contrary, the inferences CL1 perform better than the inferences CL2 in terms of the power of the test. A possible explanation might be due to the trade-off between the size and power of the test, similar to KSW2021.

Interestingly, our Monte Carlo evidence shows that the naive standard approach ST performs quite well in low dimensions (i.e., the number of inputs and outputs $p+q$) where $\kappa \geq 1/2$ (i.e., $p = 1, q = 1$ and $p = 2, q = 1$), illustrated by the fact that its performance is quite similar to that of the CL1 there. Comparing the performance of ST to that of CL2 when $\kappa = 2/5$ and $\kappa = 1/3$ (i.e., $p = 3, q = 1$ and $p = 4, q = 1$), we recognize that they are relatively analogous for small values of δ such as 0 or 0.01, while for $\delta = 0.04$, the latter slightly outperforms the former in terms of estimated coverage of confidence intervals when $n = 1000$. These findings support the conjecture in Remarks 5 and 6 in Section 7 that in some peculiar cases, the bias might be tiny (or even cancel out in some more special symmetric cases such as $\delta = 0$). Also, note that the re-centered approach, RC2, typically gave higher coverage than the ST approach, while having the same width of the estimated confidence intervals.

EC.3.3. Simulation results when $p=1, q=1$

Table EC.2 Rejection rates for test for aggregate productivity change using ξ when $p=1, q=1$.

n	δ	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.01$	
		ST	CL1	ST	CL1	ST	CL1
10	0.00	0.158	0.161	0.100	0.104	0.039	0.034
	0.01	0.169	0.182	0.110	0.112	0.043	0.050
	0.02	0.273	0.286	0.189	0.199	0.094	0.100
	0.03	0.366	0.371	0.276	0.281	0.144	0.142
	0.04	0.558	0.564	0.443	0.443	0.269	0.266
20	0.00	0.130	0.134	0.080	0.080	0.025	0.023
	0.01	0.192	0.192	0.117	0.119	0.046	0.046
	0.02	0.354	0.348	0.257	0.253	0.138	0.138
	0.03	0.558	0.558	0.450	0.453	0.256	0.253
	0.04	0.701	0.708	0.605	0.603	0.435	0.427
50	0.00	0.122	0.125	0.061	0.063	0.013	0.012
	0.01	0.255	0.254	0.163	0.160	0.065	0.063
	0.02	0.557	0.552	0.435	0.426	0.247	0.241
	0.03	0.833	0.834	0.754	0.752	0.558	0.556
	0.04	0.970	0.971	0.950	0.948	0.870	0.864
100	0.00	0.108	0.105	0.058	0.058	0.014	0.013
	0.01	0.356	0.351	0.254	0.254	0.116	0.112
	0.02	0.798	0.792	0.705	0.702	0.480	0.488
	0.03	0.978	0.977	0.963	0.961	0.888	0.888
	0.04	1.000	1.000	1.000	1.000	0.994	0.995
500	0.00	0.113	0.113	0.060	0.060	0.013	0.012
	0.01	0.878	0.878	0.807	0.808	0.589	0.590
	0.02	1.000	1.000	1.000	1.000	0.999	0.999
	0.03	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.00	0.130	0.127	0.069	0.069	0.008	0.008
	0.01	0.984	0.985	0.961	0.960	0.902	0.902
	0.02	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000

$MC=1000$, $L=10$, $\rho=\rho_\lambda=0.5$.

Table EC.3 Coverage of confidence intervals for ξ when $p=1, q=1$.

δ	n	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.01$	
		ST	CL1	ST	CL1	ST	CL1
0.00	10	0.842	0.839	0.900	0.896	0.961	0.966
	20	0.870	0.866	0.920	0.920	0.975	0.977
	50	0.878	0.875	0.939	0.937	0.987	0.988
	100	0.892	0.895	0.942	0.942	0.986	0.987
	500	0.887	0.887	0.940	0.940	0.987	0.988
	1000	0.870	0.873	0.931	0.931	0.992	0.992
0.01	10	0.851	0.848	0.912	0.905	0.968	0.967
	20	0.871	0.869	0.926	0.923	0.978	0.976
	50	0.890	0.896	0.936	0.938	0.983	0.985
	100	0.877	0.879	0.933	0.934	0.989	0.988
	500	0.897	0.898	0.943	0.943	0.993	0.993
	1000	0.887	0.887	0.936	0.936	0.991	0.991
0.02	10	0.834	0.828	0.896	0.887	0.960	0.956
	20	0.830	0.831	0.894	0.887	0.966	0.963
	50	0.883	0.885	0.950	0.945	0.992	0.992
	100	0.902	0.897	0.950	0.951	0.987	0.985
	500	0.918	0.921	0.961	0.961	0.993	0.993
	1000	0.889	0.889	0.949	0.949	0.990	0.991
0.03	10	0.847	0.841	0.904	0.903	0.962	0.961
	20	0.863	0.858	0.920	0.917	0.977	0.976
	50	0.883	0.883	0.942	0.938	0.981	0.981
	100	0.893	0.892	0.946	0.949	0.987	0.986
	500	0.912	0.909	0.954	0.954	0.988	0.989
	1000	0.919	0.922	0.953	0.953	0.991	0.991
0.04	10	0.861	0.851	0.894	0.893	0.960	0.950
	20	0.847	0.852	0.910	0.905	0.965	0.965
	50	0.893	0.891	0.938	0.937	0.984	0.985
	100	0.898	0.895	0.947	0.946	0.988	0.988
	500	0.897	0.895	0.939	0.940	0.986	0.987
	1000	0.894	0.893	0.942	0.942	0.991	0.991

