Appendix A: More on Validity Conditions of Protection Level Algorithms

In this section, we justify the necessity of the validity conditions stated in Definition 1 for PLAs.

Non-Increasing PL Function: We argue why the protection level function $p(x)$ must be non-increasing. At a high level, the goal of a PLA is to reserve or "protect" a certain amount of resources for high-reward requests. However, these protection levels must be *credible*—that is, any promise to reserve resources must be implementable and feasible under the given constraints. Suppose we are at a point where x low-reward requests have been processed. If the PLA has allocated the maximum allowable resources to low-reward requests up to this point, then the total amount allocated to low-reward requests is $m - p(x)$, where m is the total available resource capacity. The remaining $p(x)$ units are thus being reserved for high-reward requests.

Now, suppose that the PL function $p(x)$ increases at x, i.e., $p'(x) > 0$. This means that as we process more low-reward requests, we plan to reserve even more resources for high-reward requests than before. But this leads to a contradiction: we have already allocated $m - p(x)$ units to low-reward requests, so only $p(x)$ units remain. If we now try to reserve more than $p(x)$, we exceed the total available resource budget m, making the updated protection level infeasible.

This inconsistency illustrates why the PL function must be non-increasing. By ensuring that $p(x)$ does not grow with x, we guarantee that the resources promised to high-reward requests are always consistent with past allocation decisions, thereby preserving feasibility and preventing over-commitment.

Derivative Condition: We also require that the derivative of the protection level function satisfy $p'(x) \geqslant -1$. This condition ensures that protection levels do not decrease too rapidly.

To illustrate why we impose this condition, consider the same setting as before. Let m denote the total available resources, and suppose we have processed $\bar{s}$ units of low-reward requests so far, with $\bar{a}$ denoting the total amount of resources allocated to them. Now, a new low-reward request of size $\epsilon > 0$, where ϵ is very small, arrives. Assume that we have already allocated the maximum amount allowable for low-reward requests up to this point, i.e., $\bar{a} = m - p(\bar{s}+1)$, meaning only $p(\bar{s}+1)$ resources remain protected for high-reward requests.

According to the PLA rule, the amount we can allocate to the incoming ϵ-sized request is:

$$\min\{m - p(\bar{s}+1+\epsilon) - \bar{a}, \epsilon\}.$$

Substituting $\bar{a} = m - p(\bar{s}+1)$, we get:

$$\min\left\{\epsilon,\ -p(\bar{s}+1+\epsilon) + p(\bar{s}+1)\right\}.$$

If $p'(\bar{s}+1) < -1$, then for small ϵ, the difference $-p(\bar{s}+1+\epsilon)+p(\bar{s}+1)$ exceeds ϵ, and the PLA allows allocating the full request. In other words, the protection level drops so sharply that it remains ineffective. One can simply recover the same allocation outcome by setting $p'(\bar{s}+1) = -1$. Therefore, any steeper decline in the protection level function is unnecessary. Without loss of generality, we can thus assume that $p'(x) \geqslant -1$.

Appendix B: Proof of Lemma 1

Let (x, y) be fixed, and let I be the ordered sequence of arrivals such that x low-reward requests arrive first, followed by y high-reward requests. For any adaptive protection level algorithm $\mathcal{A}$, let $\tilde{x}(\mathcal{A})$ be the total number of low-reward requests that are accepted under the ordered sequence I. Then, the algorithm accepts $\min\{y, m-\tilde{x}(\mathcal{A})\}$ high-reward requests. Observe that algorithm $\mathcal{A}$ rejects the x-th low-reward request if and only if

- condition (1): the number of low-reward requests accepted so far is not less than $m - p(x)$, where $p(\cdot)$ is the protection level function, or
- condition (2): all the resources have been used.

We split the proof into two cases.

Case 1: $y < m - \tilde{x}(\mathcal{A})$. In this case, we claim that under any other ordering of the arrivals, the number of high-reward requests that are accepted cannot be smaller than y. We also claim that under any other ordering of the arrivals, the number of low-reward requests that are accepted cannot be smaller than $\tilde{x}(\mathcal{A})$. Showing these claims, complete the proof of this case.

We begin with the first part. First note that $\tilde{x}(\mathcal{A})$ is an upper bound on the total number of low-reward requests that algorithm $\mathcal{A}$ accepts under any ordering of the arrivals. This is because condition (2) never fails as $y \leqslant m - \tilde{x}(\mathcal{A})$, and hence with even accepting y high-reward requests, there are resources left for $\tilde{x}(\mathcal{A})$ low-reward requests, and condition (1) is independent of the order of high-reward requests. Moreover, since $y \leqslant m - \tilde{x}(\mathcal{A})$, any arriving high-reward request is accepted. Thus, for any ordering of the arrivals, algorithm $\mathcal{A}$ can accept y high-reward requests.

We now show the second part. Here, we want to show that under any other ordering of the arrivals, the number of low-reward requests that are accepted is greater than or equal to $\tilde{x}(\mathcal{A})$. Contrary to our claim, suppose that the total number of low-reward requests that algorithm $\mathcal{A}$ accepts under any unordered sequence is strictly less than $\tilde{x}(\mathcal{A})$. This means that at some point, condition (2) is not satisfied. Therefore, there exists a time t such that $x(t)+y(t) = m$, where $x(t)$ and $y(t)$ denote the number of low- and high-reward requests accepted up to time t, respectively.

However, we know that $x(t) < \tilde{x}(\mathcal{A})$, $y(t) \leqslant y$, and $\tilde{x}(\mathcal{A}) + y \leqslant m$. Therefore, $x(t) + y(t) = m$ cannot hold, which contradicts the assumption that algorithm $\mathcal{A}$ has accepted fewer than $\tilde{x}(\mathcal{A})$ low-reward requests. This completes the proof of the first case.

Case 2: $y \geqslant m - \tilde{x}(\mathcal{A})$. In this case, we claim that under any other ordering of the requests, the number of high-reward requests accepted cannot be smaller than $m - \tilde{x}(\mathcal{A})$. Additionally, we claim that if the number of low-reward requests accepted under any other ordering is less than $\tilde{x}(\mathcal{A})$, then the total reward generated by the algorithm is larger than the case where the number of low-reward requests accepted is $\tilde{x}(\mathcal{A})$.

We begin with the first part. We will show that under any other ordering of the requests, the number of high-reward requests accepted cannot be smaller than $m - \tilde{x}(\mathcal{A})$. As we know, high-reward requests are rejected only when all resources are used up. Therefore, the later a high-reward request arrives, the less chance we accept the request. Compare any unordered arrival sequence with the ordered arrival sequence, each high-reward request arrives earlier, which implies that the number of high-reward requests accepted cannot be smaller than $m - \tilde{x}(\mathcal{A})$.

Next, we show the second part, which is if the number of low-reward requests accepted under any other ordering is less than $\tilde{x}(\mathcal{A})$, then the total reward generated by the algorithm is larger than the case where the number of low-reward requests accepted is $\tilde{x}(\mathcal{A})$. If the total number of low-reward requests the algorithm $\mathcal{A}$ accepts is strictly less than $\tilde{x}(\mathcal{A})$, then at some point, condition (2) is not satisfied. This means that for some time t, $x(t) + y(t) = m$, where $x(t)$ and $y(t)$ denote the number of low-reward and high-reward requests accepted before time t, respectively. Since $x(t) < \tilde{x}(\mathcal{A})$, we have $y(t) > m - \tilde{x}(\mathcal{A})$. Therefore, the reward generated by the algorithm is

$$y(t)r_h + x(t)r_\ell = y(t)r_h + (m - y(t))r_\ell > (m - \tilde{x}(\mathcal{A}))r_h + \tilde{x}(\mathcal{A})r_\ell,$$

which implies that the algorithm generates a larger reward.

Therefore, we conclude that the adversary should choose the first instance where all x low reward requests arrive first and follow with all y high reward requests.

Appendix C: Proof of Statements in Section 4

Throughout the proofs, we make use of some preliminary lemmas that are presented in Section G.

C.1. Proof of Lemma 2

We first show that for any $p \leqslant \min\{m, y_1\}$, we have $\mathrm{CP}_u(p; (x, y_1)) \geqslant \mathrm{CP}_u(p; (x, y_2))$, where we recall that $y_1 \leqslant y_2$. Observe that when $p \leqslant y_1$ and $x + y_1 < m$, by Lemma 8, we have

$\text{CP}_u(p;(x,y_1)) = 1$, and hence $1 = \text{CP}_u(p;(x,y_1)) \geqslant \text{CP}_u(p;(x,y_2))$ trivially holds. Now suppose that $x + y_1 \geqslant m$. By definition, we have

$$\begin{aligned}\text{CP}_u(p;(x,y_1)) &= \frac{\max\{p,(m-x)^+\}r_h + \min\{x,m-p\}r_\ell}{\min\{y_1,m\}r_h + \min\{x,(m-y_1)^+\}r_\ell} \\ &\geqslant \frac{\max\{p,(m-x)^+\}r_h + \min\{x,m-p\}r_\ell}{\min\{y_2,m\}r_h + \min\{x,(m-y_2)^+\}r_\ell} = \text{CP}_u(p;(x,y_2)),\end{aligned}$$

where the inequality holds because $y_2 \geqslant y_1$ and $y \mapsto yr_h + (m-y)r_\ell$ is increasing in y as $r_h > r_\ell$.

Second, we show that any protection level p with $p \geqslant \min\{m,y_2\}$, we have $\text{CP}_o(p;(x,y_2)) \geqslant \text{CP}_o(p;(x,y_1))$. To show this, first consider the case where $x + y_1 < m$ and $x + y_2 < m$. Then, by the definition of CP_o in Equation (6), we have

$$\text{CP}_o(p;(x,y_2)) = \frac{\min\{y_2,m\}r_h + \min\{x,m-p\}r_\ell}{\min\{y_2,m\}r_h + \min\{x,(m-y_2)^+\}r_\ell} = \frac{y_2r_h + \min\{x,m-p\}r_\ell}{y_2r_h + xr_\ell} \tag{40}$$

$$\geqslant \frac{y_1r_h + \min\{x,m-p\}r_\ell}{y_1r_h + xr_\ell} = \text{CP}_o(p;(x,y_1)), \tag{41}$$

where the inequality holds because $y \mapsto \frac{yr_h + \min\{x,m-p\}r_\ell}{yr_h + xr_\ell}$ is increasing in y.

Now consider the case where $x + y_2 \geqslant m$. Next, we show that by fixing an x, for any y such that $x + y > m$, $\text{CP}_o(p;(x,y))$ is increasing in y. Observe that for any y with $x + y \geqslant m$ and $p \geqslant y$, we have

$$\text{CP}_o(p;(x,y)) = \frac{\min\{y,m\}r_h + \min\{x,m-p\}r_\ell}{\min\{y,m\}r_h + \min\{x,(m-y)^+\}r_\ell},$$

If $y \geqslant m$, the statement is trivial. Otherwise,

$$\begin{aligned}\frac{\partial \text{CP}_o(p;(x,y))}{\partial y} &= \frac{r_h(yr_h + (m-y)r_\ell) - (r_h - r_\ell)(yr_h + \min\{m-p,x\}r_\ell)}{(yr_h + (m-y)r_\ell)^2} \\ &\geqslant \frac{r_h(yr_h + (m-y)r_\ell) - (r_h - r_\ell)(yr_h + (m-p)r_\ell)}{(yr_h + (m-y)r_\ell)^2} \\ &\geqslant \frac{(r_h - r_\ell)(yr_h + (m-y)r_\ell) - (r_h - r_\ell)(yr_h + (m-p)r_\ell)}{(yr_h + (m-y)r_\ell)^2} \\ &= \frac{(r_h - r_\ell)(yr_h + (p-y)r_\ell)}{(yr_h + (m-y)r_\ell)^2} \geqslant 0,\end{aligned}$$

where the last inequality is because $p \geqslant y$. The chain of inequalities shows that $\text{CP}_o(p;(x,y))$ is increasing in y when $x + y \geqslant m$ and $p \geqslant y$. This implies that we have $\text{CP}_o(p;(x,y_2)) \geqslant \text{CP}_o(p;(x,y_1))$ when $x + y_i \geqslant m$, $i \in \{1,2\}$, as desired.

For the case where $x + y_2 \geqslant m$ and $x + y_1 < m$, we have

$$\text{CP}_o(p;(x,y_2)) \geqslant \text{CP}_o(p;(x,m-x)) \geqslant \text{CP}_o(p;(x,y_1)),$$

where the first inequality holds because $\frac{\partial \mathrm{CP}_o(p;(x,y))}{\partial y} \geqslant 0$ when $x+y \geqslant m$, and the second inequality holds because of Equation (40).

Finally, we show that any $x \in [\underline{x}, \bar{x}]$ and $p \geqslant 0$, we have $\min_{y \in [\underline{h}(x), \bar{h}(x)]}\{\mathrm{CP}(p;(x,y))\} = \min\left\{\mathrm{CP}(p;(x,\underline{h}(x))), \mathrm{CP}(p;(x,\bar{h}(x)))\right\}$. Suppose that $p \leqslant \underline{h}(x)$. Then, $\mathrm{CP}(p;(x,y)) = \mathrm{CP}_u(p;(x,y))$ for any $y \in [\underline{h}(x), \bar{h}(x)]$, and hence by the first result of this lemma, we have

$$\min_{y \in [\underline{h}(x), \bar{h}(x)]}\{\mathrm{CP}(p;(x,y))\} = \mathrm{CP}_u(p;(x,\bar{h}(x)))\,,$$

as desired. Now, suppose that $p \geqslant \bar{h}(x)$. Then, $\mathrm{CP}(p;(x,y)) = \mathrm{CP}_o(p;(x,y))$ for any $y \in [\underline{h}(x), \bar{h}(x)]$, and hence by the second result of this lemma, we have

$$\min_{y \in [\underline{h}(x), \bar{h}(x)]}\{\mathrm{CP}(p;(x,y))\} = \mathrm{CP}_o(p;(x,\underline{h}(x)))\,,$$

as desired. Now, suppose the final case where $p \in (\underline{h}(x), \bar{h}(x))$. Then,

$$\begin{aligned}\min_{y \in [\underline{h}(x), \bar{h}(x)]}\{\mathrm{CP}(p;(x,y))\} &= \min\left\{\min_{y \in [\underline{h}(x), p]}\{\mathrm{CP}_o(p;(x,y))\}, \min_{y \in [p, \bar{h}(x)]}\{\mathrm{CP}_u(p;(x,y))\}\right\}\\ &= \min\left\{\mathrm{CP}_o(p;(x,\underline{h}(x))), \mathrm{CP}_u(p;(x,\bar{h}(x)))\right\}\,,\end{aligned}$$

where the last inequality follows from the first and second results of this lemma.

□

C.2. Properties of functions $u(\cdot;C)$ and $l(\cdot;C)$: Lemma 6 and its Proof

LEMMA 6 **(Properties of functions $u(\cdot;C)$ and $l(\cdot;C)$).** *The functions $u(\cdot;C)$ and $l(\cdot;C)$, which are respectively defined in Equations (13) and (15), have the following properties.*

1. *For any $x \in (\underline{x}_u, \bar{x}_u)$ and $C \in [0,1]$, let $\underline{\mathcal{H}}(x) = \min\{\underline{h}(x), m\}$. When $\underline{\mathcal{H}}'(x)$ exists, we have*
$$\frac{\partial u(x;C)}{\partial x} = \begin{cases} ((1-C)\frac{r_h}{r_\ell} + C)\underline{\mathcal{H}}'(x) & \text{if } x + \underline{\mathcal{H}}(x) \geqslant m;\\ (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x) - C & \text{if } x + \underline{\mathcal{H}}(x) < m.\end{cases} \tag{42}$$
2. *For any $x \in (x_H, \bar{x}_l)$ and $C \in [0,1]$, let $\overline{\mathcal{H}}(x) = \min\{\bar{h}(x), m\}$. When $\overline{\mathcal{H}}'(x)$ exists, we have*
$$\frac{\partial l(x;C)}{\partial x} = C\overline{\mathcal{H}}'(x). \tag{43}$$
3. *For any $C \in [0,1]$, $u(x;C)$ is non-increasing in $x \in [0,\bar{x}]$ and is convex for $x \in [\underline{x}_u, \bar{x}]$.*
4. *For any $C \in [0,1]$, $l(x;C)$ is non-increasing in $x \in [0,\bar{x}]$ and is concave for $x \in [\underline{x}, \bar{x}_\ell]$.*
5. *For any $C \leqslant C^\star(\mathcal{R})$ and any $x \in [0,\bar{x}]$, we have $l(x;C) \leqslant u(x;C)$.*
6. *For any $x \in [0,\bar{x}]$, $l(x;C)$ is continuously increasing in C and $u(x;C)$ is continuously decreasing in C.*

Proof of Lemma 6 Here, we will show the following six properties.

C.2.1. Property 1 We first show Equation (57). We split the analysis into two cases: *Case 1:* $x+\underline{\mathcal{H}}(x)\geqslant m$ and *Case 2:* $x+\underline{\mathcal{H}}(x)<m$.

Case 1: ($x+\underline{\mathcal{H}}(x)\geqslant m$). By Lemma 12, we have for any $x\in(\underline{x}_u,\bar{x}_u)$, $\underline{\mathcal{H}}(x)=\underline{h}(x)$, which implies that $\underline{h}(x)\leqslant m$. Then, we take an arbitrary point $(x_1,\underline{h}(x_1))$ with $x_1\in(\underline{x}_u,\bar{x}_u)$. By definition of $u(\cdot;C)$, we should have $\text{CP}_o(u(x_1;C);(x_1,\underline{h}(x_1)))=C$, and by Lemma 14, we have such $u(x_1;C)$ always exists and $u(x_1;C)\geqslant\underline{h}(x_1)$. By Equation (6),

$$\text{CP}_o(u(x_1;C);(x_1,\underline{h}(x_1)))=\frac{\underline{h}(x_1)r_h+\min\{x_1,m-u(x_1;C)\}r_\ell}{\underline{h}(x_1)r_h+\min\{x_1,m-\underline{h}(x_1)\}r_\ell}=C. \tag{44}$$

As in this case, if $x_1+\underline{h}(x_1)\geqslant m$, we have $\min\{x_1,m-\underline{h}(x_1)\}=m-\underline{h}(x_1)$. We then argue that $\min\{x_1,m-u(x_1;C)\}=m-u(x_1;C)$. Suppose that contrary to our claim, $\min\{x_1,m-u(x_1;C)\}=x_1$. We then have

$$\text{CP}_o(u(x_1;C);(x_1,\underline{h}(x_1)))=\frac{\underline{h}(x_1)r_h+x_1r_\ell}{\underline{h}(x_1)r_h+(m-\underline{h}(x_1))r_\ell}=\text{CP}_o(m;(x_1,\underline{h}(x_1))),$$

which implies that $x_1\leqslant\underline{x}_u$. However, we define $x_1\in(\underline{x}_u,\bar{x}_u)$, therefore, this is a contradiction, and we can only have $\min\{x_1,m-u(x_1;C)\}=m-u(x_1;C)$. Then, by Equation (44), we have

$$\frac{\underline{h}(x_1)r_h+(m-u(x_1;C))r_\ell}{\underline{h}(x_1)r_h+(m-\underline{h}(x_1))r_\ell}=C,$$

which is equivalent to

$$u(x_1;C)=((1-C)\frac{r_h}{r_\ell}+C)\underline{h}(x_1)+m(1-C).$$

This implies that $\frac{\partial u(x;C)}{\partial x}=((1-C)\frac{r_h}{r_\ell}+C)\underline{h}'(x)$ when $x+\underline{h}(x)\geqslant m$. As we have $\underline{\mathcal{H}}(x)=\underline{h}(x)$ for $x\in[\underline{x}_u,\bar{x}_u]$, this implies the desired result.

Case 2 ($x+\underline{\mathcal{H}}(x)<m$). In this case, as $x+\underline{\mathcal{H}}(x)<m$, we have $\underline{\mathcal{H}}(x)<m$, which implies that $\underline{\mathcal{H}}(x)=\underline{h}(x)$. Let us take a point $(x_1,\underline{h}(x_1))$ with $x_1\in(\underline{x}_u,\bar{x}_u)$. By definition of $u(\cdot;C)$, we have $\text{CP}_o(u(x_1;C);(x_1,\underline{h}(x_1)))=C$ and by Lemma 14, we have such $u(x_1;C)$ always exists and $u(x_1;C)\geqslant\underline{h}(x_1)$. By Equation (6),

$$\text{CP}_o(u(x_1;C);(x_1,\underline{h}(x_1)))=\frac{\underline{h}(x_1)r_h+\min\{x_1,m-u(x_1;C)\}r_\ell}{\underline{h}(x_1)r_h+\min\{x_1,m-\underline{h}(x_1)\}r_\ell}=C.$$

As in this case $x_1+\underline{h}(x_1)<m$, we have $\min\{x_1,m-\underline{h}(x_1)\}=x_1$. Here, we argue that $\min\{x_1,m-u(x_1;C)\}=m-u(x_1;C)$. Contrary to our claim, suppose that $\min\{x_1,m-u(x_1;C)\}=x_1$. We then have

$$\text{CP}_o(u(x_1;C);(x_1,\underline{h}(x_1)))=\frac{\underline{h}(x_1)r_h+x_1r_\ell}{\underline{h}(x_1)r_h+x_1r_\ell}=1>C,$$

which is a contradiction. Therefore, we can only have $\min\{x_1, m-u(x_1;C)\} = m-u(x_1;C)$. Then, we have

$$\frac{\underline{h}(x_1)r_h + (m-u(x_1;C))r_\ell}{\underline{h}(x_1)r_h + x_1 r_\ell} = C,$$

which is equivalent as

$$u(x_1;C) = (1-C)\frac{r_h}{r_\ell}\underline{h}(x_1) - Cx_1 + m.$$

This implies that $\frac{\partial u(x;C)}{\partial x} = (1-C)\frac{r_h}{r_\ell}\underline{h}'(x) - C$ when $x + \underline{h}(x) < m$. As we have $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for $x \in [\underline{x}_u, \bar{x}_u]$, this implies the desired result.

