Electronic Companion
Revenue Management with Heterogeneous Resources: Unit Resource Capacities, Advance Bookings, and Itineraries over Time Intervals

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Appendix A: Offering Itineraries Only with Positive Contributions

We establish a lemma that we use throughout the paper to argue that some of our policies never offer an unavailable resource. In particular, we show that if the net revenue contribution from making a sale for a resource is nonpositive, then there exists an assortment that maximizes the net expected revenue from a customer without offering this resource. The proof of the lemma strictly uses the fact that the choice probabilities of the resources in an assortment decrease as we add more resources into the assortment; that is, $\phi_i^q(S \cup \{j\}) \leq \phi_i^q(S)$ for all $i \in S$ and $j \notin S$. For fixed net revenue contributions $\{p_i : i \in \mathcal{N}\}$, we focus on the problem

$$\max_{S \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{N}} \phi_i^q(S)\, p_i \right\}. \tag{16}$$

Lemma A.1 (Offering Only Positive Contributions) *There exists an optimal solution S^* to problem* (16) *such that $S^* \subseteq \{i \in \mathcal{N} : p_i > 0\}$.*

Proof: Letting S^* be an optimal solution to problem (16), we define the assortment $\hat{S} \subseteq S^*$ as $\hat{S} = \{i \in S^* : p_i > 0\}$. By the assumption that $\phi_i^q(S \cup \{j\}) \leq \phi_i^q(S)$ for all $i \in S$ and $j \notin S$, we have $\phi_i^q(S^*) \leq \phi_i^q(\hat{S})$ for all $i \in \hat{S}$. In this case, using the fact that $p_i \leq 0$ for all $i \in S^* \setminus \hat{S}$, along with $p_i > 0$ for all $i \in \hat{S}$, we obtain the chain of inequalities

$$\sum_{i \in S^*} \phi_i^q(S^*)\, p_i \;=\; \sum_{i \in \hat{S}} \phi_i^q(S^*)\, p_i + \sum_{i \in S^* \setminus \hat{S}} \phi_i^q(S^*)\, p_i \;\leq\; \sum_{i \in \hat{S}} \phi_i^q(S^*)\, p_i \;\leq\; \sum_{i \in \hat{S}} \phi_i^q(\hat{S})\, p_i.$$

Thus, the chain of inequalities above shows that $\hat{S}$ is also an optimal solution to problem (16). Furthermore, we have $\hat{S} \subseteq \{i \in \mathcal{N} : p_i > 0\}$. ∎

Appendix B: Performance Guarantee for Resource Based Static Policy

We give the proof of Theorem 4.1. The proof is based on giving an upper bound on the performance of the optimal policy and a lower bound on the performance of the resource based static policy.

B.1 Preliminary Lemmas

We will need three lemmas. Resource i is available for booking over interval $[s,f]$ if and only if $\prod_{\ell=s}^{f} x_{i,\ell} = 1$. In the next lemma, we give a lower bound on $\prod_{\ell=s}^{f} x_{i,\ell}$ that is linear in $\boldsymbol{x}$.

Lemma B.1 (Linear Proxy for Resource Availability Condition) *For each* $\boldsymbol{x} \in \{0,1\}^{N\times T}$*,* $i \in \mathcal{N}$*, and* $[s,f] \in \mathcal{F}$*, we have*

$$\prod_{\ell=s}^{f} x_{i,\ell} \;\geq\; \frac{\left(\sum_{\ell=s}^{f} x_{i,\ell}\right) - (f-s)}{1+f-s}.$$

Proof: First, assume that $\prod_{\ell=s}^{f} x_{i,\ell} = 1$. Thus, we have $x_{i,\ell} = 1$ for all $\ell = s,\ldots,f$, so that $\sum_{\ell=s}^{f} x_{i,\ell} = f-s+1$. In this case, the left side of the inequality in the lemma is one, whereas the right side is $\frac{1}{1+f-s}$. Because $1 \geq \frac{1}{1+f-s}$, the inequality holds whenever $\prod_{\ell=s}^{f} x_{i,\ell} = 1$. Second, assume that $\prod_{\ell=s}^{f} x_{i,\ell} = 0$. Thus, we have $x_{i,\ell} = 0$ for some $\ell = s,\ldots,f$, so that $\sum_{\ell=s}^{f} x_{i,\ell} \leq f-s$. In this case, noting that $\sum_{\ell=s}^{f} x_{i,\ell} - (f-s) \leq 0$, the left side of the inequality in the lemma is zero, whereas the right side is at most zero, so the inequality holds whenever $\prod_{\ell=s}^{f} x_{i,\ell} = 0$. ■

The above lemma requires every resource to be unique so that $\boldsymbol{x} \in \{0,1\}^{N\times T}$. The next lemma shows that the contribution of each resource to the objective value in (6) is nonnegative.

Lemma B.2 (Nonnegative Contribution of Each Resource) *For each* $i \in \mathcal{N}$*,* $[s,f] \in \mathcal{F}$*, and* $q \in \mathcal{Q}$*, we have* $\phi_i^q(A_{[s,f]}^q)\left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta_{i,h}^{q+1}\right] \geq 0$.

Proof: For notational brevity, fixing $[s,f] \in \mathcal{F}$ and $q \in \mathcal{Q}$, let $p_i = r_{i,[s,f]} - \sum_{h=s}^{f} \eta_{i,h}^{q+1}$ for each $i \in \mathcal{N}$. Suppose on the contrary that we have $\phi_k^q(A_{[s,f]}^q)\, p_k < 0$ for some $k \in \mathcal{N}$. Then, we have $p_k < 0$ and $\phi_k^q(A_{[s,f]}^q) > 0$. Noting $\phi_k^q(A_{[s,f]}^q) > 0$, since the booking probability of a resource that is not offered is zero, it must be the case that $k \in A_{[s,f]}^q$. We partition the assortment of resources $A_{[s,f]}^q$ into $A^+ = \{j \in A_{[s,f]}^q : p_j \geq 0\}$ and $A^- = \{j \in A_{[s,f]}^q : p_j < 0\}$. Using the fact that we have $k \in A^-$ and $\phi_k^q(A_{[s,f]}^q)\, p_k < 0$, we have the chain of inequalities

$$\sum_{i\in\mathcal{N}} \phi_i^q(A_{[s,f]}^q)\, p_i = \sum_{i\in A^+} \phi_i^q(A_{[s,f]}^q)\, p_i + \sum_{i\in A^-} \phi_i^q(A_{[s,f]}^q)\, p_i < \sum_{i\in A^+} \phi_i^q(A_{[s,f]}^q)\, p_i \overset{(a)}{\leq} \sum_{i\in A^+} \phi_i^q(A^+)\, p_i,$$

where (a) uses the assumption that $\phi_i^q(S\cup\{j\}) \leq \phi_i^q(S)$ for all $i \in S$ and $j \notin S$. The chain of inequalities above contradicts the fact that $A_{[s,f]}^q$ is an optimal solution to problem (6). ■

In the next lemma, we give an inequality that will become useful to lower bound the total expected revenue obtained by the resource based static policy.

Lemma B.3 (Upper Bound on Net Expected Revenue) *Letting* $\{A^q_{[s,f]} : [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ *and* $\{\eta^q_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T},\ q \in \mathcal{Q}\}$ *be computed through* (6) *and* (7)*, we have*

$$\sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i\big(A^q_{[s,f]}\big)\, \frac{f-s}{f-s+1} \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \;\le\; \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \big(\hat{J}^q_{\mathsf{L}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{e})\big).$$

Proof: Because $x/(x+1)$ is increasing in $x \in \mathbb{R}_+$ and $f-s+1 \le \mathsf{D}_{\max}$, we have $\frac{f-s}{f-s+1} \le \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}}$. Noting that $\phi^q_i(A^q_{[s,f]})\big[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h}\big] \ge 0$ for all $i \in \mathcal{N}$ and $[s,f] \in \mathcal{F}$ by Lemma B.2, we get

$$\begin{aligned}
&\sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i\big(A^q_{[s,f]}\big)\; \frac{f-s}{f-s+1} \left[r_{i,[s,f]} \;-\; \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \\
&\qquad \le\; \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \sum_{i\in\mathcal{N}} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\; \phi^q_i(A^q_{[s,f]}) \left[r_{i,[s,f]} \;-\; \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \\
&\qquad \overset{(a)}{=}\; \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\; \phi^q_i(A^q_{[s,f]})\, \frac{\mathbb{1}_{\{\ell\in[s,f]\}}}{f-s+1} \left[r_{i,[s,f]} \;-\; \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \\
&\qquad \overset{(b)}{=}\; \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} (\eta^q_{i,\ell} - \eta^{q+1}_{i,\ell}) \;\overset{(c)}{=}\; \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \big(\hat{J}^q_{\mathsf{L}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{e})\big),
\end{aligned}$$

where (a) holds by the identity $\sum_{\ell\in\mathcal{T}} \frac{\mathbb{1}_{\{\ell\in[s,f]\}}}{f-s+1} = 1$, (b) follows by (7), and (c) follows because we have $\hat{J}^q_{\mathsf{L}}(\boldsymbol{x}) = \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \eta^q_{i,\ell}\, x_{i,\ell}$.

We focus on the first part of the proof of Theorem 4.1, which constructs an upper bound on the optimal total expected revenue.

B.2 Upper Bound on the Optimal Total Expected Revenue

Letting the value functions $\{J^q : q \in \mathcal{Q}\}$ be computed through (1), it is a standard result that if, for all $\boldsymbol{x} \in \{0,1\}^{N\times T}$ and $q \in \mathcal{Q}$, the value function approximations $\{\widetilde{J}^q : q \in \mathcal{Q}\}$ satisfy

$$\widetilde{J}^q(\boldsymbol{x}) \ge \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \left(\prod_{\ell=s}^{f} x_{i,\ell} \right) \left[r_{i,[s,f]} + \widetilde{J}^{q+1}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]}) - \widetilde{J}^{q+1}(\boldsymbol{x}) \right] \right\} + \widetilde{J}^{q+1}(\boldsymbol{x}), \quad (17)$$

then we have $\widetilde{J}^q(\boldsymbol{x}) \ge J^q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \{0,1\}^{N\times T}$ and $q \in \mathcal{Q}$; see Section 5.3.3 in Bertsekas (2017). This result is known as the monotonicity of the dynamic programming operator. Note that the inequality above is the version of (1) in the greater than or equal to sense. Thus, if the value function approximations $\{\widetilde{J}_q : q \in \mathcal{Q}\}$ satisfy (1) in the greater than or equal to sense, then they form an upper bound on the optimal value functions $\{J^q : q \in \mathcal{Q}\}$ that satisfy (1) in the equality sense. In the next proposition, we use this result to show that $2\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})$ provides an upper bound on the optimal total expected revenue. Thus, we can use our linear value function approximations to come up with an upper bound on the optimal total expected revenue.

Proposition B.4 (Upper Bound on Optimal Performance) *Noting that the optimal total expected revenue is* $J^1(\boldsymbol{e})$*, we have* $J^1(\boldsymbol{e}) \le 2\,\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})$.

Proof: Using our linear value function approximations $\{\hat{J}^q_{\mathsf{L}} : q \in \mathcal{Q}\}$ computed through (6) and (7), let $\widetilde{J}^q(\boldsymbol{x}) = \hat{J}^q_{\mathsf{L}}(\boldsymbol{e}) + \hat{J}^q_{\mathsf{L}}(\boldsymbol{x})$. We show that $\{\widetilde{J}^q : q \in \mathcal{Q}\}$ satisfies (17). In particular, we have

$$
\begin{aligned}
&\sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \left(\prod_{\ell=s}^{f} x_{i,\ell} \right) \left[r_{i,[s,f]} + \widetilde{J}^{q+1}(\boldsymbol{x}) - \widetilde{J}^{q+1}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]}) \right] \right\} \\
&\qquad = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \left(\prod_{\ell=s}^{f} x_{i,\ell} \right) \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \right\} \\
&\qquad \overset{(a)}{\le} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \right\} \\
&\qquad \overset{(b)}{=} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i(A^q_{[s,f]}) \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \\
&\qquad \overset{(c)}{=} \sum_{i\in\mathcal{N}} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \phi^q_i(A^q_{[s,f]})\, \frac{\sum_{\ell\in\mathcal{T}} \mathbb{1}_{\{\ell\in[s,f]\}}}{f-s+1} \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \\
&\qquad = \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \phi^q_i(A^q_{[s,f]})\, \frac{\mathbb{1}_{\{\ell\in[s,f]\}}}{f-s+1} \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \\
&\qquad \overset{(d)}{=} \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \left(\eta^q_{i,\ell} - \eta^{q+1}_{i,\ell} \right) \overset{(e)}{\le} \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \left(\eta^q_{i,\ell} - \eta^{q+1}_{i,\ell} \right) (1 + x_{i,\ell}) = \widetilde{J}^q(\boldsymbol{x}) - \widetilde{J}^{q+1}(\boldsymbol{x}).
\end{aligned}
$$

The inequality (a) holds because Lemma A.1 in Appendix A implies that if $r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \le 0$, then there exists an optimal solution to the problem on the left side of (a) such that resource i is not offered. So, if $r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \le 0$, then we can drop resource i from consideration in this problem, but if $r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} > 0$, then we have $\left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \le r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h}$. Moreover, (b) follows from the definition of $A^q_{[s,f]}$ in (6), (c) uses the fact that $\sum_{\ell\in\mathcal{T}} \mathbb{1}_{\{\ell\in[s,f]\}} = f - s + 1$, and (d) follows from the definition of $\eta^q_{i,\ell}$ in (7). Lastly, (e) holds because we have $\phi^q_i(A^q_{[s,f]}) \left[r_{i,[s,f]} - \sum_{h=s}^{f} \eta^{q+1}_{i,h} \right] \ge 0$ by Lemma B.2, which implies that $\eta^q_{i,\ell} \ge \eta^{q+1}_{i,\ell}$ by (7).

The chain of inequalities above shows that $\{\widetilde{J}^q : q \in \mathcal{Q}\}$ satisfies (17). Therefore, we have $\widetilde{J}^q(\boldsymbol{x}) \ge J^q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \{0,1\}^{N\times T}$ and $q \in \mathcal{Q}$, so $J^1(\boldsymbol{e}) \le \widetilde{J}^1(\boldsymbol{e}) = 2\,\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})$. ■

We focus on the second part of the proof of Theorem 4.1, which constructs a lower bound on the total expected revenue of the resource based static policy.

B.3 Lower Bound on the Performance of the Static Policy

We give a dynamic program to compute the total expected revenue of the resource based static policy. Let $U^q_{\mathsf{L}}(\boldsymbol{x})$ be the total expected revenue obtained by the resource based static policy over time periods $\{q,\ldots,Q\}$ given that the state of the system at time period q is $\boldsymbol{x}$. We can compute the value functions $\{U^q_{\mathsf{L}} : q\in\mathcal{Q}\}$ through the dynamic program

$$U^q_{\mathsf{L}}(\boldsymbol{x}) = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(A^q_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \left[r_{i,[s,f]} + U^{q+1}_{\mathsf{L}}(\boldsymbol{x}-\boldsymbol{e}_{i,[s,f]}) - U^{q+1}_{\mathsf{L}}(\boldsymbol{x}) \right] \right\} + U^{q+1}_{\mathsf{L}}(\boldsymbol{x}), \quad (18)$$

with the boundary condition that $U^{Q+1}_{\mathsf{L}} = 0$. The dynamic program above is similar to that in (2), but under the resource based static policy, a customer making a booking request for interval $[s,f]$ at time period q chooses resource i with probability $\phi^q_i(A^q_{[s,f]})$. In the next proposition, we use the linear value function approximations $\{\hat{J}^q_{\mathsf{L}} : q\in\mathcal{Q}\}$ to give a lower bound on the performance of the resource based static policy. Lemma B.1 plays an important role in the proof of the next proposition. Thus, the lower bound on the performance of the resource based static policy explicitly uses the fact that the resources are unique so that each has a capacity of one.