$MC=1000,\ L=10,\ \rho=\rho_\lambda=0.5.$

EC.3.4. Simulation results when $p=2, q=1$

Table EC.4 Rejection rates for test for aggregate productivity change using ξ when $p=2, q=1$.

n	δ	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.01$	
		ST	CL1	ST	CL1	ST	CL1
10	0.00	0.215	0.290	0.136	0.206	0.050	0.114
	0.01	0.311	0.367	0.224	0.294	0.119	0.171
	0.02	0.584	0.566	0.471	0.482	0.322	0.347
	0.03	0.799	0.769	0.707	0.698	0.543	0.550
	0.04	0.928	0.878	0.889	0.849	0.775	0.749
20	0.00	0.142	0.186	0.087	0.115	0.030	0.036
	0.01	0.382	0.392	0.288	0.310	0.133	0.154
	0.02	0.749	0.735	0.672	0.642	0.451	0.464
	0.03	0.960	0.950	0.933	0.926	0.827	0.818
	0.04	0.996	0.992	0.989	0.981	0.956	0.957
50	0.00	0.100	0.109	0.051	0.056	0.009	0.012
	0.01	0.589	0.584	0.462	0.472	0.252	0.278
	0.02	0.975	0.976	0.953	0.952	0.861	0.865
	0.03	0.999	0.999	0.998	0.998	0.993	0.993
	0.04	1.000	1.000	1.000	1.000	1.000	1.000
100	0.00	0.106	0.112	0.053	0.062	0.016	0.018
	0.01	0.845	0.844	0.757	0.763	0.569	0.554
	0.02	1.000	0.999	0.999	0.999	0.994	0.994
	0.03	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000
500	0.00	0.106	0.106	0.052	0.052	0.009	0.008
	0.01	1.000	1.000	1.000	1.000	1.000	1.000
	0.02	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.00	0.097	0.097	0.053	0.054	0.015	0.015
	0.01	1.000	1.000	1.000	1.000	1.000	1.000
	0.02	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000

$MC=1000$, $L=10$, $\rho=\rho_\lambda=0.5$.

Table EC.5 Coverage of confidence intervals for ξ when $p = 2, q = 1$.

δ	n	$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.01$	
		ST	CL1	ST	CL1	ST	CL1
0.00	10	0.785	0.710	0.864	0.794	0.950	0.886
	20	0.858	0.814	0.913	0.885	0.970	0.964
	50	0.900	0.891	0.949	0.944	0.991	0.988
	100	0.894	0.888	0.947	0.938	0.984	0.982
	500	0.894	0.894	0.948	0.948	0.991	0.992
	1000	0.903	0.903	0.947	0.946	0.985	0.985
0.01	10	0.794	0.709	0.855	0.800	0.939	0.886
	20	0.849	0.813	0.914	0.883	0.971	0.955
	50	0.880	0.869	0.937	0.933	0.984	0.986
	100	0.878	0.867	0.934	0.938	0.990	0.988
	500	0.900	0.895	0.946	0.946	0.984	0.983
	1000	0.897	0.897	0.949	0.945	0.988	0.987
0.02	10	0.787	0.702	0.859	0.790	0.931	0.892
	20	0.852	0.825	0.917	0.896	0.969	0.960
	50	0.880	0.879	0.939	0.943	0.982	0.976
	100	0.895	0.888	0.950	0.949	0.991	0.990
	500	0.908	0.906	0.949	0.948	0.985	0.982
	1000	0.894	0.895	0.946	0.945	0.989	0.988
0.03	10	0.812	0.737	0.882	0.802	0.951	0.905
	20	0.848	0.813	0.905	0.877	0.969	0.956
	50	0.904	0.888	0.953	0.938	0.988	0.980
	100	0.906	0.897	0.953	0.950	0.988	0.984
	500	0.882	0.883	0.947	0.949	0.992	0.991
	1000	0.914	0.908	0.963	0.964	0.994	0.993
0.04	10	0.812	0.723	0.865	0.794	0.945	0.897
	20	0.845	0.835	0.916	0.895	0.974	0.966
	50	0.878	0.859	0.935	0.920	0.984	0.982
	100	0.897	0.890	0.952	0.948	0.992	0.990
	500	0.908	0.907	0.944	0.950	0.986	0.991
	1000	0.909	0.899	0.963	0.958	0.994	0.993

$MC = 1000,\ L = 10,\ \rho = \rho_\lambda = 0.5.$

EC.3.5. Simulation results when $p=3, q=1$

Table EC.6 Rejection rates for test for aggregate productivity change using ξ when $p=3, q=1$.