C.2.2. Property 2 We take a point $(x_1, \overline{\mathcal{H}}(x_1))$ with $x_1 \in (x_H, \bar{x}_\ell)$. By definition of $l(\cdot;C)$, we have $\mathrm{CP}_u(l(x_1;C);(x_1,\bar{h}(x_1))) = C$. As $\overline{\mathcal{H}}(x_1) = \min\{m, \bar{h}(x_1)\}$, by Lemma 10, we have $\mathrm{CP}_u(l(x_1;C);(x_1,\overline{\mathcal{H}}(x_1))) = \mathrm{CP}_u(l(x_1;C);(x_1,\bar{h}(x_1))) = C$. By Lemma 14, such $l(x_1;C)$ always exists and $l(x_1;C) \leqslant \overline{\mathcal{H}}(x_1)$. By Equation (7),

$$\mathrm{CP}_u(l(x_1;C);(x_1,\overline{\mathcal{H}}(x_1))) = \frac{\max\{l(x_1;C), \min\{\overline{\mathcal{H}}(x_1), m-x_1\}\}r_h + \min\{x_1, m-l(x_1;C)\}r_\ell}{\overline{\mathcal{H}}(x_1)r_h + \min\{x_1, m-\overline{\mathcal{H}}(x_1)\}r_\ell} = C.$$

If $\min\{x_1, m-\overline{\mathcal{H}}(x_1)\} = x_1$, by Lemma 8 and the fact that $l(x_1;C) \leqslant \overline{\mathcal{H}}(x_1)$, $\mathrm{CP}_u(l(x_1;C);(x_1,\overline{\mathcal{H}}(x_1))) = 1 \neq C$, which cannot happen, and hence $\min\{x_1, m-\overline{\mathcal{H}}(x_1)\} = m - \overline{\mathcal{H}}(x_1)$. If $\min\{x_1, m-l(x_1;C)\} = x_1$, then

$$\mathrm{CP}_u(l(x_1;C);(x_1,\overline{\mathcal{H}}(x_1))) = \frac{(m-x_1)r_h + x_1 r_\ell}{\overline{\mathcal{H}}(x_1)r_h + (m-\overline{\mathcal{H}}(x_1))r_\ell} = \mathrm{CP}_u(0;(x_1,\overline{\mathcal{H}}(x_1))),$$

which implies that $x_1 \geqslant \bar{x}_\ell$. However, we define $x_1 \in (x_H, \bar{x}_\ell)$, therefore, this is a contradiction, and we can only have $\min\{x_1, m-\overline{\mathcal{H}}(x_1)\} = m-\overline{\mathcal{H}}(x_1)$ and $\min\{x_1, m-l(x_1;C)\} = m-l(x_1;C)$. Then, we have

$$\frac{l(x_1;C)r_h + (m-l(x_1;C))r_\ell}{\overline{\mathcal{H}}(x_1)r_h + (m-\overline{\mathcal{H}}(x_1))r_\ell} = C,$$

which is equivalent to

$$l(x_1;C) = C\overline{\mathcal{H}}(x_1) - \frac{(1-C)mr_\ell}{r_h - r_\ell},$$

and verifies Equation (43).

C.2.3. Property 3 We first show that $u(x;C)$ is non-increasing for $x \in [\underline{x}, \bar{x}]$. First, by the definition of $\underline{x}_u$, we have $u(x;C) = m$ for $x \in [\underline{x}, \underline{x}_u]$, which is non-increasing.

For $x \in (\underline{x}_u, \bar{x}_u)$, first recall that

$$\bar{x}_u = \begin{cases} x_L & \text{if } x_L + y_L \geqslant m; \\ \sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\} & \text{Otherwise}, \end{cases}$$

Case 1 – $\mathbf{x_L} + \mathbf{y_L} \geqslant \mathbf{m}$. If $x_L + y_L \geqslant m$, we have $\bar{x}_u = x_L$. Since point $L = (x_L, y_L)$ is the lowest point, $\underline{h}(x)$ decreases for $x < x_L$, increases for $x > x_L$, which implies that $\underline{h}'(x) \leqslant 0$ for $x < x_L$ and $\underline{h}'(x) \geqslant 0$ for $x > x_L$. Now, recall Property 1 that we just showed:

$$\frac{\partial u(x;C)}{\partial x} = \begin{cases} ((1-C)\frac{r_h}{r_\ell} + C)\underline{\mathcal{H}}'(x) & \text{if } x + \underline{\mathcal{H}}(x) \geqslant m; \\ (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x) - C & \text{if } x + \underline{\mathcal{H}}(x) < m. \end{cases}$$

where by Lemma 12, we have $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for $x \in (\underline{x}_u, \bar{x}_u)$. This property and the fact that $\underline{h}'(x) \leqslant 0$ for $x < x_L$ and $\underline{h}'(x) \geqslant 0$ for $x > x_L$ imply that $u(x;C)$ decreases for $x < x_L$ and increases for $x > x_L$. As we force $u(x;C) = u(x_L;C)$ for $x > \bar{x}_u = x_L$, we have $u(x;C)$ is always non-increasing.

Case 2 – $\mathbf{x_L} + \mathbf{y_L} < \mathbf{m}$. If $x_L + y_L < m$, we have $\bar{x}_u = \sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{h}'(x) - C < 0\}$. For $x \in (\underline{x}_u, \bar{x}_u)$, by Lemma 12, we have $\underline{\mathcal{H}}(x) = \underline{h}(x)$. As $\underline{h}(x)$ is convex, we have its subderivative is increasing. Then, for $x < \bar{x}_u$, we have $(1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x) - C < 0$. Therefore, Property (1) that we just showed (i.e., Equation (57)) implies that $u(x;C)$ decreases for $x \in [\underline{x}_u, \bar{x}_u]$. As we force $u(x;C)$ to be a constant for $x \in [\bar{x}_u, \bar{x}]$, we have $u(x;C)$ is non-increasing for $x \in [\underline{x}, \bar{x}]$.

Finally, we show that $u(x;C)$ is convex for $x \in [\underline{x}_u, \bar{x}]$ by proving that its subderivative is increasing. As $\underline{h}(x)$ is convex, we have the subderivative of $\underline{h}(x)$ is increasing for $x \in [\underline{x}_u, \bar{x}_u]$. By Lemma 12, we have $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for $x \in [\underline{x}_u, \bar{x}_u]$. Then, we have the subderivative of $\underline{\mathcal{H}}(x)$ is increasing for $x \in [\underline{x}_u, \bar{x}_u]$. Therefore, Equation (57) implies that $u(x;C)$ has increasing subderivative, and $u(x;C)$ is convex for $x \in [\underline{x}_u, \bar{x}_u]$. As $u(x;C)$ is non-increasing and convex for $x \in [\underline{x}_u, \bar{x}_u]$ and constant for $x \in [\bar{x}_u, \bar{x}]$, we have $u(x;C)$ is convex for $x \in [\underline{x}_u, \bar{x}]$.

C.2.4. Property 4 Because H is the highest point and $\mathcal{R}$ is convex, $\bar{h}(x)$ increases for $x < x_H$ and decreases for $x > x_H$, which implies that $\bar{h}'(x) \geqslant 0$ a.e. for $x < x_H$ and $\bar{h}'(x) \leqslant 0$ a.e. for $x > x_H$. (Recall that $\bar{h}(\cdot)$ is concave and point $H = (x_H, y_H) \in \bar{\mathcal{R}}$, which lies on the upper envelope $\bar{h}(\cdot)$, is the point in set $\bar{\mathcal{R}}$ that has the highest low-reward demand, where $\bar{\mathcal{R}} = \{(x,y) \in \mathcal{R} : y = \sup_{(x',y') \in \mathcal{R}} \min\{y', m\}\}$ is a subset of region $\mathcal{R}$ under which the high-reward demand (more precisely $\min\{y', m\}$ for any point $(x', y') \in \mathcal{R}$) is maximized.) Therefore, Property (2) that we just showed, (i.e., $\frac{\partial l(x;C)}{\partial x} = C\overline{\mathcal{H}}'(x)$) implies that $l(x;C)$ increases for $x < x_H$. As we force $l(x;C) = l(x_H;C)$ for $x \geqslant x_H$, we have $l(x;C)$ is always non-increasing.

Next, we show that $l(\cdot;C)$ is concave. As stated earlier, because $\mathcal{R}$ is a convex set, we have $\bar{h}(x)$ is concave. Since both $\bar{h}(x)$ and $y = m$ are concave, we have $\overline{\mathcal{H}}(x) = \min\{\bar{h}(x), m\}$ is concave. Therefore, the subderivative of $\overline{\mathcal{H}}(x)$ is decreasing. By Equation (43), we have the subderivative of

$l(x;C)$ decreases for $x \in [\underline{x}_\ell, \bar{x}_\ell]$, which implies that $l(x;C)$ is concave for $x \in [\underline{x}_\ell, \bar{x}_\ell]$. As $l(x;C)$ is a constant for $x \in [\underline{x}, \underline{x}_\ell]$ and $l(x;C)$ is concave and non-increasing for $x \in [\underline{x}_\ell, \bar{x}_\ell]$, we have $l(x;C)$ is concave for $x \in [\underline{x}, \bar{x}_\ell]$.

C.2.5. Property 5 Here, we want to show that for any $C \leqslant C^\star(\mathcal{R})$ and any $x \in [\underline{x}, \bar{x}]$, we have $l(x;C) \leqslant u(x;C)$. For $x \in [\underline{x}, \bar{x}]$, let $p_b(x)$ be a function that

$$\text{CP}_o(p_b(x);(x,\underline{h}(x))) = \text{CP}_u(p_b(x);(x,\bar{h}(x))).$$

Lemma 17 shows that such $p_b(x)$ always exists, and $\text{CP}_o(p_b(x);(x,\underline{h}(x))) = \text{CP}_u(p_b(x);(x,\bar{h}(x))) \geqslant C^\star(\mathcal{R})$. Then, by Lemma 9, for any $C \leqslant C^\star(\mathcal{R})$, we have $\text{CP}_o(l(x;C);(x,\underline{h}(x))) = C$ implies that $l(x;C) \leqslant p_b(x)$, and $\text{CP}_u(u(x;C);(x,\bar{h}(x))) = C \leqslant C^\star(\mathcal{R})$ implies that $u(x;C) \geqslant p_b(x)$. Therefore, we have $l(x;C) \leqslant p_b(x) \leqslant u(x;C)$.

C.2.6. Property 6 Here, we would like to show for any $x \in [\underline{x}, \bar{x}]$, $l(x;C)$ is continuously increasing in C and $u(x;C)$ is continuously decreasing in C. This is because we have showed

$$l(x_1;C) = C\overline{\mathcal{H}}(x_1) - \frac{(1-C)mr_\ell}{r_h - r_\ell},$$

and

$$u(x_1;C) = (1-C)\frac{r_h}{r_\ell}\underline{h}(x_1) - Cx_1 + m.$$

From these two equations, we can simply find that for any $x \in [\underline{x}, \bar{x}]$, $l(x;C)$ is continuously increasing in C and $u(x;C)$ is continuously decreasing in C.

□

C.3. Proof of Lemma 3

First Direction. We first show that if $l(x;C) \leqslant p(x) \leqslant u(x;C)$, we have $\underline{\text{CP}}(p(x);x) \geqslant C$ for any $x \in [\underline{x}, \bar{x}]$, where we define

$$\underline{\text{CP}}(p(x);x) = \min\{\text{CP}_o(p(x);(x,\underline{h}(x))), \text{CP}_u(p(x);(x,\bar{h}(x)))\}. \tag{45}$$

This gives us the desired result because by Lemma 2, for any $x \in [\underline{x}, \bar{x}]$ and $p \geqslant 0$, we have

$$\min_{y \in [\underline{h}(x), \bar{h}(x)]} \{\text{CP}(p;(x,y))\} = \min\left\{\text{CP}(p;(x,\underline{h}(x))), \text{CP}(p;(x,\bar{h}(x)))\right\}.$$

Part 1: $\text{CP}_o(p(x);(x,\underline{h}(x))) \geqslant C$ if $p(x) \leqslant u(x;C)$. Here, we show that for any $x \in [\underline{x}, \bar{x}]$, $\text{CP}_o(p(x);(x,\underline{h}(x)))$ is greater than or equal to C as long as $p(x) \leqslant u(x;C)$. Let us first focus on

$x \in [\underline{x}, \underline{x}_u]$ and $x \in [\bar{x}_u, \bar{x}]$. By Lemma 13, for any $\underline{x} < x \leqslant \underline{x}_u$, we have $\text{CP}_o(m; (x, \underline{h}(x))) \geqslant C$. By Lemma 15, for any $x \in [\bar{x}_u, \bar{x}]$, we have $\text{CP}_o(m; (x, \underline{h}(x))) \geqslant C$. By Lemma 9, for any $p(x) \leqslant m = u(x; C)$ for $x \in [\underline{x}, \underline{x}_u]$ and $[\bar{x}_u, \bar{x}]$, we have

$$\text{CP}_o(p(x); (x, \underline{h}(x))) \geqslant \text{CP}_o(m; (x, \underline{h}(x))) \geqslant C.$$

Next, we consider $x \in [\underline{x}_u, \bar{x}_u]$. By Lemma 14, we have

$$\text{CP}_o(u(x; C); (x, \underline{h}(x))) = C,$$

and by Lemma 9, we have for any $p(x) \leqslant u(x; C)$,

$$\text{CP}_o(p(x); (x, \underline{h}(x))) \geqslant \text{CP}_o(u(x; C); (x, \underline{h}(x))) = C\,,$$

which is the desired result.

Part 2: $\mathbf{CP}_u(p(x); (x, \bar{h}(x))) \geqslant C$ if $p(x) \geqslant l(x; C)$. Here, we show that for any $x \in [\underline{x}, \bar{x}]$, $\text{CP}_u(p(x); (x, \bar{h}(x)))$ is greater than or equal to C as long as $p(x) \geqslant l(x; C)$. Let us first consider any $x \in [\underline{x}, x_H]$ and $x \in [\bar{x}_l, \bar{x}]$. By the definition of x_H and Lemma 13, for $x \in [\underline{x}, x_H]$ and $[\bar{x}_\ell, \bar{x}]$, we have $\text{CP}_u(0; (x, \bar{h}(x))) \geqslant C$. By Lemma 9, for any $p(x) \geqslant 0 = l(x; C)$ for $x \in [\underline{x}, x_H]$ and $[\bar{x}_\ell, \bar{x}]$, we then have

$$\text{CP}_u(p(x); (x, \bar{h}(x))) \geqslant \text{CP}_u(0; (x, \bar{h}(x))) \geqslant C\,,$$

which is the desired result. Now, let us consider any $x \in [x_H, \bar{x}_\ell]$, By Lemma 14, we have $\text{CP}_u(l(x; C); (x, \bar{h}(x))) = C$, and by Lemma 9, we have for any $p(x) \geqslant l(x; C)$,

$$\text{CP}_u(p(x); (x, \bar{h}(x))) \geqslant \text{CP}_u(l(x; C); (x, \bar{h}(x))) = C\,,$$

which is the desired result.

Second Direction. So far we have established that if $p(x) \in [l(x; C), u(x; C)]$, we have $\underline{\text{CP}}(p(x); x) \geqslant C$ for any $(x, y) \in \mathcal{R}$. Next, we show that if $p(x) > u(x; C)$ or $p(x) < l(x; C)$, we have $\underline{\text{CP}}(p(x); x) < C$.

Part 1: $\mathbf{CP}_o(p(x); (x, \underline{h}(x))) < C$ if $p(x) > u(x; C)$. First, as $u(x; C) = m$ for $x \in [\underline{x}, \underline{x}_u]$ and $[\bar{x}_u, \bar{x}]$, and $p(x) \leqslant m$, we cannot have $p(x) > u(x; C)$. Thus, we need to only consider $x \in [\underline{x}_u, \bar{x}_u]$. For any $x \in [\underline{x}_u, \bar{x}_u]$, by definition, $u(x; C)$ is the largest PL value such that

$$\text{CP}_o(u(x; C); (x, \underline{h}(x))) = C,$$

and by Lemma 9, if $p(x) > u(x;C)$, we have

$$\mathrm{CP}_o(p(x);(x,\underline{h}(x))) < \mathrm{CP}_o(u(x;C);(x,\underline{h}(x))) = C\,,$$

which is the desired result.

Part 2: $\mathbf{CP}_u(p(x);(x,\bar{h}(x))) < C$ if $p(x) < l(x;C)$. As $l(x;C) = 0$ for $x \in [\underline{x}, x_H]$ and $[\bar{x}_\ell, \bar{x}]$, and $p(x) \geqslant 0$, we cannot have $p(x) < l(x;C)$. Thus, we consider $x \in [x_H, \bar{x}_\ell]$. For any $x \in [x_H, \bar{x}_\ell]$, by definition, $l(x;C)$ is the smallest PL value such that

$$\mathrm{CP}_u(l(x;C);(x,\bar{h}(x))) = C,$$

and by Lemma 9, if $p(x) < l(x;C)$, we have

$$\mathrm{CP}_u(p(x);(x,\bar{h}(x))) < \mathrm{CP}_u(l(x;C);(x,\bar{h}(x))) = C\,,$$

which is the desired result.

C.4. Proof of Lemma 4

To show the result, we show the optimization problem in Equation (18) is equivalent to that in Equation (16). Since the only difference between these two problems is their first set of constraints, we only need to show that the feasible regions of these two problems are identical. To do so, we show that any feasible solution to Problem (18) is a feasible solution to Problem (16) and vice versa.

Considering a feasible solution to Problem (18) with $p(x) \in [\widetilde{l}(x;C), u(x;C)]$ for any $x \in [\underline{x}, \bar{x}]$. By Equation (17), we know that for any $C \in [0,1]$ and $x \in [\underline{x}, \bar{x}]$, $\widetilde{l}(x;C) \geqslant l(x;C)$. Therefore, $\widetilde{l}(x;C) \leqslant p(x) \leqslant u(x;C)$ implies that $l(x;C) \leqslant p(x) \leqslant u(x;C)$, as desired. Recall that $l(x;C) \leqslant p(x) \leqslant u(x;C)$ is the first constraint in Problem (16), and hence the above argument shows that any feasible solution to Problem (18) is a feasible solution to Problem (16).

Next, we show the opposite direction. Contrary to our claim, suppose that there exists a feasible solution $p(x)$ to Problem (16) with $l(x_1;C) \leqslant p(x_1) < \widetilde{l}(x_1;C)$ for some $x_1 \in [\underline{x}, \bar{x}]$. (This shows that there exists a feasible solution to Problem (16), which is not a feasible solution to Problem (18).) By Equation (17), we must have $x_1 > x_{-1}$, where x_{-1} is defined in Equation (17). As $l(x_{-1};C) = \widetilde{l}(x_{-1};C)$, we have $p(x_{-1}) \geqslant l(x_{-1};C) = \widetilde{l}(x_{-1};C)$. Then, we have

$$\frac{p(x_1) - p(x_{-1})}{x_1 - x_{-1}} < \frac{\widetilde{l}(x_1;C) - \widetilde{l}(x_{-1};C)}{x_1 - x_{-1}} = -1,$$

where the equation holds because by definition of $\widetilde{l}(x;C)$, the slope of $\widetilde{l}(x;C)$ w.r.t. x is -1 for any $x \in [x_{-1}, x_1]$. That $\frac{p(x_1)-p(x_{-1})}{x_1-x_{-1}} < -1$ implies that $p'(x) < -1$ on a positive measure set, and hence, $p(x)$ is not a valid PL function, which is a contradiction.

C.5. Proof of Lemma 5

First Direction. We first show the 'if' statement. That is, if $\underline{g}(x;R) \leqslant p(x) \leqslant \bar{g}(x;R)$, we have $\underline{\mathrm{CP}}(p(x);x) \geqslant R$ for any $x \in [0, \max\{m, \bar{x}\}]$, where with a slight abuse of notation, we define

$$\underline{\mathrm{CP}}(p(x);x) = \min\{\mathrm{CP}_o(p(x);(x,0)), \mathrm{CP}_u(p(x);(x,m))\}. \tag{46}$$

Notice that by Lemma 10, we have $\mathrm{CP}_u(p(x);(x,m)) = \mathrm{CP}_u(p(x);(x,y))$ for any $y \geqslant m$. Then, by Lemma 2, it suffices to show $\underline{\mathrm{CP}}(p(x);x) \geqslant R$ when $\underline{g}(x;R) \leqslant p(x) \leqslant \bar{g}(x;R)$.

Part 1: $\mathrm{CP}_o(p(x);(x,0)) \geqslant R$ if $p(x) \leqslant \bar{g}(x;R)$. First observe that, by Definition of CP_o in Equation (6), if we set $p(x) = \bar{g}(x;R)$, we have

$$\mathrm{CP}_o(\bar{g}(x;R);(x,0)) = \frac{0 \cdot r_h + \min\{x, m - \bar{g}(x;R)\} r_\ell}{0 \cdot r_h + \min\{x, m-0\} r_\ell} = \frac{\min\{x, m - \bar{g}(x;R)\} r_\ell}{\min\{x, m\} r_\ell}.$$

If $x \leqslant m$, we have $\bar{g}(x;R) = -Rx + m$, and we can obtain

$$\frac{\min\{x, m - \bar{g}(x;R)\} r_\ell}{\min\{x, m\} r_\ell} = \frac{\min\{x, m - (-Rx+m)\} r_\ell}{x r_\ell} = \frac{R x r_\ell}{x r_\ell} = R.$$

Otherwise, if $x > m$, we have $\bar{g}(x;R) = \bar{g}(m;R) = -Rm + m$, and we can obtain

$$\frac{\min\{x, m - \bar{g}(x;R)\} r_\ell}{m r_\ell} = \frac{\min\{x, m - (-Rm+m)\} r_\ell}{m r_\ell} = \frac{R m r_\ell}{m r_\ell} = R.$$

Then, by Lemma 9, we have for any $p(x) \leqslant \bar{g}(x;R)$, we have

$$\mathrm{CP}_o(p(x);(x,0)) \geqslant \mathrm{CP}_o(\bar{g}(x;R);(x,0)) = R\,,$$

which is the desired result.