Proposition B.5 (Lower Bound on Policy Performance) *Letting the value functions* $\{U^q_{\mathsf{L}} : q\in\mathcal{Q}\}$ *be computed through* (18), *for each* $\boldsymbol{x}\in\{0,1\}^{N\times T}$ *and* $q\in\mathcal{Q}$, *we have*

$$U^q_{\mathsf{L}}(\boldsymbol{x}) \;\geq\; \hat{J}^q_{\mathsf{L}}(\boldsymbol{x}) \;-\; \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}}\, \hat{J}^q_{\mathsf{L}}(\boldsymbol{e}).$$

Proof: We give an inequality that will be useful later in the proof. Because $\prod_{\ell=s}^{f} x_{i,\ell} \geq \frac{\sum_{\ell=s}^{f} x_{i,\ell}-(f-s)}{1+f-s}$ by Lemma B.1 and $\phi^q_i(A^q_{[s,f]})\big[r_{i,[s,f]} - \sum_{h=s}^{f}\eta^{q+1}_{i,h}\big] \geq 0$ by Lemma B.2, we have

$$\begin{aligned}
&\sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i(A^q_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \left[r_{i,[s,f]} + \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x}-\boldsymbol{e}_{i,[s,f]}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x}) \right] \\
&\quad = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i(A^q_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \left[r_{i,[s,f]} - \sum_{h=s}^{f}\eta^{q+1}_{i,h} \right] \\
&\quad \geq \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i(A^q_{[s,f]}) \; \frac{\sum_{\ell\in\mathcal{T}} \mathbb{1}_{\{\ell\in[s,f]\}}\, x_{i,\ell} - (f-s)}{f-s+1} \left[r_{i,[s,f]} - \sum_{h=s}^{f}\eta^{q+1}_{i,h} \right] \\
&\quad \overset{(a)}{\geq} \sum_{i\in\mathcal{N}}\sum_{\ell\in\mathcal{T}} x_{i,\ell} \left\{ \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \phi^q_i(A^q_{[s,f]}) \frac{\mathbb{1}_{\{\ell\in[s,f]\}}}{f-s+1} \left[r_{i,[s,f]} - \sum_{h=s}^{f}\eta^{q+1}_{i,h} \right] \right\} - \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \left(\hat{J}^q_{\mathsf{L}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{e})\right) \\
&\quad \overset{(b)}{=} \sum_{i\in\mathcal{N}}\sum_{\ell\in\mathcal{T}} x_{i,\ell}\,(\eta^q_{i,\ell} - \eta^{q+1}_{i,\ell}) - \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \left(\hat{J}^q_{\mathsf{L}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{e})\right) \\
&\quad \overset{(c)}{=} \hat{J}^q_{\mathsf{L}}(\boldsymbol{x}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x}) - \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}} \left(\hat{J}^q_{\mathsf{L}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{e})\right),
\end{aligned}$$

where (a) follows by arranging the terms on the left side of (a) and using Lemma B.3, (b) uses (7), and (c) uses the fact that $\hat{J}^q_{\mathsf{L}}(\boldsymbol{x}) = \sum_{i\in\mathcal{N}}\sum_{\ell\in\mathcal{T}} \eta_{i,\ell}\, x_{i,\ell}$.

In the rest of the proof, we show the inequality in the proposition by using induction over the time periods. At time period $Q+1$, because we have $U^{Q+1}_{\mathsf{L}} = 0 = \hat{J}^{Q+1}_{\mathsf{L}}$, the inequality holds at time period $Q+1$. Assuming that the inequality holds at time period $q+1$, we show that the inequality holds at time period q as well. Arranging the terms, the coefficient of $U_{\mathsf{L}}^{q+1}(\boldsymbol{x})$ on the right side of (18) is $1-\sum_{[s,f]\in\mathcal{F}}\lambda^q_{[s,f]}\sum_{i\in\mathcal{N}}\phi^q_i(A^q_{[s,f]})\prod_{\ell=s}^f x_{i,\ell}$. Because $\sum_{[s,f]\in\mathcal{F}}\lambda^q_{[s,f]}\le 1$ and $\sum_{i\in\mathcal{N}}\phi^q_i(A^q_{[s,f]})\le 1$, this coefficient is nonnegative. Letting $\alpha^q=\frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}}\hat{J}^q_{\mathsf{L}}(\boldsymbol{e})$ for notational brevity, by the induction assumption, we have $U^{q+1}_{\mathsf{L}}(\boldsymbol{x})\ge\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x})-\alpha^{q+1}$ for all $\boldsymbol{x}\in\{0,1\}^{N\times T}$. In this case, replacing $U^{q+1}_{\mathsf{L}}(\boldsymbol{x})$ and $U^{q+1}_{\mathsf{L}}(\boldsymbol{x}-\boldsymbol{e}_{i,[s,f]})$ on the right side of (18) with their corresponding lower bounds $\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x})-\alpha^{q+1}$ and $\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x}-\boldsymbol{e}_{i,[s,f]})-\alpha^{q+1}$, respectively, the right side of (18) gets smaller, so we get the chain of inequalities

$$\begin{aligned}U^q_{\mathsf{L}}(\boldsymbol{x}) &\ge \sum_{[s,f]\in\mathcal{F}}\lambda^q_{[s,f]}\left\{\sum_{i\in\mathcal{N}}\phi^q_i(A^q_{[s,f]})\left(\prod_{\ell=s}^f x_{i,\ell}\right)\Big[r_{i,[s,f]}+\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x}-\boldsymbol{e}_{i,[s,f]})-\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x})\Big]\right\}+\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x})-\alpha^{q+1}\\ &\overset{(d)}{\ge} \hat{J}^q_{\mathsf{L}}(\boldsymbol{x})-\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x})-(\alpha^q-\alpha^{q+1})+\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{x})-\alpha^{q+1} \;=\; \hat{J}^q_{\mathsf{L}}(\boldsymbol{x})-\alpha^q,\end{aligned}$$

where (d) uses the chain of inequalities earlier in the proof, along with $\frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}}(\hat{J}^q_{\mathsf{L}}(\boldsymbol{e})-\hat{J}^{q+1}_{\mathsf{L}}(\boldsymbol{e}))=\alpha^q-\alpha^{q+1}$. Thus, the inequality in the proposition holds at time period q. ■

Finally, we can use Propositions B.4 and B.5 to give a proof for Theorem 4.1.

Proof of Theorem 4.1:

By Proposition B.4, we have $\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})\ge\frac{1}{2}J^1(\boldsymbol{e})$. Using Proposition B.5 with $\boldsymbol{x}=\boldsymbol{e}$ and $q=1$, we have $U^1_{\mathsf{L}}(\boldsymbol{e})\ge\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})-\frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\max}}\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})=\frac{1}{\mathsf{D}_{\max}}\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})$, so we get $U^1_{\mathsf{L}}(\boldsymbol{e})\ge\frac{1}{\mathsf{D}_{\max}}\hat{J}^1_{\mathsf{L}}(\boldsymbol{e})\ge\frac{1}{2\,\mathsf{D}_{\max}}J^1(\boldsymbol{e})$. ■

Appendix C: Performance Guarantee for Itinerary Based Static Policy

In this section, we give a proof for Theorem 5.1. We will use two preliminary lemmas. The next lemma is the analogue of Lemma B.2. Its proof is identical to that of Lemma B.2 and omitted.

Lemma C.1 (Nonnegative Contribution of Each Resource) *For each* $i\in\mathcal{N}$, $[s,f]\in\mathcal{F}$ *and* $q\in\mathcal{Q}$, *we have* $\phi^q_i(B^q_{[s,f]})\big[r_{i,[s,f]}-\sum_{[a,b]\in\mathcal{F}}\big|[s,f]\cap C_{[a,b]}\big|\,\gamma^{q+1}_{i,[a,b]}\big]\ge 0$.

In the next lemma, we give an upper bound on the opportunity cost of the capacities consumed by using resource i to serve a request for interval $[s,f]$ under the polynomial approximations.

Lemma C.2 (Bound on Opportunity Cost) *For each* $i\in\mathcal{N}$, $[s,f]\in\mathcal{F}$ *and* $\boldsymbol{x}\in\{0,1\}^{N\times T}$ *such that* $\prod_{\ell=s}^f x_{i,\ell}=1$, *we have* $\hat{J}^q_{\mathsf{P}}(\boldsymbol{x})-\hat{J}^q_{\mathsf{P}}(\boldsymbol{x}-\boldsymbol{e}_{i,[s,f]})\le\sum_{[a,b]\in\mathcal{F}}\big|[s,f]\cap C_{[a,b]}\big|\,\gamma^q_{i,[a,b]}$.

Proof: Using the previous lemma, by (9), we have $\gamma^q_{i,[s,f]} \ge 0$ for all $i \in \mathcal{N}$ and $[s,f] \in \mathcal{F}$. Using the fact that $\hat{J}^q_{\mathsf{P}}(\boldsymbol{x}) = \sum_{i\in\mathcal{N}} \sum_{[s,f]\in\mathcal{F}} \gamma^q_{i,[s,f]} \prod_{\ell=s}^{f} x_{i,\ell}$, the difference $\hat{J}^q_{\mathsf{P}}(\boldsymbol{x}) - \hat{J}^q_{\mathsf{P}}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]})$ is

$$\begin{aligned}
\hat{J}^q_{\mathsf{P}}(\boldsymbol{x}) - \hat{J}^q_{\mathsf{P}}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]}) &= \sum_{[a,b]\in\mathcal{F}} \gamma^q_{i,[a,b]} \, \mathbb{1}_{\{[a,b]\cap[s,f]\neq\varnothing\}} \left(\prod_{\ell=a}^{b} x_{i,\ell} \right) \\
&\le \sum_{[a,b]\in\mathcal{F}} \gamma^q_{i,[a,b]} \, \mathbb{1}_{\{[a,b]\cap[s,f]\neq\varnothing\}} \;\le\; \sum_{[a,b]\in\mathcal{F}} \big| \, [s,f] \cap C_{[a,b]} \, \big| \, \gamma^q_{i,[a,b]},
\end{aligned}$$

where the last inequality holds because $C_{[a,b]}$ is intersection preserving, so if $[a,b] \cap [s,f] \neq \varnothing$, then $[s,f] \cap C_{[a,b]} \neq \varnothing$. In this case, we have $\mathbb{1}_{\{[a,b]\cap[s,f]\neq\varnothing\}} \le \big|[s,f] \cap C_{[a,b]}\big|$. ■

In the next proposition, we show that we can use the value function approximations $\{\hat{J}^q_{\mathsf{P}} : q \in \mathcal{Q}\}$ to come up with an upper bound on the optimal total expected revenue.

Proposition C.3 (Upper Bound on Optimal Performance) *Noting that the optimal total expected revenue is $J^1(\boldsymbol{e})$, we have $J^1(\boldsymbol{e}) \le (1 + \|\mathcal{C}\|)\, \hat{J}^1_{\mathsf{P}}(\boldsymbol{e})$.*

Proof: Letting $\beta^q_{i,\ell} = \sum_{[a,b]\in\mathcal{F}} \mathbb{1}_{\{\ell \,\in\, C_{[a,b]}\}} \gamma^q_{i,[a,b]}$ for notational brevity, we define the linear value function approximations $\{\hat{V}^q_{\mathsf{P}} : q \in \mathcal{Q}\}$ as $\hat{V}^q_{\mathsf{P}}(\boldsymbol{x}) = \hat{J}^q_{\mathsf{P}}(\boldsymbol{e}) + \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \beta^q_{i,\ell}\, x_{i,\ell}$. We have

$$\begin{aligned}
&\sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \left(\prod_{\ell=s}^{f} x_{i,\ell} \right) \Big[r_{i,[s,f]} + \hat{V}^{q+1}_{\mathsf{P}}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]}) - \hat{V}^{q+1}_{\mathsf{P}}(\boldsymbol{x}) \Big] \right\} \\
&\quad = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \left(\prod_{\ell=s}^{f} x_{i,\ell} \right) \Big[r_{i,[s,f]} - \sum_{h=s}^{f} \beta^{q+1}_{i,h} \Big] \right\} \\
&\quad \overset{(a)}{\le} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \Big[r_{i,[s,f]} - \sum_{h=s}^{f} \beta^{q+1}_{i,h} \Big] \right\} \\
&\quad \overset{(b)}{=} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S) \Big[r_{i,[s,f]} - \sum_{[a,b]\in\mathcal{F}} \Big| C_{[a,b]} \cap [s,f] \Big| \gamma^{q+1}_{i,[a,b]} \Big] \right\} \\
&\quad \overset{(c)}{=} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{i\in\mathcal{N}} \phi^q_i(B^q_{[s,f]}) \Big[r_{i,[s,f]} - \sum_{[a,b]\in\mathcal{F}} \Big| C_{[a,b]} \cap [s,f] \Big| \gamma^{q+1}_{i,[a,b]} \Big] \\
&\quad \overset{(d)}{=} \sum_{i\in\mathcal{N}} \sum_{[s,f]\in\mathcal{F}} (\gamma^q_{i,[s,f]} - \gamma^{q+1}_{i,[s,f]}) \;=\; \hat{J}^q_{\mathsf{P}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{e}) \\
&\quad \overset{(e)}{\le} \hat{J}^q_{\mathsf{P}}(\boldsymbol{e}) - \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{e}) + \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} (\beta^q_{i,\ell} - \beta^{q+1}_{i,\ell})\, x_{i,\ell} \;=\; \hat{V}^q_{\mathsf{P}}(\boldsymbol{x}) - \hat{V}^{q+1}_{\mathsf{P}}(\boldsymbol{x}).
\end{aligned}$$

Here, (a) follows from the same argument that we use to obtain the inequality (a) in the proof of Proposition B.4. In (b), we use the definition of $\beta^q_{i,\ell}$ and use the interchange of sums $\sum_{h=s}^{f} \sum_{[a,b]\in\mathcal{F}} \mathbb{1}_{\{h \in C_{[a,b]}\}} = \sum_{[a,b]\in\mathcal{F}} \sum_{h=s}^{f} \mathbb{1}_{\{h \in C_{[a,b]}\}} = \sum_{[a,b]\in\mathcal{F}} |C_{[a,b]} \cap [s,f]|$. Also, (c) follows from (8), and (d) follows from (9). To see that (e) holds, note that Lemma C.1, along with (9), implies that $\gamma^q_{i,[s,f]} \ge \gamma^{q+1}_{i,[s,f]}$, in which case, we also have $\beta^q_{i,\ell} \ge \beta^{q+1}_{i,\ell}$. The chain of inequalities above shows that the value function approximations $\{\hat{V}^q_{\mathsf{P}} : q \in \mathcal{Q}\}$ satisfy (17), in which case, by the discussion that follows (17), we have $\hat{V}^q_{\mathsf{P}}(\boldsymbol{x}) \geq J^q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \{0,1\}^{N\times T}$ and $q \in \mathcal{Q}$. Thus, we have

$$\begin{aligned} J^1(\boldsymbol{e}) \;\leq\; \hat{V}^1_{\mathsf{P}}(\boldsymbol{e}) \;&=\; \hat{J}^1_{\mathsf{P}}(\boldsymbol{e}) + \sum_{i\in\mathcal{N}}\sum_{\ell\in\mathcal{T}} \beta^1_{i,\ell} \;=\; \hat{J}^1_{\mathsf{P}}(\boldsymbol{e}) + \sum_{i\in\mathcal{N}}\sum_{[a,b]\in\mathcal{F}}\sum_{\ell\in\mathcal{T}} \mathbb{1}_{\{\ell\in C_{[a,b]}\}}\,\gamma^1_{i,[a,b]} \\ &=\; \hat{J}^1_{\mathsf{P}}(\boldsymbol{e}) + \sum_{i\in\mathcal{N}}\sum_{[a,b]\in\mathcal{F}} \left|C_{[a,b]}\right| \gamma^1_{i,[a,b]} \;\leq\; \hat{J}^1_{\mathsf{P}}(\boldsymbol{e}) + \|\mathcal{C}\| \sum_{i\in\mathcal{N}}\sum_{[a,b]\in\mathcal{F}} \gamma^1_{i,[a,b]} \;=\; (1+\|\mathcal{C}\|)\,\hat{J}^1_{\mathsf{P}}(\boldsymbol{e}), \end{aligned}$$

where the last inequality holds because $\|\mathcal{C}\| = \max_{[a,b]\in\mathcal{F}} |C_{[a,b]}|$, and the last equality follows from the fact that $\hat{J}^q_{\mathsf{P}}(\boldsymbol{e}) = \sum_{i\in\mathcal{N}}\sum_{[a,b]\in\mathcal{F}} \gamma^q_{i,[a,b]}$. ■

Let $U^q_{\mathsf{P}}(\boldsymbol{x})$ be the total expected revenue obtained by the itinerary based static policy over time periods $\{q,\ldots,Q\}$ given that the state of the system at time period q is $\boldsymbol{x}$. We can compute the value functions $\{U^q_{\mathsf{P}} : q \in \mathcal{Q}\}$ through the dynamic program in (18) after replacing the ideal assortments $\{A^q_{[s,f]} : [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ with $\{B^q_{[s,f]} : [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$. In the next proposition, we lower bound the performance of the itinerary based static policy.