n	δ	$\alpha=0.10$				$\alpha=0.05$				$\alpha=0.01$			
		ST	CL1	CL2	RC2	ST	CL1	CL2	RC2	ST	CL1	CL2	RC2
10	0.00	0.235	0.364	0.333	0.254	0.171	0.278	0.240	0.189	0.078	0.172	0.132	0.099
	0.01	0.365	0.470	0.394	0.350	0.273	0.378	0.318	0.265	0.170	0.249	0.185	0.145
	0.02	0.648	0.648	0.551	0.546	0.567	0.569	0.472	0.471	0.394	0.455	0.348	0.312
	0.03	0.878	0.816	0.713	0.750	0.827	0.765	0.654	0.674	0.681	0.648	0.514	0.511
	0.04	0.967	0.914	0.852	0.880	0.946	0.892	0.799	0.822	0.860	0.811	0.688	0.701
20	0.00	0.196	0.256	0.237	0.143	0.126	0.185	0.160	0.087	0.046	0.097	0.054	0.031
	0.01	0.460	0.491	0.371	0.316	0.360	0.391	0.273	0.219	0.203	0.236	0.149	0.099
	0.02	0.830	0.784	0.621	0.662	0.755	0.727	0.539	0.562	0.575	0.584	0.380	0.359
	0.03	0.988	0.965	0.872	0.913	0.977	0.942	0.822	0.857	0.904	0.873	0.667	0.702
	0.04	1.000	1.000	0.976	0.992	1.000	0.995	0.961	0.977	0.993	0.980	0.900	0.929
50	0.00	0.120	0.152	0.142	0.045	0.078	0.090	0.078	0.018	0.019	0.032	0.021	0.002
	0.01	0.690	0.697	0.439	0.424	0.594	0.588	0.348	0.302	0.391	0.386	0.169	0.109
	0.02	0.997	0.995	0.902	0.965	0.994	0.990	0.863	0.925	0.953	0.955	0.716	0.764
	0.03	1.000	1.000	0.996	1.000	1.000	1.000	0.987	1.000	1.000	1.000	0.964	0.986
	0.04	1.000	1.000	0.999	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.998	1.000
100	0.00	0.123	0.135	0.120	0.022	0.068	0.069	0.056	0.006	0.013	0.022	0.011	0.001
	0.01	0.915	0.917	0.655	0.722	0.879	0.868	0.538	0.542	0.730	0.732	0.309	0.219
	0.02	1.000	1.000	0.989	0.999	1.000	1.000	0.978	0.998	0.999	0.999	0.919	0.985
	0.03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
500	0.00	0.115	0.126	0.107	0.005	0.062	0.069	0.052	0.000	0.016	0.018	0.012	0.000
	0.01	1.000	1.000	0.994	1.000	1.000	1.000	0.981	1.000	1.000	1.000	0.915	0.987
	0.02	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.00	0.113	0.115	0.103	0.002	0.052	0.058	0.052	0.000	0.015	0.014	0.008	0.000
	0.01	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.995	1.000
	0.02	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

$MC=1000$, $L=10$, $\rho=\rho_\lambda=0.5$.

Table EC.7 Coverage of confidence intervals for ξ when $p=3, q=1$.

δ	n	$\alpha=0.10$				$\alpha=0.05$				$\alpha=0.01$			
		ST	CL1	CL2	RC2	ST	CL1	CL2	RC2	ST	CL1	CL2	RC2
0.00	10	0.765	0.636	0.667	0.746	0.829	0.722	0.760	0.811	0.922	0.828	0.868	0.901
	20	0.804	0.744	0.763	0.857	0.874	0.815	0.840	0.913	0.954	0.903	0.946	0.969
	50	0.880	0.848	0.858	0.955	0.922	0.910	0.922	0.982	0.981	0.968	0.979	0.998
	100	0.877	0.865	0.880	0.978	0.932	0.931	0.944	0.994	0.987	0.978	0.989	0.999
	500	0.885	0.874	0.893	0.995	0.938	0.931	0.948	1.000	0.984	0.982	0.988	1.000
	1000	0.887	0.885	0.897	0.998	0.948	0.942	0.948	1.000	0.985	0.986	0.992	1.000
0.01	10	0.778	0.650	0.675	0.759	0.851	0.731	0.744	0.811	0.939	0.828	0.859	0.906
	20	0.810	0.732	0.781	0.850	0.875	0.799	0.851	0.915	0.959	0.908	0.944	0.968
	50	0.864	0.831	0.863	0.954	0.913	0.900	0.919	0.975	0.969	0.965	0.979	0.999
	100	0.882	0.868	0.886	0.980	0.932	0.917	0.939	0.991	0.988	0.976	0.990	1.000
	500	0.902	0.893	0.908	0.998	0.937	0.934	0.956	1.000	0.989	0.984	0.994	1.000
	1000	0.888	0.883	0.892	1.000	0.950	0.937	0.951	1.000	0.991	0.988	0.988	1.000
0.02	10	0.768	0.615	0.640	0.710	0.838	0.685	0.728	0.782	0.933	0.796	0.864	0.884
	20	0.805	0.736	0.785	0.866	0.889	0.816	0.857	0.922	0.965	0.917	0.953	0.975
	50	0.860	0.840	0.860	0.961	0.923	0.903	0.917	0.986	0.987	0.969	0.979	0.998
	100	0.886	0.881	0.880	0.985	0.940	0.929	0.941	0.993	0.983	0.984	0.983	1.000
	500	0.896	0.897	0.916	0.998	0.946	0.937	0.951	1.000	0.993	0.991	0.988	1.000
	1000	0.895	0.900	0.883	0.999	0.960	0.952	0.931	1.000	0.991	0.990	0.992	1.000
0.03	10	0.779	0.636	0.702	0.736	0.846	0.708	0.774	0.809	0.932	0.820	0.874	0.897
	20	0.820	0.742	0.786	0.849	0.881	0.807	0.867	0.916	0.969	0.904	0.943	0.970
	50	0.842	0.832	0.859	0.947	0.927	0.896	0.927	0.982	0.976	0.961	0.978	0.996
	100	0.860	0.842	0.885	0.979	0.921	0.914	0.949	0.994	0.980	0.979	0.985	0.999
	500	0.889	0.885	0.913	0.997	0.940	0.933	0.950	1.000	0.987	0.987	0.993	1.000
	1000	0.900	0.894	0.897	1.000	0.949	0.952	0.941	1.000	0.994	0.995	0.989	1.000
0.04	10	0.778	0.630	0.682	0.739	0.856	0.712	0.757	0.812	0.936	0.825	0.879	0.907
	20	0.819	0.738	0.779	0.871	0.884	0.816	0.852	0.935	0.967	0.931	0.950	0.978
	50	0.858	0.834	0.886	0.952	0.926	0.902	0.932	0.977	0.980	0.962	0.983	0.994
	100	0.873	0.862	0.866	0.979	0.937	0.929	0.931	0.988	0.977	0.979	0.981	0.999
	500	0.910	0.902	0.896	0.999	0.953	0.949	0.948	1.000	0.987	0.985	0.985	1.000
	1000	0.906	0.907	0.913	0.999	0.946	0.949	0.960	0.999	0.992	0.992	0.993	1.000