Part 2: $\mathrm{CP}_u(p(x);(x,m)) \geqslant R$ if $p(x) \geqslant \underline{g}(x;R)$. By Definition of CP_u in Equation (7), we have

$$\begin{aligned}\mathrm{CP}_u(p(x);(x,m)) &= \frac{\max\{p(x), \min\{m, m-x\}\} r_h + \min\{x, m - p(x)\} r_\ell}{m r_h + \min\{x, m-m\} r_\ell} \\ &= \frac{\max\{p(x), m-x\} r_h + \min\{x, m-p(x)\} r_\ell}{m r_h}.\end{aligned}$$

We would like to show that if $p(x) \geqslant \underline{g}(x;R)$, we have $\mathrm{CP}_u(p(x);(x,m)) \geqslant R$, where $\underline{g}(x;R) = \frac{m(R - r_\ell/r_h)}{1 - r_\ell/r_h}$ for $x \in [0, \max\{m, \bar{x}\}]$. If $p(x) \geqslant m - x$, we have

$$\mathrm{CP}_u(p(x);(x,m)) = \frac{p(x) r_h + (m - p(x)) r_\ell}{m r_h} \geqslant \frac{\frac{m(R - r_\ell/r_h)}{1 - r_\ell/r_h} r_h + (m - \frac{m(R - r_\ell/r_h)}{1 - r_\ell/r_h}) r_\ell}{m r_h} = R\,,$$

where the inequality holds because $p(x) \geqslant \underline{g}(x;R) = \frac{m(R - r_\ell/r_h)}{1 - r_\ell/r_h}$. Otherwise, if $p(x) < m - x$, as $p(x) \geqslant \underline{g}(x;R)$, we have $\underline{g}(x;R) < m - x$. Then,

$$\text{CP}_u(p(x);(x,m)) = \frac{(m-x)r_h + xr_\ell}{mr_h} \geqslant \frac{p(x)r_h + (m - p(x))r_\ell}{mr_h} \geqslant R,$$

where the first inequality is because $\frac{(m-x)r_h + xr_\ell}{mr_h}$ is decreasing in x and $x < m - p(x)$. The last inequality, which is the desired result, is because

$$p(x)r_h + (m - p(x))r_\ell = p(x)(r_h - r_\ell) + mr_\ell \geqslant \underline{g}(x;R)(r_h - r_\ell) + mr_\ell\,,$$

and by some calculations, we have $\frac{\underline{g}(x;R)(r_h - r_\ell) + mr_\ell}{mr_h} = R$.

Second Direction. So far, we have established that if $p(x) \in [\underline{g}(x;R), \bar{g}(x;R)]$, we have $\underline{\text{CP}}(p(x);x) \geqslant R$ for any $(x,y) \in \mathcal{R}$. Next, we show that if $p(x) > \bar{g}(x;R)$ or $p(x) < \underline{g}(x;C)$, we have $\underline{\text{CP}}(p(x);x) < R$.

If $p(x) > \bar{g}(x;R)$ for some $x \in [0, \max\{m, \bar{x}\}]$, by Equation (19), we have if $x \in [0,m]$, $\bar{g}(x;R) = -Rx + m \geqslant -x + m$ and hence $p(x) + x > m$. If $x > m$, we have $\bar{g}(x;R) = \bar{g}(m;R)$. Then, we have

$$\text{CP}_o(p(x);(x,0)) = \frac{0 \cdot r_h + \min\{x, m - p(x)\}r_\ell}{0 \cdot r_h + \min\{x, m - 0\}r_\ell} = \frac{\min\{x, m - p(x)\}r_\ell}{\min\{x, m\}r_\ell}.$$

If $x \leqslant m$, we have

$$\frac{\min\{x, m - p(x)\}r_\ell}{\min\{x, m\}r_\ell} = \frac{\min\{x, m - p(x)\}r_\ell}{xr_\ell} < \frac{(m - \bar{g}(x;R))\,r_\ell}{xr_\ell} = R,$$

where the inequality is because $p(x) > \bar{g}(x;R)$. Otherwise, if $x > m$, we have

$$\frac{\min\{x, m - p(x)\}r_\ell}{\min\{x, m\}r_\ell} = \frac{(m - p(x))r_\ell}{mr_\ell} = \frac{m - p(x)}{m} < \frac{m - \bar{g}(m;R)}{m} = R.$$

If $p(x) < \underline{g}(x;R)$ for some $x \in [0, \max\{m, \bar{x}\}]$, as a valid PL function $p(x)$ is non-increasing, we have $p(\max\{m, \bar{x}\}) < \underline{g}(x;R)$. Therefore,

$$\begin{aligned}\text{CP}_u(p(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)) &= \frac{p(\max\{m,\bar{x}\})r_h + (m - p(\max\{m,\bar{x}\}))r_\ell}{mr_h} \\ &< \frac{\frac{m(R - r_\ell/r_h)}{1 - r_\ell/r_h} r_h + (m - \frac{m(R - r_\ell/r_h)}{1 - r_\ell/r_h})r_\ell}{mr_h} = R.\end{aligned}$$

C.6. Proof of Theorem 2

Algorithm 2 presents an optimal solution to Problem (C-Pareto-right). That is, at the optimal solution to Problem (C-Pareto-right), denoted by $p_{\text{right}}(\cdot)$, we set $p_{\text{right}}(x)$ based on Equations (23) and (25). Furthermore, the optimal objective value of Problem (C-Pareto-right), R_{right}, is given in Equation (24).

We split the proof into three cases, in each case, we first figure out the robust ratio under the PL function $p_{\text{right}}(\cdot)$ and check the feasibility and optimality of $p_{\text{right}}(\cdot)$:

- *Case 1:* $[\widetilde{l}(\bar{x};C), u(\bar{x};C)] \cap [\underline{g}(\bar{x}), \bar{g}(\bar{x})] \neq \emptyset$. In this case, if $\widetilde{l}(\bar{x};C) < \underline{g}(\bar{x})$,

$$p_{\text{right}}(\bar{x}) = \arg \min_{p \in [\widetilde{l}(\bar{x};C), u(\bar{x};C)]} |p - \underline{g}(\bar{x})| = \underline{g}(\bar{x}).$$

  If $\widetilde{l}(\bar{x};C) \geqslant \underline{g}(\bar{x})$, we have $p_{\text{right}}(\bar{x}) = \widetilde{l}(\bar{x};C)$. If $\bar{x} \geqslant m$, then as the right problem is only defined on $\bar{x}$, by definition, we have $\underline{g}(\bar{x}) = \bar{g}(\bar{x}) = \frac{1-r_\ell/r_h}{2-r_\ell/r_h} m$. Then, $p_{\text{right}}(\bar{x}) = \frac{1-r_\ell/r_h}{2-r_\ell/r_h} m$, which is feasible. By Lemma 10, we have

$$\begin{aligned} R_{\text{right}} &= \min\{\text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)), \text{CP}_u(p_{\text{right}}(\bar{x}); (\bar{x}, m))\} \\ &= \min\left\{\text{CP}_o(\frac{1-r_\ell/r_h}{2-r_\ell/r_h} m; (m, 0)), \text{CP}_u(\frac{1-r_\ell/r_h}{2-r_\ell/r_h} m; (m, m))\right\} = \frac{1}{2-r_\ell/r_h}, \end{aligned}$$

  which matches the upper bound of the robust ratio in the absence of ML advice. Therefore, $p_{\text{right}}(\bar{x})$ is optimal in this case.

  Otherwise, if $\bar{x} < m$, by Equation (22), $p_{\text{right}}(x) = \max\{-x + \bar{x} + p_{\text{right}}(\bar{x};C), \underline{g}(x)\}$ for $x \in [\bar{x}, m]$. By definition, we have $\underline{g}(x) \leqslant p_{\text{right}}(x)$. For the part where $p_{\text{right}}(x) = \underline{g}(x)$, we have $p_{\text{right}}(x) \leqslant \bar{g}(x)$ because by Equation (19), $\underline{g}(x) \leqslant \bar{g}(x)$ for any $x \in [0, m]$. Then, we check that $-x + \bar{x} + p_{\text{right}}(\bar{x};C) \leqslant \bar{g}(x)$ for $x \in [\bar{x}, m]$. Notice that $-x + \bar{x} + p_{\text{right}}(\bar{x};C)$ is a line with slope -1 and by Equation (19), $\bar{g}(x)$ is a line with slope $-R \geqslant -1$. Moreover, $-\bar{x} + \bar{x} + p_{\text{right}}(\bar{x};C) = p_{\text{right}}(\bar{x};C) = \max\{\underline{g}(\bar{x}), \widetilde{l}(\bar{x};C)\} \leqslant \bar{g}(x)$ since $[\widetilde{l}(\bar{x};C), u(\bar{x};C)] \cap [\underline{g}(\bar{x}), \bar{g}(\bar{x})] \neq \emptyset$. We have $p_{\text{right}}(x) \leqslant \bar{g}(x)$. By taking $R = \rho$ in Problem (C-Pareto-Trans), we can find $p_{\text{right}}(\cdot)$ is a feasible solution, and therefore, it achieves a robust ratio of at least ρ. By Ball and Queyranne (2009), we know ρ is the upper bound among all algorithms, and therefore, $p_{\text{right}}(\cdot)$ is optimal. In addition, notice that $p_{\text{right}}(m) = \underline{g}(m)$ and we can check $\text{CP}_u(p_{\text{right}}(\bar{x}); (m, m)) = \rho$, and hence $R_{\text{right}} = \min\{\text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)), \text{CP}_u(p_{\text{right}}(\bar{x}); (m, m))\}$.

- *Case 2:* $u(\bar{x};C) < \underline{g}(\bar{x})$. In this case, we first show $p_{\text{right}}(\cdot)$ achieves a robust ratio of $R_{\text{right}} = \text{CP}_u(p_{\text{right}}(\bar{x}); (\max\{m,\bar{x}\},m))$ and $\text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x},0)) \geqslant \text{CP}_u(p_{\text{right}}(\bar{x}); (\max\{m,\bar{x}\},m))$. Then, we show $p_{\text{right}}(\cdot)$ is feasible, and finally, we show it is optimal among all PL functions.

In this case, $p_{\text{right}}(x) = \arg\min_{p\in[\tilde{l}(\bar{x};C),u(\bar{x};C)]} |p - \underline{g}(\bar{x})| = u(\bar{x};C)$, and by definition, $p_{\text{right}}(x) = p_{\text{right}}(\bar{x})$ for $x \in [\bar{x}, \max\{m,\bar{x}\}]$. By Lemmas 2 and 10, we know the worst case is achieved on $(x,0)$ or (x,m) for some $x \in [\bar{x}, \max\{m,\bar{x}\}]$; that is, $R_{\text{right}} = \inf_{x\in[\bar{x},\max\{m,\bar{x}\}]} \min\{\text{CP}_u(p_{\text{right}}(x);(x,m)), \text{CP}_o(p_{\text{right}}(x);(x,0))\}$. As we have $p_{\text{right}}(x) \leqslant m$, by Lemma 7, we have

$$\text{CP}_u(p_{\text{right}}(x);(x,m)) \geqslant \text{CP}_u(p_{\text{right}}(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)).$$

As $p_{\text{right}}(\max\{m,\bar{x}\}) = p_{\text{right}}(\bar{x}) < \underline{g}(\bar{x}) = \underline{g}(\max\{m,\bar{x}\}) = m(1-r_\ell/r_h)/(2-r_\ell/r_h)$, where the first inequality is because $p_{\text{right}}(\bar{x}) \leqslant u(\bar{x};C) < \underline{g}(\bar{x})$, by Lemma 9, we have

$$\text{CP}_u(p_{\text{right}}(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)) < \text{CP}_u(\underline{g}(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)) = \rho.$$

As $p_{\text{right}}(\bar{x}) < \underline{g}(\bar{x})$, we also have $p_{\text{right}}(x) < \bar{g}(\bar{x})$ for any $x \in [\bar{x}, m]$. This is because by definition, for any $x \in [\bar{x}, m]$, $\underline{g}(x) \leqslant \bar{g}(x)$. Then, by Lemma 9, for $x \in [\bar{x}, m]$, we have

$$\text{CP}_o(p_{\text{right}}(x);(x,0)) \geqslant \text{CP}_o(\bar{g}(x);(x,0)) = \rho > \text{CP}_u(p_{\text{right}}(m);(m,m)),$$

where the equality is because $\bar{g}(x) = -\rho x + m$ and one can easily check $\text{CP}_o(-\rho x + m;(x,0)) = \rho$ for any $x \in [0,m]$.

Therefore, the robust ratio of $p_{\text{right}}(x)$ for $x \in [\bar{x}, m]$ is $R_{\text{right}} = \text{CP}_u(p_{\text{right}}(m);(m,m))$, and by Lemma 5, we have $\underline{g}(x;R_{\text{right}}) \leqslant p_{\text{right}}(x) \leqslant \bar{g}(x;R_{\text{right}})$. Also, $p_{\text{right}}(\cdot)$ is a constant function and is valid. We obtain $p_{\text{right}}(\cdot)$ is feasible.

Finally, we show that $p_{\text{right}}(\cdot)$ is optimal among all PL algorithms. We prove by contradiction. Suppose that a valid $p_1(x)$ can achieve a robust ratio greater than R_{right}. Then, we have

$$\inf_{x\in[\bar{x},m]} \min\{\text{CP}_o(p_1(x);(x,0)), \text{CP}_u(p_1(x);(x,m))\} > \text{CP}_u(p_{\text{right}}(m);(m,m)),$$

which implies that $\text{CP}_u(p_1(m);(m,m)) > \text{CP}_u(p_{\text{right}}(m);(m,m))$. By Lemma 9, we have $p_1(m) > p_{\text{right}}(m)$. As $p_1(\cdot)$ is valid, it is non-increasing, we have $p_1(\bar{x}) \geqslant p_1(m) > p_{\text{right}}(m) = u(\bar{x};C)$, which is a contradiction because $p_1(\bar{x}) > u(\bar{x};C)$ means it is infeasible.

- *Case 3:* $\widetilde{l}(\bar{x};C) > \bar{g}(\bar{x})$. In this case, $p_{\text{right}}(\bar{x}) = \arg\min_{p\in[\widetilde{l}(\bar{x};C),u(\bar{x};C)]} |p - \underline{g}(\bar{x})| = \widetilde{l}(\bar{x};C)$, and we have $p_{\text{right}}(x) = \bar{g}(x;R_{\text{right}})$ for $x \in [\bar{x}, \max\{m,\bar{x}\}]$ where $R_{\text{right}} = \text{CP}_o(\widetilde{l}(\bar{x};C);(\bar{x},0))$. Observe that $p_{\text{right}}(\cdot)$ is continuous at $\bar{x}$ because $p_{\text{right}}(\bar{x}) = \bar{g}(\bar{x};R_{\text{right}}) = \bar{g}(\bar{x};\text{CP}_o(\widetilde{l}(\bar{x};C);(\bar{x},0)))$, and if $\bar{x} \leqslant m$, we have $\bar{g}(\bar{x};\text{CP}_o(\widetilde{l}(\bar{x};C);(\bar{x},0))) = -\frac{m-\widetilde{l}(\bar{x};C)}{\bar{x}}\bar{x} + m = \widetilde{l}(\bar{x};C)$. Similarly, if $\bar{x} > m$, we have $\bar{g}(\bar{x};\text{CP}_o(\widetilde{l}(\bar{x};C);(\bar{x},0))) = -\frac{m-\widetilde{l}(\bar{x};C)}{m}m + m = \widetilde{l}(\bar{x};C)$. By Lemmas 2 and 10, for $x \in [\bar{x}, \max\{m,\bar{x}\}]$, the worst over- and under-protected points are $(x,0)$ and (x,m), respectively; that is, $R_{\text{right}} = \inf_{x\in[\bar{x},\max\{m,\bar{x}\}]} \min\{\text{CP}_o(p_{\text{right}}(x);(x,0)), \text{CP}_u(p_{\text{right}}(x);(x,m))\}$. So, we first claim that $p_{\text{right}}(\cdot)$ achieves a robust ratio of $R_{\text{right}} = \text{CP}_o(\widetilde{l}(\bar{x};C);(\bar{x},0))$ by showing that for any $x \in [\bar{x}, \max\{m,\bar{x}\}]$, $\text{CP}_o(p_{\text{right}}(x);(x,0)) \geqslant R_{\text{right}}$ and $\text{CP}_u(p_{\text{right}}(x);(x,m)) \geqslant R_{\text{right}}$. Then, we show its feasibility and optimality.

By Equation (6), we have for any $x \in [\bar{x}, \max\{m,\bar{x}\}]$,

$$\text{CP}_o(p_{\text{right}}(x);(x,0)) = \frac{0\cdot r_h + \min\{x, m - p_{\text{right}}(x)\}r_\ell}{0\cdot r_h + \min\{x, m-0\}r_\ell} = \frac{\min\{x, m - p_{\text{right}}(x)\}r_\ell}{\min\{x,m\}r_\ell}.$$

If $x \leqslant m$, we have

$$\begin{aligned}\frac{\min\{x, m - p_{\text{right}}(x)\}r_\ell}{\min\{x,m\}r_\ell} &= \frac{(m - p_{\text{right}}(x))r_\ell}{xr_\ell}\\ &= \frac{(m - \bar{g}(x;R_{\text{right}}))r_\ell}{xr_\ell}\\ &= \frac{m - \bar{g}(x;R_{\text{right}})}{x} = R_{\text{right}},\end{aligned}$$

where the second equality is because $p_{\text{right}}(x) = \bar{g}(x;R_{\text{right}}) = -R_{\text{right}}x + m$ and $m - (-R_{\text{right}}x + m) = R_{\text{right}}x \leqslant x$. If $x > m$, we have

$$\begin{aligned}\frac{\min\{x, m - p_{\text{right}}(x)\}r_\ell}{\min\{x,m\}r_\ell} &= \frac{(m - p_{\text{right}}(x))r_\ell}{mr_\ell}\\ &= \frac{(m - \bar{g}(x;R_{\text{right}}))r_\ell}{mr_\ell}\\ &= \frac{m - \bar{g}(m;R_{\text{right}})}{m} = R_{\text{right}}.\end{aligned}$$

For any $x \in [\bar{x}, \max\{m,\bar{x}\}]$, by Lemma 7, we have

$$\text{CP}_u(p_{\text{right}}(x);(x,m)) \geqslant \text{CP}_u(p_{\text{right}}(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)).$$

As we have $\text{CP}_u(\underline{g}(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)) = \rho$, and $p_{\text{right}}(\max\{m,\bar{x}\}) = \bar{g}(\max\{m,\bar{x}\}) \geqslant \underline{g}(\max\{m,\bar{x}\})$, by Lemma 9, we have

$$\text{CP}_u(p_{\text{right}}(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)) \geqslant \rho \geqslant R_{\text{right}}.$$

Therefore, the robust ratio of $p_{\text{right}}(x)$ is R_{right}. As $p_{\text{right}}(x) = \bar{g}(x; R_{\text{right}})$, we have $\underline{g}(x; R_{\text{right}}) \leqslant p_{\text{right}}(x) \leqslant \bar{g}(x; R_{\text{right}})$. Also, since $p_{\text{right}}(\bar{x}) = \tilde{l}(\bar{x}; C)$, we have $\tilde{l}(\bar{x}; C) \leqslant p_{\text{right}}(\bar{x}) \leqslant u(\bar{x}; C)$. In addition, $p_{\text{right}}(x)$ has slope $-R_{\text{right}} \leqslant -1$, which means it is valid. We have $p_{\text{right}}(\cdot)$ is a feasible solution.

Finally, we show that $p_{\text{right}}(\cdot)$ is optimal among all PL algorithms. We prove by contradiction. Suppose that a valid $p_1(x)$ can achieve a robust ratio greater than R_{right}. Then, we have

$$\inf_{x \in [\bar{x}, \max\{m, \bar{x}\}]} \min\{\text{CP}_o(p_1(x); (x, 0)), \text{CP}_u(p_1(x); (x, m))\} > \text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)),$$

which implies that $\text{CP}_o(p_1(\bar{x}); (\bar{x}, 0)) > \text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0))$. By Lemma 9, we have $p_1(\bar{x}) < p_{\text{right}}(\bar{x})$. However, as we have $p_{\text{right}}(\bar{x}) = \tilde{l}(\bar{x}; C)$, we obtain $p_1(\bar{x}) < \tilde{l}(\bar{x}; C)$, which means $p_1(\cdot)$ is infeasible and forms a contradiction.

□

C.7. Proof of Theorem 3

We first show that $(p_{\text{left}}(\cdot), R_{\text{left}})$ is a feasible solution to Problem (C-Pareto-left), where $p_{\text{left}}(x) = \max\{\tilde{l}(x; C), p_{\text{right}}(\bar{x})\}, x \in [0, \bar{x}]$ and $R_{\text{left}} = \min\{\text{CP}_u(p_{\text{left}}(\bar{x}); (\bar{x}, m))), \inf_{x \in [0, \bar{x}]} \text{CP}_o(p_{\text{left}}(x); (x, 0))\}$. Under the PL $p_{\text{left}}(\cdot)$, we first note that by Lemma 2, we only need to consider the points $(x, 0)$ and (x, m) for any $x \in [0, \bar{x}]$. That is,

$$R_{\text{left}} = \min\left\{\inf_{x \in [0, \bar{x}]} \text{CP}_o(p_{\text{left}}(x); (x, 0)), \inf_{x \in [0, \bar{x}]} \text{CP}_u(p_{\text{left}}(x); (x, m))\right\},$$

By Lemma 7, we then have

$$\text{CP}_u(p_{\text{left}}(x); (x, m)) \geqslant \text{CP}_u(p_{\text{left}}(\bar{x}); (\bar{x}, m)) \quad \Rightarrow \quad \inf_{x \in [0, \bar{x}]} \text{CP}_u(p_{\text{left}}(x); (x, m)) = \text{CP}_u(p_{\text{left}}(\bar{x}); (\bar{x}, m)).$$

This implies that $R_{\text{left}} = \min\{\text{CP}_u(p_{\text{left}}(\bar{x}); (\bar{x}, m)), \inf_{x \in [0, \bar{x}]} \text{CP}_o(p_{\text{left}}(x); (x, 0))\}$, as desired.