Proposition C.4 (Lower Bound on Policy Performance) *Letting $\{U^q_{\mathsf{P}} : q \in \mathcal{Q}\}$ be the value functions of the itinerary based static policy, for each $\boldsymbol{x} \in \{0,1\}^{N\times T}$ and $q \in \mathcal{Q}$, $U^q_{\mathsf{P}}(\boldsymbol{x}) \;\geq\; \hat{J}^q_{\mathsf{P}}(\boldsymbol{x})$.*

Proof: We show the result by using induction over the time periods. At time period $Q+1$, since we have $U^{Q+1}_{\mathsf{P}} = 0 = \hat{J}^{Q+1}_{\mathsf{P}}$, the inequality holds at time period $Q+1$. Assuming that the inequality holds at time period $q+1$, we show that the inequality holds at time period q as well. By the induction hypothesis, $U^{q+1}_{\mathsf{P}}(\boldsymbol{x})$ and $U^{q+1}_{\mathsf{P}}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]})$ are, respectively, lower bounded by $\hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x})$ and $\hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]})$, so by (18), we get

$$\begin{aligned} U^q_{\mathsf{P}}(\boldsymbol{x}) \;&\geq\; \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(B^q_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \left[r_{i,[s,f]} + \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x} - \boldsymbol{e}_{i,[s,f]}) - \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x}) \right] \right\} + \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x}) \\ &\overset{(a)}{\geq}\; \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(B^q_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \left[r_{i,[s,f]} \;-\; \sum_{[a,b]\in\mathcal{F}} \left| [s,f] \cap C_{[a,b]} \right| \gamma^{q+1}_{i,[a,b]} \right] \right\} + \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x}) \\ &\overset{(b)}{=}\; \sum_{i\in\mathcal{N}}\sum_{[s,f]\in\mathcal{F}} (\gamma^q_{i,[s,f]} \;-\; \gamma^{q+1}_{i,[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) + \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x}) \;\overset{(c)}{=}\; [\hat{J}^q_{\mathsf{P}}(\boldsymbol{x}) \;-\; \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x})] + \hat{J}^{q+1}_{\mathsf{P}}(\boldsymbol{x}) \;=\; \hat{J}^q_{\mathsf{P}}(\boldsymbol{x}), \end{aligned}$$

where (a) follows from Lemma C.2, (b) holds by (9), and (c) holds by the definition of $\hat{J}^q_{\mathsf{P}}(\boldsymbol{x})$. The chain of inequalities above establishes the result. ■

We can use Propositions C.3 and C.4 to give a proof for Theorem 5.1.

Proof of Theorem 5.1:

By Proposition C.3, we have $\hat{J}^1_{\mathsf{P}}(\boldsymbol{e}) \geq \frac{1}{1+\|\mathcal{C}\|} J^1(\boldsymbol{e})$. Using Proposition C.4 with $\boldsymbol{x} = \boldsymbol{e}$ and $q = 1$, we have $U^1_{\mathsf{P}}(\boldsymbol{e}) \geq \hat{J}^1_{\mathsf{P}}(\boldsymbol{e})$. So, we get $U^1_{\mathsf{P}}(\boldsymbol{e}) \geq \hat{J}^1_{\mathsf{P}}(\boldsymbol{e}) \geq \frac{1}{1+\|\mathcal{C}\|} J^1(\boldsymbol{e})$. ■

Appendix D: Tightness of the Performance Guarantee

We show that our performance guarantee for the itinerary based static policy is tight. Consider a problem instance with $N+1$ resources indexed by $\mathcal{N}=\{1,\ldots,N+1\}$, where we assume that $N\geq 2$. The resources are available for use during the days $\mathcal{T}=\{1,\ldots,T\}$. The set of possible intervals of use are $\mathcal{F}=\{[t,t]:t\in\mathcal{T}\}\cup\{[1,T]\}$. Thus, there are $T+1$ possible intervals of use. Each of the first T possible intervals of use occupies a resource for only one day, whereas the last possible interval of use occupies the resource for the whole interval $[1,T]$. The revenue from booking any resource i over the interval $[t,t]$ is $r_{i,[t,t]}=\frac{N-1}{N\,(N+1)}$, whereas the revenue from booking any resource i over the interval $[1,T]$ is $r_{i,[1,T]}=1$. Note that the revenues from the booking requests depend on the interval of use, but not on the specific resource that is used to serve the booking request. Because $N\geq 2$, we have $r_{i,[t,t]}>0$. The booking requests arrive over the time periods $\mathcal{Q}=\{1,\ldots,NT+1\}$. There is one customer arrival at each time period. For $t=1,\ldots,T$, at each of the time periods $\{(t-1)\,N+1,\ldots,tN\}$, we have a booking request for the interval $[t,t]$ with probability one, whereas at the last time period $NT+1$, we have a booking request for the interval $[1,T]$ with probability one. In other words, the booking requests for the interval $[t,t]$ arrive only at time periods $\{(t-1)\,N+1,\ldots,tN\}$. There is a single booking request for the interval $[1,T]$ and it arrives at the last time period. At each of the first NT time periods, given that we offer the assortment S of resources, the customer chooses each offered resource with an equal probability of $\frac{1}{|S|}$, never leaving the system without choosing any of the offered resources as along as a resource is offered to this customer. At the last time period $NT+1$, given that we offer the assortment S of resources, the customer chooses among the offered resources with an equal probability of $\frac{N}{1+N\,|S|}$, leaving the system without choosing any of the offered resources with a probability of $\frac{1}{1+N\,|S|}$. At each of the first NT time periods, the customer never leaves without choosing an offered resource, so offering a single resource ensures that the customer chooses this resource.

We compute the total expected revenue of a benchmark policy, which provides a lower bound on the optimal total expected revenue. The benchmark policy uses one of the first N resources for the customers arriving at the first NT time periods, leaving the last resource untouched for the customer arriving at the last time period. For $t=1,\ldots,T$, the customers arriving at each of the time periods $\{(t-1)\,N+1,\ldots,tN\}$ make a booking request for the interval $[t,t]$. Thus, there are N customers making a booking request for the interval $[t,t]$. The benchmark policy offers only one of the first N resources to each one of these customers. As discussed earlier, the customers arriving at one of the first NT time periods always chooses the resource offered to them. Thus, the benchmark policy uses the first N resources to serve the booking requests from the customers arriving at time periods $\{(t-1)\,N+1,\ldots,tN\}$, in which case, noting that the revenue from a booking request for the interval $[t,t]$ is $\frac{N-1}{N\,(N+1)}$, the benchmark policy obtains a total expected revenue of $N\frac{N-1}{N(N+1)}$ over time periods $\{(t-1)\,N+1,\ldots,tN\}$. Thus, the benchmark policy obtains a total expected revenue of $TN\frac{N-1}{N(N+1)}$ from the first NT customers. The last resource is left untouched for the customer arriving at the last time period $NT+1$. The benchmark policy offers only this resource to the customer at the last time period and the customer chooses the resource with probability $\frac{N}{1+N}$. The revenue from a booking request for the interval $[1,T]$ is one, so the benchmark policy obtains a total expected revenue of $\frac{N}{1+N}$ at the last time period. Thus, the total expected revenue of the benchmark policy is $T\,\frac{N-1}{N+1}+\frac{N}{1+N}$, which is arbitrarily close to $T+1$ as N gets large. Therefore, letting OPT be the optimal total expected revenue, $\mathsf{OPT}\ge T+1-\frac{1}{N}$ for large enough N.

The intersection preserving subset for the interval $[1,T]$ must be $C_{[1,T]}=\{1,\ldots,T\}$. If $t\notin C_{[1,T]}$ for some $t\in\mathcal{T}$, then we get $[1,T]\cap[t,t]\neq\varnothing$ and $C_{[1,T]}\cap[t,t]=\varnothing$, which violates the definition of an intersection preserving subset. The only intersection preserving subset for the interval $[t,t]$ is $C_{[t,t]}=\{t\}$. Thus, there is a unique collection $\mathcal{C}$ of intersection preserving subsets for our problem instance and its norm is T. Consider (8) for the interval $[s,f]=[1,T]$ at time period $q=NT+1$. We have $B^{NT+1}_{[1,T]}=\arg\max_{S\subseteq\mathcal{N}}\sum_{i\in\mathcal{N}}\mathbf{1}(i\in S)\frac{N}{1+N\,|S|}=\arg\max_{S\subseteq\mathcal{N}}\frac{N\,|S|}{1+N\,|S|}$. Because $\frac{N\,|S|}{1+N\,|S|}$ is increasing in $|S|$, we get $B^{NT+1}_{[1,T]}=\mathcal{N}$. Plugging the assortment $B^{NT+1}_{[1,T]}=\mathcal{N}$ into (9), noting that $\phi^{NT+1}_i(\mathcal{N})=\frac{N}{1+N\,(N+1)}$ and $|\mathcal{N}|=N+1$, we have $\gamma^{NT+1}_{i,[1,T]}=\frac{N}{1+N\,(N+1)}$. We get a booking request for the interval $[1,T]$ only at the last time period, so $\gamma^{q}_{i,[1,T]}=\frac{N}{1+N\,(N+1)}$ for all $q\in\mathcal{Q}$.

Consider time period tN. This time period is the last one at which we have a booking request for the interval $[t,t]$. Therefore, by (8), we have

$$\begin{aligned} B^{tN}_{[t,t]} &= \arg\max_{S\subseteq\mathcal{N}}\left\{\sum_{i\in\mathcal{N}}\phi^q_i(S)\Big[r_{i,[t,t]}\;-\;\gamma^{tN+1}_{i,[t,t]}-\gamma^{tN+1}_{i,[1,T}\Big]\right\}.\\ &= \arg\max_{S\subseteq\mathcal{N}}\left\{\sum_{i\in\mathcal{N}}\mathbf{1}(S\neq\varnothing)\,\frac{1}{|S|}\left[\frac{N-1}{N\,(N+1)}\;-\;0-\frac{N}{1+N\,(N+1)}\right]\right\}\;=\;\varnothing, \end{aligned}$$

where the last equality holds because $\frac{N-1}{N\,(N+1)}-\frac{N}{1+N\,(N+1)}<0$, so any non-empty solution to the problem above yields a strictly negative objective value. Plugging the assortment $B^{tN}_{[t,t]}=\varnothing$ into (9) yields $\gamma^{tN}_{i,[t,t]}=0$. Moving backwards over the time periods and continuing in the same fashion, it follows that $B^q_{[t,t]}=\varnothing$ and $\gamma^q_{i,[t,t]}=0$ for all $i\in\mathcal{N}$ and $q\in\{(t-1)N+1,\ldots,tN\}$. Therefore, the itinerary based static policy offers the empty assortment of resources to all of the booking requests for a single day and offers the full assortment of resources to the booking request for the interval $[1,T]$ that arrives at the last time period. In this case, letting STA be the total expected revenue of the itinerary based static policy, noting that $\phi^{NT+1}_i(\mathcal{N})=\frac{N}{1+N\,(N+1)}$, we get $\mathsf{STA}=\sum_{i\in\mathcal{N}}\phi^{NT+1}_i(\mathcal{N})=\frac{N\,(N+1)}{1+N\,(N+1)}$, which is no larger than one. Thus, we have $\mathsf{STA}\le 1$.

Noting that $\mathsf{OPT}\ge T+1-\frac{1}{N}$ for large enough N, we get $\frac{\mathsf{STA}}{\mathsf{OPT}}\le\frac{1}{T+1-\frac{1}{N}}=\frac{1}{1+\|\mathcal{C}\|-\frac{1}{N}}$. By Theorem 5.1, we have $\frac{\mathsf{STA}}{\mathsf{OPT}}\ge\frac{1}{1+\|\mathcal{C}\|}$. Thus, $\frac{\mathsf{STA}}{\mathsf{OPT}}$ is arbitrarily close to $\frac{1}{1+\|\mathcal{C}\|}$ as N gets large.

Appendix E: Norm of a Collection of Intersection Preserving Subsets

In this section, we give a proof for Theorem 5.2. To see that the first statement holds, we construct a feasible solution to problem (10) that provides an objective value of $1 + \lceil (\mathsf{D}_{\max} - 1)/\mathsf{D}_{\min} \rceil$. In particular, we set $\hat{t} = 1 + \lceil (\mathsf{D}_{\max} - 1)/\mathsf{D}_{\min} \rceil$. Furthermore, for each $[s,f] \in \mathcal{F}$, we set

$$\hat{C}_{[s,f]} = \left\{ s + k\,\mathsf{D}_{\min} \;:\; k = 0, 1, 2, \ldots, \left\lceil \frac{f-s}{\mathsf{D}_{\min}} \right\rceil - 1 \right\} \cup \{f\}. \tag{19}$$

The solution $\{\hat{C}_{[s,f]} : [s,f] \in \mathcal{F}\}$ and $\hat{t}$ provides an objective value of $\hat{t} = 1 + \lceil (\mathsf{D}_{\max} - 1)/\mathsf{D}_{\min} \rceil$ for problem (10). We proceed to arguing that this solution is feasible for problem (10). Using the fact that $f - s + 1 \le \mathsf{D}_{\max}$ for all $[s,f] \in \mathcal{F}$, we have $\left|\hat{C}_{[s,f]}\right| \le \left\lceil \frac{f-s}{\mathsf{D}_{\min}} \right\rceil + 1 \le \left\lceil \frac{\mathsf{D}_{\max}-1}{\mathsf{D}_{\min}} \right\rceil + 1 = \hat{t}$. Thus, the first constraint is satisfied. The smallest element of $\hat{C}_{[s,f]}$ is s. Noting that $s + \left(\left\lceil \frac{f-s}{\mathsf{D}_{\min}} \right\rceil - 1 \right) \mathsf{D}_{\min} \le s + \frac{f-s}{\mathsf{D}_{\min}} \mathsf{D}_{\min} = f$, none of the elements of $\hat{C}_{[s,f]}$ exceeds f, so $\hat{C}_{[s,f]} \subseteq [s,f]$. Thus, the third constraint is satisfied. To check the second constraint, consider any $[a,b] \in \mathcal{F}$ such that $[s,f] \cap [a,b] \ne \varnothing$. We show that $\hat{C}_{[s,f]} \cap [a,b] \ne \varnothing$. We consider three cases.

First, consider the case $[s,f] \subseteq [a,b]$. Because $\hat{C}_{[s,f]} \subseteq [s,f]$ by the earlier discussion in the proof, we get $\hat{C}_{[s,f]} \subseteq [s,f] \subseteq [a,b]$, so $\hat{C}_{[s,f]} \cap [a,b] \ne \varnothing$, as desired.

Second, consider the case $[s,f] \not\subseteq [a,b]$ and $[s,f] \not\supseteq [a,b]$. Because $[s,f] \cap [a,b] \ne \varnothing$, we must have $f \in [a,b]$ or $s \in [a,b]$. Noting that $s \in \hat{C}_{[s,f]}$ and $f \in \hat{C}_{[s,f]}$, we get $\hat{C}_{[s,f]} \cap [a,b] \ne \varnothing$, as desired.

