Electronic Companion: Throttling Equilibria in Auction Markets

EC.1. Appendix: Examples of Irrationality and the Nature of Approximation

First-Price Auctions: First, we give an example for which the unique *first-price* throttling equilibrium is irrational.

EXAMPLE EC.1. Define a throttling game as follows: There are 2 goods and 2 buyers, i.e., $m = 2$ and $n = 2$; $b_{11} = b_{12} = 2$ and $b_{21} = 1, b_{22} = 3$; $B_1 = 2$ and $B_2 = 1$. Suppose, in equilibrium, the buyers use the throttling parameters θ_1 and θ_2. Then the payments of buyer 1 and buyer 2 are given by $2\theta_1 + 2(1-\theta_2)\theta_1$ and $3\theta_2 + (1-\theta_1)\theta_2$ respectively. Therefore, for this game, in any throttling equilibrium, we have $0 < \theta_1, \theta_2 < 1$, which implies $2\theta_1 + 2(1-\theta_2)\theta_1 = 2$ and $3\theta_2 + (1-\theta_1)\theta_2 = 1$. Substituting $\theta_1 = 1/(2-\theta_2)$ from the first equation into the second yields

$$3\theta_2 + \theta_2 \cdot \frac{1-\theta_2}{2-\theta_2} = 1$$

which implies $2\theta_2^2 - 4\theta_2 + 1 = 0$. As $\theta_2 < 1$, solving the quadratic gives $\theta_2 = (2-\sqrt{2})/2$.

EXAMPLE EC.2 (A δ-APPROXIMATE THROTTLING EQUILIBRIUM CAN BE FAR IN ℓ_∞). Fix any $\delta \in (0, 1/2)$ and consider a *first-price* throttling game with $n = 2$ buyers and $m = 1$ good. Let the bids be $b_{11} = 2$, $b_{21} = 1$, and the budgets be $B_1 = 4 \cdot (1-\delta)$, $B_2 = \frac{\delta^2}{2(1-\delta)}$. For $\theta = (\theta_1, \theta_2) \in [0,1]^2$, the expected payments on the (single) good are

$$p(\theta)_{11} = 2\theta_1, \qquad p(\theta)_{21} = \theta_2(1-\theta_1).$$

Since $B_1 > 2$, buyer 1 spends less than the budget B_1 for every $\theta_1 \in [0,1]$: $2\theta_1 \le 2 < B_1$. Thus, by the "no unnecessary throttling" condition, any throttling equilibrium must satisfy $\theta_1 = 1$. When $\theta_1 = 1$, buyer 2's expected payment is $0 < B_2$, so any throttling equilibrium must also satisfy $\theta_2 = 1$. Therefore, $\theta^* = (1,1)$ is the unique throttling equilibrium.

However, there exists a δ-approximate throttling equilibrium, namely $\hat{\theta} := (1-\delta,\ \delta/2)$, which is far away from θ^* in ℓ_∞ distance. To see $\hat{\theta}$ is indeed a δ-approximate throttling equilibrium, note that buyer 1's expected payment is $2(1-\delta) \le B_1$, and buyer 2's expected payment is

$$\frac{\delta}{2}(1-(1-\delta)) = \frac{\delta}{2} \cdot \delta = \frac{\delta^2}{2} \le \frac{\delta^2}{2(1-\delta)} = B_2,$$

so $\hat{\theta}$ is budget-feasible. Moreover, buyer 1 satisfies $\hat{\theta}_1 = 1-\delta$, and buyer 2 satisfies

$$p(\hat{\theta})_{21} = \frac{\delta^2}{2} = (1-\delta)B_2,$$

so $\hat{\theta}$ satisfies the "not too much unnecessary throttling" condition in the definition of a δ-approximate throttling equilibrium. Hence, $\hat{\theta}$ is a δ-approximate throttling equilibrium. However,

$$\begin{aligned}\|\hat{\theta} - \theta^*\|_\infty &= \max\{\,|1-(1-\delta)|,\ |1-\delta/2|\,\}\\ &= 1-\delta/2,\end{aligned}$$

which can be arbitrarily close to 1 for small δ.

REMARK EC.1. For the instance in Example EC.2, Algorithm 1 (with parameter δ) initializes

$$\theta_1 = \min\left\{\frac{B_1}{2\sum_j b_{1j}}, 1\right\} = 1-\delta,$$

and buyer 1 is never updated (since the update condition requires $\theta_i < 1-\delta$). Buyer 2 is increased multiplicatively until its expected payment reaches $(1-\delta)B_2$. Thus, the algorithm terminates with

$$\theta_2 \in \left[\frac{\delta}{2},\ \frac{\delta}{2(1-\delta)}\right].$$

Therefore, this example shows that Algorithm 1 can output a δ-approximate throttling equilibrium at ℓ_∞-distance at least $1-\delta/(2(1-\delta))$ from the unique exact throttling equilibrium $\theta^* = (1,1)$.

Second-Price Auctions: Next, we give an example for which all *second-price* throttling equilibria are irrational.

EXAMPLE EC.3. Define a throttling game as follows:

- There are 4 goods and 3 buyers, i.e., $m = 4$ and $n = 3$
- $b_{11} = b_{12} = 2$, $b_{14} = 1$, $b_{23} = b_{24} = 4$, $b_{22} = 1$, $b_{31} = 1$ and $b_{33} = 2$
- $B_1 = B_2 = 1$ and $B_3 = \infty$

For this game, in any throttling equilibrium, we have $0 < \theta_1, \theta_2 < 1$ and $\theta_3 = 1$. Hence, if θ is a throttling equilibrium, then it satisfies $\theta_1 + \theta_1\theta_2 = 1$ and $2\theta_2 + \theta_2\theta_1 = 1$. Substituting $\theta_1 = 1/(1+\theta_2)$ from the first equation into the second equation yields

$$2\theta_2 + \theta_2 \cdot \frac{1}{1+\theta_2} = 1$$

which further implies $2\theta_2^2 + 2\theta_2 - 1 = 0$. As $\theta_2 > 0$, solving the quadratic gives $\theta_2 = (\sqrt{3}-1)/2$.

EXAMPLE EC.4 (A δ-APPROXIMATE EQUILIBRIUM CAN BE FAR IN ℓ_∞). Fix any $\delta \in (0, 1/2)$ and consider a *second-price* throttling game with $n = 3$ buyers and $m = 1$ good. Let the bids be

$b_{11} = 3,\ b_{21} = 2,\ b_{31} = 1$, and the budgets be $B_1 = 3,\ B_3 = 1,\ B_2 = \frac{\delta^2}{2(1-\delta)}$. For $\theta = (\theta_1, \theta_2, \theta_3) \in [0,1]^3$, the expected payments under second-price are

$$p(\theta)_{11} = \theta_1\Big(2\theta_2 + 1 \cdot (1-\theta_2)\theta_3\Big),$$
$$p(\theta)_{21} = (1-\theta_1)\theta_2\theta_3 \cdot 1, \quad p(\theta)_{31} = 0.$$

Since $p(\theta)_{11} \le 2\theta_1 \le 2 < B_1$ for all θ, buyer 1 always spends less than her budget. Hence, any throttling equilibrium must satisfy $\theta_1 = 1$. Also $p(\theta)_{31} = 0 < B_3$, so any throttling equilibrium must satisfy $\theta_3 = 1$. With $\theta_1 = 1$, buyer 2 never wins and thus $p(\theta)_{21} = 0 < B_2$, implying $\theta_2 = 1$. Therefore, $\theta^* = (1,1,1)$ is the unique throttling equilibrium.