$MC=1000$, $L=10$, $\rho=\rho_\lambda=0.5$.

EC.3.6. Simulation results when $p=4, q=1$

Table EC.8 Rejection rates for test for aggregate productivity change using ξ when $p=4, q=1$.

n	δ	$\alpha=0.10$				$\alpha=0.05$				$\alpha=0.01$			
		ST	CL1	CL2	RC2	ST	CL1	CL2	RC2	ST	CL1	CL2	RC2
10	0.00	0.239	0.465	0.359	0.293	0.170	0.387	0.285	0.213	0.086	0.262	0.178	0.118
	0.01	0.458	0.547	0.428	0.393	0.382	0.480	0.354	0.305	0.241	0.368	0.243	0.196
	0.02	0.810	0.727	0.610	0.605	0.755	0.676	0.530	0.519	0.617	0.568	0.411	0.390
	0.03	0.975	0.888	0.785	0.814	0.951	0.860	0.728	0.753	0.884	0.792	0.596	0.594
	0.04	0.993	0.955	0.890	0.915	0.992	0.941	0.856	0.886	0.976	0.905	0.771	0.803
20	0.00	0.197	0.351	0.236	0.140	0.120	0.274	0.156	0.078	0.049	0.155	0.083	0.019
	0.01	0.618	0.602	0.399	0.363	0.523	0.524	0.325	0.265	0.346	0.388	0.190	0.115
	0.02	0.964	0.903	0.720	0.780	0.941	0.874	0.637	0.689	0.868	0.802	0.477	0.474
	0.03	1.000	0.991	0.919	0.950	0.999	0.984	0.888	0.925	0.987	0.960	0.797	0.831
	0.04	1.000	1.000	0.988	0.997	1.000	1.000	0.973	0.994	1.000	0.999	0.939	0.973
50	0.00	0.179	0.249	0.154	0.028	0.123	0.177	0.088	0.011	0.041	0.071	0.030	0.001
	0.01	0.912	0.875	0.533	0.549	0.862	0.828	0.434	0.385	0.716	0.700	0.262	0.132
	0.02	1.000	1.000	0.926	0.990	1.000	0.998	0.877	0.964	0.999	0.994	0.731	0.841
	0.03	1.000	1.000	0.996	1.000	1.000	1.000	0.991	1.000	1.000	1.000	0.971	0.999
	0.04	1.000	1.000	0.999	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.996	1.000
100	0.00	0.122	0.176	0.132	0.005	0.074	0.110	0.079	0.002	0.019	0.036	0.022	0.000
	0.01	0.995	0.991	0.656	0.796	0.989	0.979	0.541	0.610	0.956	0.931	0.333	0.230
	0.02	1.000	1.000	0.987	1.000	1.000	1.000	0.979	1.000	1.000	1.000	0.912	0.997
	0.03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
500	0.00	0.111	0.123	0.105	0.000	0.061	0.069	0.051	0.000	0.018	0.016	0.016	0.000
	0.01	1.000	1.000	0.980	1.000	1.000	1.000	0.952	1.000	1.000	1.000	0.832	0.997
	0.02	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.00	0.106	0.119	0.103	0.000	0.044	0.056	0.052	0.000	0.010	0.009	0.009	0.000
	0.01	1.000	1.000	0.999	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.984	1.000
	0.02	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.03	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.04	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

$MC=1000$, $L=10$, $\rho=\rho_\lambda=0.5$.

Table EC.9 Coverage of confidence intervals for ξ when $p = 4, q = 1$.