Third, consider the case $[s,f] \supseteq [a,b]$. To get a contradiction, suppose, on the contrary, that $\hat{C}_{[s,f]} \cap [a,b] = \varnothing$. Since $[s,f] \supseteq [a,b]$, we have $s \le a \le b \le f$, but noting that $\hat{C}_{[s,f]} \cap [a,b] = \varnothing$ and $s, f \in \hat{C}_{[s,f]}$, there are two successive days c_1, c_2 in $\hat{C}_{[s,f]}$ such that $c_1 < a \le b < c_2$. Thus, there are at least $b - a + 1$ days in between days c_1 and c_2. Because $b - a + 1 \ge \mathsf{D}_{\min}$, there must be at least $\mathsf{D}_{\min}$ days in between days c_1 and c_2. On the other hand, by our construction of $\hat{C}_{[s,f]}$ in (19), there are at most $\mathsf{D}_{\min} - 1$ days in between two successive days in $\hat{C}_{[s,f]}$, which is a contradiction! Therefore, the first statement in the theorem holds.

To see that the second statement holds, we write the second constraint in problem (10) as $|\,C_{[s,f]} \cap [a,b]\,| \;\ge\; \mathbb{1}_{\{[s,f]\cap[a,b]\ne\varnothing\}}$ for all $[s,f], [a,b] \in \mathcal{F}$. Define the constant $\mathsf{Z}^*_{[s,f]}$ as

$$\mathsf{Z}^*_{[s,f]} \;=\; \min\left\{ \left| C_{[s,f]} \right| \;:\; \left| C_{[s,f]} \cap [a,b] \right| \;\ge\; \mathbb{1}_{\{[s,f]\cap[a,b]\ne\varnothing\}} \quad \forall\,[a,b] \in \mathcal{F}, \quad C_{[s,f]} \subseteq [s,f] \right\}, \tag{20}$$

where the only decision variable is $C_{[s,f]}$. In this case, problem (10) becomes equivalent to $\min\{t : t \ge \mathsf{Z}^*_{[s,f]} \;\; \forall\,[s,f] \in \mathcal{F}\}$, which has the optimal objective value $\max_{[s,f]\in\mathcal{F}} \mathsf{Z}^*_{[s,f]}$.

In the rest of the proof, we will show that we can compute $\mathsf{Z}^*_{[s,f]}$ in (20) by solving a minimization linear program with $O(\mathsf{D}_{\max})$ decision variables and $O(\mathsf{D}^2_{\max})$ constraints. Thus, letting χ^* be the optimal objective value of problem (10), by the discussion in the previous paragraph, we have $\chi^* = \max_{[s,f]\in\mathcal{F}} \mathsf{Z}^*_{[s,f]}$, which implies that χ^* is given by the maximum of the optimal objective values of $|\mathcal{F}| = O(\mathsf{D}_{\max} T)$ minimization linear programs, each of the linear programs having $O(\mathsf{D}_{\max})$ decision variables are $O(\mathsf{D}^2_{\max})$ constraints. In this case, it immediately follows that we can compute the maximum of the optimal objective values of these linear programs by solving a single linear program with $O(|\mathcal{F}|\,\mathsf{D}_{\max}) = O(\mathsf{D}^2_{\max} T)$ decision variables and $O(|\mathcal{F}|\,\mathsf{D}^2_{\max}) = O(\mathsf{D}^3_{\max} T)$ constraints. To compute $\mathsf{Z}^*_{[s,f]}$ in (20) by solving a linear program, we use the decision variables $\{x_\ell : \ell = s,\ldots,f\} \in \{0,1\}^{f-s+1}$, where $x_\ell = 1$ if and only if day ℓ is included in the intersection preserving subset $C_{[s,f]}$. We write problem (20) as

$$\begin{aligned} \min \quad & \sum_{\ell=s}^{f} x_\ell \\ \text{st} \quad & \sum_{\ell=s}^{f} \mathbb{1}_{\{\ell\in[a,b]\}}\, x_\ell \ \geq\ \mathbb{1}_{\{[s,f]\cap[a,b]\neq\varnothing\}} \qquad \forall\,[a,b]\in\mathcal{F} \\ & x_\ell \in \{0,1\} \qquad \forall\,\ell = s,\ldots,f. \end{aligned}$$

Each row of the constraint matrix above includes only consecutive ones. Such a matrix is called an interval matrix and it is totally unimodular; see Corollary 2.10 in Chapter III.1 in Nemhauser and Wolsey (1988). Thus, we can relax the integrality requirements without an integrality gap. Also, the problem above has a covering constraint and the right side of the constraint never exceeds one, which implies that even if we did not have an upper bound of one on the decision variables, these decision variables would never take a value greater than one. Thus, we can drop the constraints $x_\ell \leq 1$ for all $\ell = s,\ldots,f$. Lastly, the right side of the constraint is nonzero for all $[a,b]\in\mathcal{F}$ such that $[s,f]\cap[a,b]\neq\varnothing$ and there are only $O(\mathsf{D}^2_{\max})$ such constraints. Thus, the problem above actually has $f - s + 1 = O(\mathsf{D}_{\max})$ decision variables and $O(\mathsf{D}^2_{\max})$ constraints.

Appendix F: Examples of Intersection Preserving Subsets

By the discussion in Section 5, setting $C_{[s,f]} = \{s, s+1,\ldots,f\}$ trivially yields an intersection preserving subset for the interval $[s,f]$. In this case, we immediately obtain a collection of intersection preserving subsets $\{\{s,s+1,\ldots,f\} : [s,f]\in\mathcal{F}\}$. The norm of this collection is $\max_{[s,f]\in\mathcal{F}}\{f - s + 1\} = \mathsf{D}_{\max}$. Also, by Theorem 5.2, we always have a collection of intersection preserving subsets with norm at most $1 + \lceil(\mathsf{D}_{\max} - 1)/\mathsf{D}_{\min}\rceil$. These results hold for any arbitrary set of intervals $\mathcal{F}$ for which we may get booking requests. In this section, we give examples of intersection preserving subsets when there is a special structure in the set of intervals for which we may get booking requests. We may end up with collections of intersection preserving subsets with smaller norms when there is a special structure in the set of intervals. Smaller norms for the collection of intersection preserving subsets translate into better performance guarantees for the itinerary based static policy. We consider two special settings.

First, consider the setting where there exists a set of days $\{\tau_1, \ldots, \tau_K\}$ such that each booking request starts on one of these days. In this case, we argue that setting $C_{[s,f]} = [s,f] \cap \{\tau_1, \ldots, \tau_K\}$ yields an intersection preserving subset for the interval $[s,f]$. With this definition of an intersection preserving subset, we clearly have $C_{[s,f]} \subseteq [s,f]$, so the first property of an intersection preserving subset is satisfied. To check that the second property is satisfied, let the interval $[a,b] \in \mathcal{F}$ be such that $[s,f] \cap [a,b] \neq \varnothing$. Consider the case $s \leq a$. Because $[s,f] \cap [a,b] \neq \varnothing$, having $s \leq a$ implies that $f \geq a$, yielding $s \leq a \leq f$. Thus, we have $a \in [s,f]$. Noting that all booking requests start on one of the days $\{\tau_1, \ldots, \tau_K\}$, we have $a \in \{\tau_1, \ldots, \tau_K\}$. In this case, we obtain $a \in [s,f] \cap \{\tau_1, \ldots, \tau_K\}$, so $a \in C_{[s,f]}$. Thus, we have $C_{[s,f]} \cap [a,b] \neq \varnothing$, so the second property of an intersection preserving subset is satisfied. We can use the same approach to show that $C_{[s,f]} \cap [a,b] \neq \varnothing$ when $s \geq a$.

Second, consider the setting where each booking request is for the same number of days, so $s - f = L$ for all $[s,f] \in \mathcal{F}$. In this case, we argue that setting $C_{[s,f]} = \{s,f\}$ yields an intersection preserving subset for the interval $[s,f]$. With this definition of an intersection preserving subset, we clearly have $C_{[s,f]} = \{s,f\} \subseteq [s,f]$, so the first property of an intersection preserving subset is satisfied. To check that the second property is satisfied, assume on the contrary that there exists an interval $[a,b]$ such that $[s,f] \cap [a,b] \neq \varnothing$ and $C_{[s,f]} \cap [a,b] = \varnothing$. Because $C_{[s,f]} = \{s,f\}$, having $C_{[s,f]} \cap [a,b] = \varnothing$ implies that $s \notin [a,b]$ and $f \notin [a,b]$, in which case, having $[s,f] \cap [a,b] \neq \varnothing$ implies that $[a,b] \subseteq [s+1, f-1]$. If $[a,b] \subseteq [s+1, f-1]$, then the booking requests for the interval $[a,b]$ are at most for $f - s - 1$ days, but the booking requests for the interval $[s,f]$ are for $f - s + 1$ days, contradicting the fact that each booking request is for the same number of days.

Appendix G: Upper Bound on the Optimal Policy Performance

In this section, we give the proofs of the three results in Section 6, along with Lemma G.1 that we use in that section. Here is the proof of Proposition 6.1.

Proof of Proposition 6.1:

We will use a simple manipulation in the proof. In particular, for fixed values $\{\varphi_i : i \in \mathcal{N}\}$, we have the chain of equalities

$$\sum_{j \in \mathcal{N}} \varphi_j \sum_{i \in \mathcal{N} \setminus \{j\}} \beta^q_{j,[s,f] \to i} = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \varphi_j \, \beta^q_{j,[s,f] \to i} = \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{N} \setminus \{j\}} \varphi_i \, \beta^q_{i,[s,f] \to j}, \tag{21}$$

where the first equality holds by interchanging the order of sums and the second equality holds by interchanging the roles of i and j. We show the proposition by using induction over the time periods. At time period $Q+1$, we have $J^{Q+1}=0=\sum_{j\in\mathcal{N}} V^{Q+1}_{\boldsymbol{\beta},j}$, so the result holds at time period $Q+1$. Assuming that the result holds at time period $q+1$, we show that the result holds at time period q as well. Fixing $\boldsymbol{x}\in\{0,1\}^{N\times T}$ and $q\in\mathcal{Q}$, let $S^*_{[s,f]}$ be an optimal solution to the maximization problem on the right side of (1). By the induction hypothesis, $J^{q+1}(\boldsymbol{x}-\boldsymbol{e}_{j,[s,f]})\le \sum_{i\in\mathcal{N}\setminus\{j\}} V^{q+1}_{\boldsymbol{\beta},i}(\boldsymbol{x}_i)+V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j-\boldsymbol{e}_{[s,f]})$ and $J^{q+1}(\boldsymbol{x})\le\sum_{j\in\mathcal{N}} V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)$. Using (1), we get

$$
\begin{aligned}
J^q(\boldsymbol{x}) \;&=\; \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{j\in\mathcal{N}} \phi^q_j(S^*_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{j,\ell}\right) \Big[r_{j,[s,f]} + J^{q+1}(\boldsymbol{x}-\boldsymbol{e}_{j,[s,f]})) - J^{q+1}(\boldsymbol{x})\Big] + J^{q+1}(\boldsymbol{x}) \\
&\overset{(a)}{\le} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{j\in\mathcal{N}} \phi^q_j(S^*_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{j,\ell}\right) \Big[r_{j,[s,f]} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j-\boldsymbol{e}_{[s,f]})) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)\Big] + \sum_{j\in\mathcal{N}} V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \\
&\overset{(b)}{=} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \Bigg\{ \sum_{j\in\mathcal{N}} \phi^q_j(S^*_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{j,\ell}\right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j-\boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)\Big] \\
&\qquad\qquad + \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S^*_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \beta^q_{i,[s,f]\to j} \Bigg\} + \sum_{j\in\mathcal{N}} V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \\
&\overset{(c)}{=} \sum_{j\in\mathcal{N}} \Bigg\{ \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \Bigg\{ \phi^q_j(S^*_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{j,\ell}\right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j-\boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)\Big] \\
&\qquad\qquad + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S^*_{[s,f]}) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \beta^q_{i,[s,f]\to j} \Bigg\} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Bigg\} \\
&\le \sum_{j\in\mathcal{N}} \Bigg\{ \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \Bigg\{ \phi^q_j(S) \left(\prod_{\ell=s}^{f} x_{j,\ell}\right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j-\boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)\Big] \\
&\qquad\qquad + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S) \left(\prod_{\ell=s}^{f} x_{i,\ell}\right) \beta^q_{i,[s,f]\to j} \Bigg\} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Bigg\} \\
&\overset{(d)}{\le} \sum_{j\in\mathcal{N}} \Bigg\{ \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \Bigg\{ \phi^q_j(S) \left(\prod_{\ell=s}^{f} x_{j,\ell}\right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j-\boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)\Big] \\
&\qquad\qquad + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \Bigg\} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Bigg\} \\
&\overset{(e)}{=} \sum_{j\in\mathcal{N}} V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j).
\end{aligned}
$$

In the chain of inequalities above, (a) uses the induction hypothesis. To see that (b) holds, we note that $r_{j,[s,f]}=\sum_{i\in\mathcal{N}}\beta^q_{j,[s,f]\to i}=\beta^q_{j,[s,f]\to j}+\sum_{i\in\mathcal{N}\setminus\{j\}}\beta^q_{j,[s,f]\to i}$ and use (21) after identifying φ_j with $\phi^q_j(S^*_{[s,f]})\prod_{\ell=s}^{f} x_{j,\ell}$. Also, (c) holds by rearranging the order of the sums. To get (d), we use the same argument that we use to obtain inequality (a) in the proof of Proposition B.4. Lastly, (e) follows from (12). The chain of inequalities above completes the induction argument. ■

Next, we state and prove Lemma G.1.

Lemma G.1 (Equivalence of Linear Programs) *Problems* (11) *and* (13) *have the same optimal objective value.*

Proof: We let $\hat{\boldsymbol{h}} = \{\hat{h}^q_{[s,f]}(S) : S \subseteq \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ and $\hat{\boldsymbol{y}} = \{\hat{y}_{i,[s,f]} : i \in \mathcal{N},\ [s,f] \in \mathcal{F}\}$ be an optimal solution to problem (11). For each $j \in \mathcal{N}$, we set $\widetilde{h}^q_{[s,f]\to j}(S) = \hat{h}^q_{[s,f]}(S)$. In this case, we observe that the solution $\widetilde{\boldsymbol{h}} = \{\widetilde{h}^q_{[s,f]\to j}(S) : j \in \mathcal{N},\ S \subseteq \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ and $\hat{\boldsymbol{y}}$ is feasible to problem (13) and provides the same objective value as the optimal objective value of problem (11). Therefore, the optimal objective value of problem (13) is at least as large as that of problem (11). In the rest of the proof, we show that the reverse inequality also holds. We let $\widetilde{\boldsymbol{h}} = \{\widetilde{h}^q_{[s,f]\to j}(S) : j \in \mathcal{N},\ S \subseteq \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ and $\widetilde{\boldsymbol{y}} = \{\widetilde{y}_{i,[s,f]} : i \in \mathcal{N},\ [s,f] \in \mathcal{F}\}$ be an optimal solution to problem (13). We define $\hat{h}^q_{[s,f]}(S)$ as

$$\hat{h}^q_{[s,f]}(S) = \frac{1}{N} \sum_{j \in \mathcal{N}} \widetilde{h}^q_{[s,f]\to j}(S).$$

We establish that the solution $\hat{\boldsymbol{h}} = \big\{\hat{h}^q_{[s,f]}(S) : S \subseteq \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\big\}$ and $\widetilde{\boldsymbol{y}}$ is feasible to problem (11). Using the definition of $\hat{h}^q_{[s,f]}(S)$ above, we have

$$\sum_{S \subseteq \mathcal{N}} \hat{h}^q_{[s,f]}(S) \ = \ \frac{1}{N} \sum_{j \in \mathcal{N}} \sum_{S \subseteq \mathcal{N}} \widetilde{h}^q_{[s,f]\to j}(S) \ \overset{(a)}{=} \ \frac{1}{N} \sum_{j \in \mathcal{N}} \lambda^q_{[s,f]} \ = \ \lambda^q_{[s,f]},$$

where (a) holds because the solution $(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the second constraint in problem (13). Thus, the solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the second constraint in problem (11).