However, there exists an approximate throttling equilibrium, namely $\hat{\theta} := (1-\delta,\ \delta/2,\ 1)$, which is far from θ^*. To see this, note that

$$p(\hat{\theta})_{21} = (1-(1-\delta)) \cdot \frac{\delta}{2} \cdot 1 = \frac{\delta^2}{2} = (1-\delta)B_2,$$

so buyer 2 is budget-feasible and (despite $\hat{\theta}_2 < 1-\delta$) satisfies the "not too much unnecessary throttling" condition of a δ-approximate throttling equilibrium. Buyers 1 and 3 have $\hat{\theta}_1 = 1-\delta$ and $\hat{\theta}_3 = 1$, so they satisfy the same condition trivially. Budget-feasibility holds for buyer 1 as well since

$$\begin{aligned} p(\hat{\theta})_{11} &= (1-\delta)\Big(2 \cdot \frac{\delta}{2} + (1-\frac{\delta}{2}) \cdot 1\Big) \\ &= (1-\delta)\Big(1 + \frac{\delta}{2}\Big) \le 3 = B_1, \end{aligned}$$

and for buyer 3 since $p(\hat{\theta})_{31} = 0 \le B_3$. Thus $\hat{\theta}$ is a δ-approximate throttling equilibrium. Finally,

$$\begin{aligned} &\|\hat{\theta} - \theta^*\|_\infty \\ &\quad = \max\{|1-(1-\delta)|,\ |1-\delta/2|,\ |1-1|\} \\ &\quad = 1-\delta/2, \end{aligned}$$

which can be arbitrarily close to 1 for small δ.

EC.2. Appendix: Missing Proofs

EC.2.1. Proof of Lemma 3

Before proceeding onto the proof of Lemma 3, we establish some key lemmas. Since we are considering a particular θ, we will suppress the dependence on θ of the payment made by buyer B on good G and simply denote it by $p(B,G)$. The next lemma notes the payment terms of buyers in $\mathcal{I}_{\mathcal{G}}$.

LEMMA EC.1. *For all $i \in [n]$, we have*

1. $p(T(i), G(i,j)) = \left(1 - \theta_{S(j)}\right) \theta_{T(i)} \theta_{S(i)} 4$, *for all* $j \in N_i$; *and* $p(T(i), R(i)) = 0$.
2. $p(S(i), R(i)) = \theta_{S(i)} \theta_{T(i)} M |O_i|$; $p(S(i), G(i,j)) = 0$, *for all* $j \in N_i$; *and*

$$\begin{aligned} &p(S(i), G(j,i)) \\ &\quad = \theta_{S(i)} \left[\theta_{T(j)} 5 \ + \ (1 - \theta_{T(j)}) \theta_{S(j)} 4\right], \\ &\text{for all } j \in O_i. \end{aligned}$$

In the next lemma, we bound the total payment made by strategy buyer $S(i)$ on the neighbor goods and provide lower bounds on the throttling parameters:

LEMMA EC.2. *For all $i \in [n]$, we have $\theta_{S(i)} \geq (1 - 2\delta)/2$, $\theta_{T(i)} \geq 1/32$, and the total payment of $S(i)$ on the neighbor goods satisfies:*

$$\begin{aligned} |O_i| \theta_{S(i)} &\leq \sum_{j \in N_i} p(S(i), G(i,j)) \\ &\quad + \sum_{j \in O_i} p(S(i), G(j,i)) \\ &\leq 5 |O_i| \theta_{S(i)} \end{aligned}$$

Proof. For $i \in [n]$, using Lemma EC.1, we get

$$\begin{aligned} &\sum_{j \in N_i} p(S(i), G(i,j)) + \sum_{j \in O_i} p(S(i), G(j,i)) \\ &\quad = \sum_{j \in O_i} \theta_{S(i)} \left[\theta_{T(j)} 5 \ + \ (1 - \theta_{T(j)}) \theta_{S(j)} 4\right] \\ &\quad \leq 5 |O_i| \theta_{S(i)}. \end{aligned}$$

Suppose $\theta_{S(i)} < (1 - 2\delta)/2$ for some $i \in [n]$. Then, the total payment made by $S(i)$ is at most

$$\begin{aligned} &\theta_{S(i)} \theta_{T(i)} M |O_i| + 5 |O_i| \theta_{S(i)} \\ &\quad < \frac{(1 - 2\delta) M |O_i|}{2} + 5 |O_i| \\ &\quad < (1 - \delta) \cdot \frac{M |O_i|}{2} \end{aligned}$$

using $M > 10/\delta$ by our choice of M. This contradicts the definition of δ-approximate throttling equilibrium because $S(i)$ has a budget of $M|O_i|/2$. Therefore, we have $\theta_{S(i)} \geq (1-2\delta)/2$ and in particular, $\theta_{S(i)} \geq 1/4$ for all $i \in [n]$ using $\delta \leq 1/4$. Hence,

$$\begin{aligned}
&\sum_{j\in N_i} p(S(i), G(i,j)) + \sum_{j\in O_i} p(S(i), G(j,i)) \\
&= \sum_{j\in O_i} \theta_{S(i)} \left[\theta_{T(j)} 5 \;+\; (1-\theta_{T(j)})\theta_{S(j)} 4\right] \\
&> \sum_{j\in O_i} \theta_{S(i)} \left[\theta_{T(j)}\theta_{S(j)} 4 \;+\; (1-\theta_{T(j)})\theta_{S(j)} 4\right] \\
&= 4\theta_{S(i)} \cdot \sum_{j\in O_i} \theta_{S(j)} \geq |O_i|\theta_{S(i)}.
\end{aligned}$$

Suppose $\theta_{T(i)} < 1/32$. By Lemma EC.1 the total payment of threshold buyer $T(i)$ is at most

$$\sum_{j\in N_i} 4\theta_{T(i)} \leq 12\theta_{T(i)} < \frac{3}{8} \leq \frac{1-\delta}{2}$$

using $|N_i| \leq 3$ and $\delta \leq 1/4$. Hence, we have obtained a contradiction with the definition of δ-approximate throttling equilibria. Therefore, $\theta_{T(i)} \geq 1/32$. □

We are now ready to prove Lemma 3.