δ	n	$\alpha = 0.10$				$\alpha = 0.05$				$\alpha = 0.01$			
		ST	CL1	CL2	RC2	ST	CL1	CL2	RC2	ST	CL1	CL2	RC2
0.00	10	0.761	0.535	0.641	0.707	0.830	0.613	0.715	0.787	0.914	0.738	0.822	0.882
	20	0.803	0.649	0.764	0.860	0.880	0.726	0.844	0.922	0.951	0.845	0.917	0.981
	50	0.821	0.751	0.846	0.972	0.877	0.823	0.912	0.989	0.959	0.929	0.970	0.999
	100	0.878	0.824	0.868	0.995	0.926	0.890	0.921	0.998	0.981	0.964	0.978	1.000
	500	0.889	0.877	0.895	1.000	0.939	0.931	0.949	1.000	0.982	0.984	0.984	1.000
	1000	0.894	0.881	0.897	1.000	0.956	0.944	0.948	1.000	0.990	0.991	0.991	1.000
0.01	10	0.738	0.513	0.606	0.680	0.816	0.601	0.687	0.760	0.920	0.704	0.821	0.867
	20	0.791	0.633	0.766	0.868	0.870	0.718	0.845	0.920	0.948	0.848	0.935	0.975
	50	0.840	0.752	0.837	0.967	0.893	0.843	0.898	0.987	0.967	0.923	0.975	0.999
	100	0.876	0.822	0.873	0.995	0.932	0.884	0.936	0.999	0.982	0.959	0.986	1.000
	500	0.895	0.874	0.903	1.000	0.945	0.942	0.955	1.000	0.989	0.984	0.988	1.000
	1000	0.901	0.897	0.905	1.000	0.948	0.942	0.955	1.000	0.993	0.991	0.995	1.000
0.02	10	0.754	0.509	0.625	0.686	0.814	0.573	0.687	0.770	0.906	0.710	0.820	0.882
	20	0.758	0.642	0.729	0.847	0.836	0.722	0.804	0.904	0.944	0.826	0.924	0.962
	50	0.827	0.732	0.848	0.968	0.906	0.818	0.910	0.991	0.974	0.933	0.969	0.999
	100	0.858	0.798	0.854	0.995	0.922	0.868	0.935	0.998	0.976	0.960	0.986	1.000
	500	0.897	0.871	0.889	1.000	0.946	0.935	0.946	1.000	0.990	0.990	0.990	1.000
	1000	0.912	0.901	0.892	1.000	0.957	0.948	0.947	1.000	0.990	0.991	0.991	1.000
0.03	10	0.759	0.519	0.633	0.684	0.827	0.598	0.702	0.759	0.920	0.709	0.822	0.877
	20	0.787	0.601	0.747	0.840	0.855	0.691	0.827	0.900	0.934	0.817	0.916	0.968
	50	0.833	0.739	0.843	0.969	0.892	0.822	0.904	0.985	0.962	0.918	0.968	1.000
	100	0.869	0.817	0.879	0.994	0.928	0.897	0.935	0.999	0.980	0.955	0.983	1.000
	500	0.891	0.861	0.899	1.000	0.941	0.929	0.947	1.000	0.990	0.989	0.991	1.000
	1000	0.875	0.880	0.892	1.000	0.939	0.943	0.943	1.000	0.989	0.983	0.989	1.000
0.04	10	0.736	0.495	0.606	0.659	0.804	0.560	0.684	0.733	0.898	0.684	0.804	0.863
	20	0.776	0.641	0.751	0.855	0.855	0.725	0.835	0.916	0.938	0.840	0.925	0.974
	50	0.821	0.735	0.859	0.963	0.896	0.813	0.902	0.985	0.973	0.918	0.969	0.998
	100	0.855	0.815	0.870	0.994	0.923	0.896	0.932	0.998	0.984	0.958	0.983	1.000
	500	0.891	0.877	0.898	1.000	0.938	0.923	0.951	1.000	0.989	0.982	0.995	1.000
	1000	0.900	0.889	0.902	1.000	0.943	0.938	0.950	1.000	0.986	0.981	0.986	1.000

$MC = 1000,\ L = 10,\ \rho = \rho_\lambda = 0.5.$

EC.3.7. A procedure to generate correlated random numbers for Monte Carlo experiments

Here we describe how we generate correlated random numbers in our Monte Carlo simulations. Basically, the method is inspired by Kneip et al. (2021). Assume that we want to generate a dataset of n observations from random variables $V_j^{(t)}$ $(j=1,\dots,p; t=1,2)$, where $V_j^{(t)} \sim \text{Uniform}(d_{j,min}^{(t)}, d_{j,max}^{(t)})$ and $corr(V_j^{(1)}, V_j^{(2)}) = \rho$ for all j. The details are as follows.

1. Construct the matrix

$$U = \begin{bmatrix} I_p & \rho I_p \\ 0 & \sqrt{1-\rho^2} I_p \end{bmatrix} \tag{EC.25}$$

   where I_p is the identity matrix of order p. It can be seen that $U'U = C$ where

$$C = \begin{bmatrix} I_p & \rho I_p \\ \rho I_p & I_p \end{bmatrix}. \tag{EC.26}$$

2. Generate an $n \times 2p$ matrix R of iid $\mathcal{N}(0,1)$.
3. Compute the matrix $D = \Phi(RU)$ where $\Phi(.)$ is the cdf of the standard normal distribution. Partition D into $D = [D^{(1)}, D^{(2)}]$, where $D^{(t)} = [D_1^{(t)}, \dots, D_p^{(t)}]$ for $t = 1,2$. Then it is clear that $D_j^{(t)}$ follows uniform distribution on $(0,1)$ and $corr(D_j^{(1)}, D_j^{(2)}) = \rho$ $(j=1,\dots,p; t=1,2)$.
4. For each $j = 1,\dots,p$, transform $D_j^{(t)}$ to $V_j^{(t)} = (d_{j,max}^{(t)} - d_{j,min}^{(t)}) D_j^{(t)} + d_{j,min}^{(t)}$. We then have numbers $V_j^{(t)}$ follow uniform distribution on $(d_{j,min}^{(t)}, d_{j,max}^{(t)})$ and $corr(V_j^{(1)}, V_j^{(2)}) = \rho$.
5. If we want to change the distribution of $V_j^{(t)}$ for some t and j to a new one other than uniform distribution (e.g., half normal distribution plus unity: $1 + |N(0,\sigma^2)|$), we just simply need to replace the transformation in step 4 by $V_j^{(t)} = F^{-1}(D_j^{(t)})$ where F^{-1} is the quantile function of the desired distribution. By this way, $corr(V_j^{(1)}, V_j^{(2)})$ might not be exactly ρ but is determined via a function of ρ which in turn depends on the new distribution. However, this does not raise any issue since our purpose in this paper is to ensure that the numbers generated are correlated and we can control the degree of correlation via the parameter ρ.