To check that the solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the third constraint in problem (11), using the definition of $\hat{h}^q_{[s,f]}(S)$ once more, we have

$$\sum_{S \subseteq \mathcal{N}} \phi^q_i(S)\, \hat{h}^q_{[s,f]}(S) \ = \ \frac{1}{N} \sum_{j \in \mathcal{N}} \sum_{S \subseteq \mathcal{N}} \phi^q_i(S)\, \widetilde{h}^q_{[s,f]\to j}(S) \ \overset{(b)}{=} \ \frac{1}{N} \sum_{j \in \mathcal{N}} \widetilde{y}^q_{i,[s,f]} \ = \ \widetilde{y}^q_{i,[s,f]}, \tag{22}$$

where (b) holds because the solution $(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the third constraint in problem (13). Thus, the solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the third constraint in problem (11).

Lastly, we check that the solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the first constraint in problem (11). In particular, we have the chain of inequalities

$$\begin{aligned} \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \sum_{S \subseteq \mathcal{N}} \mathbb{1}_{\{\ell \in [s,f]\}}\ \phi^q_i(S)\ \hat{h}^q_{[s,f]}(S) \ &= \ \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{\ell \in [s,f]\}} \sum_{S \subseteq \mathcal{N}} \phi^q_i(S)\ \hat{h}^q_{[s,f]}(S) \\ &\overset{(c)}{=} \ \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{\ell \in [s,f]\}}\, \widetilde{y}^q_{i,[s,f]} \ \overset{(d)}{=} \ \sum_{q \in \mathcal{Q}} \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{\ell \in [s,f]\}} \sum_{S \subseteq \mathcal{N}} \phi^q_i(S)\, \widetilde{h}_{[s,f]\to i}(S) \ \overset{(e)}{\leq} \ 1, \end{aligned}$$

where (c) follows from (22), (d) follows from the fact that the solution $(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the third constraint in problem (13) with $i = j$, and (e) holds because the solution $(\widetilde{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ also satisfies the first constraint in problem (13). Therefore, the solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ satisfies the first constraint in problem (11). Thus, the solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ is feasible to problem (11). The solution $(\hat{\boldsymbol{h}}, \widetilde{\boldsymbol{y}})$ provides the objective value $\sum_{q\in\mathcal{Q}} \sum_{i\in\mathcal{N}} \sum_{[s,f]\in\mathcal{F}} r_{i,[s,f]}\, \widetilde{y}^q_{i,[s,f]}$, which is the optimal objective value of problem (13). So, the optimal objective value of problem (11) is at least as large as that of problem (13). ■

To write the dual of problem (13), associating the dual variables $\boldsymbol{\mu} = \{\mu_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T}\}$, $\boldsymbol{\sigma} = \{\sigma^q_{[s,f]\to j} : j \in \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ and $\boldsymbol{\beta} = \{\beta^q_{i,[s,f]\to j} : i, j \in \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$, we have

$$\begin{aligned}
\min \quad & \sum_{i\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \mu_{i,\ell} + \sum_{q\in\mathcal{Q}} \sum_{j\in\mathcal{N}} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \sigma^q_{[s,f]\to j} \\
\text{st} \quad & \sigma^q_{[s,f]\to j} \geq \sum_{i\in\mathcal{N}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} - \phi^q_j(S) \sum_{\ell=s}^{f} \mu_{j,\ell} \quad \forall\, j \in \mathcal{N},\ [s,f] \in \mathcal{F},\ S \subseteq \mathcal{N},\ q \in \mathcal{Q} \\
& \sum_{j\in\mathcal{N}} \beta^q_{i,[s,f]\to j} = r_{i,[s,f]} \quad \forall\, i \in \mathcal{N},\ [s,f] \in \mathcal{F} \\
& \boldsymbol{\mu} \geq \mathbf{0},\ \boldsymbol{\sigma}, \boldsymbol{\beta} \text{ free.}
\end{aligned} \tag{23}$$

It is simple to check that problem (13) is feasible and bounded, so by strong duality, problem (23) also has the optimal objective value $\mathsf{Z}^*_{\mathsf{LP}}$. We use problem (23) to give a proof for Theorem 6.2.

Proof of Theorem 6.2:

Let $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\beta}})$ be an optimal solution to problem (23). For notational brevity, also letting $\hat{\alpha}^q_j = \sum_{k=q}^{Q} \sum_{[s,f]\in\mathcal{F}} \lambda^k_{[s,f]}\, \hat{\sigma}^k_{[s,f]\to j}$, we will use induction over the time periods to establish that $V^q_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{x}_j) \leq \hat{\alpha}^q_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell}$ for all $\boldsymbol{x}_j \in \{0,1\}^T$, $j \in \mathcal{N}$ and $q \in \mathcal{Q}$. In this case, using this result with $q = 1$ and $\boldsymbol{x}_j = \boldsymbol{e}'$, we obtain $\sum_{j\in\mathcal{N}} V^1_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{e}') \leq \sum_{j\in\mathcal{N}} \hat{\alpha}^1_j + \sum_{j\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell} = \mathsf{Z}^*_{\mathsf{LP}}$, where the equality uses the fact that $\sum_{j\in\mathcal{N}} \hat{\alpha}^1_j + \sum_{j\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell} = \sum_{j\in\mathcal{N}} \sum_{q\in\mathcal{Q}} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \hat{\sigma}^q_{[s,f]\to j} + \sum_{j\in\mathcal{N}} \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}$ and the last quantity is the optimal objective value of problem (23), which is $\mathsf{Z}^*_{\mathsf{LP}}$. Thus, the desired result follows. In the rest of the proof, we focus on the induction argument. Noting that $\hat{\boldsymbol{\mu}} \geq 0$, for each $\boldsymbol{x}_j \in \{0,1\}^T$, at time period $Q+1$, we have $V^{Q+1}_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{x}_j) = 0 \leq \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell}$, so the result holds at time period $Q+1$. Assuming that the result holds at time period $q+1$, we show that the result holds at time period q as well. By the induction hypothesis $V^{q+1}_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{x}_j)$ and $V^{q+1}_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]})$ are upper bounded by $\hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell}$ and $\hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell} - \sum_{\ell=s}^{f} \hat{\mu}_{j,\ell}$. So, by (12),

$$\begin{aligned}
V^q_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{x}_j) \leq \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \left[\beta^q_{j,[s,f]\to j} - \sum_{\ell=s}^{f} \hat{\mu}_{j,\ell} \right] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\} \\
+ \hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell}.
\end{aligned}$$

Following the same argument that we used to obtain the inequality (a) in the proof of Proposition B.4, we can drop the product $\prod_{\ell=s}^{f} x_{j,\ell}$ on the right side above to make the right side of the inequality even larger. After dropping the product $\prod_{\ell=s}^{f} x_{j,\ell}$, let $S^*_{[s,f]\to j}$ be an optimal solution to the resulting maximization problem. In this case, using the inequality above, we get

$$\begin{aligned}
V^q_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{x}_j) \;&\le\; \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left[\beta^q_{j,[s,f]\to j} - \sum_{\ell=s}^{f} \hat{\mu}_{j,\ell} \right] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\} \\
&\qquad\qquad + \hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell} \\
&\overset{(a)}{=} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \left\{ \phi^q_j(S^*_{[s,f]\to j}) \left[\beta^q_{j,[s,f]\to j} - \sum_{\ell=s}^{f} \hat{\mu}_{j,\ell} \right] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S^*_{[s,f]\to j})\, \beta^q_{i,[s,f]\to j} \right\} + \hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell} \\
&\overset{(b)}{=} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \left\{ \sum_{i\in\mathcal{N}} \phi^q_i(S^*_{[s,f]\to j})\, \beta^q_{i,[s,f]\to j} - \phi^q_j(S^*_{[s,f]\to j}) \sum_{\ell=s}^{f} \hat{\mu}_{j,\ell} \right\} + \hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell} \\
&\overset{(c)}{\le} \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \hat{\sigma}^q_{[s,f]\to j} + \hat{\alpha}^{q+1}_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell} \;\overset{(d)}{=}\; \hat{\alpha}^q_j + \sum_{\ell\in\mathcal{T}} \hat{\mu}_{j,\ell}\, x_{j,\ell},
\end{aligned}$$

where (a) holds because $S^*_{[s,f]\to j}$ is an optimal solution to the maximization problem on the left side of (a), (b) follows by arranging terms, (c) holds because $(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\beta}})$ satisfies the first constraint in (23), and (d) follows because $\hat{\alpha}^q_j = \hat{\alpha}^{q+1}_j + \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]}\, \hat{\sigma}^q_{[s,f]\to j}$. So, the induction is complete. ■

We focus on the proof of Theorem 6.3. To give an alternative representation of the value functions $\{\Gamma^q_{\boldsymbol{\beta},j} : q\in\mathcal{Q}\}$ in (14), for each $\ell\in\mathcal{T}$, define the value function

$$\Psi^q_{\boldsymbol{\beta},j}(\ell) \;=\; \sum_{k=q}^{Q} \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s=\ell\}}\, \lambda^k_{[s,f]} \max_{S\subseteq\mathcal{N}\setminus\{j\}} \left\{ \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^k_i(S)\, \beta^k_{i,[s,f]\to j} \right\}. \tag{24}$$

Directly by comparing (24) with (14), observe that the value functions $\{\Gamma^q_{\boldsymbol{\beta},j} : q\in\mathcal{Q}\}$ and $\{\Psi^q_{\boldsymbol{\beta},j} : q\in\mathcal{Q}\}$ satisfy the relationship $\Gamma^q_{\boldsymbol{\beta},j}(a,b) = \sum_{\ell=a}^{b} \Psi^q_{\boldsymbol{\beta},j}(\ell)$ for each interval $[a,b]$. Given that the state of resource j is $\boldsymbol{x}_j\in\{0,1\}^T$, let $\mathcal{K}(\boldsymbol{x}_j)$ be the set of unavailable days for this resource; that is $\ell\in\mathcal{K}(\boldsymbol{x}_j)$ if and only if $x_{j,\ell}=0$. Note that $\mathcal{K}(\boldsymbol{x}_j)$ is the union of the maximal unavailable intervals with respect to $\boldsymbol{x}_j$. In other words, we have $\mathcal{K}(\boldsymbol{x}_j) = \cup_{[a,b]\in\mathcal{H}(\boldsymbol{x}_j)}[a,b]$. In this case, by the discussion earlier in this paragraph, we obtain the identity

$$\sum_{[a,b]\in\mathcal{H}(\boldsymbol{x}_j)} \Gamma^q_{\boldsymbol{\beta},j}(a,b) \;=\; \sum_{[a,b]\in\mathcal{H}(\boldsymbol{x}_j)} \sum_{\ell\in[a,b]} \Psi^q_{\boldsymbol{\beta},j}(\ell) \;=\; \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \Psi^q_{\boldsymbol{\beta},j}(\ell),$$

where the last equality holds since $\mathcal{K}(\boldsymbol{x}_j) = \cup_{[a,b]\in\mathcal{H}(\boldsymbol{x}_j)}[a,b]$. Thus, to establish Theorem 6.3, it is enough to show that $V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) = \sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \Theta^q_{\boldsymbol{\beta},j}(a,b) + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \Psi^q_{\boldsymbol{\beta},j}(\ell)$.

Proof of Theorem 6.3:

In the proof, we will use an induction argument over the time periods to establish that $V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) = \sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \Theta^q_{\boldsymbol{\beta},j}(a,b) + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \Psi^q_{\boldsymbol{\beta},j}(\ell)$ for all $\boldsymbol{x}_j\in\{0,1\}^T$ and $q\in\mathcal{Q}$. At time period

$Q+1$, we have $V^{Q+1}_{\boldsymbol{\beta},j} = \Theta^{Q+1}_{\boldsymbol{\beta},j} = \Psi^{Q+1}_{\boldsymbol{\beta},j} = 0$, so the result holds at time period $Q+1$. Assuming that the result holds at time period $q+1$, we show that the result holds at time period q as well. For each $\boldsymbol{x}_j \in \{0,1\}^T$, the collection of maximal available intervals $\mathcal{I}(\boldsymbol{x}_j)$ and the set of unavailable days $\mathcal{K}(\boldsymbol{x}_j)$ collectively cover $\mathcal{T}$; that is, $\mathcal{T} = \big(\cup_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)}[a,b]\big) \cup \mathcal{K}(\boldsymbol{x}_j)$. Thus, for each $s \in \mathcal{T}$, we have $\sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \mathbb{1}_{\{s\in[a,b]\}} + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \mathbb{1}_{\{s=\ell\}} = 1$. In this case, by (12), we get

$$\begin{aligned}
V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \;&=\; V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) + \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \left(\sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \mathbb{1}_{\{s\in[a,b]\}} + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \mathbb{1}_{\{s=\ell\}} \right) \times \\
&\qquad \max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Big] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\} \\
&= \sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \left(\Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) \;+\; \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s\in[a,b]\}}\, \lambda^q_{[s,f]} \times \right. \\
&\qquad \left. \max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Big] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\} \right) \\
&\quad + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \left(\Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) \;+\; \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s=\ell\}}\, \lambda^q_{[s,f]} \times \right. \\
&\qquad \left. \max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Big] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\} \right),
\end{aligned}$$

where the second equality follows by using the induction hypothesis to note that $V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) = \sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell)$ and rearranging the order of sums.

Noting the right side of the chain of equalities above, in the rest of the proof, for each $[a,b]\in\mathcal{I}(\boldsymbol{x}_j)$ and $\ell\in\mathcal{K}(\boldsymbol{x}_j)$, we will establish the following two equalities

$$\begin{aligned}
\Theta^q_{\boldsymbol{\beta},j}(a,b) \;&=\; \Theta^{q+1}_{\boldsymbol{\beta},j}([a,b]) + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s\in[a,b]\}}\, \lambda^q_{[s,f]} \times \\
&\max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Big] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\}, \quad (25) \\
\Psi^q_{\boldsymbol{\beta},j}(\ell) \;&=\; \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s=\ell\}}\, \lambda^q_{[s,f]} \times \\
&\max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \Big[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Big] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\, \beta^q_{i,[s,f]\to j} \right\}. \quad (26)
\end{aligned}$$

Thus, by the first displayed equality in the proof, $V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) = \sum_{[a,b]\in\mathcal{I}(\boldsymbol{x}_j)} \Theta^q_{\boldsymbol{\beta},j}(a,b) + \sum_{\ell\in\mathcal{K}(\boldsymbol{x}_j)} \Psi^q_{\boldsymbol{\beta},j}(\ell)$, completing the induction argument.