Proof of Lemma 3. Fix an $i \in [n]$. First consider the case when $\sum_{j\in N_i} x_j > 0.5 + \epsilon$. Assume for a contradiction that $x_i > \epsilon$. Then, $\theta_{S(i)} < 1 - (\epsilon/2) \leq 1 - \delta$ using $\delta \leq \epsilon/2$. By the definition of δ-approximate throttling equilibrium, the total payment of strategy buyer $S(i)$ is at least $(1-\delta)M|O_i|/2$. Combining this observation with Lemma EC.1 and Lemma EC.2, we get

$$\theta_{S(i)} \left[\theta_{T(i)} M|O_i| + 5|O_i|\right] \geq (1-\delta) \cdot \frac{M|O_i|}{2}$$

and thus (using $\theta_{T(i)} \geq 1/32$ from Lemma EC.2 and our choice of $M = 160/\delta$),

$$\begin{aligned}
\theta_{S(i)} &\geq \frac{1-\delta}{2\theta_{T(i)} + (10/M)} \geq \frac{1}{2\theta_{T(i)}} \cdot \frac{1-\delta}{1+\delta} \\
&\implies \theta_{T(i)}\theta_{S(i)} \geq \frac{1}{2(1+2\epsilon)}
\end{aligned}$$

using $\delta \leq \epsilon/2$ and $\epsilon < 1$. Moreover, note that $\sum_{j\in N_i} x_j > 0.5 + \epsilon$ implies

$$\sum_{j\in N_i} 2\left(1 - \theta_{S(j)}\right) > (1+2\epsilon)/2$$

Combining the above statements allows us to bound the total payment of buyer $T(i)$:

$$\sum_{j\in N_i}\left(1-\theta_{S(j)}\right)\theta_{T(i)}\theta_{S(i)}4 \geq \frac{4}{2(1+2\epsilon)}\cdot\sum_{j\in N_i}\left(1-\theta_{S(j)}\right)>\frac{1}{2}.$$

This yields a contradiction because $T(i)$ has budget $1/2$. Hence $x_i \leq \epsilon$ when $\sum_{j\in N_i} x_j > 0.5+\epsilon$. Next consider the case of $\sum_{j\in N_i} x_j < 0.5-\epsilon$. The budget constraint of $S(i)$ and Lemma EC.2 yield

$$\theta_{S(i)}\left[\theta_{T(i)}M|O_i|+|O_i|\right]\leq\frac{M|O_i|}{2}$$

which implies that

$$\begin{aligned}&\theta_{S(i)}\left[\theta_{T(i)}M|O_i|+\theta_{T(i)}|O_i|\right]\leq\frac{M|O_i|}{2}\\ \implies\quad&\theta_{T(i)}\theta_{S(i)}\leq\frac{1}{2(1+(1/M))}<\frac{1}{2}.\end{aligned}\tag{EC.1}$$

By Lemma EC.2, we have $\theta_{S(j)}\geq(1-2\delta)/2$ and thus, $2(1-\theta_{S(j)})\leq 1+2\delta$. Multiplying both sides by $(1-2\delta)$ yields $2(1-\theta_{S(j)})(1-2\delta)\leq 1-4\delta^2<1$. In other words, we have

$$\begin{aligned}2(1-\theta_{S(j)})(1-2\delta)&<\min\{2(1-\theta_{S(j)}),1\}\\&=x_j\end{aligned}\tag{EC.2}$$

for every $j\in[n]$. This together with $\sum_{j\in N_i}x_j<0.5-\epsilon$ implies

$$(1-2\delta)\sum_{j\in N_i}2\left(1-\theta_{S(j)}\right)<(1-2\epsilon)/2.$$

Therefore, we get that the total payment of $T(i)$ satisfies the following bound

$$\begin{aligned}&\sum_{j\in N_i}\left(1-\theta_{S(j)}\right)\theta_{T(i)}\theta_{S(i)}4\\&<\sum_{j\in N_i}2\left(1-\theta_{S(j)}\right)<\frac{1-2\epsilon}{2(1-2\delta)}\leq(1-\delta)\cdot\frac{1}{2}\end{aligned}$$

using $\delta\leq\epsilon/2$. As a consequence of the definition of δ-approximate throttling equilibria, we have that $\theta_{T(i)}\geq 1-\delta$. Finally, using (EC.1) and (EC.2), we have

$$\begin{aligned}x_i&>2(1-\theta_{S(i)})(1-2\delta)\geq 2\left(1-\frac{1}{2\theta_{T(i)}}\right)(1-2\delta)\\&\geq\frac{(1-2\delta)^2}{1-\delta}>\frac{1-4\delta}{1-\delta}\geq 1-\epsilon,\end{aligned}$$

where the last inequality follows from $\delta\leq\epsilon/(3+\epsilon)$. □

EC.2.2. Proof of Theorem 6

Consider a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$ and an approximation parameter $\delta \in (0, 1/2)$. Define $f : [0,1]^n \to [0,1]^n$ as

$$\begin{aligned} f_i(\theta) &= \min\left\{\frac{(1-\delta/2)B_i}{\sum_j p(1,\theta_{-i})_{ij}}, 1\right\} \\ &= \min\left\{\frac{(1-\delta/2)B_i}{\max\{\sum_j p(1,\theta_{-i})_{ij}, B_i/2\}}, 1\right\} \\ &\quad \forall \theta \in [0,1]^n \end{aligned}$$

First, we prove that f is L-Lipschitz continuous with Lipschitz constant $L = 2mn\overline{B}\underline{B}^{-2}\overline{b}$, where $\overline{b} = \max_{i,j} b_{ij}$, $\overline{B} = \max_i B_i$, and $\underline{B} = \min_i B_i$. To achieve this, we will repeatedly use the following facts about Lipschitz functions. For Lipschitz continuous functions f and g with Lipschitz constants L_1 and L_2 respectively,

- $f + g$ is $L_1 + L_2$-Lipschitz continuous
- If f and g are bounded above by M, then $f \cdot g$ is $M(L_1 + L_2)$-Lipschitz continuous
- If f is bounded below by c, then $1/f$ is L_1/c^2-Lipschitz continuous
- For a constant C, $\max\{f, C\}$ and $\min\{f, C\}$ are both L_1-Lipschitz continuous

Observe that

$$p(1,\theta_{-i})_{ij} = \sum_{\ell : b_{\ell j} < b_{ij}} b_{\ell j}\theta_\ell \prod_{k \neq i : b_{kj} > b_{\ell j}} (1-\theta_k)$$

Therefore, for all $i \in [n]$, $\theta \mapsto p(1,\theta_{-i})_{ij}$ is $(2n\overline{b})$-Lipschitz continuous, which further implies that $\theta \mapsto \sum_j p(1,\theta_{-i})_{ij}$ is $2mn\overline{b}$-Lipschitz continuous. Finally, due to the second equality in the definition of f, we get that f is $(2mn\overline{B}\underline{B}^{-2}\overline{b})$-Lipschitz continuous.