EC.3.8. Computing the true parameters of interest

Similar to KSW2021, we only know the efficiency scores measured toward the true underlying production technology sets (i.e., $\lambda(.|\Psi^t)$). Meanwhile those measured toward the conical hull of the production technology sets (i.e., $\lambda_C(.|\Psi^t)$) are unknown (see step 3 of the data generating process mentioned in Section EC.3.1). We show how to obtain these figures below.

Assume that we have to measure the efficiency of a DMU represented by (x,y) toward $\mathcal{C}(\Psi^t)$ $(t=1,2)$. Taking advantage of the fact that $y\in\mathbb{R}^1_+$, we can transform the efficiency score as follows.

$$\begin{aligned}\lambda_C(x,y|\Psi^t) &= \sup_{\lambda}\{\lambda:(x,\lambda y)\in\mathcal{C}(\Psi^t)\} = \sup_{\lambda}\{\lambda:(x,\lambda y)=(a\tilde{x},a\tilde{y}),(\tilde{x},\tilde{y})\in\Psi^t,a\in\mathbb{R}^1_+\}\\ &= \sup_{\lambda}\{\lambda:(x,\lambda y)=(a\tilde{x},a\tilde{y}),\tilde{y}\le\psi^t(\tilde{x}),\tilde{x}\in\mathbb{R}^p_+,a\in\mathbb{R}^1_+\}\\ &= \sup_{\lambda}\{\lambda:(x,\lambda y)=(a\tilde{x},a\psi^t(\tilde{x})),\tilde{x}\in\mathbb{R}^p_+,a\in\mathbb{R}^1_+\}\\ &= \sup_{a,\tilde{x}}\left\{\frac{a\psi^t(\tilde{x})}{y}:a\tilde{x}=x,\tilde{x}\in\mathbb{R}^p_+,a\in\mathbb{R}^1_+\right\} = \sup_{a}\left\{\frac{a\psi^t(x/a)}{y}:a\in\mathbb{R}^1_+\right\}\\ &= \frac{1}{y}\max_{a>0}\left(a\psi^t\left(\frac{x}{a}\right)\right).\end{aligned} \tag{EC.27}$$

Now one can apply available optimization solvers (e.g., "fmincon" in MATLAB®) to solve for $\max_{a>0}\left(a\psi^t(\frac{x}{a})\right)$ and then obtain $\lambda_C(x,y|\Psi^t)$. It is noteworthy that there is a potential risk of receiving a local optimum instead of the global one. Our further analysis below will demonstrate that the global optimum will be achieved.

We fix (x,y) and consider the function

$$\varphi(a) := a\psi^t(x/a) = a\Upsilon^t(x_1/a-b_1^t)^{\beta_1^t}(x_2/a-b_2^t)^{\beta_2^t}\dots(x_p/a-b_p^t)^{\beta_p^t} \tag{EC.28}$$

with the domain $\mathbb{D}=\left(0,\min_{j=1,\dots,p}\dfrac{x_j}{b_j^t}\right)$. Its first derivative is presented below.

$$\begin{aligned}\frac{d\varphi}{da} &= \Upsilon^t(x_1/a-b_1^t)^{\beta_1^t}(x_2/a-b_2^t)^{\beta_2^t}\dots(x_p/a-b_p^t)^{\beta_p^t}\\ &\quad - a\Upsilon^t\left[\beta_1^t(x_1/a-b_1^t)^{\beta_1^t-1}x_1a^{-2}\right](x_2/a-b_2^t)^{\beta_2^t}\dots(x_p/a-b_p^t)^{\beta_p^t}\\ &\quad - a\Upsilon^t(x_1/a-b_1^t)^{\beta_1^t}\left[\beta_2^t(x_2/a-b_2^t)^{\beta_2^t-1}x_2a^{-2}\right]\dots(x_p/a-b_p^t)^{\beta_p^t}\dots\end{aligned}$$

$$
\begin{aligned}
&- a\Upsilon^t (x_1/a - b_1^t)^{\beta_1^t} (x_2/a - b_2^t)^{\beta_2^t} \dots \left[\beta_p^t (x_p/a - b_p^t)^{\beta_p^t - 1} x_p a^{-2}\right] \\
&= \Upsilon^t (x_1/a - b_1^t)^{\beta_1^t} (x_2/a - b_2^t)^{\beta_2^t} \dots (x_p/a - b_p^t)^{\beta_p^t} \left(1 - \frac{x_1 \beta_1^t}{a(x_1/a - b_1^t)} - \dots - \frac{x_p \beta_p^t}{a(x_p/a - b_p^t)}\right) \\
&= \psi^t(x/a) \left(1 - \sum_{j=1}^{p} \frac{x_j \beta_j^t}{x_j - a b_j^t}\right) = \psi^t(x/a)\iota(a), \qquad \text{(EC.29)}
\end{aligned}
$$

where

$$
\iota(a) = 1 - \sum_{j=1}^{p} \frac{x_j \beta_j^t}{x_j - a b_j^t}. \qquad \text{(EC.30)}
$$