R_{left} is feasible because the range of compatible ratio is $[0, 1]$. To show that $p_{\text{left}}(x)$ is feasible, first, as we have shown it achieves a robust ratio of R_{left}, by Lemma 5, we have $\underline{g}(x; R_{\text{left}}) \leqslant p_{\text{left}}(x) \leqslant \bar{g}(x; R_{\text{left}})$. Second, we show that $p_{\text{left}}(\cdot)$ is a valid PL function and $p_{\text{left}}(x) \in [\tilde{l}(x; C), u(x; C)]$. Observe that $\tilde{l}(x; C)$ is a valid PL function by definition, and hence we have $p_{\text{left}}(x)$ is also valid. Third, we show that $\tilde{l}x; C \leqslant p_{\text{left}}(x) \leqslant u(x; C)$ for $x \in [0, \bar{x}]$. As $p_{\text{left}}(x) = \max\{\tilde{l}(x; C), p_{\text{right}}(\bar{x})\}$, we have $\tilde{l}(x; C) \leqslant p_{\text{left}}(x)$ for any $x \in [0, \bar{x}]$. In addition, we have $p_{\text{left}}(\bar{x}) = p_{\text{right}}(\bar{x}) \leqslant u(\bar{x}; C)$, and by Lemma 6, we have $u(x; C)$ is a non-increasing function in

x. This implies that $u(x;C) \geqslant p_{\text{left}}(\bar{x})$ for any $x \in [0,\bar{x}]$. Also, as C is assumed to be less than $C^\star(\mathcal{R})$, by Lemma 4, we have $\widetilde{l}(x;C) \leqslant u(x;C)$ for $x \in [0,\bar{x}]$. Therefore, we have $p_{\text{left}}(x) = \max\{p_{\text{left}}(\bar{x}), \widetilde{l}(x;C)\} \leqslant u(x;C)$, which is the desired result.

Second, we show that $p_{\text{left}}(\cdot)$ is optimal. To do so, we argue that (i) by Lemma 5, we have

$$\underline{g}(x;R_{\text{left}}) \leqslant p_{\text{left}}(x) \leqslant \bar{g}(x;R_{\text{left}}),$$

and (ii) there does not exist any other valid PL function that achieves a higher robust ratio than R_{left} while satisfying the consistency lower and upper bounds.

Recall that $R_{\text{left}} = \min\{\text{CP}_u(p_{\text{left}}(\bar{x});(\bar{x},m)), \inf_{x\in[0,\bar{x}]}\text{CP}_o(p_{\text{left}}(x);(x,0))\}$. As Problem (C-Pareto-left) restricts the value of $p(\bar{x})$, we have any PL algorithm has the same worst under-protected ratio, i.e. $\text{CP}_u(p_{\text{left}}(\bar{x});(\bar{x},m))$. Then, we show that any PL algorithm cannot get a larger worst over-protected ratio. For over-protected case, we let

$$\widetilde{R_{\text{left}}} = \inf_{x\in[0,\bar{x}]}\text{CP}_o(p_{\text{left}}(x);(x,0)),$$

and let the infimum is achieved on x_1, i.e. $\text{CP}_o(p_{\text{left}}(x_1);(x_1,0)) = \widetilde{R_{\text{left}}}$. If there exists a PL algorithm $p(x)$ such that $\inf_{x\in[0,\bar{x}]}\text{CP}_o(p(x);(x,0)) > R_{\text{left}}$, then $\text{CP}_o(p(x_1);(x_1,0)) > \text{CP}_o(p_{\text{left}}(x_1);(x_1,0))$. By Lemma 9, we have $p(x_1) < p_{\text{left}}(x_1)$. This implies that either $p(x_1) < p_{\text{right}}(\bar{x})$ or $p(x_1) < \tilde{\ell}(x;C)$. However, if $p(x_1) < p_{\text{right}}(\bar{x})$, as $p(\cdot)$ is non-increasing, we have $p(\bar{x}) < p_{\text{right}}(\bar{x})$, which implies $p(\cdot)$ is infeasible. If $p(x_1) < \tilde{\ell}(x;C)$, this immediately contradicts to $\tilde{\ell}(x;C) \leqslant p(x) \leqslant u(x;C)$. Therefore, such $p(x)$ does not exist.

□

Appendix D: Proof of Theorem 1

The proof is naturally divided into three parts.

D.1. Result 1: $R^\star = \min\{R_{\textbf{right}}, R_{\textbf{left}}\}$

First, we show that $R^\star = \min\{R_{\text{right}}, R_{\text{left}}\}$, where Theorems 2 and 3 show that

$$R_{\text{left}} = \min\{\text{CP}_u(p_{\text{left}}(\bar{x});(\bar{x},m)), \inf_{x\in[0,\bar{x}]}\text{CP}_o(p_{\text{left}}(x);(x,0))\}.$$

and

$$R_{\text{right}} = \min\{\text{CP}_o(p_{\text{right}}(\bar{x});(\bar{x},0)), \text{CP}_u(p_{\text{right}}(\bar{x});(\max\{m,\bar{x}\},m))\}.$$

Let us denote $\widetilde{R_{\text{left}}} = \inf_{x\in[0,\bar{x}]} \text{CP}_o(p_{\text{left}}(x);(x,0))$, and note that by Lemma 7, we have $\text{CP}_u(p_{\text{right}}(\bar{x});(\bar{x},m)) \geqslant \text{CP}_u(p_{\text{right}}(\bar{x});(\max\{m,\bar{x}\},m))$, where by construction, we have $\text{CP}_u(p_{\text{right}}(\bar{x});(\bar{x},m)) = \text{CP}_u(p_{\text{left}}(\bar{x});(\bar{x},m))$. Therefore, we have

$$\begin{aligned}
&\min\{R_{\text{right}}, R_{\text{left}}\}\\
&= \min\Big\{\text{CP}_o(p_{\text{right}}(\bar{x});(\bar{x},0)), \text{CP}_u(p_{\text{right}}(\bar{x});(\max\{m,\bar{x}\},m)), \text{CP}_u(p_{\text{left}}(\bar{x});(\bar{x},m)), \inf_{x\in[0,\bar{x}]} \text{CP}_o(p_{\text{left}}(x);(x,0))\\
&= \min\Big\{\text{CP}_o(p_{\text{right}}(\bar{x});(\bar{x},0)), \text{CP}_u(p_{\text{right}}(\bar{x});(\max\{m,\bar{x}\},m)), \text{CP}_u(p_{\text{right}}(\bar{x});(\bar{x},m)), \inf_{x\in[0,\bar{x}]} \text{CP}_o(p_{\text{left}}(x);(x,0)\\
&= \min\Big\{\text{CP}_o(p_{\text{right}}(\bar{x});(\bar{x},0)), \text{CP}_u(p_{\text{right}}(\bar{x});(\max\{m,\bar{x}\},m)), \inf_{x\in[0,\bar{x}]} \text{CP}_o(p_{\text{left}}(x);(x,0))\Big\} = \min\{R_{\text{right}}, \widetilde{R_{\text{left}}}\},
\end{aligned}$$

where the second equality is because $p_{\text{right}}(\bar{x}) = p_{\text{left}}(\bar{x})$, and the third equality is because $\text{CP}_u(p_{\text{right}}(\bar{x});(\bar{x},m)) \geqslant \text{CP}_u(p_{\text{right}}(\bar{x});(\max\{m,\bar{x}\},m))$.

Therefore, showing $R^\star = \min\{R_{\text{right}}, R_{\text{left}}\}$ is equivalent to show that $R^\star = \min\{R_{\text{right}}, \widetilde{R_{\text{left}}}\}$. As Theorems 2 and 3 show, if we set $p(\bar{x})$ optimally, our $p^\star(\cdot)$, presented in Algorithm 1, achieves an optimal robust ratio. Therefore, it suffices to show that for any valid $\widehat{p}(x)$ such that $\widehat{p}(\bar{x}) \neq p_{\text{right}}(\bar{x})$, we have $\text{ROBUST}(p(\cdot)) \leqslant \text{ROBUST}(p^\star(\cdot))$. Here, with a slight abuse of notation, $\text{ROBUST}(p(\cdot))$ is the robust ratio of a PLA with PL of $p(\cdot)$. We split the analysis into two cases based on the value of $\widetilde{R_{\text{left}}}$ and R_{right}, where in case 1, we have $R_{\text{right}} \leqslant \widetilde{R_{\text{left}}}$, and in case 2, we have $R_{\text{right}} > \widetilde{R_{\text{left}}}$.

- $R_{\text{right}} \leqslant \widetilde{R_{\text{left}}}$. In this case, $\text{ROBUST}(p^\star(\cdot)) = \min\{\widetilde{R_{\text{left}}}, R_{\text{right}}\} = R_{\text{right}}$. By Theorem 2, we know that no PL can achieve a robust ratio greater than R_{right} for $x \in [\bar{x}, \max\{m,\bar{x}\}]$. Therefore, we have $\text{ROBUST}(\widehat{p}(\cdot)) \leqslant \text{ROBUST}(p^\star(\cdot)) = R_{\text{right}}$, which is the desired result.
- $\widetilde{R_{\text{left}}} < R_{\text{right}}$. As is shown in Theorem 3, by fixing $p^\star(\bar{x}) = p_{\text{right}}(\bar{x})$, no PL can achieve a robust ratio greater than $\widetilde{R_{\text{left}}}$. Then, in this part, we show that if a valid and feasible PL function $\hat{p}(x)$ for $x \in [0,\bar{x}]$ does not have restriction on $\bar{x}$, it can still not achieve a robust ratio greater than $\widetilde{R_{\text{left}}}$. To show this, we define $\widehat{x} \in [0,\bar{x}]$ as such that $\widetilde{l}(\widehat{x};C) = p_{\text{right}}(\bar{x})$. We start with showing that such $\widehat{x}$ always exists, and then we show that $p_{\text{left}}(\cdot)$ achieves $\widetilde{R_{\text{left}}}$ in $[0,\widehat{x}]$ and no PL function $\hat{p}(x)$ can outperform $\widetilde{R_{\text{left}}}$ in $[0,\widehat{x}]$.

  To show the existence, we use a contradiction argument. Contrary to our claim, suppose that there does not exist any $\widehat{x} \in [0,\bar{x}]$ such that $\widetilde{l}(\widehat{x};C) = p_{\text{right}}(\bar{x})$. Observe that $p_{\text{right}}(\bar{x})$ is a constant and is greater than $\widetilde{l}(\bar{x};C)$ by feasibility of $p_{\text{right}}(\cdot)$. Further, note that by Lemma 16, $\widetilde{l}(x;C)$ is non-increasing in x. Then, when $\widehat{x}$ does not exist, we must have $p_{\text{right}}(\bar{x}) > \widetilde{l}(x;C)$

for any $x \in [0, \bar{x}]$. By Theorem 3, in this case, $p_{\text{left}}(x) = p_{\text{right}}(\bar{x})$ is a constant function. (Recall that $p_{\text{left}}(x) = \max\{\widetilde{l}(x;C), p_{\text{right}}(\bar{x})\}$ for any $x \in [0, \bar{x}]$.) By Lemma 18, we have

$$\widetilde{R_{\text{left}}} = \inf_{x \in [0,\bar{x}]} \text{CP}_o(p_{\text{right}}(\bar{x}); (x, 0)) = \text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)).$$

However, as $\text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)) \geqslant \min\{\text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)), \text{CP}_u(p_{\text{right}}(\bar{x}); (\max\{m, \bar{x}\}, m))\}$, we have $\widetilde{R_{\text{left}}} \geqslant R_{\text{right}}$, which is a contradiction. Therefore, such $\widehat{x}$ exists.

Then, $p_{\text{left}}(x) = p_{\text{right}}(\bar{x})$ for $x \in [\widehat{x}, \bar{x}]$ and $p_{\text{left}}(x) = \widetilde{l}(x;C)$ for $x \in [0, \widehat{x}]$. By definition of $\widetilde{R_{\text{left}}}$, we have

$$\widetilde{R_{\text{left}}} = \min\{\inf_{x \in [0,\widehat{x}]} \text{CP}_o(\widetilde{l}(x;C); (x, 0)), \inf_{x \in [\widehat{x},\bar{x}]} \text{CP}_o(p_{\text{right}}(\bar{x}); (x, 0))\}.$$

By Lemma 18, we have

$$\inf_{x \in [\widehat{x},\bar{x}]} \text{CP}_o(p_{\text{right}}(\bar{x}); (x, 0)) = \text{CP}_o(p_{\text{right}}(\bar{x}); (\bar{x}, 0)) \geqslant R_{\text{right}}.$$

Given that $\widetilde{R_{\text{left}}} < R_{\text{right}}$, we have

$$\widetilde{R_{\text{left}}} = \inf_{x \in [0,\widehat{x}]} \text{CP}_o(\widetilde{l}(x;C); (x, 0)).$$

Finally, we show that no valid and feasible PL $\hat{p}(x)$ can outperform $\widetilde{R_{\text{left}}}$ in $[0, \widehat{x}]$. Suppose that $x_1 = \arg\min_{x \in [0,\widehat{x}]} \text{CP}_o(\widetilde{l}(x;C); (x, 0))$, which means that $\text{CP}_o(\widetilde{l}(x_1;C); (x_1, 0)) = \widetilde{R_{\text{left}}}$. If $\hat{p}(x)$ outperforms $\widetilde{R_{\text{left}}}$, we have

$$\inf_{x \in [0,\widehat{x}]} \text{CP}_o(\hat{p}(x); (x, 0)) > \widetilde{R_{\text{left}}},$$

which implies that $\text{CP}_o(\hat{p}(x_1); (x_1, 0)) > \widetilde{R_{\text{left}}}$. By Lemma 9, we have $\hat{p}(x_1) < \widetilde{l}(x_1;C)$, which shows that $\hat{p}$ is not a feasible PL function and forms a contradiction.

□

D.2. Result 2: $p^\star$ is an Optimal Solution

Second, it is trivial that Algorithm 1 presents an optimal solution to Problem (C-Pareto-Trans). The reason is by Theorem 2, we have $p_{\text{right}}(x)$ achieves R_{right} for Problem (C-Pareto-right) and by Theorem 3, we have $p_{\text{left}}(x)$ achieves R_{left} for Problem (C-Pareto-left). By Equation (20), we know that $\text{ROBUST}(p^\star(\cdot)) = \min\{R_{\text{right}}, R_{\text{left}}\}$, and we just showed that $\min\{R_{\text{right}}, R_{\text{left}}\} = R^\star$. The remaining thing is to show that $p^\star(\cdot)$ is feasible and valid. As we have shown in Theorems 2 and 3 that $p_{\text{right}}(\cdot)$ and $p_{\text{left}}(\cdot)$ are both feasible and valid for any $x \geqslant \bar{x}$ and $x \in [0, \bar{x}]$, respectively. Further, $p_{\text{right}}(\bar{x}) = p_{\text{left}}(\bar{x})$, which means that $p^\star(\cdot)$ is continuous. Therefore, $p^\star(\cdot)$ is feasible.

D.3. Result 3: No Algorithm Can Outperform $p^\star$

Here, we show that a PLA with the PL function of $p^\star(\cdot)$ is an optimal solution to Problem (3). That is, among any online algorithms Π, the aforementioned algorithm maximizes the robust ratio while ensuring its consistent ratio is at least C. Recall that we just showed

$$R^\star = \min\{R_{\text{right}}, R_{\text{left}}\} = \min\{\text{CP}_u(p^\star(\max\{m,\bar{x}\}); (\max\{m,\bar{x}\}, m)), \text{CP}_o(p^\star(\bar{x}), (\bar{x}, 0)), \inf_{x\in[0,\bar{x}]} \text{CP}_o(p^\star(x); (x \tag{47}$$

Then, we split the proof into three parts, where in each parts, we discuss each term in Equation (47) is the minimum value.

Part 1: $R^\star = CP_u(p^\star(\max\{m,\bar{x}\}); (\max\{m,\bar{x}\}, m))$. Here, we show that no deterministic or randomized algorithm can achieve a robust ratio more than $\text{CP}_u(p^\star(\max\{m,\bar{x}\}); (\max\{m,\bar{x}\}, m))$. We define two (ordered) input sequences: In the first input sequence, I_1, $\bar{x}_u \leqslant \bar{x}$ low-reward requests arrive first, followed with $\underline{h}(\bar{x}_u)$ high-reward requests. In the second input sequence, I_2, $\max\{m,\bar{x}\}$ low-reward requests arrive first, followed m high-reward requests. Before receiving $\bar{x}_u$ low reward requests, any deterministic or randomized algorithm cannot differentiate the two input sequences and has to decide to accept how many low-reward requests in expectation. If there exists a deterministic or randomized algorithm $\mathcal{A}$, which can achieve a consistent ratio of at least C, and a robust ratio higher than $\text{CP}_u(p^\star(\max\{m,\bar{x}\}); (\max\{m,\bar{x}\}, m))$, it should satisfy

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_1)]}{\text{OPT}(I_1)} \geqslant C,$$

Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A}, I_1)$, $\ell(\mathcal{A}, I_1)$, respectively. Then,

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_1)]}{\text{OPT}(I_1)} = \frac{h(\mathcal{A}, I_1)r_h + \ell(\mathcal{A}, I_1)r_\ell}{\underline{h}(\bar{x}_u)r_h + \min\{\bar{x}_u, m - \underline{h}(\bar{x}_u)\}r_\ell} \geqslant C.$$

By the definition of $u(\bar{x}_u; C)$, we have $u(\bar{x}_u; C) = \sup\{p : \text{CP}_o(p; (\bar{x}_u, \underline{h}(\bar{x}_u))) = C\}$, and by Equation (6), $\frac{\min\{\underline{h}(\bar{x}_u),m\}r_h + \min\{\bar{x}_u, m-u(\bar{x}_u;C)\}r_\ell}{\min\{\underline{h}(\bar{x}_u),m\}r_h + \min\{\bar{x}_u,(m-\underline{h}(\bar{x}_u))^+\}r_\ell} = C$, and by Lemma 19, we have $\frac{\min\{\underline{h}(\bar{x}_u),m\}r_h + (m-u(\bar{x}_u;C))r_\ell}{\min\{\underline{h}(\bar{x}_u),m\}r_h + \min\{\bar{x}_u,(m-\underline{h}(\bar{x}_u))^+\}r_\ell} = C$. Therefore, we have

$$\frac{h(\mathcal{A}, I_1)r_h + \ell(\mathcal{A}, I_1)r_\ell}{\min\{\underline{h}(\bar{x}_u), m\}r_h + \min\{\bar{x}_u, (m - \underline{h}(\bar{x}_u))^+\}r_\ell} \geqslant \frac{\min\{\underline{h}(\bar{x}_u), m\}r_h + (m - u(\bar{x}_u; C))r_\ell}{\min\{\underline{h}(\bar{x}_u), m\}r_h + \min\{\bar{x}_u, (m - \underline{h}(\bar{x}_u))^+\}r_\ell},$$

as $h(\mathcal{A}, I_1) \leqslant \min\{\underline{h}(\bar{x}_u), m\}$, we have $\ell(\mathcal{A}, I_1) \geqslant m - u(\bar{x}_u; C)$, which implies that we should accept at least $m - u(\bar{x}_u; C)$ low-reward requests in expectation.

However, to achieve a robust ratio higher than $\text{CP}_u(p^\star(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m))$ for sequence I_2, we have

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_2)]}{\text{OPT}(I_2)} > \text{CP}_u(p^\star(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m)).$$

By Equation (7), we have

$$\begin{aligned}\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_2)]}{\text{OPT}(I_2)} &= \frac{h(\mathcal{A},I_2)r_h+\ell(\mathcal{A},I_2)r_\ell}{mr_h} > \text{CP}_u(p^\star(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m))\\ &= \frac{p^\star(\max\{m,\bar{x}\})r_h+(m-p^\star(\max\{m,\bar{x}\}))r_\ell}{mr_h},\end{aligned}$$

as $h(\mathcal{A},I_2)\leqslant(m-\ell(\mathcal{A}),I_2)$, we have $\ell(\mathcal{A},I_2)<m-p^\star(\max\{m,\bar{x}\})$, which implies that we should accept less than $m-p^\star(\max\{m,\bar{x}\})$ low-reward requests in expectation. However, when $R^\star=\text{CP}_u(p^\star(\max\{m,\bar{x}\});(\max\{m,\bar{x}\},m))$, Part 1 of the proof of Theorem 2 shows that $p^\star(\max\{m,\bar{x}\})=p_{\text{right}}(\max\{m,\bar{x}\})=u(\bar{x};C)$. By Equation (13), we have $u(\bar{x};C)=u(\bar{x}_u;C)$. Therefore, under sequence I_2, we have $\ell(\mathcal{A},I_2)<m-u(\bar{x}_u;C)$, which is a contradiction because once we observed $\bar{x}_u$ low-reward requests, we have already accept at least $m-u(\bar{x}_u;C)$ of them in expectation.

□

Part 2: $R^\star=\mathit{CP}_o(p^\star(\bar{x}),(\bar{x},0))$. To show the result, we split the analysis into two sub parts based on the value of $p^\star(\bar{x})$ and $\widetilde{l}(x_{-1};C)$. First, we show that when $l(x_{-1};C)>p^\star(\bar{x})$, no deterministic or randomized algorithm can achieve a robust ratio more than $\text{CP}_o(p^\star(\bar{x}),(\bar{x},0))$. Here, we recall that $x_{-1}=\sup\{x\in[x_H,\bar{x}]:\frac{\partial l(x^-;C)}{\partial x}\leqslant -1\}$. Then, we show the same result when $l(x_{-1};C)\leqslant p^\star(\bar{x})$.