Since BROUWER is in PPAD (Chen et al. 2009), to complete the proof, it suffices to show that a $(\delta\underline{B}/4m\overline{b})$-approximate fixed point θ^* of f, i.e., θ^* such that $\|f(\theta^*) - \theta^*\|_\infty \le \delta\underline{B}/4m\overline{b}$, is a δ-approximate throttling equilibrium. First, note that $p(1,\theta_{-i})_{ij} \le \overline{b}$ for all $i \in [n], j \in [m]$. Therefore, $f(\theta)_i \ge \underline{B}/2m\overline{b}$ for all $i \in [n]$. Hence, for $i \in [n]$, we have

$$\left|1 - \frac{\theta_i^*}{f_i(\theta^*)}\right| \le \frac{\delta\underline{B}}{f_i(\theta^*) \cdot 4m\overline{b}} \le \frac{\delta}{2}$$

As a consequence, we get $\theta_i^* \le (1+\delta/2) f_i(\theta^*)$ and $\theta_i^* \ge (1-\delta/2) f_i(\theta^*)$. The first inequality implies

$$\sum_j p(\theta^*)_{ij} = \theta_i^* \cdot \sum_j p(1,\theta_{-i}^*)_{ij}$$

$$\leq (1+\delta/2)(1-\delta/2)B_i \leq B_i$$

and the second one implies that if $\theta_i^* < 1-\delta/2$, then

$$\begin{aligned}\sum_j p(\theta^*)_{ij} &= \theta_i^* \cdot \sum_j p(1,\theta_{-i}^*)_{ij} \\ &\geq (1-\delta/2)^2 B_i \geq (1-\delta)B_i\end{aligned}$$

Hence, θ^* is a δ-approximate throttling equilibrium, thereby completing the proof. □

EC.2.3. Proof of Theorem 7

Consider an instance of 3-SAT with variables $\{x_1,\ldots,x_n\}$ and clauses $\{C_1,\ldots,C_m\}$. Our goal is to define an instance $\mathcal{I}$ of REV (a throttling game G and a target revenue R) which always has the same solution (Yes or No) as the 3-SAT instance, and has a size of the order $\text{poly}(n,m)$. We do so next, starting with an informal description to build intuition. To better understand the core motivations behind the gadgets, we will restrict our attention to exact throttling equilibria ($\delta = 0$) in the informal discussion that follows. As we will see in the formal proof, the target revenue R can be chosen carefully to ensure that only exact throttling equilibria can achieve the revenue R.

Reciprocal Gadget: Fix $i \in [n]$. Corresponding to variable x_i, there are two goods $\mathbb{A}_i$ and $\mathbb{B}_i$, and two buyers V_i^+ and V_i^- in the throttling game G. Each buyer bids 1 for one of the goods and bids 2 for the other, with both buyers bidding differently on each good. Furthermore, we set the budgets of both buyers to be 1/2, and ensure that they do not spend any non-zero amount on goods other than $\mathbb{A}_i$ and $\mathbb{B}_i$. In equilibrium, this forces the throttling parameter of V_i^+ (which we denote by θ_i^+) to be half of the reciprocal of the throttling parameter of V_i^- (which we denote by θ_i^-) and vice-versa. As a consequence, both throttling parameters lie in the interval $[1/2,1]$.

Binary Gadget: For each variable x_i, there are two additional goods $\mathbb{S}_i$ and $\mathbb{T}_i$, which receive a bid of 1 from buyers V_i^+ and V_i^- respectively. The throttling game G also has one unbounded buyer U who has an infinite budget, and bids 2 on both goods $\mathbb{S}_i$ and $\mathbb{T}_i$. By the definition of throttling equilibria (Definition 1), the throttling parameter of U is always 1 in equilibrium. Therefore, buyer U wins both $\mathbb{S}_i$ and $\mathbb{T}_i$ with probability one, and pays $\theta_i^+ + \theta_i^- = \theta_i^+ + 1/2\theta_i^+$ for them. Finally, observe that $t \mapsto t + 1/2t$, when restricted to $t \in [1/2,1]$, is maximized at $t = 1$ or $t = 1/2$. Therefore, by appropriately choosing the target revenue R, we can ensure that revenue R is only achieved by throttling equilibria in which exactly one of the following holds: $(\theta_i^+ = 1, \theta_i^- = 1/2)$ or $(\theta_i^+ = 1/2, \theta_i^- = 1)$. This allows us to interpret $\theta_i^+ = 1$ as setting $x_i = 1$ and $\theta_i^- = 1$ as setting $x_i = 0$.

Clause Gadget: For each clause C_j, there is a good $\mathbb{C}_j$. If C_j contains a non-negated literal x_i, then buyer V_i^+ bids 1 on good $\mathbb{C}_j$, and if it contains a negated literal $\neg x_i$, then buyer V_i^- bids 1 on good $\mathbb{C}_j$. Furthermore, the unbounded buyer U bids 2 on good $\mathbb{C}_j$, thereby always winning it. Hence, the total payment on good $\mathbb{C}_j$ is 1 if some literal is satisfied (corresponding throttling parameter is 1), and is strictly less than 1 if no literal is satisfied (corresponding throttling parameters are 1/2). The rest of the reduction boils down to choosing R appropriately.

Proof of Theorem 7. Guided by the informal intuition described above, we proceed with the formal definition of the instance $\mathcal{I}$, which involves specifying the throttling game G and the target revenue R. The throttling game G consists of the following goods:

- **Reciprocal Gadget:** For each variable x_i, there are two goods $\mathbb{A}_i$ and $\mathbb{B}_i$.
- **Binary Gadget:** For each variable x_i, there are two binary goods $\mathbb{S}_i$ and $\mathbb{T}_i$.
- **Clause Gadget:** For each clause C_j, there is a good $\mathbb{C}_j$.

Moreover, G has the following set of buyers:

- Corresponding to each variable x_i, there are two buyers V_i^+ and V_i^- with non-zero bids only for the following goods:
  - $b(V_i^+, \mathbb{A}_i) = 2$ and $b(V_i^+, \mathbb{B}_i) = 1$
  - $b(V_i^-, \mathbb{A}_i) = 1$ and $b(V_i^-, \mathbb{B}_i) = 2$
  - $b(V_i^+, \mathbb{S}_i) = 1$
  - $b(V_i^-, \mathbb{T}_i) = 1$
  - $b(V_i^+, \mathbb{C}_j) = 1$ if x_i is a literal in C_j
  - $b(V_i^-, \mathbb{C}_j) = 1$ if $\neg x_i$ is a literal in C_j

  Moreover, the budget of both V_i^+ and V_i^- is $1/2$ for all $i \in [n]$.
- There is one unbounded buyer U with $b(U, \mathbb{C}_j) = 2$ for all $j \in [m]$ and $b(U, \mathbb{S}_i) = b(U, \mathbb{T}_i) = 2$ for all $i \in [n]$. Moreover, U has a budget of ∞.