Therefore, the sign of $d\varphi/da$ is the same as that of $\iota(a)$ and furthermore, $d\varphi/da = 0$ if and only if $\iota(a) = 0$. Since $\frac{d\iota}{da} = -\sum_{j=1}^{p} \frac{x_j \beta_j^t b_j^t}{(x_j - a b_j^t)^2} < 0 \ \forall a > 0$, $\iota(a)$ is strictly decreasing. In addition,

$$
\lim_{a \to 0^+} \iota(a) = 1 - \sum_{j=1}^{p} \beta_j^t > 0, \quad \lim_{a \to \frac{x_j}{b_j^t}^-} \iota(a) = -\infty, \quad \lim_{a \to \frac{x_j}{b_j^t}^+} \iota(a) = +\infty, \lim_{a \to +\infty} \iota(a) = 1 > 0.
$$

Hence, $\iota(a) = 0$ has only one solution in the domain $\mathbb{D}$, say a^*. Moreover, the above findings show that $\iota(a)$ changes sign from positive to negative when passing a^* in the left-right direction, and hence, so is $d\varphi/da$. This implies that a^* is the global maximum of $\varphi(a)$ in the domain $\mathbb{D}$ as desired.

Table EC.10 True values of parameter ξ.

δ	$p=1, q=1$	$p=2, q=1$	$p=3, q=1$	$p=4, q=1$
0.00	0.00000	0.00000	0.00000	0.00000
0.01	0.02528	0.03916	0.05088	0.06385
0.02	0.05050	0.07820	0.10189	0.12791
0.03	0.07567	0.11745	0.15292	0.19215
0.04	0.10079	0.15663	0.20415	0.25657

$n = 10 \times 10^6$, $\rho = \rho_\lambda = 0.5$.

The true values of the parameter ξ are reported in Table EC.10. Note that we set $\xi = 0$ for $\delta = 0$ since the production frontier functions $\psi^1(x)$ and $\psi^2(x)$ are identical here and hence there is no productivity change. Prior Monte Carlo simulations (with $n = 10 \times 10^6$) also justify this point with some acceptable noises due to finite n, e.g., $\xi = 7 \times 10^{-8}$ for $p = 1$.

Appendix EC.4: Empirical data and computational codes

Table EC.11: Data Used in the Empirical Illustration (from PWT10)

COUNTRY	Labor 1990 (*millions*)	Capital 1990 (*mil. 2017US$*)	GDP 1990 (*mil. 2017US$*)	Labor 2019 (*millions*)	Capital 2019 (*mil. 2017US$*)	GDP 2019 (*mil. 2017US$*)	D
Albania	1.324	61056.902	12096.719	1.076	229818.328	36103.043	0
Argentina	11.811	688641.750	204212.688	20.643	3374818.500	977420.562	0
Armenia	1.833	58440.266	24350.492	0.966	98945.867	43582.574	0
Australia	7.845	2247378.750	507695.969	12.863	5899094.000	1364677.750	1
Austria	3.560	992590.125	205574.156	4.550	2875459.750	477705.500	1
Azerbaijan	3.878	111656.461	76027.461	5.027	273638.750	159346.000	0
Belgium	3.861	1150535.500	261314.031	4.922	3496175.000	517419.750	1
Bulgaria	4.097	86723.477	106472.703	3.420	458279.156	149432.422	0
Belarus	5.126	712582.625	145477.172	4.306	656323.125	204795.344	0
Bolivia (Pl. State of)	2.890	34102.156	16026.933	5.544	215585.516	98835.711	0
Brazil	61.632	3226204.500	954572.000	93.957	13677290.000	3080048.500	0
Canada	13.290	3838143.500	941469.375	19.299	8721109.000	1866214.875	1
Switzerland	3.833	1428866.500	281175.219	5.011	3225664.000	646919.625	1
Chile	4.323	340115.500	106364.055	8.100	2075795.000	440683.344	0
China	660.477	5404394.500	2856361.250	798.808	101544168.000	20257660.000	0
Colombia	10.505	908843.438	252406.266	21.200	2194572.500	707659.625	0
Costa Rica	1.085	50398.020	26685.379	2.256	236104.750	93491.828	0
Czech Republic	5.414	1184608.250	250625.656	5.481	2375330.250	401072.219	0
Germany	39.548	9745840.000	2221450.250	44.795	20907856.000	4275312.000	1
Denmark	2.634	607417.000	143015.422	2.972	1594880.000	311838.000	1
Dominican Republic	2.106	90115.703	36452.879	5.127	774568.875	192514.500	0
Ecuador	3.336	214565.938	63273.090	8.247	1162897.125	195204.016	0
Spain	14.039	3213968.500	728396.125	19.872	11827519.000	1886595.250	1
Estonia	0.829	73727.727	19404.123	0.673	219444.094	44876.348	0
Finland	2.473	696621.375	137362.797	2.674	1281196.500	248551.547	1
France	23.660	6785194.000	1581094.750	28.533	17987252.000	2946958.250	1
United Kingdom	26.704	4443014.500	1445536.750	32.982	15302674.000	2989895.500	1
Greece	4.063	1109733.500	181550.156	4.235	2565751.000	284893.625	1
Guatemala	3.119	67479.258	37128.180	7.093	450848.750	136605.016	0
China, Hong Kong SAR	2.731	403845.594	187216.141	3.864	2566180.000	407575.750	1
Honduras	1.444	28410.459	16123.038	3.833	235830.922	52902.539	0
Croatia	2.178	198289.812	65532.152	1.818	602383.250	107392.039	0
Hungary	5.126	459231.188	147425.906	4.712	1428009.375	283145.156	0
Indonesia	73.691	989806.875	625933.562	131.171	18173848.000	3137931.000	0
India	345.387	2664443.750	1248876.875	497.616	35324124.000	9170555.000	0
Ireland	1.215	231919.500	66850.164	2.260	1868599.250	501053.594	1
Iceland	0.138	45228.027	8657.975	0.192	93709.984	17972.625	1
Israel	1.735	385235.938	105208.664	4.229	1226272.125	328529.531	1
Italy	22.803	6936038.000	1541241.875	25.596	18855818.000	2466327.500	1
Jamaica	0.896	35057.539	12786.100	1.245	152178.188	24867.324	0
Japan	65.104	14416034.000	3658643.500	69.977	26102876.000	5036891.000	1
Kazakhstan	7.556	635239.000	201472.328	8.821	1148598.750	525124.062	0
Kenya	9.065	102847.047	54920.566	25.074	540344.688	222738.469	0
Kyrgyzstan	1.743	62718.914	36275.133	2.671	87622.602	39366.727	0
Republic of Korea	18.206	1737276.500	593101.812	26.799	11164507.000	2162705.250	1
Sri Lanka	5.043	97555.164	58180.059	8.180	828180.250	283395.000	0
Lithuania	1.706	118734.898	54314.094	1.380	379209.500	89642.180	0
Latvia	1.254	140698.047	44794.594	0.897	460683.312	56079.445	0
Morocco	7.549	351582.594	112781.969	11.522	1531662.875	288999.750	0