First, we show that (25) holds for each $[a,b]\in\mathcal{I}(\boldsymbol{x}_j)$. So, throughout this portion of the discussion, we focus on the maximal available intervals $[a,b]\in\mathcal{I}(\boldsymbol{x}_j)$. In (25), we consider $s\in[a,b]$. Note that if

$s \in [a,b]$ and $f \in [a,b]$, then because $[a,b] \in \mathcal{I}(\boldsymbol{x}_j)$, we get $x_{j,h} = 1$ for all $h = s, \ldots, f$, so $\prod_{h=s}^{f} x_{j,h} = 1$. Furthermore, because $s \in [a,b]$, we have $f \in [a,b]$ if and only if $[s,f] \subseteq [a,b]$. On the other hand, if $s \in [a,b]$ and $f \notin [a,b]$, then since the interval $[a,b]$ is a maximal available interval, there must be some $h = s+1, \ldots, f$ such that $x_{j,h} = 0$, so $\prod_{h=s}^{f} x_{j,h} = 0$. Furthermore, because $s \in [a,b]$, we have $f \notin [a,b]$ if and only if $[s,f] \not\subseteq [a,b]$. Thus, using $\mathbb{1}_{\{s \in [a,b]\}} = \mathbb{1}_{\{s \in [a,b],\ [s,f] \subseteq [a,b]\}} + \mathbb{1}_{\{s \in [a,b],\ [s,f] \not\subseteq [a,b]\}}$, we equivalently express the right side of (25) as

$$
\begin{aligned}
&\Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) + \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{s \in [a,b],\ [s,f] \subseteq [a,b]\}} \, \lambda^q_{[s,f]} \times \\
&\qquad \max_{S \subseteq \mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \left[\beta^q_{j,[s,f] \to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi^q_i(S) \, \beta^q_{i,[s,f] \to j} \right\} \\
&\qquad\quad + \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{s \in [a,b],\ [s,f] \not\subseteq [a,b]\}} \, \lambda^q_{[s,f]} \times \\
&\qquad \max_{S \subseteq \mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \left[\beta^q_{j,[s,f] \to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi^q_i(S) \, \beta^q_{i,[s,f] \to j} \right\} \\
&\overset{(a)}{=} \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) + \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{s \in [a,b],\ [s,f] \subseteq [a,b]\}} \, \lambda^q_{[s,f]} \times \\
&\qquad \max_{S \subseteq \mathcal{N}} \left\{ \phi^q_j(S) \left[\beta^q_{j,[s,f] \to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \right] + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi^q_i(S) \, \beta^q_{i,[s,f] \to j} \right\} \\
&\qquad\quad + \sum_{[s,f] \in \mathcal{F}} \mathbb{1}_{\{s \in [a,b],\ [s,f] \not\subseteq [a,b]\}} \, \lambda^q_{[s,f]} \max_{S \subseteq \mathcal{N} \setminus \{j\}} \left\{ \sum_{i \in \mathcal{N} \setminus \{j\}} \phi^q_i(S) \, \beta^q_{i,[s,f] \to j} \right\}, \qquad (27)
\end{aligned}
$$

where (a) holds because if $[s,f] \subseteq [a,b]$, then $\prod_{h=s}^{f} x_{j,h} = 1$, whereas if $s \in [a,b]$ and $[s,f] \not\subseteq [a,b]$, then $\prod_{h=s}^{f} x_{j,h} = 0$, as discussed right before the chain of equalities just above.

From Section 3, recall that $\mathcal{I}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) = \big(\mathcal{I}(\boldsymbol{x}_j) \setminus [a,b]\big) \cup \big\{[a,s-1],\, [f+1,b]\big\}$ for $[s,f] \subseteq [a,b]$. By definition of $\mathcal{K}(\boldsymbol{x}_j)$, $\mathcal{K}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) = \mathcal{K}(\boldsymbol{x}_j) \cup \{s, \ldots, f\}$ So, by the induction hypothesis, we get

$$
\begin{aligned}
&V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \\
&\qquad = \sum_{[a,b] \in \mathcal{I}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]})} \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) \ + \sum_{\ell \in \mathcal{K}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]})} \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) \ - \sum_{[a,b] \in \mathcal{I}(\boldsymbol{x}_j)} \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) \ - \sum_{\ell \in \mathcal{K}(\boldsymbol{x}_j)} \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) \\
&\qquad = \Theta^{q+1}_{\boldsymbol{\beta},j}(a,s-1) \ + \ \Theta^{q+1}_{\boldsymbol{\beta},j}(f+1,b) \ - \ \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) \ + \ \sum_{\ell=s}^{f} \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) \\
&\qquad \overset{(b)}{=} \Theta^{q+1}_{\boldsymbol{\beta},j}(a,s-1) \ + \ \Theta^{q+1}_{\boldsymbol{\beta},j}(f+1,b) \ - \ \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) \ + \ \Gamma^{q+1}_{\boldsymbol{\beta},j}(s,f), \qquad (28)
\end{aligned}
$$

where (b) follows from the discussion right before the proof of the theorem, which shows that the value functions $\{\Gamma^q_{\boldsymbol{\beta},j} : q \in \mathcal{Q}\}$ and $\{\Psi^q_{\boldsymbol{\beta},j} : q \in \mathcal{Q}\}$ computed, respectively, through (14) and (24) satisfy the identity $\Gamma^{q+1}_{\boldsymbol{\beta},j}(s,f) = \sum_{\ell=s}^{f} \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell)$. In this case, plugging (28) into (27), we equivalently express the right side of (25) as

$$\begin{aligned}
&\Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s\in[a,b],\ [s,f]\subseteq[a,b]\}}\, \lambda^q_{[s,f]} \times \\
&\qquad \max_{S\subseteq\mathcal{N}} \Bigg\{ \phi^q_j(S) \left[\beta^q_{j,[s,f]\to j} + \Theta^{q+1}_{\boldsymbol{\beta},j}(a,s-1) \ + \ \Theta^{q+1}_{\boldsymbol{\beta},j}(f+1,b) \ - \ \Theta^{q+1}_{\boldsymbol{\beta},j}(a,b) \ + \ \Gamma^{q+1}_{\boldsymbol{\beta},j}(s,f) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\ \beta^q_{i,[s,f]\to j} \Bigg\} \\
&\qquad + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s\in[a,b],\ [s,f]\not\subseteq[a,b]\}}\, \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}\setminus\{j\}} \left\{ \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\ \beta^q_{i,[s,f]\to j} \right\} \\
&\overset{(c)}{=} \ \Theta^q_{\boldsymbol{\beta},j}(a,b),
\end{aligned}$$

where (c) follows from (15). By the equality above, the right side of (25) is equal to $\Theta^q_{\boldsymbol{\beta},j}(a,b)$, establishing the equality in (25).

Second, we show that (26) holds for each $\ell \in \mathcal{K}(\boldsymbol{x}_j)$. Thus, we focus on the unavailable days $\ell \in \mathcal{K}(\boldsymbol{x}_j)$. For $\ell \in \mathcal{K}(\boldsymbol{x}_j)$, we have $x_{j,\ell} = 0$, so $\prod_{h=s}^{f} x_{j,h} = 0$ for $s = \ell$. So, the right side of (26) reads

$$\begin{aligned}
&\Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s=\ell\}}\, \lambda^q_{[s,f]} \times \\
&\qquad \max_{S\subseteq\mathcal{N}} \left\{ \phi^q_j(S) \left(\prod_{h=s}^{f} x_{j,h} \right) \left[\beta^q_{j,[s,f]\to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \right] + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\ \beta^q_{i,[s,f]\to j} \right\} \\
&= \ \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s=\ell\}}\, \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\ \beta^q_{i,[s,f]\to j} \right\} \\
&\overset{(d)}{=} \ \Psi^{q+1}_{\boldsymbol{\beta},j}(\ell) + \sum_{[s,f]\in\mathcal{F}} \mathbb{1}_{\{s=\ell\}}\, \lambda^q_{[s,f]} \max_{S\subseteq\mathcal{N}\setminus\{j\}} \left\{ \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S)\ \beta^q_{i,[s,f]\to j} \right\} \ \overset{(e)}{=} \ \Psi^q_{\boldsymbol{\beta},j}(\ell),
\end{aligned}$$

where (d) holds because resource j does not appear in the objective function of the maximization problem on the left side of (d), and (e) follows by (24). Thus, the equality in (26) holds. ■

Appendix H: Connection of the Upper Bound to Lagrangian Relaxation

In this section, we show that we can obtain the dynamic program in (12) through Lagrangian relaxation in the dynamic program in (1). In this way, we also relate our revenue allocations to Lagrange multipliers. In the maximization problem in (1), the decision variable S corresponds to the assortment of resources that we offer to a customer making a booking request for the interval

$[s,f]$. Using the decision variables $\{S_j : j \in \mathcal{N} \cup \{0\}\}$ instead of the decision variable S in the maximization problem in (1), we write the dynamic program in (1) equivalently as

$$J^q(\boldsymbol{x}) = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{\substack{S_j \subseteq \mathcal{N} \\ \forall\, j \in \mathcal{N}\cup\{0\}}} \left\{ \sum_{j\in\mathcal{N}} \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \Big\{ \phi^q_j(S_0)\, r_{j,[s,f]} + \phi^q_j(S_j) \Big[J^{q+1}(\boldsymbol{x} - \boldsymbol{e}_{j,[s,f]}) - J^{q+1}(\boldsymbol{x}) \Big] \Big\} \right.$$
$$\left. : \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \phi^q_j(S_0) = \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \phi^q_j(S_i) \quad \forall\, i,j \in \mathcal{N} \right\} + J^{q+1}(\boldsymbol{x}). \quad (29)$$

We claim that the maximization problems on the right sides of (1) and (29) have the same optimal objective value. In particular, if S^* is an optimal solution to the maximization problem on the right side of (1), then setting $\hat{S}_j = S^*$ for all $j \in \mathcal{N} \cup \{0\}$ provides a feasible solution to the maximization problem on the right side of (29). Furthermore, the objective values of the two maximization problems at these two solutions match. On the other hand, if $\{S^*_j : j \in \mathcal{N} \cup \{0\}\}$ is an optimal solution to the maximization problem on the right side of (29), then setting $\hat{S} = S^*_0$ provides a feasible solution to the maximization problem on the right side of (1). Furthermore, noting that $\prod_{\ell=s}^{f} x_{j,\ell}\, \phi^q_j(\hat{S}) = \prod_{\ell=s}^{f} x_{j,\ell}\, \phi^q_j(S^*_0) = \prod_{\ell=s}^{f} x_{j,\ell}\, \phi^q_j(S^*_j)$ for all $j \in \mathcal{N}$ by the constraints in (29), the objective values of the two maximization problems at these two solutions match. Thus, the claim holds, which implies that the dynamic programs in (1) and (29) are equivalent to each other. Intuitively speaking, the dynamic program in (29) uses $N+1$ copies of the decision variable S in (1). These copies are given by $\{S_j : j \in \mathcal{N} \cup \{0\}\}$. Even though the dynamic program in (29) uses $N+1$ copies of the decision variable S in (1), the constraints in (29) ensure that the choice probabilities of all resources that are feasible to offer match under the different copies. We solve the maximization problem on the right side of (29) for each $[s,f] \in \mathcal{F}$. Thus, associating the Lagrange multipliers $\{\beta^q_{j,[s,f]\to i} : i,j \in \mathcal{N}\}$ with the constraints on the right side of (29), relaxing these constraints and arranging the terms, we obtain the dynamic program

$$\widetilde{J}^q_{\boldsymbol{\beta}}(\boldsymbol{x}) = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \max_{S_0 \in \mathcal{N}} \left\{ \sum_{j\in\mathcal{N}} \phi^q_j(S_0) \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \Big[r_{j,[s,f]} - \sum_{i\in\mathcal{N}} \beta^q_{j,[s,f]\to i} \Big] \right\}$$
$$+ \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{j\in\mathcal{N}} \max_{S_j \in \mathcal{N}} \left\{ \phi^q_j(S_j) \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \Big[\beta^q_{j,[s,f]\to j} + \widetilde{J}^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x} - \boldsymbol{e}_{j,[s,f]}) - \widetilde{J}^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x}) \Big] \right.$$
$$\left. + \sum_{i\in\mathcal{N}\setminus\{j\}} \phi^q_i(S_j) \left(\prod_{\ell=s}^{f} x_{i,\ell} \right) \beta^q_{i,[s,f]\to j} \right\} + \widetilde{J}^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x}), \quad (30)$$

where we use the fact that once we relax the constraints in (29), the maximization problem on the right side of (29) decomposes by the elements of $\mathcal{N} \cup \{0\}$.

Note that we make the dependence of the value functions $\{\widetilde{J}^q_{\boldsymbol{\beta}} : q \in \mathcal{Q}\}$ in (30) on the Lagrange multipliers $\boldsymbol{\beta} = \{\beta^q_{i,[s,f]\to j} : i,j \in \mathcal{N},\ [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ explicit. The dynamic program in (30) is a relaxed version of the dynamic program in (29). It is a standard result that the value functions in (30) provide upper bounds on those in (29) for any choice of the Lagrange multipliers; see, for example, Adelman and Mersereau (2008). In other words, for any $\boldsymbol{\beta} \in \mathbb{R}^{N^2 \times |\mathcal{F}| \times |\mathcal{Q}|}$, we have $\widetilde{J}^q_{\boldsymbol{\beta}}(\boldsymbol{x}) \geq J^q(\boldsymbol{x})$ for each $\boldsymbol{x} \in \{0,1\}^{N \times T}$ and $q \in \mathcal{Q}$. Because of the terms $\left\{\prod_{\ell=s}^f x_{j,\ell} : j \in \mathcal{N}\right\}$ in (30), the value functions $\{\widetilde{J}^q_{\boldsymbol{\beta}} : q \in \mathcal{Q}\}$ can still be computationally difficult to compute. We carry out two additional forms of relaxation in (30), while making sure that we still obtain upper bounds on the value functions $\{J^q : q \in \mathcal{Q}\}$. First, recalling that we have $\widetilde{J}^q_{\boldsymbol{\beta}}(\boldsymbol{x}) \geq J^q(\boldsymbol{x})$ for any choice of the Lagrange multipliers, we choose our Lagrange multipliers such that $\sum_{i \in \mathcal{N}} \beta^q_{j,[s,f]\to i} = r_{j,[s,f]}$ for all $j \in \mathcal{N}$, $[s,f] \in \mathcal{F}$ and $q \in \mathcal{Q}$. In this case, the objective function of the first maximization problem in (30) becomes zero, so we can drop this maximization problem. Second, considering the second maximization problem in (30), by Lemma A.1, if $\beta^q_{i,[s,f]\to j} \leq 0$ for some $i \in \mathcal{N} \setminus \{j\}$ in the summation in the objective function of this maximization problem, then there exists an optimal solution to this maximization problem that does not include resource i. On the other hand, if we have $\beta^q_{i,[s,f]\to j} > 0$ for some $i \in \mathcal{N} \setminus \{j\}$, then $\prod_{\ell=s}^f x_{i,\ell}\, \beta^q_{i,[s,f]\to j} \leq \beta^q_{i,[s,f]\to j}$. Thus, if we drop the term $\prod_{\ell=s}^f x_{i,\ell}$ in the summation in the objective function of the second maximization problem in (30), then the optimal objective value of this maximization problem stays at least as large. In this case, focusing on Lagrange multipliers that satisfy $\sum_{i \in \mathcal{N}} \beta^q_{j,[s,f]\to i} = r_{j,[s,f]}$ for all $j \in \mathcal{N}$, $[s,f] \in \mathcal{F}$ and $q \in \mathcal{Q}$ and dropping the term $\prod_{\ell=s}^f x_{i,\ell}$ in the summation in the objective function of the second maximization problem in (30), we get the dynamic program

$$\Theta^q_{\boldsymbol{\beta}}(\boldsymbol{x}) = \sum_{[s,f]\in\mathcal{F}} \lambda^q_{[s,f]} \sum_{j \in \mathcal{N}} \max_{S_j \in \mathcal{N}} \left\{ \phi^q_j(S_j) \left(\prod_{\ell=s}^f x_{j,\ell} \right) \left[\beta^q_{j,[s,f]\to j} + \Theta^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x} - \boldsymbol{e}_{j,[s,f]}) - \Theta^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x}) \right] \right.$$
$$\left. + \sum_{i \in \mathcal{N}\setminus\{j\}} \phi^q_i(S_j)\, \beta^q_{i,[s,f]\to j} \right\} + \Theta^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x}). \quad (31)$$

By the discussion in this paragraph, the value functions $\{\Theta^q_{\boldsymbol{\beta}} : q \in \mathcal{Q}\}$ in (31) provide upper bounds on the value functions $\{\widetilde{J}^q_{\boldsymbol{\beta}} : q \in \mathcal{Q}\}$ in (30), as long as $\sum_{i \in \mathcal{N}} \beta^q_{j,[s,f]\to i} = r_{j,[s,f]}$.