Set the target revenue to be $R = n + m + (3n/2)$. Suppose there exists a δ-approximate throttling equilibrium Θ, for some $\delta \in [0, 1)$, with revenue greater than or equal to R. Let θ_i^+ and θ_i^- denote the throttling parameters of V_i^+ and V_i^- in Θ. Then, $\theta_i^+ \theta_i^- \leq 1/2$ by virtue of the budget constraints. Therefore, the revenue from goods $\{\mathbb{A}_i\}_{i=1}^n \cup \{\mathbb{B}_i\}_{i=1}^n$ is at most n. Furthermore, it is easy to see that the revenue from goods $\{\mathbb{C}_j\}_{j=1}^m$ is at most m. Additionally, the total payment by buyer U on goods $\mathbb{S}_i$ and $\mathbb{T}_i$ is at most $\theta_i^+ + \theta_i^- \leq \theta_i^+ + (1/2\theta_i^+)$. Note that $\theta_i^+ + (1/2\theta_i^+)$ is maximized at $\theta_i^+ = 1/2$ or $\theta_i^+ = 1$, with a value of $\theta_i^+ + (1/2\theta_i^+) = 3/2$. Therefore, the revenue from goods $\{\mathbb{S}_i\}_{i=1}^n \cup \{\mathbb{T}_i\}_{i=1}^n$ is at most $3n/2$. Hence, the total payment made on all the goods is at most R.

For the total revenue under Θ to be greater than or equal to R, the revenue from $\{\mathbb{S}_i\}_{i=1}^n \cup \{\mathbb{T}_i\}_{i=1}^n$ must be at least $3n/2$ and the revenue from $\{\mathbb{C}_j\}_{j=1}^m$ must be at least m. Hence, under Θ, buyer U has a throttling parameter of 1, and for each $i \in [n]$, either $(\theta_i^+ = 1, \theta_i^- = 1/2)$ or $(\theta_i^+ = 1/2, \theta_i^- = 1)$. Furthermore, the payment made by buyer U on $\mathbb{C}_j$ is 1 for every $j \in [m]$. This allows us to assign values to the variables as follows: set $x_i = 1$ if $\theta_i^+ = 1$ and $x_i = 0$ if $\theta_i^- = 1$. With this assignment of the variables, each clause is satisfied since the payment made by buyer U on $\mathbb{C}_j$ is 1 for all $j \in [m]$. Hence, we have shown that if there exists a δ-approximate throttling equilibrium with revenue R or greater, then there exists a satisfying assignment for the 3-SAT instance.

Conversely, note that if there exists a satisfying assignment for the 3-SAT instance, then setting $\theta_i^+ = 1, \theta_i^- = 1/2$ if $x_i = 1$ and $\theta_i^+ = 1/2, \theta_i^- = 1$ if $x_i = 0$ yields a throttling equilibrium with revenue equal to R. To complete the proof, observe that the size of the instance $|\mathcal{I}| = \text{poly}(n, m)$. □

EC.2.4. Proof of Theorem 8

In this appendix, we analyze the correctness and runtime of Algorithm 2. To do so, we will make repeated use of the following crucial observation:

$$p(\theta)_{ij} = \begin{cases} \theta_i \theta_k b_{kj} & \text{if } b_{ij} > b_{kj} > 0 \\ & \text{for some } k \in [n] \\ 0 & \text{otherwise} \end{cases} \tag{EC.3}$$

In particular, this observation implies that $p(1, \theta_{-i})_{ij}$ is a linear function of θ.

The following lemma makes a step towards the proof of correctness of the algorithm by showing that the budget constraints are always satisfied.

LEMMA EC.3. *At the start of each iteration of the while loop, we have $\sum_j p(\theta)_{ij} \le B_i$ for all $i \in [n]$.*

Proof. We will use induction on the number of iterations of the while loop to prove this lemma. By our choice of initialization of θ, the budget constraints are satisfied before the first iteration of the while loop. Suppose the constraints are satisfied before the start of the t-th iteration and the value of θ at that stage is $\theta^{(0)}$. We will use $\theta^{(1)}$ and $\theta^{(2)}$ to denote the value of θ after step 1 and step 2 of the t-th iteration respectively. Consider a buyer i such that $\sum_j p(\theta^{(1)})_{ij} > B_i$. By (EC.3), we get

$$B_i < \sum_j p(\theta^{(1)})_{ij} \le \left(\sum_j p(\theta^{(0)})_{ij} \right) / (1-\gamma)^2$$

which further implies $\sum_j p(\theta^{(0)})_{ij} > (1-\gamma)^2 B_i$. Therefore, the throttling parameter of buyer i was not changed in step 1 of the t-th iteration, i.e., $\theta_i^{(0)} = \theta_i^{(1)}$. As a consequence, we get

$$\sum_j p(\theta^{(1)})_{ij} \leq \left(\sum_j p(\theta^{(0)})_{ij}\right)/(1-\gamma)$$

After step 2 of the t-th iteration, we get $\theta_i^{(2)} = (1-\gamma)\theta_i^{(1)}$. Hence,

$$\begin{aligned}\sum_j p(\theta^{(2)})_{ij} &\leq (1-\gamma)\sum_j p(\theta^{(1)})_{ij} \\ &\leq \left(\sum_j p(\theta^{(0)})_{ij}\right) \leq B_i\end{aligned}$$

where the last inequality follows from our inductive hypothesis. As $\theta^{(2)}$ is the value of θ after the t-th iteration, the lemma follows by induction. □

The next lemma establishes that the algorithm never loses any progress, i.e., any buyer who satisfies the 'Not too much unnecessary throttling condition' of Definition 2 at the beginning of some iteration of the while loop continues to do so at the end of it.

LEMMA EC.4. *If $\sum_j p(\theta)_{ij} \geq (1-\gamma)^3 B_i$ or $\theta_i \geq 1-\gamma$ at the start of some iteration of the while loop, then $\sum_j p(\theta)_{ij} \geq (1-\gamma)^3 B_i$ or $\theta_i \geq 1-\gamma$ at the end of that iteration.*

Proof. Consider an iteration of the while loop which starts with $\theta = \theta^{(0)}$. We will use $\theta^{(1)}$ and $\theta^{(2)}$ to denote the value of θ after step 1 and step 2 of this iteration. If $\sum_j p(\theta^{(0)})_{ij} \geq (1-\gamma)^2 B_i$ at the beginning of the iteration, then $\sum_j p(\theta^{(2)})_{ij} \geq (1-\gamma)^3 B_i$ because

$$\sum_j p(\theta^{(1)})_{ij} \geq (1-\gamma)^2 B_i \quad \text{and}$$
$$\sum_j p(\theta^{(2)})_{ij} \geq (1-\gamma)\sum_j p(\theta^{(1)})_{ij}$$

Suppose $(1-\gamma)^3 B_i \leq \sum_j p(\theta^{(0)})_{ij} < (1-\gamma)^2 B_i$ and $\theta_i^{(0)} < 1-\gamma$ at the start of the iteration. Then, after step 1, we have $(1-\gamma)^2 B_i \leq \sum_j p(\theta^{(1)})_{ij} \leq B_i$. Hence, after step 2, we get $(1-\gamma)^3 B_i \leq \sum_j p(\theta^{(2)})_{ij}$.