For the case where $l(x_{-1};C)>p^\star(\bar{x})$, we again define two (ordered) input sequences: In the first input sequence, I_1, x_{-1} low-reward requests arrive first, followed with $\bar{h}(x_{-1})$ high-reward requests. In the second input sequence, I_2, $\bar{x}$ low-reward requests arrive first, followed by 0 high-reward requests. Before receiving x_{-1} low reward requests, any deterministic or randomized algorithm cannot differentiate the two input sequences and has to decide to accept how many low-reward requests in expectation. If there exists a deterministic or randomized algorithm $\mathcal{A}$, which can achieve a consistent ratio of at least C, and a robust ratio more than $\text{CP}_o(p^\star(\bar{x}),(\bar{x},0))$, it should satisfy

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_1)]}{\text{OPT}(I_1)}\geqslant C,$$

Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A}, I_1)$, $\ell(\mathcal{A}, I_1)$, respectively. Then,

$$\frac{\mathbf{E}[\mathrm{Rew}(\mathcal{A}, I_1)]}{\mathrm{OPT}(I_1)} = \frac{h(\mathcal{A}, I_1)r_h + \ell(\mathcal{A}, I_1)r_\ell}{\bar{h}(x_{-1})r_h + \min\{x_{-1}, m - \bar{h}(x_{-1})\}r_\ell} \geqslant C.$$

By Equation (15), we have $\mathrm{CP}_u(l(x_{-1}; C); (x_{-1}, \bar{h}(x_{-1})) = C$. By Equation (7), we have

$$\mathrm{CP}_u(l(x_{-1}; C); (x_{-1}, \bar{h}(x_{-1})) = \frac{\max\{l(x_{-1}; C), \min\{\bar{h}(x_{-1}), m - x_{-1}\}\}r_h + \min\{x_{-1}, m - l(x_{-1}; C)\}r_\ell}{\bar{h}(x_{-1})r_h + \min\{\bar{x}_u, m - \bar{h}(x_{-1})\}r_\ell}.$$

By Lemma 19, we have $\max\{l(x_{-1}; C), \min\{\bar{h}(x_{-1}), m - x_{-1}\}\} = l(x_{-1}; C)$ and $\min\{x_{-1}, m - l(x_{-1}; C)\} = m - l(x_{-1}; C)$. Therefore, we have

$$\mathrm{CP}_u(l(x_{-1}; C); (x_{-1}, \bar{h}(x_{-1})) = \frac{l(x_{-1}; C)r_h + (m - l(x_{-1}; C))r_\ell}{\bar{h}(x_{-1})r_h + \min\{\bar{x}_u, m - \bar{h}(x_{-1})\}r_\ell} = C,$$

which implies that

$$\frac{h(\mathcal{A}, I_1)r_h + \ell(\mathcal{A}, I_1)r_\ell}{\bar{h}(x_{-1})r_h + \min\{\bar{x}_u, m - \bar{h}(x_{-1})\}r_\ell} \geqslant \frac{l(x_{-1}; C)r_h + (m - l(x_{-1}; C))r_\ell}{\bar{h}(x_{-1})r_h + \min\{\bar{x}_u, m - \bar{h}(x_{-1})\}r_\ell}.$$

As $h(\mathcal{A}, I_1) \leqslant m - \ell(\mathcal{A}, I_1)$, we have $\ell(\mathcal{A}, I_1) \leqslant m - l(x_{-1}; C)$, which implies that it should accept no more than $m - l(x_{-1}; C)$ low-reward requests in expectation. However, to achieve a robust ratio more than $\mathrm{CP}_o(p^\star(\bar{x}), (\bar{x}, 0))$ for sequence I_2, we have

$$\frac{\mathbf{E}[\mathrm{Rew}(\mathcal{A}, I_2)]}{\mathrm{OPT}(I_2)} > \mathrm{CP}_o(p^\star(\bar{x}), (\bar{x}, 0)).$$

Let the expected total amount of low-reward requests being accepted under I_2 by $\mathcal{A}$ be $\ell(\mathcal{A}, I_2)$, respectively. Then,

$$\frac{\mathbf{E}[\mathrm{Rew}(\mathcal{A}, I_1)]}{\mathrm{OPT}(I_1)} = \frac{\ell(\mathcal{A})}{\bar{x}} > \mathrm{CP}_o(p^\star(\bar{x}), (\bar{x}, 0)).$$

By Equation (6), we have

$$\mathrm{CP}_o(p^\star(\bar{x}), (\bar{x}, 0)) = \frac{\min\{\bar{x}, m - p^\star(\bar{x})\}}{\bar{x}}.$$

If $\bar{x} \leqslant m - p^\star(\bar{x})$, we have $\mathrm{CP}_o(p^\star(\bar{x}), (\bar{x}, 0)) = 1$, which means no algorithm can achieve a higher robust ratio. Otherwise, we have

$$\mathrm{CP}_o(p^\star(\bar{x}), (\bar{x}, 0)) = \frac{m - p^\star(\bar{x})}{\bar{x}},$$

and we have

$$\frac{\mathbf{E}[\mathrm{Rew}(\mathcal{A}, I_1)]}{\mathrm{OPT}(I_1)} = \frac{\ell(\mathcal{A}, I_2)}{\bar{x}} > \frac{m - p^\star(\bar{x})}{\bar{x}},$$

which implies that $\ell(\mathcal{A}) > m - p^\star(\bar{x})$, and we should accept more than $m - p^\star(\bar{x})$ low-reward requests in expectation. However, when $R^\star = \text{CP}_o(p^\star(\bar{x}), (\bar{x}, 0))$, by the proof of Theorem 2, we have in this case $p^\star(\bar{x}) = p_{\text{right}}(\bar{x}) = \widetilde{l}(\bar{x}; C)$. However, given that $l(x_{-1}; C) > p^\star(\bar{x}) = \widetilde{l}(\bar{x}; C)$, we have $x_{-1} < \bar{x}$. Otherwise, $l(x_{-1}; C) = l(\bar{x}; C) = \widetilde{l}(\bar{x}; C)$. Between x_{-1} and $\bar{x}$, there are at most $\bar{x} - x_{-1}$ low-reward requests arriving, and any algorithm can accept at most $\bar{x} - x_{-1}$ low-reward requests. But we accept no more than $m - l(x_{-1}; C)$ low-reward requests in expectation under I_1 and more than $m - p^\star(\bar{x})$ low-reward requests in expectation under I_2, and since $\widetilde{l}(x; C)$ is a line with slope -1 for $x \geqslant x_{-1}$, we have $m - p^\star(\bar{x}) - (m - l(x_{-1}; C)) = \widetilde{l}(x_{-1}; C) - \widetilde{l}(\bar{x}; C) = \bar{x} - x_{-1}$, which is a contradiction.

Now, let us consider the case where $l(x_{-1}; C) \leqslant p^\star(\bar{x})$, where we recall that here by assumption $R^\star = \text{CP}_o(p^\star(\bar{x}), (\bar{x}, 0))$. By the proof of Theorem 2, when $R_{\text{right}} = \text{CP}_o(p^\star(\bar{x}), (\bar{x}, 0))$, we have $p^\star(\bar{x}) = p_{\text{right}}(\bar{x}) = \widetilde{l}(\bar{x}; C)$. In addition, we have $l(x_{-1}; C) = \widetilde{l}(x_{-1}; C)$, and $l(x_{-1}; C) \leqslant p^\star(\bar{x})$ is equivalent to $\widetilde{l}(x_{-1}; C) \leqslant \widetilde{l}(\bar{x}; C)$. Due to $\widetilde{l}(\cdot; C)$ is non-increasing, we have $x_{-1} = \bar{x}$ in this case.

To show the result, we again define two (ordered) input sequences: In the first input sequence, I_1, $\bar{x}$ low-reward requests arrive first, followed by $\bar{h}(\bar{x})$ high-reward requests. In the second input sequence, I_2, $\bar{x}$ low-reward requests arrive first, followed by 0 high-reward requests. If there exists a deterministic or randomized algorithm $\mathcal{A}$, which can achieve a consistent ratio of at least C, and a robust ratio more than $\text{CP}_o(p^\star(\bar{x}), (\bar{x}, 0))$, it should satisfy

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_1)]}{\text{OPT}(I_1)} \geqslant C.$$

Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A}, I_1)$, $\ell(\mathcal{A}, I_1)$, respectively. Then,

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_1)]}{\text{OPT}(I_1)} = \frac{h(\mathcal{A}, I_1) r_h + \ell(\mathcal{A}, I_1) r_\ell}{\bar{h}(\bar{x}) r_h + \min\{\bar{x}, m - \bar{h}(\bar{x})\} r_\ell} \geqslant C.$$

By Equation (17), we have $\text{CP}_u(l(\bar{x}; C); (\bar{x}, \bar{h}(\bar{x}))) = C$. By Equation (7),

$$\text{CP}_u(l(\bar{x}; C); (\bar{x}, \bar{h}(\bar{x}))) = \frac{\max\{l(\bar{x}; C), \min\{\bar{h}(\bar{x}), m - \bar{x}\} r_h + \min\{\bar{x}, m - l(\bar{x}; C)\} r_\ell}{\bar{h}(\bar{x}) r_h + \min\{\bar{x}, m - \bar{h}(\bar{x})\} r_\ell}.$$

As we have shown $\bar{x} = x_{-1}$ at the beginning of this case, by Lemma 19, we have

$\max\{l(\bar{x}; C), \min\{\bar{h}(\bar{x}), m - \bar{x}\} = l(\bar{x}; C)$ and $\min\{\bar{x}, m - l(\bar{x}; C)\} = m - l(\bar{x}; C)$. Therefore, we have

$$\frac{l(\bar{x}; C) r_h + (m - l(\bar{x}; C)) r_\ell}{\bar{h}(\bar{x}) r_h + \min\{\bar{x}, m - \bar{h}(\bar{x})\} r_\ell} = C,$$

and this implies

$$\frac{h(\mathcal{A},I_1)r_h+\ell(\mathcal{A},I_1)r_\ell}{\bar{h}(\bar{x})r_h+\min\{\bar{x},m-\bar{h}(\bar{x})\}r_\ell}\geqslant\frac{l(\bar{x};C)r_h+(m-l(\bar{x};C))r_\ell}{\bar{h}(\bar{x})r_h+\min\{\bar{x},m-\bar{h}(\bar{x})\}r_\ell}.$$

As $h(\mathcal{A},I_1)=m-\ell(\mathcal{A},I_1)$, we have $\ell(\mathcal{A},I_1)\leqslant m-l(\bar{x};C)$, which implies that we should accept no more than $m-l(\bar{x};C)$ low-reward requests in expectation. However, to achieve a robust ratio more than the $\text{CP}_o(p^\star(\bar{x}),(\bar{x},0))$ for sequence I_2, we have

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_2)]}{\text{OPT}(I_2)}>\text{CP}_o(p^\star(\bar{x}),(\bar{x},0)).$$

This implies that we should accept more than $m-p^\star(\bar{x})$ low-reward requests in expectation, which is shown in the previous case. However, as is shown at the beginning of this case, here, $\bar{x}=x_{-1}$, and $p^\star(\bar{x})=l(\bar{x};C)$, which shows that upon receiving $\bar{x}$ low-reward requests, we should accept no more than $m-p^\star(\bar{x})$ low-reward requests and more than $m-p^\star(\bar{x})$ low-reward requests, which is obviously a contradiction.

Part 3: $R^\star=\inf_{x\in[0,\bar{x}]}CP_o(p^\star(x);(x,0))$. Define $\widehat{x}=\text{argmin}_{x\in[0,\bar{x}]}\text{CP}_o(p^\star(x);(x,0))$. If $\widehat{x}$ is not unique, we randomly the one with smallest x value. Let us first consider the case where $l(x_{-1};C)>p^\star(\bar{x})$ and $\widehat{x}\in[x_{-1},x_{-1}+l(x_{-1};C)-p^\star(\bar{x})]$. In this case, we replace I_2 in the proof of Part 2 under the case where $l(x_{-1};C)>p^\star(\bar{x})$ by: we define I_2 as $\widehat{x}$ low-reward requests arrive first, followed 0 high-reward requests, and we can have $p^\star(x)$ is optimal among all algorithms in this case.

For any other cases, we replace I_1, I_2 in the proof of Part 2 under the case where $l(x_{-1};C)\leqslant p^\star(\bar{x})$ by: we define I_1 as $\widehat{x}$ low-reward requests arrive first, followed $\underline{h}(\widehat{x})$ high-reward requests; we define I_2 as $\widehat{x}$ low-reward requests arrive first, followed 0 high-reward requests, and we can have $p^\star(x)$ is optimal among all algorithms in this case.

Therefore, we have $p^\star(x)$ for $x\in[0,m]$ is optimal among all deterministic and randomized algorithms.

□

D.4. Lemma 7 and its Proof

LEMMA 7. *Given an arbitrary valid PL function $p(\cdot)$, we have for any $x\in[0,\max\{m,\bar{x}\}]$, $CP_u(p(x);(x,m))$ is a non-incresing function in x. That is, for any $x\in[0,\max\{m,\bar{x}\}]$, with $p(x)\leqslant m$,*

$$CP_u(p(m);(\max\{m,\bar{x}\},m))\leqslant CP_u(p(x);(x,m)).$$

Proof of Lemma 7 Take any valid PL function $p(x)$ for $x \in [0,m]$, by Equation (7),

$$\begin{aligned}\mathrm{CP}_u(p(x);(x,m)) &= \frac{\max\{p(x),\min\{m,m-x\}\}r_h + \min\{x,m-p(x)\}r_\ell}{mr_h + \min\{x,m-m\}r_\ell} \\ &= \frac{\max\{p(x),m-x\}r_h + \min\{x,m-p(x)\}r_\ell}{mr_h}.\end{aligned}$$

Let $x_1 = \sup\{x : p(x)+x \leqslant m\}$, first, as a valid PL function satisfies that $p'(x) \geqslant -1$ a.e., and the line $x+y=m$ has slope -1, once $x+p(x) \geqslant m$, we have for all $x' > x$, we have $x'+p(x') \geqslant m$. Therefore, for $x \in [0,x_1]$, we have $p(x)+x \leqslant m$, and

$$\mathrm{CP}_u(p(x);(x,m)) = \frac{(m-x)r_h + xr_\ell}{mr_h},$$

which is a monotone decreasing function with x.

For $x \in [x_1,m]$,

$$\mathrm{CP}_u(p(x);(x,m)) = \frac{p(x)r_h + (m-p(x))r_\ell}{mr_h},$$

which is also a non-increasing function with x due to $p(x)$ is non-increasing. Therefore, it is monotone decreasing, and we have for any valid PL function $p(x)$,

$$\mathrm{CP}_u(p(m);(m,m)) \leqslant \mathrm{CP}_u(p(x);(x,m)),$$

for $x \in [0,m]$. If $\bar{x} > m$, by Lemma 10, we have $\mathrm{CP}_u(p(\bar{x});(\bar{x},m)) = \mathrm{CP}_u(p(x);(x,m))$. Therefore, we have

$$\mathrm{CP}_u(p(m);(\max\{m,\bar{x}\},m)) \leqslant \mathrm{CP}_u(p(x);(x,m)),$$

for any $x \in [0,\max\{m,\bar{x}\}]$. □

Appendix E: Proof of Statements in Section 6

In this section, we provide the proof of statements in Section 6.In Section E.1, we prove Proposition 1 and shows the computational complexity and accuracy of the bisection algorithm. In Section E.2, we prove Theorem 5, which provides several properties of an optimal $C^\star(\mathcal{R})$. In Section E.3, we prove Theorem 6, which states that Algorithm 5 returns $C^\star(\mathcal{R})$. Finally, in Section E.4, we prove Theorem 4, which shows that the optimal PL function is optimal among all deterministic and randomized algorithms.

E.1. Proof of Proposition 1

We first show that the feasibility (i.e., determining if for any $x \in [\underline{x}, \bar{x}]$, and a given C, we have $u(x;C) \geqslant \widetilde{l}(x;C)$) check can be performed by a polynomial time algorithm. For any $C \in [0,1]$, checking whether for any $x \in [\underline{x}, \bar{x}]$, $u(x;C) \geqslant \widetilde{l}(x;C)$ is equivalent to checking the following condition

$$\min_{x \in [\underline{x}, \bar{x}]} u(x;C) - \widetilde{l}(x;C) \geqslant 0. \tag{48}$$

Notice that by definition, for $x \in [\underline{x}, \underline{x}_u]$, $u(x;C) = m$ and $u(x;C) - \widetilde{l}(x;C)$ is always non-negative and for $x \in [\tilde{x}_\ell, \bar{x}]$, $\widetilde{l}(x;C) = 0$ and $u(x;C) - \widetilde{l}(x;C)$ is always non-negative, where $\tilde{x}_\ell = \inf\{x_{-1} < x < \bar{x} : \widetilde{l}(x;C) = 0\}$. Therefore, in Equation (48), we can ignore these two intervals. For any $x \in [\underline{x}_u, \tilde{x}_\ell]$, by Lemmas 6 and 16, $u(\cdot;C)$ is convex and $\widetilde{l}(\cdot;C)$ is concave, which implies that $u(x;C) - \widetilde{l}(x;C)$ is convex. Therefore, Problem (48) is a convex optimization problem on a compact set, which can be solved by polynomial-time algorithms.

With this result, Algorithm 4 has exactly the same structure with the classical bisection method which is a root-finding method that applies to any continuous function. Here, we make an analogy to the classical bisection method for finding the root (a single zero point). Compare to the classical bisection method for finding a single zero point, we can treat $C^\star(\mathcal{R})$ as the zero point, and treat the feasibility check as whether the value is positive or negative. It is well known that the classical bisection method can return a $x \in [x_0 - \epsilon, x_0 + \epsilon]$ in $O(\log(1/\epsilon))$ time, where x_0 is the zero point. Therefore, we have Algorithm 4 can return a $C_0 \in [C^\star(\mathcal{R}) - \epsilon, C^\star(\mathcal{R}) + \epsilon]$ in $O(\log(1/\epsilon))$ time.

E.2. Proof of Theorem 5

We split the proof into three parts. In part 1, we show that if for any $x \in \mathcal{V}$, we have $\widetilde{l}(x;C) \leqslant u(x;C)$, then such a C is feasible. That is, $C^\star(\mathcal{R}) \leqslant C$. In part 2, we show that a feasible $C = C^\star(\mathcal{R})$ is optimal if and only if there exists $\widehat{x} \in [\underline{x}, \bar{x}]$, such that $\widetilde{l}(\widehat{x}; C^\star(\mathcal{R})) = u(\widehat{x}; C^\star(\mathcal{R}))$. In part 3, we show that such $\widehat{x}$ must belong to $\mathcal{V}$.

Part 1: Feasibility of C. By Lemma 4, C is feasible if and only if $u(x;C) \geqslant \widetilde{l}(x;C)$ for any $x \in [\underline{x}, \bar{x}]$. Here, we show that by only checking $u(x;C)$ and $\widetilde{l}(x;C)$ on $x \in \mathcal{V}$ is enough to check the feasibility of C given that $\mathcal{R}$ is a polyhedron. As $\mathcal{R}$ is a polyhedron, we have both $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are piece-wise linear functions. By the first two properties of Lemma 6 and Equation (17), we have both $\widetilde{l}(\cdot;C)$ and $u(\cdot;C)$ are also piece-wise linear functions. We want to show that $\widetilde{l}(x;C) \leqslant u(x;C)$ for any $x \in \mathcal{V}$, implies that $\widetilde{l}(x;C) \leqslant u(x;C)$ for any $x \in [\underline{x}, \bar{x}]$

Suppose that for all $x \in \mathcal{V}$, $\widetilde{l}(x;C) \leqslant u(x;C)$. We take any $x_1 \in [\underline{x}, \bar{x}] - \mathcal{V}$. Let $[\underline{x}_1, \bar{x}_1]$ be the smallest interval contains x_1 such that $\underline{x}_1, \bar{x}_1 \in \mathcal{V}$. As it is the smallest interval, it does not contain any other x-vertices, which implies that for any $x \in [\underline{x}_1, \bar{x}_1]$, both $\widetilde{l}(x;C)$ and $u(x;C)$ are linear. As $\widetilde{l}(\underline{x}_1;C) \leqslant u(\underline{x}_1;C)$ and $\widetilde{l}(\bar{x}_1;C) \leqslant u(\bar{x}_1;C)$, we have $\widetilde{l}(x_1;C) \leqslant u(x_1;C)$ due to linearity and continuity of $\widetilde{l}(x;C)$ and $u(x;C)$.

Part 2: Optimality of C. We first prove the 'if' statement. That is, if there exists $\widehat{x} \in [\underline{x}, \bar{x}]$ and $C \in [\rho, 1]$, such that $\widetilde{l}(\widehat{x};C) = u(\widehat{x};C)$, then we have $C = C^\star(\mathcal{R})$ is optimal. We prove by contradiction. Contrary to our claim, suppose that there exists a feasible $\widehat{C} > C$ under which for any $x \in [\underline{x}, \bar{x}]$, we have $\widetilde{l}(x;\widehat{C}) \leqslant u(x;\widehat{C})$.

By the sixth property of Lemma 6 and Lemma 16, we have $\widetilde{l}(\widehat{x};\widehat{C}) \geqslant \widetilde{l}(\widehat{x};C) = u(\widehat{x};C) \geqslant u(\widehat{x};\widehat{C})$.