Summing up the development so far, if the Lagrange multipliers satisfy $\sum_{i \in \mathcal{N}} \beta^q_{j,[s,f]\to i} = r_{j,[s,f]}$ for all $j \in \mathcal{N}$, $[s,f] \in \mathcal{F}$ and $q \in \mathcal{Q}$, then we have $\Theta^q_{\boldsymbol{\beta}}(\boldsymbol{x}) \geq \widetilde{J}^q_{\boldsymbol{\beta}}(\boldsymbol{x}) \geq J^q(\boldsymbol{x})$ for each $\boldsymbol{x} \in \{0,1\}^{N \times T}$ and $q \in \mathcal{Q}$. Therefore, we can obtain an upper bound on the value functions $\{J^q : q \in \mathcal{Q}\}$ in (1) by using the value functions $\{\Theta^q_{\boldsymbol{\beta}} : q \in \mathcal{Q}\}$ in (31). Furthermore, we obtain the dynamic program in (31) by using Lagrangian relaxation on the dynamic program in (1). In the next lemma, we show that the dynamic program in (31) is equivalent to the one in (12). Thus, we can obtain the dynamic program in (12), which we use to compute an upper bound on the optimal total expected revenue in Section 6, by using Lagrangian relaxation on the dynamic program in (1). In the next lemma, recall that we use $\boldsymbol{x}_j = (x_{j,1}, \ldots, x_{j,T}) \in \{0,1\}^T$ to capture the state of resource j, where $x_{j,\ell} = 1$ if and only if resource j is available for use on day ℓ.

Lemma H.1 (Equivalence of Dynamic Programs) *Letting the value functions* $\{V^q_{\boldsymbol{\beta},j} : q \in \mathcal{Q}\}$ *be computed through* (12) *and the value functions* $\{\Theta^q_{\boldsymbol{\beta}} : q \in \mathcal{Q}\}$ *be computed through* (31), *for each* $\boldsymbol{x} = (\boldsymbol{x}_j : j \in \mathcal{N}) \in \{0,1\}^{N \times T}$ *and* $q \in \mathcal{Q}$, *we have*

$$\Theta^q_{\boldsymbol{\beta}}(\boldsymbol{x}) = \sum_{j \in \mathcal{N}} V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j).$$

Proof: We show the result by using induction over the time periods. At time period $Q+1$, we have $\Theta^{Q+1}_{\boldsymbol{\beta}} = 0 = V^{Q+1}_{\boldsymbol{\beta},j}$, so the result holds at time period $Q+1$. Assuming that the result holds at time period $q+1$, we show that the result holds at time period q as well. By the induction argument, we have $\Theta^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x}) = \sum_{j \in \mathcal{N}} V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)$, so $\Theta^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x} - \boldsymbol{e}_{j,[s,f]}) - \Theta^{q+1}_{\boldsymbol{\beta}}(\boldsymbol{x}) = V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j)$, in which case, using the dynamic program in (31), we obtain

$$\begin{aligned}\Theta^q_{\boldsymbol{\beta}}(\boldsymbol{x}) \; = \sum_{[s,f] \in \mathcal{F}} \lambda^q_{[s,f]} \sum_{j \in \mathcal{N}} \max_{S_j \in \mathcal{N}} \Bigg\{ \phi^q_j(S_j) \left(\prod_{\ell=s}^{f} x_{j,\ell} \right) \Big[\beta^q_{j,[s,f] \to j} + V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j - \boldsymbol{e}_{[s,f]}) - V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \Big] \\ + \sum_{i \in \mathcal{N} \setminus \{j\}} \phi^q_i(S_j)\, \beta^q_{i,[s,f] \to j} \Bigg\} + \sum_{j \in \mathcal{N}} V^{q+1}_{\boldsymbol{\beta},j}(\boldsymbol{x}_j) \; = \; \sum_{j \in \mathcal{N}} V^q_{\boldsymbol{\beta},j}(\boldsymbol{x}_j),\end{aligned}$$

where the last equality follows by arranging the terms and using (12). Thus, the result holds at time period q as well, completing the induction argument. ■

By the lemma above, the dynamic program in (31) decomposes by the resources and the dynamic program that we solve for each resource j corresponds to the one in (12). Noting that we obtained the dynamic program in (31) by using Lagrangian relaxation on the dynamic program in (1), the dynamic program that we solve in Section 6 to compute an upper bound on the optimal total expected revenue can be obtained by using Lagrangian relaxation on the dynamic program in (1). The revenue allocations $\{\beta^q_{i,[s,f] \to j} : i,j \in \mathcal{N},\; [s,f] \in \mathcal{F},\; q \in \mathcal{Q}\}$ in Section 6 correspond to the Lagrange multipliers employed when using Lagrangian relaxation on (1).

Appendix I: Performance of the Constraint Splitting Policy on Synthetic Datasets

In Baek and Ma (2021), the authors consider general revenue management problems with non-unit resource capacities, as well as booking requests not necessarily over intervals of days. They split the resource constraints into two groups. In the first group, a booking request consumes the capacities of at most L different resources. In the second group, they have a matroid characterization of the capacity consumptions of the booking requests. The authors give a policy that is guaranteed to obtain at least $\frac{1}{2(1+L)}$ fraction of the optimal total expected revenue. We refer to the policy proposed by Baek and Ma (2021) as the constraint splitting policy. The performance guarantee for the constraint splitting policy applies to general revenue management

Param.	Total Exp. Revenue				
$(\mathsf{D}_{\max}, \rho, \delta)$	LINR	POLR	LPR	COS1	COS5
$(6, 1.0, 0.9)$	78.00	78.17	78.55	67.57	64.82
$(6, 1.0, 0.7)$	83.41	83.54	83.87	71.83	68.97
$(6, 1.2, 0.9)$	81.05	80.29	81.11	65.40	63.16
$(6, 1.2, 0.7)$	73.00	73.04	73.97	64.92	61.33
$(6, 1.4, 0.9)$	75.51	76.04	75.77	67.66	65.62
$(6, 1.4, 0.7)$	80.83	80.22	81.01	72.13	67.84
$(8, 1.0, 0.9)$	79.13	78.41	79.40	63.34	59.94
$(8, 1.0, 0.7)$	78.15	77.01	78.64	59.47	57.66
$(8, 1.2, 0.9)$	78.23	77.84	79.29	62.82	61.58
$(8, 1.2, 0.7)$	71.07	71.09	73.44	59.86	56.76
$(8, 1.4, 0.9)$	81.04	79.34	80.41	66.64	65.12
$(8, 1.4, 0.7)$	78.26	76.56	78.33	63.52	63.79
$(10, 1.0, 0.9)$	70.53	70.43	71.54	54.67	54.70
$(10, 1.0, 0.7)$	77.11	74.66	76.79	58.87	54.54
$(10, 1.2, 0.9)$	73.89	71.43	74.33	53.93	54.13
$(10, 1.2, 0.7)$	73.09	72.90	75.02	57.47	58.17
$(10, 1.4, 0.9)$	80.92	76.82	80.46	61.49	55.58
$(10, 1.4, 0.7)$	80.51	77.52	80.34	62.32	61.72
Avg.	77.43	76.41	77.90	63.00	60.86

Table EC.1 Total expected revenues obtained by the constraint splitting policies.

problems, not requiring the resources to be unique or the booking requests to consume capacities on consecutive days. Furthermore, this performance guarantee is independent of the number of constraints in the second group. The constraint splitting policy, when applied to our revenue management problem, provides a performance guarantee of $\frac{1}{1+\mathsf{D}_{\max}}$. Our polynomial value function approximations improve upon this performance guarantee when $\mathsf{D}_{\min} > 1$. In this paper, exploiting the unit capacities of the resources and the interval structure of the booking requests, we can further perform rollout on static policies, which dramatically improves the performance of the static policy on hand. For our synthetic datasets, we compute the parameters of the constraint splitting policy once or five times over the selling horizon. We refer to the resulting policies, respectively, as COS1 and COS5, where we emphasize that these policies are based on constraint splitting. In Table EC.1, we compare the total expected revenues obtained by COS1 and COS5 with those obtained by LINR, POLR, and LPR. We express the total expected revenue of each benchmark policy as a percentage of the upper bound on the optimal total expected revenue. The total expected revenues that we report for LINR, POLR, and LPR in Table EC.1 are taken from Table 2.

Our results indicate that LINR, POLR, and LPR perform significantly better than COS1 and COS5. On average, the total expected revenues obtained by COS5 lag behind those obtained by LINR, POLR, and LPR by, respectively, 27.24%, 25.55%, and 28.02%. Thus, exploiting the structure of our revenue management problem, our rollout policies provide substantial improvements over COS5. Similarly, the performance of LINR, POLR and LPR is better than that of COS1 by, respectively, 22.91%, 21.29% and 23.66%, on average. Comparing the performance of COS5 with that of LIN5, POL5, and LP5 from Table 2, the improvements of LIN5, POL5, and LP5 over COS5 are still significant, reaching, respectively, 24.23%, 14.66%, and 20.46%, on average.

Param.	Upp.		Total Exp. Revenue										
$(\mathsf{D}_{\max}, \rho, \delta)$	Bnd.	$\mathsf{Z}^*_{\mathsf{LP}}$	LIN1	POL1	LP1	LIN5	POL5	LP5	LINR	POLR	LPR	GUB	DEC
$(6, 1.0, 0.9)$	3,337	3,990	2,485	2,376	2,449	2,570	2,421	2,515	2,603	2,608	2,621	2,616	2,470
$(6, 1.0, 0.7)$	1,195	1,526	964	900	934	983	919	961	996	998	1,002	1,015	962
$(6, 1.2, 0.9)$	2,525	2,914	2,001	1,899	1,944	2,043	1,903	1,970	2,047	2,027	2,048	2,016	1,951
$(6, 1.2, 0.7)$	4,120	4,615	2,891	2,825	2,824	2,936	2,877	2,871	3,008	3,010	3,048	3,001	2,788
$(6, 1.4, 0.9)$	6,029	7,138	4,316	4,297	4,249	4,460	4,388	4,286	4,553	4,585	4,568	4,639	4,344
$(6, 1.4, 0.7)$	3,674	4,580	2,826	2,597	2,760	2,897	2,683	2,864	2,970	2,947	2,976	3,020	2,832
$(8, 1.0, 0.9)$	3,735	4,749	2,836	2,510	2,720	2,897	2,565	2,807	2,955	2,928	2,965	3,029	2,886
$(8, 1.0, 0.7)$	3,365	4,227	2,483	2,324	2,447	2,546	2,348	2,446	2,630	2,591	2,646	2,615	2,556
$(8, 1.2, 0.9)$	4,607	5,788	3,468	3,049	3,404	3,600	3,192	3,497	3,604	3,586	3,653	3,678	3,492
$(8, 1.2, 0.7)$	3,663	4,232	2,579	2,407	2,456	2,644	2,436	2,511	2,604	2,604	2,690	2,648	2,506
$(8, 1.4, 0.9)$	4,150	5,179	3,140	2,829	3,107	3,228	2,909	3,145	3,363	3,293	3,337	3,374	3,191
$(8, 1.4, 0.7)$	5,226	6,507	3,811	3,588	3,826	3,995	3,669	3,861	4,090	4,001	4,093	4,084	3,889
$(10, 1.0, 0.9)$	6,884	7,523	4,615	4,526	4,300	4,737	4,626	4,410	4,856	4,849	4,925	4,594	4,494
$(10, 1.0, 0.7)$	5,317	6,387	3,898	3,552	3,780	3,977	3,656	3,845	4,100	3,969	4,082	4,017	3,895
$(10, 1.2, 0.9)$	5,259	5,986	3,608	3,291	3,484	3,719	3,336	3,574	3,886	3,757	3,909	3,752	3,684
$(10, 1.2, 0.7)$	5,403	6,102	3,780	3,596	3,621	3,777	3,700	3,727	3,949	3,939	4,054	3,864	3,715
$(10, 1.4, 0.9)$	5,708	7,316	4,357	3,562	4,221	4,496	3,680	4,346	4,619	4,385	4,593	4,702	4,393
$(10, 1.4, 0.7)$	3,414	4,421	2,546	2,315	2,555	2,586	2,345	2,578	2,749	2,647	2,743	2,783	2,531

Table EC.2 Absolute values of the upper bounds and total expected revenues for all benchmark policies.

Appendix J: Absolute Values of Upper Bounds and Total Expected Revenues

In Table 1, we give the gap between the upper bound from our approach and the upper bound from the linear program in (11). Similarly, in Table 2, we give the total expected revenues of the benchmark policies as a percentage of the upper bound. In particular, these tables do not give the absolute values of the upper bounds or total expected revenues. To ensure that our computational results are transparent, in Table EC.2, we give the absolute values of all upper bounds and total expected revenues for all of our test problems. In this table, the first and second columns, respectively, give the upper bounds obtained by using the dynamic program in (12) and the linear programming approximation in (11). The last eleven columns give the total expected revenues obtained by the benchmark policies in Section 7.1. Thus, the entries in Table 1 correspond to the percent gap between the entries in the first and second columns in Table EC.2, whereas the entries in each column in Table 2 correspond to the entries in each of last eleven columns in Table EC.2 expressed as a percentage of the entries in the first column in Table EC.2.

Appendix K: Heuristic Modification of the Resource Based Static Policy

In our discussion of the resource based static policy, we intuitively interpreted the fraction $\frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1}$ in (7) as a way of spreading the net revenue from a booking request for interval $[s, f]$ evenly over each day $\ell \in \{s, \dots, f\}$. This interpretation is merely a heuristic and the main reason that this fraction appears in (7) is algebraic, resting on the the fact that we have the relationship $\prod_{\ell=s}^{f} x_{i,\ell} \geq \frac{\left(\sum_{\ell=s}^{f} x_{i,\ell}\right) - (f-s)}{1+f-s}$ for any $\boldsymbol{x} \in \{0,1\}^{N \times T}$, as established in Lemma B.1. In particular, we use this relationship in the proof of Proposition B.5 to lower bound the total expected revenue of the resource based static policy. Even though the main motivation for the fraction $\frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1}$ in (7) is algebraic, an interesting question is whether we can use different fractions in (7) to come up with variants of the resource based static policy. These variants would be heuristics and would not necessarily inherit the performance guarantee of $\frac{1}{1+2\,\mathsf{D}_{\max}}$. We experimented with a variant of the resource based static policy, where we replace the fraction $\frac{\mathbb{1}_{\{\ell \in [s,f]\}}}{f-s+1}$ with $\frac{\mathbb{1}_{\{\ell \in [s,f]\}} \zeta_{i,\ell}}{\sum_{t=s}^{f} \zeta_{i,t}}$, where $\zeta_{i,\ell}$ is, roughly speaking, the weight that we attach to resource i on day ℓ. Letting $\{\hat{\mu}_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T}\}$ be the optimal values of the dual variables associated with the first constraint in problem (11), we can use $\hat{\mu}_{i,\ell}$ to capture the opportunity cost of the capacity for resource i on day ℓ. Thus, we set the weights $\{\zeta_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T}\}$ as $\zeta_{i,\ell} = \hat{\mu}_{i,\ell}$ for all $i \in \mathcal{N}$ and $\ell \in \mathcal{T}$. We refer to this policy as the resource based static policy with weights. Once again, we emphasize that the resource based static policy with weights is a heuristic extension and does not necessarily have a performance guarantee, because we cannot use the fraction $\frac{\mathbb{1}_{\{\ell \in [s,f]\}} \zeta_{i,\ell}}{\sum_{t=s}^{f} \zeta_{i,t}}$ to lower bound the total expected revenue of the resource based static policy with weights, as is done in the proof of Proposition B.5.

In our implementation of the resource based static policy with weights, we compute the opportunity costs $\{\eta_{i,\ell}^{q} : i \in \mathcal{N},\ \ell \in \mathcal{T},\ q \in \mathcal{Q}\}$ once and five times over the selling horizon, yielding the benchmark policies LW1 and LW5. We also perform rollout on the resource based static policy with weights, yielding the benchmark policy LWR. In Table EC.3, we compare the performance of the resource based static policy with weights against the performance of the original resource based static policy. Recall that we compute the opportunity costs $\{\eta_{i,\ell}^{q} : i \in \mathcal{N},\ \ell \in \mathcal{T},\ q \in \mathcal{Q}\}$ for the original resource based static policy once and five times over the selling horizon, yielding the benchmark policies LIN1 and LIN5. We also perform rollout on the original resource based static policy, yielding the benchmark policy LINR. We express all total expected revenues as a percentage of the upper bound on the optimal total expected revenue. Our results indicate that LW1 and LW5 lag behind LIN1 and LIN5 slightly, but once we perform rollout, the resource based static policy with weights is comparable to the original resource based static policy, even though the resource based static policy with weights does not necessarily have a performance guarantee.