Finally, suppose $(1-\gamma)^3 B_i \leq \sum_j p(\theta^{(0)})_{ij} < (1-\gamma)^2 B_i$ and $\theta_i^{(0)} \geq 1-\gamma$ at the start of the iteration. Then, after step 1, we have $\sum_j p(\theta^{(1)})_{ij} \leq B_i$. Hence, after step 2, we still have $\theta_i^{(2)} \geq (1-\gamma)$. This completes the proof of the lemma. □

Finally, we combine the above lemmas to establish the correctness and polynomial-runtime of the algorithm.

Proof of Theorem 8. Let θ^* be the vector of throttling parameters returned by the algorithm. Lemma EC.3 implies that θ^* satisfies the budget constraints of every buyer. Furthermore, upon combining $(1-\gamma)^3 \geq 1-3\gamma$ with the termination condition of the while loop, we get that either $\theta_i^* \geq 1-\gamma$ or $\sum_j p(\theta^*)_{ij} \geq (1-3\gamma)B_i$ for all $i \in [n]$, which makes θ^* a 3γ-approximate throttling equilibrium.

Next, we bound the running time of the algorithm. Define $c = \min_i \min\{B_i/(2\sum_j b_{ij}), 1\}$. Note that $c \leq \theta_i \leq 1$ for all $i \in [n]$ for the entire run of the algorithm. Based on Lemma EC.4, we define

$$A(\theta) := \{i \in [n] \mid \sum_j p(\theta)_{ij} \geq (1-\gamma)^3 B_i \text{ or } \theta_i \geq 1-\gamma\}$$

Then Lemma EC.4 simply states that if $i \in A(\theta)$ at the start of iteration T of the while loop, then $i \in A(\theta)$ at the start of all future iterations $t \geq T$. Moreover, recall that the while loop terminates when $A(\theta) = [n]$.

Observe that, in each iteration of the while loop, $\theta_i \leftarrow \theta_i/(1-\gamma)$ for some $i \notin A(\theta)$. Hence, the total number of iterations of the while loop T satisfies the following equivalent statements:

$$\frac{c}{(1-\gamma)^{T/n}} \leq 1 \iff T \leq \frac{n\log(1/c)}{\log(1/(1-\gamma))} \leq \frac{n\log(1/c)}{\gamma}$$

This completes the proof because each iteration takes polynomially many steps. □

EC.2.5. Proof of Theorem 9

Proof of Theorem 9. Consider a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$. Let $\theta = (\theta_i)_i$ be the unique throttling equilibrium (TE) and $\alpha = (\alpha_i)_i$ be the unique pacing equilibrium (PE) for this game. We will use $p_j(\theta)$ and $p_j(\alpha)$ to denote the (expected) payment made to the seller on good j under the TE and PE respectively. Then, REV(TE) $= \sum_j p_j(\theta)$ and REV(PE) $= \sum_j p_j(\alpha)$.

First, we show that REV(PE) $\leq 2 \times$ REV(TE). Let $N := \{i \in [n] \mid \theta_i = 1\}$ be the set of buyers who are not budget constrained under the TE. Moreover, define

$$M := \{j \in [m] \mid \exists\, i \in N \text{ such that } i \text{ wins a non-zero fraction of } j \text{ under the PE } \alpha\}$$

Note that, since $\theta_i = 1$ and $\alpha_i \le 1$ for all $i \in N$, we get that the TE yields a higher revenue for the seller on all goods in the set M, i.e., $p_j(\theta) \ge p_j(\alpha)$ for all $j \in M$. Therefore, REV(TE) $\ge \sum_{j\in M} p_j(\theta) \ge \sum_{j\in M} p_j(\alpha)$. Furthermore, the definition of throttling equilibrium implies that every buyer $i \notin N$ spends her entire budget B_i under the TE. Hence, by our choice of M, we get $\sum_{j\notin M} p_j(\alpha) \le \sum_{i\notin N} B_i \le$ REV(TE). Combining the two statements yields REV(PE) $\le 2\times$REV(TE), as desired.

Next, we complete the proof by showing that REV(TE) $\le 2\times$REV(PE). Let $S = \{i \in [n] \mid \alpha_i = 1\}$ be the set of buyers who are not budget constrained under the PE. Note that, for all $i \in S$ and $j \in [m]$, buyer i bids b_{ij} under the PE, which implies $p_j(\alpha) \ge \max_{i\in S} b_{ij}$ for all goods $j \in [m]$. Therefore, for any good $j \in [m]$, the total payment made by buyers in the set S under the TE is at most $p_j(\alpha)$. As a consequence, the total payment made by buyers in S under the TE is at most REV(PE). Furthermore, the buyers not in S completely spend their budget under the PE, so the total payment made by buyers not in S under the TE is also at most REV(PE). Hence, we have the desired inequality REV(TE) $\le 2\times$REV(PE). □

EC.2.6. Proof of Theorem 10

Fix a throttling equilibrium $\theta \in \Theta^*$. Recall that we use $X = (X_1, \ldots, X_n)$ to capture the random profile of buyers who participate in the auctions, where $X_i = 1$ if and only if buyer i participates in the auctions, and $\Pr(X_i = 1) = \theta_i$. Let $y_{ij}(X)$ be the indicator random variable which equals 1 if and only if good j is allocated to buyer i under the participation profile $X = (X_1, \ldots, X_n)$, and is zero otherwise. Moreover, let $p_j(X)$ denote the price of item j under the participation profile $X = (X_1, \ldots, X_n)$. Here, the price is the highest/second-highest bid for first-price/second-price auctions respectively, and is interpreted to be 0 if no buyers bid in an auction. Observe that

$$p_{ij}(\theta) = \mathbb{E}\left[p_j(X) y_{ij}(X)\right].$$

Fix a benchmark allocation $y = \{y_{ij}\}$. We begin by establishing the following lemma, which will play a critical role in the proof of the theorem.

LEMMA EC.5. *For all $i \in [n]$, we have*

$$\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \ge \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right].$$

Proof. We consider two cases. First assume that $\theta_i < 1$. Then, the no-unnecessary-throttling condition implies that $\sum_{j=1}^{m} p_{ij}(\theta) = B_i$. Now, observe that $y_{ij}(X) > 0$ only if $b_{ij} \geq p_j(X)$. Consequently, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right] &\geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right] \\ &= \sum_{j=1}^{m} p_{ij}(\theta) = B_i\,.\end{aligned}$$

Hence, we get

$$\begin{aligned}\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} &= B_i \\ &\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\ &\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right],\end{aligned}$$

thereby establishing the required lemma statement for a buyer i such that $\theta_i < 1$.