First, consider the case where $\widehat{x} \in [\underline{x}, \underline{x}_u]$, where we recall that $\underline{x}_u = \sup\{\underline{x} < x < \bar{x}_u : \text{CP}_o(m;(x, \underline{h}(x))) \geqslant C\}$; that is, we have $u(\widehat{x};C) = m$ if and only if $x \in [\underline{x}, \underline{x}_u]$. We then have

$$\widetilde{l}(\widehat{x};\widehat{C}) \geqslant \widetilde{l}(\widehat{x};C) = u(\widehat{x};C) = m \geqslant u(\widehat{x};\widehat{C}),$$

which implies that $\widetilde{l}(\widehat{x};\widehat{C}) = \widetilde{l}(\widehat{x};C) = m$. However, by definition, we have $\text{CP}_u(\widetilde{l}(\widehat{x};\widehat{C});(\widehat{x}, \bar{h}(\widehat{x}))) \geqslant \widehat{C}$ and $\text{CP}_u(\widetilde{l}(\widehat{x};C);(\widehat{x}, \bar{h}(\widehat{x}))) \geqslant C$, and to be in the under-protecting case, given that $\widetilde{l}(\widehat{x};\widehat{C}) = \widetilde{l}(\widehat{x};C) = m$, we must have $\bar{h}(\widehat{x}) \geqslant m$. By putting $p = m$ and $y = m$ into Equation (7), we have $\text{CP}_u(\widetilde{l}(\widehat{x};\widehat{C});(\widehat{x}, \bar{h}(\widehat{x}))) = \text{CP}_u(\widetilde{l}(\widehat{x};C);(\widehat{x}, \bar{h}(\widehat{x}))) = 1$, which implies that $C = \widehat{C} = 1$, which contradicts to $C >$
$widehatC$.

Now consider the case where $\widehat{x} \in [\underline{x}_u, \bar{x}]$. We have $u(\widehat{x};C) < m$ and $\widetilde{l}(\widehat{x};\widehat{C}) \geqslant \widetilde{l}(\widehat{x};C) = u(\widehat{x};C) > u(\widehat{x};\widehat{C})$, where the last strict inequality is by Lemma 20 which states that $u(\widehat{x};\widehat{C}) = u(\widehat{x};C)$ only happens when both of them equals m. This forms a contradiction because the chain of inequalities implies $\widetilde{l}(\widehat{x};\widehat{C}) > u(\widehat{x};\widehat{C})$.

Next, we prove the 'only if' statement. That is, if $\widetilde{l}(x;C) < u(x;C)$ for all $x \in [\underline{x}, \bar{x}]$, then C is not optimal. That is, there exists $\widehat{C} > C$ such that $\widetilde{l}(x;C) \leqslant u(x;C)$ for all $x \in [\underline{x}, \bar{x}]$

In this case, by the sixth property of Lemma 6 and Lemma 16, for any $x \in [\underline{x}, \bar{x}]$, $\widetilde{l}(x;C)$ is continuously increasing in C and $u(x;C)$ is continuously decreasing in C. Therefore, by continuity of C, there exists $\delta > 0$ such that with $\widehat{C} = C + \delta$, we have $\widetilde{l}(x;\widehat{C}) < u(x;\widehat{C})$ for all $x \in [\underline{x}, \bar{x}]$. By Lemma 4, we have C is feasible and C is not optimal.

Part 3: $\widehat{x} \in \mathcal{V}$. . As we defined, $\mathcal{V}$ is the set containing the x value of all vertices of $\mathcal{R}$ (i.e., $\underline{h}(\cdot)$ and $\bar{h}(\cdot)$) and the set $\mathcal{R}_0$, where $\mathcal{R}_0 = \{(x, \underline{h}(x) : x \in [\underline{x}, \bar{x}]\} \cap \{(x,y) : x + y = m\}$. In Lemma 25, we show that all of x-vertices of $u(\cdot;C)$ and $\widetilde{l}(\cdot;C)$ are a subset of $\mathcal{V}$ and the elements of $\mathcal{R}_0$ might be x-vertices of $u(\cdot;C)$. By Lemma 27, we have there exists $\widehat{x} \in \mathcal{V}$ such that $u(\widehat{x}; C^\star(\mathcal{R})) \geqslant \widetilde{l}(\widehat{x}; C^\star(\mathcal{R}))$, which completes the proof.

□

E.3. Proof of Theorem 6

Given a polyhedron $\mathcal{R}$, let $C^\star(\mathcal{R})$ be the optimal consistent ratio among all PLAs. First, it is easy to see that the computational complexity is $O(|\mathcal{V}|^3)$ because we enumerate at most $|\mathcal{V}|(|\mathcal{V}|-1)/2$ pairs of x-vertices, and recall that

$$C^\star(\mathcal{R}) = \max\{C \in \mathcal{S} : \widetilde{l}(x;C) \leqslant u(x;C) \quad \text{for any } x \in \mathcal{V}\},$$

for each pair of vertices, if $\widetilde{l}(x;C) \leqslant u(x;C)$, for any $x \in \mathcal{V}$, we will add C into the set $\mathcal{S}$. By doing this, for each pair of vertices, we compare the value of $\widetilde{l}(x;C)$ and $u(x;C)$ for $x \in \mathcal{V}$ with at most $|\mathcal{V}|$ complexity. To summarize, the total complexity is bounded by $|\mathcal{V}|(|\mathcal{V}|-1)/2 \cdot |\mathcal{V}|$, which is $O(|\mathcal{V}|^3)$.

Second, Algorithm 5 cannot return an output $C > C^\star(\mathcal{R})$. This is because it will return

$$C^\star(\mathcal{R}) = \max\{C \in \mathcal{S} : \widetilde{l}(x;C) \leqslant u(x;C) \quad \text{for any } x \in \mathcal{V}\}.$$

and by Theorem 5, any C such that $\widetilde{l}(x;C) \leqslant u(x;C)$ for any $x \in \mathcal{V}$ implies $C \leqslant C^\star(\mathcal{R})$.

Finally, we show that by enumerating vertices as in Algorithm 5, we can find $C^\star(\mathcal{R})$. To show this, we need to split the proof into three cases according to the location of $\widehat{x}$, where $\widehat{x}$ is the intersection point of $u(\cdot; C^\star(\mathcal{R}))$ and $\widetilde{l}(\cdot; C^\star(\mathcal{R}))$. By Theorem 5, such $\widehat{x}$ always exists. If there are multiple intersection points, we take the one with the smallest x value. Before we divide into three cases, we highlight that enumerating vertices is only for reducing computational complexity, and all the following statements not related to $\mathcal{V}$ are correct for any general convex set $\mathcal{R}$.

- *Case 1:* $\widehat{x} \leqslant x_H$ In this case, we first show that $\widehat{x} = \bar{x}_u$ and $\bar{x}_u \leqslant x_H$. Second, as H is a vertex, we have $x_H \in \mathcal{V}$, and by Lemma 22, we have $\bar{x}_u \in \mathcal{V}$. We show that by balancing H and $(\bar{x}_u, \underline{h}(\bar{x}_u))$, we obtain a C_1 in Algorithm 5 according to Equation (28).

  For any $C_1 \leqslant C^\star(\mathcal{R})$, by the definition of $\widetilde{l}(\cdot;C_1)$, we have $\widetilde{l}(x;C_1) = \widetilde{l}(x_H;C_1)$ for $x \in [\underline{x}, x_H]$. As $u(\widehat{x};C_1) = \widetilde{l}(\widehat{x};C_1) = \widetilde{l}(x_H;C_1)$ and $u(x;C_1) \geqslant \widetilde{l}(x;C_1)$ for any $x \in [\underline{x}, \bar{x}]$, we have

$$\widehat{x} \in \text{argmin}_{x \in [\underline{x}, \bar{x}]} u(x;C_1).$$

By Lemma 21, we have $\widehat{x} \in [\bar{x}_u, x_H]$. As we have defined that if there are multiple intersection points, we take $\widehat{x}$ as the one with the smallest x value. That is, $\widehat{x} = \bar{x}_u$. Moreover, we must have $\bar{x}_u \leqslant x_H$ in this case because otherwise, if $\bar{x}_u > x_H$, we have $u(x; C_1) = m$ for any $x \in [\underline{x}, x_H]$, and as $\widetilde{l}(x; C_1) < m$, we have there does not exist $\widehat{x}$ such that $u(\widehat{x}; C_1) = \widetilde{l}(\widehat{x}; C_1)$, which is a contradiction.

Next, we show that by balancing H and $(\bar{x}_u, \underline{h}(\bar{x}_u))$ according to Equation (28), we can get C_1. Recall that in Equation (28), given two points x_1 and x_2 (here x_H and $\bar{x}_u$), we find p (here $\widetilde{l}(x_H; C_1)$) such $\text{CP}_u(p; (x_1, \bar{h}(x_1))) = \text{CP}_o(p; (x_2, \underline{h}(x_2))$. This is because, by definition, we have $\text{CP}_u(\widetilde{l}(x_H; C_1); H) = C_1$, and $\text{CP}_o(\widetilde{l}(\bar{x}_u; C_1); (\bar{x}_u, \underline{h}(\bar{x}_u))) = C_1$, which implies that

$$\text{CP}_u(\widetilde{l}(x_H; C_1); H) = \text{CP}_o(\widetilde{l}(\bar{x}_u; C_1); (\bar{x}_u, \underline{h}(\bar{x}_u))) = C_1.$$

Finally, as $\widetilde{l}(\widehat{x}; C_1) = \widetilde{l}(x_H; C_1)$, we have $\widetilde{l}(\widehat{x}; C_1) = u(\widehat{x}; C_1)$, and by Theorem 5, we know C_1 is optimal.

- *Case 2:* $x_H \leqslant \widehat{x} \leqslant x_{-1}$ In this case, by Theorem 5, we have $\widehat{x} \in \mathcal{V}$. We first show that by balancing $(\widehat{x}, \bar{h}(\widehat{x}))$ and $(\widehat{x}, \underline{h}(\widehat{x}))$, we obtain a C_1 in Algorithm 5 according to Equation (28). Moreover, by definition, we have $u(\widehat{x}; C_1) = \widetilde{l}(\widehat{x}; C_1)$, which shows the optimality of C_1 and the algorithm can return such a C_1. That is, let

$$\widehat{p} = \{p : \text{CP}_u(p; (\widehat{x}, \bar{h}(\widehat{x}))) = \text{CP}_o(p; (\widehat{x}, \underline{h}(\widehat{x})))\} \quad \text{and} \quad C_1 = \text{CP}_u(\widehat{p}; (\widehat{x}, \bar{h}(\widehat{x}))).$$

We will show that $\widehat{p} = \widetilde{l}(\widehat{x}; C_1)$.

As $x_H \leqslant \widehat{x} \leqslant x_{-1}$, by definition, we have $\text{CP}_u(\widetilde{l}(\widehat{x}; C_1); (\widehat{x}, \bar{h}(\widehat{x}))) = C_1$. In addition, by Lemma 23, we have $\widehat{x} \in [\underline{x}_u, \bar{x}_u]$, and by definition, $\text{CP}_o(u(\widehat{x}; C_1); (\widehat{x}, \underline{h}(\widehat{x}))) = C_1$, which implies that

$$\text{CP}_u(\widetilde{l}(\widehat{x}; C_1); (\widehat{x}, \bar{h}(\widehat{x}))) = \text{CP}_o(u(\widehat{x}; C_1); (\widehat{x}, \underline{h}(\widehat{x}))) = C_1.$$

This shows that $\widehat{p} = \widetilde{l}(\widehat{x}; C_1) = u(\widehat{x}; C_1)$, as desired.

- *Case 3:* $\widehat{x} > x_{-1}$ In this case, by Theorem 5, we have $\widehat{x} \in \mathcal{V}$, and by Lemma 24, we have $x_{-1} \in \mathcal{V}$. First, we show that by balancing $(\widehat{x}, \underline{h}(\widehat{x}))$ and $(x_{-1}, \bar{h}(x_{-1}))$, we obtain a C_1 in Algorithm 5 according to Equation (29). Finally, we show that under this C_1, $u(\widehat{x}; C_1) = \widetilde{l}(\widehat{x}; C_1)$, which shows the optimality of C_1 and the algorithm can return such a C_1. Let

$$\widehat{p} = \{p : \text{CP}_u(p; (x_{-1}, \bar{h}(x_{-1}))) = \text{CP}_o(p - (\widehat{x} - x_{-1}); (\widehat{x}, \underline{h}(\widehat{x})))\} \quad \text{and} \quad \text{CP}_u(\widehat{p}; (x_{-1}, \bar{h}(x_{-1}))) = C_1\,.$$

We will show that $\widehat{p} = u(\widehat{x}; C_1) + (\widehat{x} - x_{-1}) = \widetilde{l}(\widehat{x}; C_1) + (\widehat{x} - x_{-1}) = \widetilde{l}(x_{-1}; C_1)$.

By Lemma 23, we have $\widehat{x} \in [\underline{x}_u, \bar{x}_u]$, and by definition, $\text{CP}_o(u(\widehat{x}; C_1); (\widehat{x}, \underline{h}(\widehat{x}))) = C_1$. We futher observe that, by Equation (17), we have $\widetilde{l}(x_{-1}; C_1) = l(x_{-1}; C_1)$, and we have

$$\text{CP}_u(l(x_{-1}; C_1); (x_{-1}, \bar{h}(x_{-1}))) = \text{CP}_u(\widetilde{l}(x_{-1}; C_1); (x_{-1}, \bar{h}(x_{-1}))) = C_1,$$

which implies that

$$\text{CP}_u(\widetilde{l}(x_{-1}; C_1); (x_{-1}, \bar{h}(x_{-1}))) = \text{CP}_o(u(\widehat{x}; C_1); (\widehat{x}, \underline{h}(\widehat{x}))) = C_1.$$

Finally, as $\widetilde{l}(\widehat{x}; C_1)) = \widetilde{l}(x_{-1}; C_1) - (\widehat{x} - x_{-1}) = u(\widehat{x}; C_1)$, we have $\widehat{p} = u(\widehat{x}; C_1) + (\widehat{x} - x_{-1}) = \widetilde{l}(\widehat{x}; C_1) + (\widehat{x} - x_{-1}) = \widetilde{l}(x_{-1}; C_1)$.

E.4. Proof of Theorem 4

To show that the optimal consistent ratio among all PLAs is optimal among any deterministic and randomized algorithm, we still split the proof into three cases which are the same three cases as in the Proof of Theorem 6. Before presenting the proof, we highlight that although we used many properties from Theorem 5, here in the proof of Theorem 6, we do NOT assume $\mathcal{R}$ is a polyhedron.

Case 1: $\widehat{x} \leqslant x_H$ In this case, in case 1 of the Proof of Theorem 6, we have established that $\widehat{x} = \bar{x}_u$, and $C^\star(\mathcal{R}) = \text{CP}_u(p; H) = \text{CP}_o(p; (\widehat{x}, \underline{h}(\widehat{x})))$, where $p = l(x_H; C^\star(\mathcal{R})) = u(\bar{x}_u; C^\star(\mathcal{R}))$. Our goal here is to show that there does not exist any deterministic or randomized algorithm with a consistent ratio greater than $C^\star(\mathcal{R})$. Let us define two (ordered) input sequences: In the first input sequence, I_1, $\bar{x}_u$ low-reward requests arrive first, followed with $\underline{h}(\bar{x}_u)$ high-reward requests. In the second input sequence, I_2, x_H low-reward requests arrive first, followed y_H high-reward requests. We note that $\widehat{x} = \bar{x}_u < x_H$ and $\underline{h}(\widehat{x}) < y_H$ because H is the highest point of $\mathcal{R}$.

Observe that before receiving $\widehat{x}$ low-reward requests, any algorithm cannot differentiate the two input sequences. Hence, any algorithm should decide how many low-reward requests to accept in expectation among the first $\widehat{x}$ ones.

Next, we prove by contradiction. Suppose that there exists an algorithm $\mathcal{A}$, which can be either deterministic or randomized, and has a consistent ratio larger than $C^\star(\mathcal{R})$. Then, we have

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_1)]}{\text{OPT}(I_1)} > C^\star(\mathcal{R}),$$

where the expectation is taken on the randomization of the algorithm. Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A}, I_1)$, $\ell(\mathcal{A}, I_1)$, respectively. Replace C by $C^\star(\mathcal{R})$ in Part 1 of Section D.3 where we show that any algorithm facing this instance should accept at least $m - u(\bar{x}_u; C)$ low-reward requests, we have $\ell(\mathcal{A}, I_1) > m - u(\bar{x}_u; C^\star(\mathcal{R}))$.

Next, if $\mathcal{A}$ achieves a consistent ratio greater than $C^\star(\mathcal{R})$, it should also satisfy that

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_2)]}{\text{OPT}(I_2)} > C^\star(\mathcal{R}).$$

Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A}, I_2)$, $\ell(\mathcal{A}, I_2)$, respectively. Then,

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A}, I_2)]}{\text{OPT}(I_2)} = \frac{h(\mathcal{A}, I_2) r_h + \ell(\mathcal{A}, I_2) r_\ell}{y_H r_h + \min\{x_H, m - y_H\} r_\ell} > C^\star(\mathcal{R}).$$

Recall that by the definition of $l(\cdot; C^\star(\mathcal{R}))$, we have $\text{CP}_u(l(x_H; C^\star(\mathcal{R})); H) = C^\star(\mathcal{R})$ because $x_H \in [x_H, x_{-1}]$. Then, by Equation (7), we have

$$\text{CP}_u(l(x_H; C^\star(\mathcal{R})); H) = \frac{\max\{l(x_H; C^\star(\mathcal{R})), \min\{y_H, m - x_H\}\} r_h + \min\{x_H, m - l(x_H; C^\star(\mathcal{R}))\} r_\ell}{y_H r_h + \min\{x_H, m - y_H\} r_\ell},$$

and by Lemma 19, we have $\min\{x_H, m - l(x_H; C^\star(\mathcal{R}))\} = m - l(x_H; C^\star(\mathcal{R}))$ and $\max\{l(x_H; C^\star(\mathcal{R})), \min\{y_H, m - x_H\} = l(x_H; C^\star(\mathcal{R}))$. Therefore, we have

$$\frac{l(x_H; C^\star(\mathcal{R})) r_h + (m - l(x_H; C^\star(\mathcal{R}))) r_\ell}{y_H r_h + \min\{x_H, m - y_H\} r_\ell} = C^\star(\mathcal{R}).$$

This implies that

$$\frac{h(\mathcal{A}, I_2) r_h + \ell(\mathcal{A}, I_2) r_\ell}{y_H r_h + \min\{x_H, m - y_H\} r_\ell} > \frac{l(x_H; C^\star(\mathcal{R})) r_h + (m - l(x_H; C^\star(\mathcal{R}))) r_\ell}{y_H r_h + \min\{x_H, m - y_H\} r_\ell}.$$

As $h(\mathcal{A}, I_2) \leqslant m - \ell(\mathcal{A}, I_2)$, we have $\ell(\mathcal{A}, I_2) < m - l(x_H; C^\star(\mathcal{R}))$. Recall that as mentioned at the beginning, we have $l(x_H; C^\star(\mathcal{R})) = u(\bar{x}_u; C^\star(\mathcal{R}))$. Therefore, we have

$$\ell(\mathcal{A}, I_2) < m - l(x_H; C^\star(\mathcal{R})) = m - u(\bar{x}_u; C^\star(\mathcal{R})) < \ell(\mathcal{A}, I_1),$$

which is a contradiction, because before receiving $\widehat{x}$ low-reward requests, any algorithm cannot differentiate the two input sequences, and this implies that $\ell(\mathcal{A}, I_2) \geqslant \ell(\mathcal{A}, I_1)$.

Case 2: $x_H \leqslant \widehat{x} \leqslant x_{-1}$. We replace the instances in the proof of case 1 to get the proof for this case. We replace the first instance by: $\widehat{x}$ low-reward requests arrive first, followed with $\underline{h}(\widehat{x})$ high-reward requests. We replace the second instance by: $\widehat{x}$ low-reward requests arrive first, followed with $\bar{h}(\widehat{x})$ high-reward requests. Then, we use similar arguments in case 1 to show the result.

Case 3: $\widehat{x} > x_{-1}$. In this case, as we showed in case 3 of the Proof of Theorem 6, we have that

$$C^\star(\mathcal{R}) = \text{CP}_u(p;(x_{-1},\bar{h}(x_{-1}))) = \text{CP}_o(p-(\widehat{x}-x_{-1});(\widehat{x},\underline{h}(\widehat{x}))),$$

where $p = l(x_{-1};C^\star(\mathcal{R}))$, and $u(\widehat{x};C^\star(\mathcal{R})) = p-(\widehat{x}-x_{-1})$. Let us define two (ordered) input sequences: In the first input sequence, I_1, x_{-1} low-reward requests arrive first, followed with $\bar{h}(x_{-1})$ high-reward requests. In the second input sequence, I_2, $\widehat{x}$ low-reward requests arrive first, followed $\underline{h}(\widehat{x})$ high-reward requests. In this case, we have $x_{-1} < \widehat{x}$. Before receiving x_{-1} low-reward requests, any deterministic or randomized algorithm cannot differentiate the two input sequences. Hence, any algorithm should decide how many low-reward requests to accept among the first x_{-1} ones in expectation.

Next, we prove by contradiction. Suppose that there exists an algorithm $\mathcal{A}$ which has a consistent ratio larger than $C^\star(\mathcal{R})$. As

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_1)]}{\text{OPT}(I_1)} > C^\star(\mathcal{R}),$$

we have $\mathcal{A}$ should accept less than $m-\widetilde{l}(x_{-1};C^\star(\mathcal{R}))$ low-reward requests in expectation. Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A},I_1)$, $\ell(\mathcal{A},I_1)$, respectively. Replace C by $C^\star(\mathcal{R})$ in Part 2 of Section D.3 which shows that any algorithm facing this instance should accept less than $m-l(x_{-1};C)$ low-reward requests, we have $\ell(\mathcal{A},I_1) < m-l(x_{-1};C^\star(\mathcal{R}))$.