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Table EC.11 – *continued from previous page*

COUNTRY	Labor 1990 (*millions*)	Capital 1990 (*mil. 2017US$*)	GDP 1990 (*mil. 2017US$*)	Labor 2019 (*millions*)	Capital 2019 (*mil. 2017US$*)	GDP 2019 (*mil. 2017US$*)	D
Republic of Moldova	2.058	101708.656	27612.031	1.429	116100.477	38162.605	0
Madagascar	5.151	17904.010	16515.250	13.945	101824.883	41507.461	0
Mexico	28.223	2886328.250	1039442.875	54.994	10933568.000	2390322.500	0
North Macedonia	0.959	39136.957	13045.395	0.859	138411.672	32876.156	0
Mauritius	0.404	14596.421	12508.182	0.583	97370.297	29758.160	0
Malawi	3.874	16439.533	9986.730	7.950	24313.982	21635.066	0
Malaysia	6.832	571888.000	168216.734	15.118	3598296.250	822237.375	0
Nigeria	29.960	908735.312	92692.852	73.021	3203823.500	1001537.313	0
Netherlands	6.808	1743256.375	425511.375	9.457	4745302.500	950077.938	1
Norway	2.061	636545.375	139160.562	2.854	1723067.250	396253.875	1
New Zealand	1.521	274668.281	78282.383	2.506	656662.812	198604.609	1
Panama	0.701	34511.410	17177.756	1.920	524603.062	126883.188	0
Peru	7.723	227863.094	81677.820	16.890	1466834.875	397821.031	0
Philippines	22.012	547520.938	253960.750	42.425	2753356.000	913426.812	0
Poland	15.083	1251424.875	324730.812	16.159	3148161.000	1211846.250	0
Portugal	4.467	795612.188	156697.391	4.962	2913985.250	325160.031	1
Paraguay	1.544	60377.164	22101.969	3.358	288482.906	86788.555	0
Romania	10.901	435373.125	192325.672	8.680	1805810.125	540045.750	0
Russian Federation	75.279	14899058.000	2493859.500	71.671	19423886.000	4161194.500	0
Singapore	1.530	206869.391	86285.148	3.760	2148978.000	477907.875	1
Sierra Leone	1.475	2868.818	6813.255	2.536	16960.162	14651.260	0
Slovakia	2.450	394413.062	104279.203	2.469	777340.312	149921.438	0
Slovenia	1.129	195457.500	40194.348	1.047	493944.844	70868.516	0
Sweden	4.604	1200367.125	259599.234	5.002	2743785.000	526240.812	1
Syrian Arab Republic	3.307	24667.023	18222.824	4.761	407660.312	123085.008	0
Thailand	28.705	837414.500	335173.781	37.541	5630091.000	1191732.875	0
Tajikistan	1.943	87706.352	37711.543	2.683	357036.719	36293.750	0
Turkey	17.739	1649038.375	617188.000	28.087	10235724.000	2248225.750	0
Taiwan	8.649	953378.375	455792.906	11.500	4258740.000	1103379.750	1
Ukraine	25.068	3192392.250	614836.375	16.499	7085837.500	578290.500	0
Uruguay	1.264	104580.078	30050.559	1.635	318946.031	71122.789	0
United States	123.046	38223704.000	9978170.000	158.300	69059088.000	20595844.000	1
Venezuela (B. Rep. of)	6.405	766267.312	166738.797	11.694	217905.094	7160.107	0
Zambia	2.389	39799.594	10770.718	5.225	303724.188	56783.715	0
Zimbabwe	4.568	37124.984	55290.629	6.831	69588.242	40826.570	0

This data is originally from the Penn World Tables (version 10) for the sample of countries considered by Badunenko et al (2008). D = 1 if "developed", D = 0 if "developing".

MATLAB codes to replicate the empirical illustration are available at:

https://www.dropbox.com/s/2uqfchkw8w53rl6/MATLAB%20codes%20for%20EmpIllustr_PSZ2023_OpRe_v1.zip?dl=0.