Next, consider a buyer i such that $\theta_i = 1$, i.e., buyer i always participates. Since $p_j(X) > b_{ij}$ whenever $y_{ij}(X) < 1$, we have

$$0 \geq \mathbb{E}\left[(b_{ij} - p_j(X))(1 - y_{ij}(X))y_{ij}\right]\,.$$

Moreover, we also have

$$\mathbb{E}\left[b_{ij} y_{ij}(X)\right] \geq \mathbb{E}\left[(b_{ij} - p_j(X)) y_{ij}(X) y_{ij}\right]$$

Adding the two inequalities, we get

$$\begin{aligned}\mathbb{E}\left[b_{ij} y_{ij}(X)\right] &\geq \mathbb{E}\left[(b_{ij} - p_j(X))(1 - y_{ij}(X))y_{ij}\right] \\ &\quad + \mathbb{E}\left[(b_{ij} - p_j(X)) y_{ij}(X) y_{ij}\right] \\ &= \mathbb{E}\left[(b_{ij} - p_j(X)) y_{ij}\right]\,.\end{aligned}$$

Summing over all goods $j \in [m]$ yields

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right] \\
&\quad \geq \sum_{j=1}^{m} b_{ij} y_{ij} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] \\
&\quad \geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right]
\end{aligned}$$

Additionally, we also have

$$\begin{aligned}
B_i &\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\
&\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
&\quad \geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right].
\end{aligned}$$

This concludes the lemma by establishing it for buyers i with $\theta_i = 1$. □

With Lemma EC.5 in hand, we are ready to prove the theorem. First, note that

$$\begin{aligned}
&\sum_{i=1}^{n} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
&\quad \geq \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\
&\qquad - \sum_{i=1}^{n} \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] \\
&\quad = \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\
&\qquad - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) \sum_{i=1}^{n} y_{ij}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X)\right] \\
&\geq \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\
&\quad - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) \sum_{i=1}^{n} y_{ij}(X)\right] \\
&= \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\
&\quad - \sum_{i=1}^{n} \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right]
\end{aligned}
$$

where the second inequality follows from the observation that a good is always allocated whenever it has a positive bid, i.e., $\sum_{i=1}^{n} y_{ij}(X) = 1$ whenever $p_j(X) > 0$. Hence, if we can show that

$$
\begin{aligned}
&\sum_{i=1}^{n} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
&\qquad \geq \sum_{i=1}^{n} \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right],
\end{aligned} \tag{EC.4}
$$

we will get

$$
\begin{aligned}
&\sum_{i=1}^{n} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
&\qquad \geq \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \\
&\qquad\quad - \sum_{i=1}^{n} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
\iff & 2 \cdot \sum_{i=1}^{n} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
&\qquad \geq \sum_{i=1}^{n} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\}
\end{aligned}
$$

and thereby complete the proof, because the benchmark allocation y and the throttling equilibrium θ are both arbitrary. In the remainder, we establish (EC.4).

Since $y_{ij}(X) > 0$ only when $b_{ij} \geq p_j(X)$, we have

$$\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right] \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right].$$

Moreover, the budget constraint of buyer i implies

$$B_i \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right].$$

Combining the two inequalities, we get:

$$\begin{aligned}\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\ \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right].\end{aligned}$$

Summing over all buyers $i \in [n]$ yields (EC.4), as required. Thus we have established the desired bound of PoA ≤ 2.

Next, we show that the upper bound on PoA developed above is tight for both first-price and second-price auctions. We do so by demonstrating particular instances for which the bound is tight, starting with the second-price auction format.

EXAMPLE EC.5. Consider a second-price auction with $m+1$ buyers and m goods for some $m \in \mathbb{Z}_+$. Each of the first m buyers bids 1 for the m goods respectively and has a budget of ∞, i.e., for $i \in [m]$, we have

$$b_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

and $B_i = \infty$ (any $B > 2m$ would suffice). The last buyer has bid $b_{m+1,j} = m$ for each of the goods $j \in [m]$ and has a budget of $B_{m+1} = m + \epsilon$ for some small $\epsilon > 0$. In any throttling equilibrium $\theta \in \Theta^*$, we have $\theta_i = 1$ for all $i \in [m]$ because of the no-unnecessary-throttling condition. Since the sum of all the second-highest bids is m and buyer $m+1$ has the highest bid for every good, she cannot possibly spend her entire budget of $B_{m+1} = m + \epsilon$ and we must also have $\theta_{m+1} = 1$ by the

no-unnecessary throttling condition. Therefore, there is a unique throttling equilibrium θ such that $\theta_i = 1$ for all $i \in [m+1]$ and it has liquid welfare given by

$$\begin{aligned}\mathrm{LW}(y(\theta)) &= \left(\sum_{i=1}^{m} \min\{y_{ii}(\theta), B_i\}\right) \\ &\quad + \min\left\{\sum_{j=1}^{m} m \cdot y_{m+1,j}(\theta), m+\epsilon\right\} \\ &= m + \epsilon\end{aligned}$$

because $y_{m+1,j}(\theta) = 1$ for all $j \in [m]$. On the other hand, consider the allocation y such that $y_{ii} = 1$ for all $i \in [m-1]$ and $y_{m+1,m} = 1$. It has liquid welfare given by

$$\begin{aligned}\mathrm{LW}(y) &= \left(\sum_{i=1}^{m-1} \min\{y_{ii}, B_i\}\right) \\ &\quad + \min\left\{m \cdot y_{m+1,m}, m+\epsilon\right\} \\ &= m - 1 + m = 2m - 1\,.\end{aligned}$$

Hence, the PoA is at least $(2m-1)/(m+\epsilon)$. As m and ϵ were arbitrary, we can consider the limit when $m \to \infty$ and $\epsilon \to 0$, which yields the required lower bound of $\mathrm{PoA} \geq 2$.

Observe that in the previous example none of the buyers were throttled ($\theta_i = 1$), which indicates that the lower bound is driven more by the second-price auction format than the specific budget management method, and applies to other methods like pacing. Next, we show that our bound is tight for first-price auctions.

EXAMPLE EC.6. Consider a first-price auction with $m+1$ buyers and $m+1$ goods, for some $m \in \mathbb{Z}_+$. Each of the first m buyers bids 1 for the first m goods respectively and bids m on good $m+1$, and has a budget of 1, i.e., for each $i \in [m]$, we have

$$b_{ij} = \begin{cases} 1 & \text{if } j = i \\ m & \text{if } j = m+1 \\ 0 & \text{otherwise} \end{cases},$$

and $B_i = 1$. Moreover, buyer $m+1$ has value $b_{m+1,m+1} = m$ for the $(m+1)$-th good and $b_{m+1,j} = 0$ for all $j \in [m]$, with $B_{m+1} = \infty$.