Next, if $\mathcal{A}$ achieves a consistent ratio greater than $C^\star(\mathcal{R})$, it should also satisfy that

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_2)]}{\text{OPT}(I_2)} > C^\star(\mathcal{R}).$$

Let the expected total amount of high-reward, low-reward requests being accepted by $\mathcal{A}$ be $h(\mathcal{A},I_2)$, $\ell(\mathcal{A},I_2)$, respectively. Then,

$$\frac{\mathbf{E}[\text{Rew}(\mathcal{A},I_2)]}{\text{OPT}(I_2)} = \frac{h(\mathcal{A},I_2)r_h+\ell(\mathcal{A},I_2)r_\ell}{\underline{h}(\widehat{x})r_h+\min\{\widehat{x},m-\underline{h}(\widehat{x})\}r_\ell} > C^\star(\mathcal{R}).$$

As is mentioned at the beginning of this case, we have $\text{CP}_o(u(\widehat{x};C^\star(\mathcal{R}));(\widehat{x},\underline{h}(\widehat{x}))) = C^\star(\mathcal{R})$. By Equation (6), we have

$$\text{CP}_o(u(\widehat{x};C^\star(\mathcal{R}));(\widehat{x},\underline{h}(\widehat{x}))) = \frac{\underline{h}(\widehat{x})r_h+\min\{\widehat{x},m-u(\widehat{x};C^\star(\mathcal{R}))\}r_\ell}{\underline{h}(\widehat{x})r_h+\min\{\widehat{x},m-\underline{h}(\widehat{x})\}r_\ell}.$$

Here, we must have $\min\{\widehat{x},m-u(\widehat{x};C^\star(\mathcal{R}))\} = m-u(\widehat{x};C^\star(\mathcal{R}))$ because otherwise, if $\min\{\widehat{x},m-u(\widehat{x};C^\star(\mathcal{R}))\} = \widehat{x}$, then due to $u(\widehat{x};C^\star(\mathcal{R})) \geqslant \underline{h}(\widehat{x})$, we have $\min\{\widehat{x},m-\underline{h}(\widehat{x})\} = \widehat{x}$, and

$\text{CP}_o(u(\widehat{x};C^\star(\mathcal{R}));(\widehat{x},\underline{h}(\widehat{x}))) = 1 \neq C^\star(\mathcal{R})$. Therefore, by taking $\min\{\widehat{x}, m-u(\widehat{x};C^\star(\mathcal{R}))\} = m-u(\widehat{x};C^\star(\mathcal{R}))$, we obtain

$$\frac{\underline{h}(\widehat{x})r_h + (m-u(\widehat{x};C^\star(\mathcal{R})))r_\ell}{\underline{h}(\widehat{x})r_h + \min\{\widehat{x}, m-\underline{h}(\widehat{x})\}r_\ell} = C^\star(\mathcal{R}).$$

This implies that

$$\frac{h(\mathcal{A},I_2)r_h + \ell(\mathcal{A},I_2)r_\ell}{\underline{h}(\widehat{x})r_h + \min\{\widehat{x}, m-\underline{h}(\widehat{x})\}r_\ell} > \frac{\underline{h}(\widehat{x})r_h + (m-u(\widehat{x};C^\star(\mathcal{R})))r_\ell}{\underline{h}(\widehat{x})r_h + \min\{\widehat{x}, m-\underline{h}(\widehat{x})\}r_\ell}.$$

As $h(\mathcal{A},I_2) \leqslant \underline{h}(\widehat{x})$, we have $\ell(\mathcal{A},I_2) > m-u(\widehat{x};C^\star(\mathcal{R}))$. However, in this case $\widehat{x} > x_{-1}$, between x_{-1} and $\widehat{x}$, there are at most $\widehat{x}-x_{-1}$ low-reward requests arriving, and this implies that $\ell(\mathcal{A},I_2) - \ell(\mathcal{A},I_1) \leqslant \widehat{x}-x_{-1}$.

However, as is mentioned at the beginning, we have $u(\widehat{x};C^\star(\mathcal{R})) = l(x_{-1};C^\star(\mathcal{R})) - (\widehat{x}-x_{-1})$. This implies that

$$\begin{aligned}
\ell(\mathcal{A},I_2) - \ell(\mathcal{A},I_1) &> m-u(\widehat{x};C^\star(\mathcal{R})) - (m-l(x_{-1};C^\star(\mathcal{R}))) && (49)\\
&= m-(l(x_{-1};C^\star(\mathcal{R})) - (\widehat{x}-x_{-1})) - (m-l(x_{-1};C^\star(\mathcal{R}))) && (50)\\
&= \widehat{x}-x_{-1}, && (51)
\end{aligned}$$

which is a contradiction.

Appendix F: Appendix of Section 8

F.1. Proof of Proposition 2

To show Proposition 2, consider constraints listed in (35) and (36). We can express the CP function in these constraints as follows:

$$\text{CP}(p_2(\mathbf{x}_2),\ldots,p_n(x_n);\mathbf{x}) = \frac{\sum_{i=1}^n A_i(\mathbf{x})r_i}{\text{OPT}(\mathbf{x})},$$

where $\text{OPT}(\mathbf{x})$ is the offline optimal value, which is a fixed quantity based on $\mathbf{x}$ and does not depend on the protection level functions. Here, $A_i(\mathbf{x})$ represents the effective demand served to the i-th highest reward request based on the protection level functions and available capacity. The value of $A_i(\mathbf{x})$ is computed recursively:

$$A_n(\mathbf{x}) = \min\{x_n, (m-p_{n-1}(\mathbf{x}))^+\},$$

where the term $(m - p_{n-1}(\mathbf{x}))^+$ ensures that the demand served to the n-th highest reward request (i.e., lowest reward request) is based on the available capacity after considering the commitments for higher reward requests. For any $i \in [n-1]$, we have:

$$A_i(\mathbf{x}) = \min\{x_i, (m - p_{i-1}(\mathbf{x}) - \sum_{j=i+1}^{n} A_j(\mathbf{x}))^+\} = \min\{x_i, \widetilde{A}_i(\mathbf{x})\}.$$

Here, $\widetilde{A}_i(\mathbf{x}) = (m - p_{i-1}(\mathbf{x}) - \sum_{j=i+1}^{n} A_j(\mathbf{x}))^+$. We convert Problem (C-Pareto-n-Types) into a

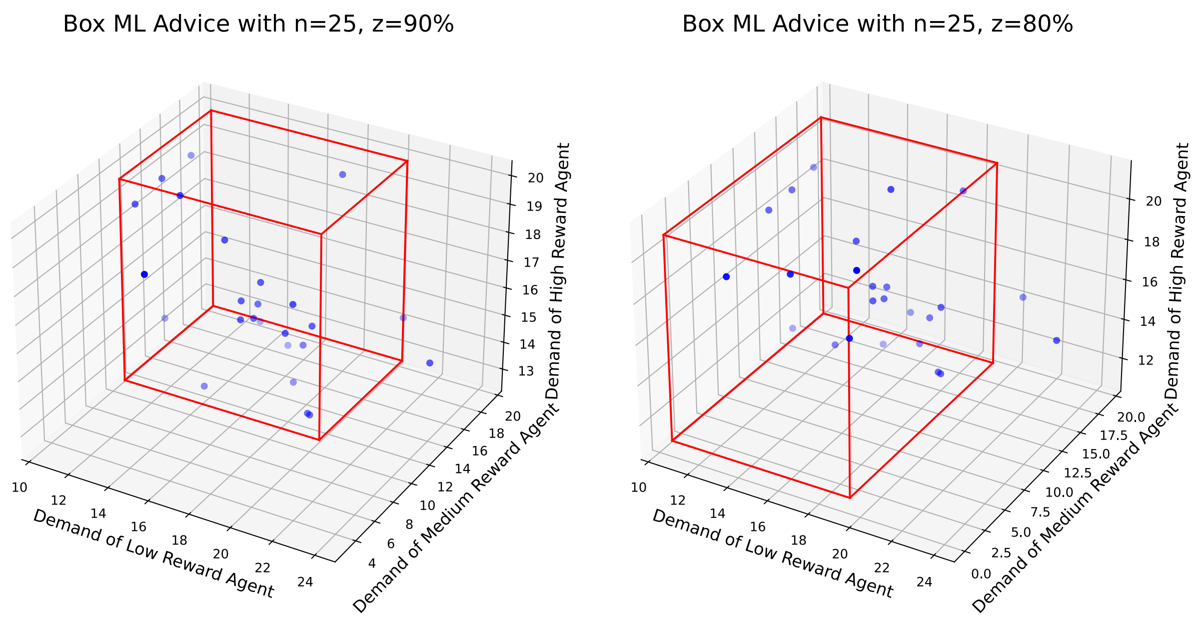


Figure 6 Constructing the box advice with $n = 25$ samples with $z = 90\%$ and $z = 80\%$ for the uniform demand model in Equations (39), respectively.

mixed integer program (MIP) by writing the constraints in a unified form:

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} A_i(\mathbf{x}) r_i}{\mathrm{OPT}(\mathbf{x})} \geqslant C, \qquad \mathbf{x} \in \mathcal{V} \cup \mathcal{V}', \\
&A_i(\mathbf{x}) \leqslant x_i, \qquad A_i(\mathbf{x}) \leqslant \widetilde{A}_i(\mathbf{x}), \qquad i \in [n], \\
&A_i(\mathbf{x}) \geqslant \widetilde{A}_i(\mathbf{x}) + (x_i - m) w_i(\mathbf{x}), \qquad i \in [n], \\
&A_i(\mathbf{x}) \geqslant x_i w_i(\mathbf{x}), \qquad i \in [n], \\
&\widetilde{A}_i(\mathbf{x}) \geqslant m - p_{i-1}(\mathbf{x}) - \sum_{j=i+1}^{n} A_j(\mathbf{x}), \qquad i \in [n], \\
&\widetilde{A}_i(\mathbf{x}) \geqslant 0, \qquad i \in [n], \\
&\widetilde{A}_i(\mathbf{x}) \leqslant m(1 - \widetilde{w}_i(\mathbf{x})), \qquad i \in [n],
\end{aligned}$$

$$\widetilde{A}_i(\mathbf{x}) \leqslant m - p_{i-1}(\mathbf{x}) - \sum_{j=i+1}^{n} A_j(\mathbf{x}) + nm\widetilde{w}_i(\mathbf{x}), \qquad i \in [n].$$

Here, $w_i(\mathbf{x}) \in \{0,1\}$, and $\widetilde{w}_i(\mathbf{x}) \in \{0,1\}$. When $w_i(\mathbf{x}) = 1$, we have $A_i(\mathbf{x}) = \min\{x_i, \widetilde{A}_i(\mathbf{x})\} = x_i$, indicating that the full demand of type i-th request is served. When $w_i(\mathbf{x}) = 0$, we have $A_i(\mathbf{x}) = \widetilde{A}_i(\mathbf{x})$. Similarly, when $\widetilde{w}_i(\mathbf{x}) = 1$, we have $\widetilde{A}_i(\mathbf{x}) = (m - p_{i-1}(\mathbf{x}) - \sum_{j=i+1}^{n} A_j(\mathbf{x}))^+ = 0$, and when $\widetilde{w}_i(\mathbf{x}) = 0$, $\widetilde{A}_i(\mathbf{x}) = m - p_{i-1}(\mathbf{x}) - \sum_{j=i+1}^{n} A_j(\mathbf{x})$. A similar transformation applies to $\frac{\sum_{i=1}^{n} A_i(\mathbf{x}) r_i}{\mathsf{OPT}(\mathbf{x})} \geqslant R,\ \mathbf{x} \in \mathcal{V}'$.

The formulation contains $2n|\mathcal{V} \cup \mathcal{V}'|$ continuous variables: $A_i(\mathbf{x}), \widetilde{A}_i(\mathbf{x})$. It also contains $2n|\mathcal{V} \cup \mathcal{V}'|$ integer variables: $w_i(\mathbf{x}), \widetilde{w}_i(\mathbf{x})$. In addition, there are $n|\mathcal{V} \cup \mathcal{V}'|$ continuous variables for the protection levels $p_i(\mathbf{x})$. Finally, R is a single continuous variable. Therefore, the total number of continuous variables is $3n|\mathcal{V} \cup \mathcal{V}'| + 1$, and the total number of binary integer variables is $2n|\mathcal{V} \cup \mathcal{V}'|$.

F.2. Algorithm 6: Nested Interpolation for Adaptive PLAs

As stated earlier, after solving the MILP problem (C-Pareto-n-Types), we obtain the optimal values of $p_2(\mathbf{x}_2), \ldots, p_n(\mathbf{x}_n)$ only at the points $\mathbf{x} \in \mathcal{V}' \cup \mathcal{V}$. To extend the adaptive protection level functions $p_i(\mathbf{x}_i)$ across the full space $\mathbb{N}^{n-i+1}$ for each $i \in 2, 3, \ldots, n$, we use the interpolation heuristic outlined in Algorithm 6, where $\mathbb{N}$ denotes the set of natural numbers.

Algorithm 6 Nested Interpolation for Adaptive PLAs

Input: Sets of vertices $\mathcal{V}$ (ML polytope) and $\mathcal{V}'$ (vertices of $[0,m]^n$), and the optimal values $\{p_i(\mathbf{x}_i)\}_{i=2}^n$ available only for $\mathbf{x} \in \mathcal{V} \cup \mathcal{V}'$.

Output: Extended protection level functions $\tilde{p}_i : \mathbb{N}^{n-i+1} \to [0,m]$ for every $i \in \{2, \ldots, n\}$.

1. **Initialize $\tilde{p}_n$.**

— For every integer $x_n \in \mathbb{N}$, define the interpolation:

$$\tilde{p}_n(x_n) = \min_{v_n \in \mathcal{U}_n} \Big\{ p_n(v_n) + [v_n - x_n]_+ \Big\}, \qquad \mathcal{U}_n = \{v_n \mid \mathbf{v} = (v_1, \ldots, v_n) \in \mathcal{V} \cup V'\}.$$

(Here, $[z]_+ := \max\{0, z\}$.)

2. **Iterate downward for $i = n-1, \ldots, 2$.**

(a) *Raw envelope.* For each integer vector $\mathbf{x}_i = (x_i, \ldots, x_n) \in \mathbb{N}^{n-i+1}$, set

$$p_i^{\text{env}}(\mathbf{x}_i) = \min_{\mathbf{v}_i \in \mathcal{U}_i} \Big\{ p_i(\mathbf{v}_i) + \sum_{j=i}^{n} [v_j - x_j]_+ \Big\}, \quad \mathcal{U}_i = \{\mathbf{v}_i = (v_i, \ldots, v_n) \mid \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathcal{V} \cup \mathcal{V}'\}.$$

(b) *Enforce nesting with the next-higher PLA.*

$$\tilde{p}_i(\mathbf{x}_i) = \min \Big\{ p_i^{\text{env}}(\mathbf{x}_i),\ \tilde{p}_{i+1}(x_{i+1}, \ldots, x_n) \Big\} \qquad \forall \mathbf{x}_i \in \mathbb{N}^{n-i+1}.$$

3. **Return** $\{\tilde{p}_i\}_{i=2}^n$.

F.3. Multi-type ML Regions

In this section, we present examples of three-dimensional ML advice constructed from 25 random samples drawn from the distribution in Equation (39). Figure 6 illustrates the box advice under different parameter settings: the left plot captures 90% of the samples, while the right plot captures 80% of the samples. In Figure 7, we construct three-dimensional polyhedra based on the approach in Bertsimas et al. (2018), varying the parameters ϵ, θ, and ϕ. We observe that as θ and ϕ increase, the number of vertices increases. Additionally, as ϵ decreases, the ML region expands, becoming more conservative and encompassing more outliers.

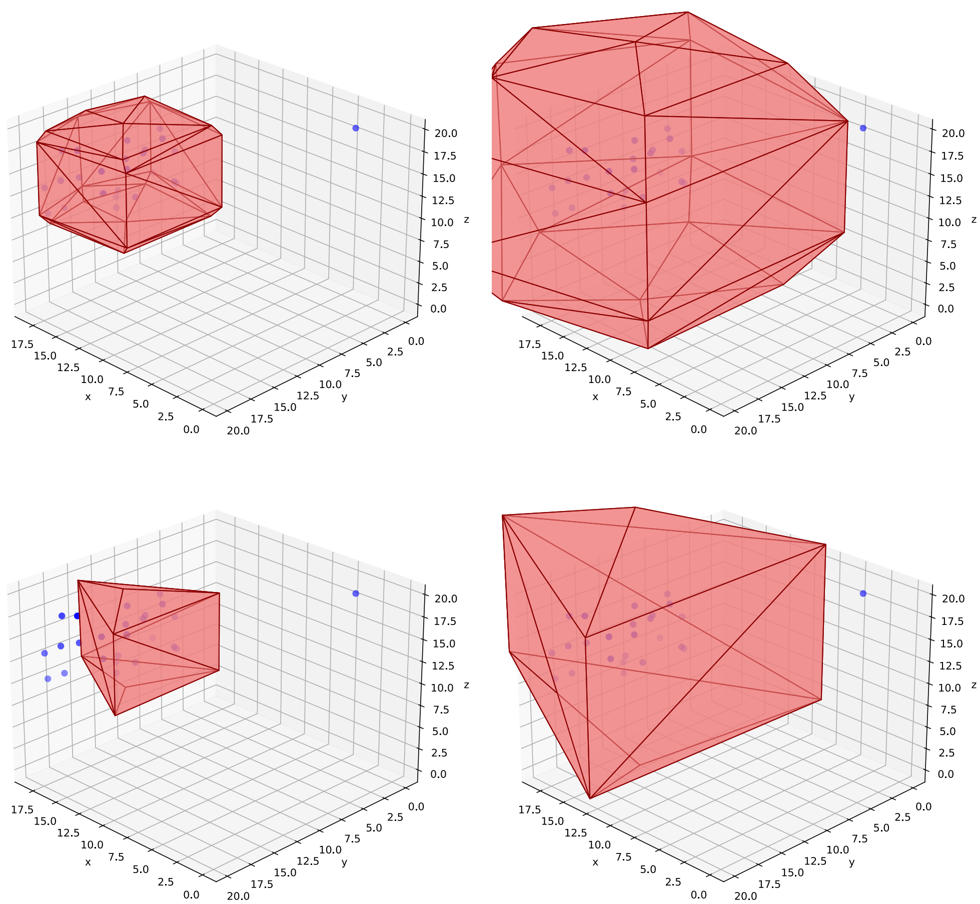


Figure 7 Constructing the box advice with $n = 25$ samples for the uniform demand model in Equations (39), respectively. **Upper-left:** $\epsilon = 0.9$, $\theta = \phi = 6$; **Upper-right:** $\epsilon = 0.1$, $\theta = \phi = 6$; **Lower-left:** $\epsilon = 0.9$, $\theta = \phi = 4$; **Lower-right:** $\epsilon = 0.1$, $\theta = \phi = 4$.

Appendix G: Preliminary Lemmas

Here, we introduce several small lemmas which will be used in several proofs in this paper.

G.1. Lemma 8 and its Proof

LEMMA 8 (**Not Enough Demand**). *For any $A = (x, y)$ with $x + y < m$ and $p < y$, we have $CP_u(p; A = (x, y)) = 1$.*

Proof of Lemma 8 By Equation (7), we have

$$\begin{aligned}\mathrm{CP}_u(p; A = (x,y)) &= \frac{\max\{p, \min\{y, m-x\}\}r_h + \min\{x, m-p\}r_\ell}{yr_h + \min\{x, m-y\}r_\ell} \\ &= \frac{\max\{p, y\}r_h + \min\{x, m-p\}r_\ell}{yr_h + xr_\ell} = \frac{yr_h + xr_\ell}{yr_h + xr_\ell} = 1\,,\end{aligned}$$

where in the first equation, we used the assumption that $x + y < m$, and in the second equation, we used the assumption that $x + y < m$ and $p < y$. The last equation is the desired result.

G.2. Lemma 9 and its Proof

LEMMA 9 (**Monotonicity of the Compatible Ratio CP**$(p; (x, y))$ **w.r.t.** p). *For any $0 \leqslant y \leqslant p_1 \leqslant p_2$, and any $x \geqslant 0$, we have*

$$CP_o(p_1; (x, y)) \geqslant CP_o(p_2; (x, y)).$$

Further, for any $p_1 \leqslant p_2 \leqslant y$, and any $x \geqslant 0$, we have

$$CP_u(p_2; (x, y)) \geqslant CP_u(p_1; (x, y)).$$

That is, the compatible ratio of a point (x, y) increases when the gap between the protection level p and y, i.e., $|p - y|$, gets smaller. In addition, if $p > m - x$, we have the strong monotonicity for both of $CP_o(p; (x, y))$ and $CP_u(p; (x, y))$.

Proof of Lemma 9 By the definition of CP_o in Equation (6) and the assumption that $y \leqslant p_1 \leqslant p_2$, we have

$$\mathrm{CP}_o(p_1; (x,y)) = \frac{\min\{y, m\}r_h + \min\{x, m-p_1\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} \geqslant \frac{\min\{y, m\}r_h + \min\{x, m-p_2\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} = \mathrm{CP}_o(p_2; (x,y$$

The last inequality is the desired result. Moreover, if $p_2 > p_1 > m - x$, we have the strong monotonicity.

By the definition of CP_u in Equation (7) and the assumption that $p_1 \leqslant p_2 \leqslant y$, we have

$$\begin{aligned}\mathrm{CP}_u(p_2; (x,y)) &= \frac{\max\{p_2, \min\{y, (m-x)^+\}\}r_h + \min\{x, m-p_2\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} \\ &\geqslant \frac{\max\{p_1, \min\{y, (m-x)^+\}\}r_h + \min\{x, m-p_1\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} = \mathrm{CP}_u(p_1; (x,y)),\end{aligned}$$

where the inequality holds because for $p_2 \geqslant p_1$, $\max\{p_2, \min\{y, m-x\}\} \geqslant \max\{p_1, \min\{y, m-x\}\}$ and $\min\{x, m-p_2\} \geqslant \min\{x, m-p_1\}$. Moreover, if $p_2 > p_1 > m - x$, we have the strong monotonicity.