Appendix L: Additional Details for the Boutique Hotel Dataset

We explain the details of the parameter estimates for the boutique hotel dataset. Following this discussion, we check the performance of the policy in Baek and Ma (2021).

Parameter Estimation: Recalling that the values of $\{\theta_k : k = 1, \ldots, 171\}$ are the same when k falls in each of the seven intervals $[1,3]$, $[4,7]$, $[8,14]$, $[15,28]$, $[29,42]$, $[43,56]$ and $[57,171]$, we need to estimate seven parameters to come up with $\{\theta_k : k = 1, \ldots, 171\}$. The length of stay for a customer ranges between one and eight days, so we need to estimate eight parameters for $\{\eta_d : d = 1, \ldots, 8\}$.

Param.	Total Exp. Revenue					
$(\mathsf{D}_{\max}, \rho, \delta)$	LIN1	LW1	LIN5	LW5	LINR	LWR
$(6, 1.0, 0.9)$	74.46	74.92	77.02	75.03	78.00	77.95
$(6, 1.0, 0.7)$	80.66	79.46	82.33	78.69	83.41	83.41
$(6, 1.2, 0.9)$	79.23	77.33	80.89	77.55	81.05	79.62
$(6, 1.2, 0.7)$	70.17	70.69	71.27	71.60	73.00	73.95
$(6, 1.4, 0.9)$	71.58	70.26	73.98	72.03	75.51	75.53
$(6, 1.4, 0.7)$	76.93	75.62	78.84	77.72	80.83	80.90
$(8, 1.0, 0.9)$	75.93	75.13	77.58	74.07	79.13	79.22
$(8, 1.0, 0.7)$	73.78	71.99	75.65	71.80	78.15	78.34
$(8, 1.2, 0.9)$	75.27	74.64	78.14	74.93	78.23	78.62
$(8, 1.2, 0.7)$	70.39	69.67	72.16	70.54	71.07	72.54
$(8, 1.4, 0.9)$	75.66	75.07	77.78	74.35	81.04	81.18
$(8, 1.4, 0.7)$	72.94	71.97	76.44	72.89	78.26	78.02
$(10, 1.0, 0.9)$	67.04	67.43	68.81	65.38	70.53	70.29
$(10, 1.0, 0.7)$	73.32	71.57	74.80	71.04	77.11	76.74
$(10, 1.2, 0.9)$	68.61	69.59	70.72	68.07	73.89	72.44
$(10, 1.2, 0.7)$	69.95	69.66	69.90	68.75	73.09	74.03
$(10, 1.4, 0.9)$	76.34	75.47	78.76	74.57	80.92	80.90
$(10, 1.4, 0.7)$	74.57	73.78	75.74	73.98	80.51	78.19
Avg.	73.71	73.01	75.60	72.94	77.43	77.33

Table EC.3 Total expected revenues obtained by the resource based static policy with weights.

Lastly, there are six rooms, which implies that we need to estimate the six preference weights $\{v_i : i = 1, \ldots, 6\}$. Thus, the total number of parameters that we need to estimate is 21. For each one of the time periods in the selling horizon, the dataset provides the assortment of rooms that was on offer to the customers and indicates whether there was a booking at the time period. If there was a booking, then the dataset also provides the room chosen in the booking and the interval of stay for the booking. Using the dataset, we use standard maximum likelihood estimation to estimate the parameters of our model. We provide summary statistics for the estimated values of the parameters. The largest values for θ_k occur when k falls in one of the intervals $[1, 3]$, $[4, 7]$ and $[8, 14]$. More than 60% of the booking requests are for intervals of three or fewer days. Lastly, the largest and smallest preference weights for a room differ by a factor of 2.28.

We carry out five-fold cross-validation for our arrival probability and preference weight estimates. Recall that we have 1190 time periods in our selling horizon. Some time periods have bookings, some do not. We split the 1190 time periods in the selling horizon into five equal segments. After estimating the parameters of our model using four-fifths of the dataset, we validate the ability of our parameter estimates to predict the arrivals and customer choices in the remaining one-fifth holdout dataset. To validate the estimated arrival probabilities and preference weights, we use our model parameters to predict the expected number of weekly bookings made within the assortments offered in the holdout dataset, and compare our predictions with the actual numbers of bookings. Over five holdout datasets, the average percent deviation between the predicted and actual bookings is 23%. For each time period with a reservation in the holdout dataset, we also

Load Fact.	Total Exp. Revenue				
	LINR	POLR	LPR	COS1	COS5
0.8	83.96	84.43	83.00	78.07	78.22
1.0	86.91	87.26	85.88	81.97	81.93
1.2	88.18	88.70	87.16	82.93	83.64
1.4	89.38	89.36	88.31	83.86	84.45
1.6	90.18	89.69	88.85	85.18	84.77
Avg.	87.72	87.89	86.64	82.40	82.60

Table EC.4 Total expected revenues obtained by the constraint splitting policies.

order the offered rooms according to their choice probabilities from the fitted choice model, and count the fraction of times that the booked room is one of the r rooms with the largest choice probabilities. We refer to this fraction as the r-hit rate. For example, the 2-hit rate is the fraction of times that the booked room had one of the top two choice probabilities. The 1-hit and 2-hit rates, averaged over five holdout datasets, are, respectively, 0.58 and 0.82. Therefore, more than 80% of the time, the booked room is one of the two options with the largest choice probabilities. Similarly, more than 50% of the time, the booked room is indeed the one with the largest choice probability. Lastly, the average rank of the booked room, averaged over all holdout datasets and bookings in each holdout dataset, is 1.75.

Performance of the Constraint Splitting Policy: As discussed in Appendix I, we compute the parameters of the constraint splitting policy in Baek and Ma (2021) once and five times over the selling horizon, yielding the policies COS1 and COS5. In Table EC.4, we compare the performance of these two policies with LINR, POLR, and LPR. Our results indicate that each of the policies LINR, POLR, and LPR provides noticeable improvements over COS1 and COS5.

Appendix M: Test Problems with a Single Resource

We test the performance of our policies on test problems with a single resource. Our experimental setup follows the one in Section 7.1 except for the fact that we have a single resource, so that $N = 1$. This resource is available for use during the days indexed by $\mathcal{T} = \{1, \ldots, 70\}$, corresponding to 10 weeks in the booking horizon. Customers arrive over time periods $\mathcal{Q} = \{1, \ldots, 700\}$. Recalling that $\mathsf{D}_{\max}$ is the maximum number of days that the resource can be booked for, ρ is the load factor, and δ is the discount parameter for booking the resource on a weekday, we vary the parameters $\mathsf{D}_{\max} \in \{3, 6\}$, $\rho \in \{1.2, 1.6, 2.0\}$ and $\delta \in \{0.7, 0.9\}$ to obtain 12 parameter configurations for our test problems. Since we have one resource, the approach described in Section 6 computes the optimal total expected revenue for our test problems. In particular, recalling that $\boldsymbol{e}' \in \{0,1\}^T$ is the vector of all ones and indexing the single resource by j, if we set the revenue allocations $\hat{\boldsymbol{\beta}} = \{\hat{\beta}^q_{j,[s,f]\to j} : [s,f] \in \mathcal{F},\ q \in \mathcal{Q}\}$ as $\hat{\beta}^q_{j,[s,f]\to j} = r_{j,[s,f]}$ and compute the value functions $\{V^q_{\hat{\boldsymbol{\beta}},j} : q \in \mathcal{Q}\}$ through the dynamic program in (12), then $V^1_{\hat{\boldsymbol{\beta}},j}(\boldsymbol{e}')$ is the optimal total expected revenue. We give our computational results in Table EC.5. The first column in this table gives the parameter

Param.	Gap in	Total Exp. Revenue								
$(\mathsf{D}_{\max}, \rho, \delta)$	Bounds	LIN1	POL1	LP1	LIN5	POL5	LP5	LINR	POLR	LPR
$(3, 1.2, 0.9)$	47.79	94.42	99.87	95.09	94.45	99.89	94.40	100.00	100.00	100.00
$(3, 1.2, 0.7)$	47.77	94.52	100.00	96.73	94.44	100.00	95.52	100.00	100.00	100.00
$(3, 1.6, 0.9)$	44.37	97.60	99.38	94.88	97.51	99.62	94.60	99.77	100.00	99.91
$(3, 1.6, 0.7)$	44.17	96.64	98.67	95.08	96.54	98.85	94.82	99.61	99.75	99.43
$(3, 2.0, 0.9)$	41.81	97.09	97.75	85.84	96.99	98.04	92.13	99.11	99.84	99.64
$(3, 2.0, 0.7)$	41.71	97.14	98.02	86.78	96.72	98.25	94.31	99.88	100.00	100.00
$(6, 1.2, 0.9)$	50.39	95.66	95.87	91.40	94.89	96.44	88.05	100.00	100.00	100.00
$(6, 1.2, 0.7)$	50.27	94.66	94.99	91.41	94.11	95.34	87.93	99.84	99.63	99.76
$(6, 1.6, 0.9)$	47.84	97.21	92.78	94.62	96.74	93.57	93.94	100.00	100.00	100.00
$(6, 1.6, 0.7)$	47.63	95.96	92.08	93.85	95.34	92.97	92.63	99.67	99.59	99.86
$(6, 2.0, 0.9)$	45.14	96.55	89.82	91.69	96.01	90.84	92.80	100.00	100.00	100.00
$(6, 2.0, 0.7)$	45.02	95.54	87.95	90.34	94.88	89.36	91.74	99.28	99.02	99.35
Avg.	46.16	96.08	95.64	92.31	95.72	96.14	92.74	99.76	99.82	99.83

Table EC.5 Computational results for the test problems with a single resource.

configuration for each test problem. The second column shows the percent gap between the upper bounds obtained by using the dynamic program in (12) and the linear program in (11). Since we can obtain the optimal total expected revenue through the dynamic program in (12), this column shows how far the upper bound provided by the linear program in (11) is from the optimal total expected revenue. The remaining columns give the total expected revenues obtained by the benchmark policies expressed as a percentage of the optimal total expected revenue.

Our results indicate that the total expected revenues obtained by our rollout policies are within only a fraction of a percent of the optimal total expected revenue even in the worst case. For many test problems, our rollout policies are optimal. One may conjecture that the rollout policies are provably optimal when there is a single resource, but it is not difficult to construct counterexamples to show that this conjecture is not correct. The performance of the policies that do not use rollout may lag behind the optimal total expected revenue by as much as 14.16%. Lastly, the upper bounds provided by the linear program in (11) can be rather loose. The gaps between these upper bounds and the optimal total expected revenue can reach 50.39%.

Appendix N: Heuristic Extension to Multiple Units of Resource Capacity

The performance guarantee for the resource based static policy, as well as our results for rolling out a static policy, use the fact that each resource has unit capacity. It appears to be nontrivial to extend these results to the case with multiple units of capacity for a resource. In this section, we give a heuristic modification of the resource based static policy when there are multiple units of a resource and test this modification in computational experiments. In particular, we consider the case where there are $C_{i,\ell}$ units of resource i available on day ℓ. In this case, we compute the opportunity costs $\{\eta^q_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T},\ q \in \mathcal{Q}\}$ for the resource based static policy after replacing $\phi^q_i(A^q_{[s,f]})$ in (7) with $\frac{1}{C_{i,\ell}}\, \phi^q_i(A^q_{[s,f]})$. The heuristic reasoning behind this modification is that $\eta^q_{i,\ell}$

characterizes the opportunity cost of a unit of capacity for resource i on day ℓ. Given that we offer the assortment $A^q_{[s,f]}$ at time period q, a customer chooses resource i with probability $\phi^q_i(A^q_{[s,f]})$. Because there are $C_{i,\ell}$ units of resource i available on day ℓ, each unit of resource faces demand with probability $\frac{1}{C_{i,\ell}}\,\phi^q_i(A^q_{[s,f]})$. At time period q, there actually may not be $C_{i,\ell}$ units of capacity of resource i available on day ℓ because of the booking requests at the previous time periods, so this reasoning is a heuristic. Also, Lemma B.1 is critical in establishing a performance guarantee for the resource based static policy and it holds only when the capacity of each resource is one.

Our computational experiments are for a boutique hotel. The set of resources corresponds to a set of room types. We have $N = 3$ room types. We have K units of each room type available on each day, so we have $C_{i,\ell} = K$ for all $i \in \mathcal{N}$ and $\ell \in \mathcal{T}$. We generate our test problems by using the same approach in Section 7.1. In all of our computational experiments, we set the discount parameter as $\delta = 0.7$ and the maximum duration of stay parameter as $\mathsf{D}_{\max} = 6$. Noting that the total available capacity that we have in all rooms on all days is KNT, the load factor parameter satisfies $\rho = \frac{\lambda^q_{[s,f]}}{\gamma^q_{[s,f]}}\,\frac{\mathsf{Demand}}{KNT}$ for each interval $[s,f]$. We use the same approach in Section 7.1 to generate the arrival probabilities for the booking requests for different intervals and the parameters of the multinomial logit model. Varying the parameters $\rho \in \{1.2, 1.6, 2.0\}$ and $K = \{1, 3, 5\}$, we obtain nine parameter configurations for our test problems.

We compute the opportunity costs $\{\eta^q_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T},\ q \in \mathcal{Q}\}$ for the resource based static policy once and five times over the selling horizon, yielding the benchmark policies LIN1 and LIN5. In the linear program in (11), we replace the right side of the first constraint with $\{C_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T}\}$, in which case, letting $\{\hat{\mu}_{i,\ell} : i \in \mathcal{N},\ \ell \in \mathcal{T}\}$ be the optimal values of the dual variables associated with this constraint, we construct a benchmark policy by using $\hat{\mu}_{i,\ell}$ as the opportunity cost of the unit of capacity of resource i on day ℓ. We compute the opportunity costs once and five times over the selling horizon, yielding the benchmark policies LP1 and LP5. We also use the constraint splitting policy in Baek and Ma (2021), where we compute the parameters of this policy once and five times over the selling horizon, yielding the benchmark policies COS1 and COS5. Lastly, we implement the dynamic programming decomposition method, yielding the benchmark policy DEC.

We give our computational results in Table EC.6. The first column in this table gives the parameter configuration for each test problem. The remaining columns give the total expected revenues obtained by the benchmark policies expressed as a percentage of the upper bound on the optimal total expected revenue. We use the optimal objective value of the linear program in (11) as an upper bound on the optimal total expected revenue, because we can efficiently implement the approach in Section 6 to obtain an upper bound only when we have unit capacities for all of the resources. Our results indicate that LIN5, LP5 and DEC are the strongest benchmark policies. For

Param.	Total Exp. Revenue						
(ρ, K)	LIN1	LP1	COS1	LIN5	LP5	COS5	DEC
$(1.2, 1)$	71.97	73.69	70.34	75.34	72.87	67.98	72.83
$(1.6, 1)$	79.31	72.53	77.67	80.81	78.79	73.29	81.14
$(2.0, 1)$	80.47	76.54	76.33	82.97	80.96	74.34	80.85
$(1.2, 5)$	75.61	77.13	66.62	79.53	78.95	64.29	78.56
$(1.6, 5)$	75.07	73.87	74.26	77.66	79.03	70.20	78.49
$(2.0, 5)$	76.40	75.94	75.01	80.57	80.90	70.53	79.68
$(1.2, 10)$	70.58	72.63	62.14	73.12	75.14	58.04	74.75
$(1.6, 10)$	52.34	55.49	49.44	55.47	57.35	46.25	56.76
$(2.0, 10)$	43.19	43.19	43.48	45.58	46.38	40.14	45.86
Avg.	69.44	69.00	66.14	72.34	72.26	62.78	72.10

Table EC.6 Total expected revenues obtained by the benchmark policies under multiple units of resource capacity.

this problem class, the constraint splitting policy in Baek and Ma (2021) continues to provide a performance guarantee of $\frac{1}{1+\mathsf{D}_{\max}}$, but our modification of the resource based static policy is a heuristic. Nevertheless, LIN1 and LIN5 perform noticeably better than COS1 and COS5.

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