Consider a throttling equilibrium $\theta \in \Theta^*$. We begin by showing that $\theta_i < 1$ for all $i \in [m]$. For contradiction, suppose not. Let i be the smallest index such that $\theta_i = 1$. Then, buyer i spends 1 on good i and spends $m \cdot \prod_{k=1}^{i-1}(1-\theta_k) > 0$ on good $m+1$ (we use the lexicographic tie-breaking rule), which makes her total expenditure strictly greater than her budget of $B_i = 1$, thereby yielding the required contradiction. Hence, $\theta_i < 1$ for all buyers $i \in [m]$, and consequently, the no-unnecessary-throttling condition implies that their total expected expenditure is exactly 1, i.e., the following equivalent statements hold

$$\begin{aligned} &\theta_i \cdot 1 + \left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot \theta_i \cdot m = 1 \\ \iff \quad &\theta_i = \frac{1}{1 + \left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot m}\,. \end{aligned} \tag{EC.5}$$

Moreover, since their expenditure is $B_i = 1$, that is also their contribution towards the liquid welfare. Let $g(i) \coloneqq \prod_{k=1}^{i}(1-\theta_k)$ denote the probability that the first i buyers do not participate. Next, observe that $\theta_{m+1} = 1$ because of the no-unnecessary-throttling condition and $B_{m+1} = \infty$. Therefore, due to the lexicographic tie-breaking rule, buyer $m+1$ wins good $m+1$ with probability $g(m) = \prod_{k=1}^{m}(1-\theta_k)$. Hence, the liquid welfare of θ is given by

$$\mathrm{LW}(y(\theta)) = m \cdot 1 + g(m) \cdot m\,.$$

On the other hand, the allocation y which awards good i to buyer i for all $i \in [m+1]$ has $\mathrm{LW}(y) = m + m = 2m$. Consequently, we have

$$\mathrm{PoA} \geq \frac{2m}{m + g(m) \cdot m} = \frac{2}{1 + g(m)}\,.$$

To show $\mathrm{PoA} \geq 2$, it suffices to show $\lim_{m\to\infty} g(m) = 0$, which is what we do next. First, observe that (EC.5) implies the following recursion for $g(\cdot)$:

$$\begin{aligned} g(i) &= (1-\theta_i)g(i-1) \\ &= \frac{\left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot m}{1 + \left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot m} \cdot g(i-1) \\ &= \frac{g(i-1)^2 \cdot m}{1 + g(i-1) \cdot m}\,. \end{aligned}$$

We will inductively show that $g(i) \le 1 - i/(m+\sqrt{m})$. Set $b = 1/(m+\sqrt{m})$. The base case $i = 1$ follows because $\theta_1 = 1/(1+m)$ (see (EC.5)). Suppose $g(i-1) \le 1 - (i-1)/(m+\sqrt{m})$ for some $i \in [m]$. Then, we have

$$\begin{aligned} g(i) &= \frac{g(i-1)^2 \cdot m}{1 + g(i-1)\cdot m} = \frac{m}{\frac{1}{g(i-1)^2} + \frac{m}{g(i-1)}} \\ &\le \frac{m}{\frac{1}{(1-b(i-1))^2} + \frac{m}{1-b(i-1)}} \\ &= \frac{m\cdot(1-bi+b)^2}{1+m(1-bi+b)}. \end{aligned}$$

To complete the induction, it suffices to show:

$$\begin{aligned} &\frac{m\cdot(1-bi+b)^2}{1+m(1-bi+b)} \le 1-bi \\ \iff & m(1+b^2i^2+b^2-2bi+2b-2b^2i) \\ &\qquad \le 1-bi+m(1+b^2i^2-2bi+b-b^2i) \\ \iff & 1-bi+m(b^2i-b-b^2) \ge 0 \\ \iff & 1-m\left(\frac{1}{m+\sqrt{m}}+\frac{1}{(m+\sqrt{m})^2}\right) \\ &\qquad -i\left(\frac{1}{m+\sqrt{m}}-\frac{m}{(m+\sqrt{m})^2}\right) \ge 0 \\ \iff & 1-m\left(\frac{1}{m+\sqrt{m}}+\frac{1}{(m+\sqrt{m})^2}\right) \\ &\qquad -i\left(\frac{\sqrt{m}}{(m+\sqrt{m})^2}\right) \ge 0 \end{aligned}$$

To see why the last inequality in the above equivalence chain holds, observe that:

$$\begin{aligned} &1-m\left(\frac{1}{m+\sqrt{m}}+\frac{1}{(m+\sqrt{m})^2}\right) \\ &\quad -i\left(\frac{\sqrt{m}}{(m+\sqrt{m})^2}\right) \\ \ge & 1-m\left(\frac{1}{m+\sqrt{m}}+\frac{1}{(m+\sqrt{m})^2}\right) \\ &\quad -m\left(\frac{\sqrt{m}}{(m+\sqrt{m})^2}\right) \\ = & \frac{(m+\sqrt{m})^2 - m(m+\sqrt{m}) - m - m\sqrt{m}}{(m+\sqrt{m})^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{m^2 + m + 2m\sqrt{m} - m^2 - m\sqrt{m} - m - m\sqrt{m}}{(m+\sqrt{m})^2} \\
&= 0
\end{aligned}
$$

which completes the induction. Hence, $g(m) \le 1 - m/(m+\sqrt{m})$ and $\lim_{m\to\infty} g(m) = 0$, as required.

EC.3. Appendix: Examples for Section 5

First, we provide an example to show that the inequality REV(TE) $\le 2 \times$ REV(PE) is tight.

EXAMPLE EC.7. Consider the throttling game in which there is 1 good and 2 buyers. The bids are given by $b_{11} = 1/\epsilon$, $b_{21} = 1 - \epsilon$ for $\epsilon > 0$ and the budgets are given by $B_1 = 1$, $B_2 = \infty$. Then, in the unique pacing equilibrium, we have $\alpha_1 = \epsilon$ and $\alpha_2 = 1$, whereas in the unique throttling equilibrium, we have $\theta_1 = \epsilon$ and $\theta_2 = 1$. Hence, REV(PE) = 1 and REV(TE) = $1 + (1-\epsilon)^2$. Since this is true for arbitrarily small ϵ, we get that the inequality REV(TE) $\le 2 \times$ REV(PE) established in Theorem 9 is tight.

Next, we give a family of examples for which REV(PE) is arbitrarily close to $(4/3) \times$ REV(TE).

EXAMPLE EC.8. Consider a throttling game with 2 goods and 2 buyers. Fix $\epsilon > 0$. The bids are given by $b_{11} = 1 + \epsilon$, $b_{12} = 1$ and $b_{21} = 1$. Moreover, the budgets are given by $B_1 = 1 - \epsilon$ and $B_2 = \infty$. Then, the unique pacing equilibrium is given by $\alpha_1 = 1 - \epsilon$, $\alpha_2 = 1$, and the unique throttling equilibrium is given by $\theta_1 = (1-\epsilon)/(2+\epsilon)$, $\theta_2 = 1$. Since ϵ was arbitrary, we can take it to be arbitrarily small. In this case, we get REV(PE) $\simeq 2$ and REV(TE) $\simeq 1.5$, as desired.