G.3. Lemma 10 and its Proof

LEMMA 10. *For any $y > m$ and $p \in [0, m]$, we have*

$$CP_u(p; (x, y)) = CP_u(p; (x, m)).$$

For any $x > m$ and $p \in [0, m]$, we have

$$CP_o(p; (x, y)) = CP_o(p; (m, y)),$$

and

$$CP_u(p; (x, y)) = CP_u(p; (m, y)).$$

Proof of Lemma 10 We first prove that $\mathrm{CP}_u(p; (x, y)) = \mathrm{CP}_u(p; (x, m))$ for any $y > m$. By Equation (7), we have

$$\begin{aligned}\mathrm{CP}_u(p; (x, y)) &= \frac{\max\{p, \min\{y, (m-x)^+\}\}r_h + \min\{x, m-p\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} \\ &= \frac{\max\{p, \min\{m, (m-x)^+\}\}r_h + \min\{x, m-p\}r_\ell}{mr_h} = \mathrm{CP}_u(p; (x, m)).\end{aligned}$$

Next, we show that $\mathrm{CP}_o(p; (x, y)) = \mathrm{CP}_o(p; (m, y))$ for any $x > m$. By Equation (6), we have

$$\begin{aligned}\mathrm{CP}_o(p; (x, y)) &= \frac{\min\{y, m\}r_h + \min\{x, m-p\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} \\ &= \frac{\min\{y, m\}r_h + \min\{m, m-p\}r_\ell}{\min\{y, m\}r_h + \min\{m, (m-y)^+\}r_\ell} = \mathrm{CP}_o(p; (m, y)).\end{aligned}$$

By Equation (7), we have

$$\begin{aligned}\mathrm{CP}_u(p; (x, y)) &= \frac{\max\{p, \min\{y, (m-x)^+\}\}r_h + \min\{x, m-p\}r_\ell}{\min\{y, m\}r_h + \min\{x, (m-y)^+\}r_\ell} \\ &= \frac{\max\{p, \min\{y, 0\}\}r_h + \min\{m, m-p\}r_\ell}{\min\{y, m\}r_h + \min\{m, (m-y)^+\}r_\ell} = \mathrm{CP}_u(p; (m, y)).\end{aligned}$$

□

G.4. Lemma 11 and its Proof

LEMMA 11. *Recall that $\underline{h}(x)$, defined in Equation (5), is the lower envelop of $\mathcal{R}$. Let $\underline{\mathcal{H}}(x) = \min\{\underline{h}(x), m\}$. If there exists $x_0 \in [\underline{x}, \bar{x}]$ such that $\underline{h}(x_0) < m$, then either $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for all $x \in [\underline{x}, \bar{x}]$ or there exists $\underline{x} \leqslant x_1 < x_2 \leqslant \bar{x}$ such that $\underline{\mathcal{H}}(x) = m$ for $x \in [\underline{x}, x_1] \cup [x_2, \bar{x}]$ and $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for $x \in [x_1, x_2]$. In other words, $\underline{\mathcal{H}}(x) = \underline{h}(x)$ on a connected interval.*

Proof of Lemma 11 We will prove this statement by contradiction. Let us suppose that $\underline{\mathcal{H}}(x) = \underline{h}(x)$ on $[x_3, x_4] \cup [x_5, x_6]$. Then, let us randomly pick $x_7 \in (x_4, x_5)$. We recall that $\underline{\mathcal{H}}(x) = \min\{\underline{h}(x), m\}$.

Now, we observe that $\underline{h}(x_4) < m$, $\underline{h}(x_5) < m$, and $\underline{h}(x_7) \geqslant m$. However, this contradicts the fact that $\underline{h}(\cdot)$ is a convex function. Therefore, our initial assumption that $\underline{\mathcal{H}}(x) = \underline{h}(x)$ on $[x_3, x_4] \cup [x_5, x_6]$ is false

□

G.5. Lemma 12 and its Proof

LEMMA 12. *Recall that $\bar{x}_u$ is defined in Equation (14), and $\underline{x}_u = \sup\{\underline{x} < x < \bar{x}_u : CP_o(m; (x, \underline{h}(x))) \geqslant C\}$. Then, for $x \in (\underline{x}_u, \bar{x}_u)$, we have $\underline{\mathcal{H}}(x) = \underline{h}(x)$.*

Proof of Lemma 12 By Lemma 11, we have either $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for all $x \in [\underline{x}, \bar{x}]$ or there exists $\underline{x} \leqslant x_1 < x_2 \leqslant \bar{x}$ such that $\underline{\mathcal{H}}(x) = m$ for $x \in [\underline{x}, x_1] \cup [x_2, \bar{x}]$ and $\underline{\mathcal{H}}(x) = \underline{h}(x)$ for $x \in [x_1, x_2]$. In the first case, the statement is trivial.

In the second case, we prove by contradiction. Consider any $x \in (\underline{x}_u, \bar{x}_u)$ and assume that $x \in [\underline{x}, x_1]$. We then have $\underline{\mathcal{H}}(x) = m$, which implies that $\underline{h}(x) \geqslant m$. By Equation (6), we have

$$\mathrm{CP}_o(m; (x, \underline{h}(x))) = 1 \geqslant C,$$

which implies that $x < \underline{x}_u$, which contradicts the fact that $x \in (\underline{x}_u, \bar{x}_u)$.

Now, consider any $x \in (\underline{x}_u, \bar{x}_u)$ and assume that $x \in [x_2, \bar{x}]$. We then have $\underline{\mathcal{H}}(x) = m$, which implies that $\underline{h}(x) \geqslant m$. Recall that

$$\bar{x}_u = \begin{cases} x_L & \text{if } x_L + y_L \geqslant m; \\ \sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\} & \text{Otherwise}, \end{cases} \tag{52}$$

In this case, as $\underline{h}(x) < m$ for $x \in [x_1, x_2]$, we have $x_L \in [x_1, x_2]$. Therefore, $\bar{x}_u \neq x_L$ because otherwise, we have $\bar{x}_u = x_L \leqslant x_2$, and there does not exist $x \in (\underline{x}_u, \bar{x}_u)$ and $x \in [x_2, \bar{x}]$. Then, we have $\bar{x}_u = \sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\}$. As we assume that $\underline{h}(x) \geqslant m$ for $x \in [x_2, \bar{x}]$, then we have $\underline{\mathcal{H}}'(\bar{x}^-) = 0$ because $\underline{\mathcal{H}}(x) = m$ for $x \in [x_2, \bar{x}]$. Therefore, $\bar{x}_u = \bar{x}$. In addition, as we have $\underline{h}(\bar{x}) \geqslant m$, by Equation (6), we have

$$\mathrm{CP}_o(m; (\bar{x}, \underline{h}(\bar{x}))) = 1 \geqslant C,$$

which implies that $\underline{x}_u = \bar{x}_u$. Therefore, there does not exist $x \in (\underline{x}_u, \bar{x}_u)$, which is a contradiction.

□

G.6. Lemma 13 and its Proof

LEMMA 13. *Fix any $C \in (0,1)$. Recall that $\underline{x}_u = \sup\{\underline{x} < x < \bar{x}_u : CP_o(m; (x, \underline{h}(x))) \geqslant C\}$, and $\bar{x}_l = \inf\{x_H < x < \bar{x} : CP_u(0; (x, \overline{h}(x))) \geqslant C\}$. Then, for $x \in [\bar{x}_\ell, \bar{x}]$, we have*

$$CP_u(0; (x, \overline{h}(x))) \geqslant C.$$

Similarly, for any $\underline{x} < x \leqslant \underline{x}_u$, we have

$$CP_o(m; (x, \underline{h}(x))) \geqslant C.$$

Proof of Lemma 13 *Part 1:* We first define the region $\mathcal{R}_1$ such that for any $(x,y) \in \mathcal{R}_1$, $\mathrm{CP}_u(p(x) = 0; (x,y)) = C$. Obviously, $\{x + y \leqslant m\} \notin \mathcal{R}_1$ because if $x + y \leqslant m$ and $p(x) = 0$, by Lemma 8, $\mathrm{CP}_u(p(x) = 0; (x,y)) = 1$. Next, we find $(x,y) \in \{x + y > m\}$, which belongs to $\mathcal{R}_1$.

We solve the following equation: for $x + y > m$

$$\mathrm{CP}_u(0; (x,y)) = C.$$

Notice that for $x > x_H$, by the definition of H, we have $y < m$, so we can obtain

$$\frac{(m-x)r_h + xr_\ell}{yr_h + (m-y)r_\ell} = C,$$

which is equivalent as

$$(m-x)r_h + xr_\ell = C(yr_h + (m-y)r_\ell). \tag{53}$$

We take derivative on both side of Equation (53), and we get

$$y'(x) = -1/C.$$

Let $\mathcal{L}(x)$ be the line with slope $-1/C$ and across $(\bar{x}_\ell, \bar{h}(\bar{x}_\ell))$. We have $\mathrm{CP}_u(0; (x, \mathcal{L}(x))) = C$. Recall that $\bar{x}_\ell = \inf\{x_H < x < \bar{x} : \mathrm{CP}_u(0; (x, \overline{h}(x))) \geqslant C\}$. Then, for any $x < \bar{x}_\ell$, we have $\mathrm{CP}_u(0; (x, \bar{h}(x))) < C$. By Lemma 2, we have $\bar{h}(x) > \mathcal{L}(x)$, and by Lemma 26, we have for $x > \bar{x}_\ell$, $\bar{h}(x) < \mathcal{L}(x)$. By Lemma 2 again, we have $\mathrm{CP}_u(0; (x, \bar{h}(x)) \geqslant C$.

Part 2: We define the region $\mathcal{R}_2$ such that for any $(x,y) \in \mathcal{R}_2$, $\mathrm{CP}_o(p(x) = m; (x,y)) = C$. For $(x,y) \in \{x + y \geqslant m\}$, we solve

$$\mathrm{CP}_o(m; (x,y)) = C.$$

Notice that we must have $y < m$ because otherwise, $\mathrm{CP}_o(m; (x,y)) = 1 \neq C$. Then, we can obtain

$$\frac{yr_h}{yr_h + (m-y)r_\ell} = C,$$

and we can get $y = \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m$. Therefore, in the area $\{x + y \geqslant m\}$, $\mathcal{R}_2$ is a line with slope 0.

Next, we explore the part $(x, y) \in \{x + y < m\}$, this time

$$\text{CP}_o(m; (x, y)) = C,$$

implies that

$$\frac{yr_h}{yr_h + xr_\ell} = C,$$

which is equivalent as

$$y(1 - C)r_h = Cr_\ell x. \tag{54}$$

Then, we take derivative on both side of Equation (54), and we get

$$y'(x) = \frac{Cr_\ell}{(1 - C)r_h},$$

which means that, in the area $\{x + y < m\}$, $\mathcal{R}_2$ is a line with positive slope $\frac{Cr_\ell}{(1-C)r_h}$, for $x < m - \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m$, and across $(m - \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m, \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m)$.

Combined with the results above, we start to prove this lemma. Let $\mathcal{L}(x)$ be a piecewise linear line segment such that $(x, \mathcal{L}(x)) \in \mathcal{R}_2$ for all $x \in [0, m]$. That is, for $x \in [0, m - \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m]$, $\mathcal{L}(x)$ has slope $\frac{Cr_\ell}{(1-C)r_h}$ and across $(m - \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m, \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m)$. For $x \in (m - \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m, m]$, $\mathcal{L}(x) = \frac{Cr_\ell}{r_h - C(r_h - r_\ell)} m$. Moreover, for any $x \in [0, m]$, we have $\text{CP}_o(m; (x, \mathcal{L}(x))) = C$.

Recall that $\underline{x}_u = \sup\{\underline{x} < x < \bar{x}_u : \text{CP}_o(m; (x, \underline{h}(x))) \geqslant C\}$. We first show the case where $\underline{x}_u \leqslant x_L$. As $\mathcal{L}$ is an non-decreasing function, and for $x \leqslant x_L$, $\underline{h}(\cdot)$ is a decreasing function. Therefore, as we have $\underline{h}(\underline{x}_u) = \mathcal{L}(\underline{x}_u)$, we have $\underline{h}(x) > \mathcal{L}(x)$ for any $x \in [\underline{x}, \underline{x}_u]$. Therefore, by Lemma 2, we have for any $x \in [\underline{x}, \underline{x}_u]$,

$$\text{CP}_o(m; (x, \underline{h}(x))) \geqslant C.$$

Second, we show the case where $\underline{x}_u > x_L$. In this case, $\underline{h}(\cdot)$ is decreasing for $x < x_L$, and increasing for $x > x_L$. If there exists $x_1 \in [\underline{x}, \underline{x}_u]$ such that $\text{CP}_o(m; (x_1, \underline{h}(x_1))) < C$, then, by Lemma 2, this implies that $\underline{h}(x_1) < \mathcal{L}(x_1)$. As the lowest point L is below the line $\mathcal{L}$ and $\underline{h}(x)$ is convex, we have there are at most two intersections of $\underline{h}(x)$ and $\mathcal{L}$. One is in the left of L and the other is $(\underline{x}_u, \underline{h}(\underline{x}_u))$. As $\underline{h}(x)$ is convex increasing for $x \in [x_L, \bar{x}_u]$ and given that $\mathcal{L}(x)$ is concave increasing for $x \in [x_L, \bar{x}_u]$, we have

$$\underline{h}'(\underline{x}_u^+) \geqslant \underline{h}'(\underline{x}_u^-) \geqslant \mathcal{L}(\underline{x}_u^+),$$

which implies that there exists $\epsilon > 0$, such that $\underline{h}(\underline{x}_u + \epsilon) \geqslant \mathcal{L}(\underline{x}_u + \epsilon)$. By Lemma 2, we have

$$\text{CP}_o(m; (\underline{x}_u + \epsilon, \underline{h}(\underline{x}_u + \epsilon))) \geqslant \text{CP}_o(m; (\underline{x}_u + \epsilon, \langle(\underline{x}_u + \epsilon))) = C,$$

which contradicts to the definition of $\underline{x}_u$. Therefore, for any $x \in [\underline{x}, \underline{x}_u]$,

$$\text{CP}_o(m; (x, \underline{h}(x))) \geqslant C.$$

□

G.7. Lemma 14 and its Proof

LEMMA 14. *Fix any $C > 0$. For any $x \in [\underline{x}_u, \bar{x}_u]$, $CP_o(p; (x, \underline{h}(x))) = C$ has a solution $p \geqslant \min\{\underline{h}(x), m\}$. For any $x \in [x_H, \bar{x}_\ell]$, $CP_u(p; (x, \bar{h}(x))) = C$ has a solution $p \leqslant \min\{\bar{h}(x), m\}$.*

Proof of Lemma 14 Fix any $C > 0$. Take $p = \min\{\underline{h}(x), m\}$, by Equation (6), we have $\text{CP}_o(p; (x, \underline{h}(x))) = 1$. If $\underline{h}(x) \geqslant m$, then we claim that $x \notin [\underline{x}_u, \bar{x}_u]$. This is because take $p = m$, by Lemma 10, we have $\text{CP}_o(m; (x, \underline{h}(x))) = \text{CP}_o(m; (x, m)) = 1$, which contradicts the definition of $\underline{x}_u$. Otherwise, if $\underline{h}(x) < m$, we have $\text{CP}_o(m; (x, \underline{h}(x))) < C$ and $\text{CP}_o(\underline{h}(x); (x, \underline{h}(x))) = 1$. Therefore, by mean value theorem, we have there exists $p_1 \in [\underline{h}(x), m]$ such that $\text{CP}_o(p_1; (x, \underline{h}(x))) = C$.

Similarly, take $p = \min\{\bar{h}(x), m\}$, by Equation (7), we have $\text{CP}_u(p; (x, \bar{h}(x))) = 1$. Take $p = 0$, by the definition of $\underline{x}_\ell = \sup\{\underline{x} < x < x_H : \text{CP}_u(0; (x, \bar{h}(x))) = C\}$, $\quad \bar{x}_l = \inf\{x_H < x < \bar{x} : \text{CP}_u(0; (x, \bar{h}(x))) \geqslant C\}$, we have $\text{CP}_u(p; (x, \bar{h}(x))) < C$. From Equation (7), we can simply check that $\text{CP}_u(p; (x, \bar{h}(x)))$ is continuous in p for any x. Therefore, by mean value theorem, we have there exists $p_1 \in [0, \min\{\bar{h}(x), m\}]$ such that $\text{CP}_u(p_1; (x, \bar{h}(x))) = C$.

G.8. Lemma 15 and its Proof

LEMMA 15. *Recall that*

$$\bar{x}_u = \begin{cases} x_L & \textit{if } x_L + y_L \geqslant m; \\ \sup\{x \in [x_L, \bar{x}] : (1 - C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\} & \textit{Otherwise}. \end{cases} \tag{55}$$

Then, for any $x \in [\bar{x}_u, \bar{x}]$, we have $CP_o(u(x; C); (x, \underline{h}(x))) \geqslant C$.

Proof of Lemma 15 Define $\hat{u}(x; C)$ as the original $u(x; C)$ without forcing to be constant after $\bar{x}_u$. More precisely, for any $C \in [0, 1]$, we define

$$\hat{u}(x; C) = \sup\left\{p \in [0, m] : \text{CP}_o(p; (x, \underline{h}(x))) = C\right\} \qquad x \in [\underline{x}_u, \bar{x}] \tag{56}$$

while we set $\hat{u}(x;C) = m$ for any $x \in [0, \underline{x}_u]$. Note while $u(x;C) \neq \hat{u}(x;C)$ for any $x \in (\bar{x}_u, \bar{x}]$, we have $u(x;C) = \hat{u}(x;C)$ for any $x \in [0, \bar{x}_u]$. Then, we can check that for any $x \in [\underline{x}_u, \bar{x}]$, Equation (57) is satisfied, which means for any $x \in [\underline{x}_u, \bar{x})$ and $C \in [0,1]$, when $\underline{\mathcal{H}}'(x)$ exists, we have

$$\frac{\partial \hat{u}(x;C)}{\partial x} = \begin{cases} ((1-C)\frac{r_h}{r_\ell} + C)\underline{\mathcal{H}}'(x) & \text{if } x + \underline{\mathcal{H}}(x) \geqslant m; \\ (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x) - C & \text{if } x + \underline{\mathcal{H}}(x) < m\,, \end{cases} \tag{57}$$

where $\underline{\mathcal{H}}(x) = \min\{\underline{h}(x), m\}$. By the definition of $\bar{x}_u$, as $\frac{\partial \hat{u}(x;C)}{\partial x}|_{x=\bar{x}_u^-} < 0$ and $\frac{\partial \hat{u}(x;C)}{\partial x}|_{\bar{x}_u^+} \geqslant 0$, we obtain

$$\inf_{x \in [\underline{x}, \bar{x}]} \hat{u}(x;C) = \hat{u}(\bar{x}_u; C),$$

which is because the derivative of $\hat{u}(x;C)$ is linear, and $\frac{\partial \hat{u}(x;C)}{\partial x}|_{x=\bar{x}_u^-} < 0$ and $\frac{\partial \hat{u}(x;C)}{\partial x}|_{\bar{x}_u^+} \geqslant 0$ implies that $\frac{\partial \hat{u}(x;C)}{\partial x} \geqslant 0$ for all $x \geqslant \bar{x}_u$. Therefore, by any $x \geqslant \bar{x}_u$, $\hat{u}(x;C) \geqslant u(x;C)$ as we force $u(x;C)$ to be a constant value of $u(\bar{x}_u;C)$. The definition of $\hat{u}(\cdot;C)$ implies that $\mathrm{CP}_o(\hat{u}(x;C);(x,\underline{h}(x))) = C$. By Lemma 9, for $x \geqslant \bar{x}_u$, as $\hat{u}(x;C) \geqslant u(x;C)$, we have

$$\mathrm{CP}_o(u(x;C);(x,\underline{h}(x))) \geqslant \mathrm{CP}_o(\hat{u}(x;C);(x,\underline{h}(x))) \geqslant C.$$

G.9. Lemma 16 and its Proof

LEMMA 16. *For any $C \in [0,1]$, $\widetilde{l}(x;C)$ is decreasing and is concave for $x \in [\underline{x}, \tilde{x}_\ell]$, where $\tilde{x}_\ell = \inf\{x_{-1} < x < \bar{x} : \widetilde{l}(x;C) = 0\}$. For any $x \in [\underline{x}, \bar{x}]$, $\widetilde{l}(x;C)$ is continuously increasing in C.*

Proof of Lemma 16. By Equation (17), as $l(x;C)$ is decreasing for any $C \in [0,1]$, we have $\widetilde{l}(x;C)$ is also decreasing for any $C \in [0,1]$. Next, by Lemma 6, we have $l(x;C)$ is concave for $x \in [\underline{x}, \bar{x}_\ell]$. Therefore, $\frac{\partial l(x^+;C)}{\partial x}$ is non-increasing. As $\widetilde{l}(x;C) = l(x;C)$ for $x \in [\underline{x}, x_{-1}]$, we have $\frac{\partial \widetilde{l}(x^+;C)}{\partial x}$ is non-increasing for $x \in [\underline{x}, x_{-1}]$. By the definition of x_{-1}, we have $\frac{\partial \widetilde{l}(x_{-1}^-;C)}{\partial x} \geqslant \frac{\partial \widetilde{l}(x_{-1}^+;C)}{\partial x} = -1$, and for $x_{-1} < x \leqslant \tilde{x}$, we have $\widetilde{l}(x;C)$ is a line with slope -1 and $\frac{\partial \widetilde{l}(x^+;C)}{\partial x} = -1$, which is non-increasing. Therefore, $\widetilde{l}(x;C)$ is concave for $x \in [\underline{x}, \tilde{x}_\ell]$.

Fix any $x_1 \in [\underline{x}, \bar{x}]$, by Lemma 6, we know that $l(x_1;C)$ is a continuous increasing function in C. If $x_1 \in [\underline{x}, x_{-1}]$, as $\widetilde{l}(x_1;C) = l(x_1;C)$, we have $\widetilde{l}(x_1;C)$ is also continuously increasing in C. Otherwise, define $\mathcal{L}(x;C) = (-x + x_{-1} + l(x_{-1};C), 0)^+$. Then, $\widetilde{l}(x_1;C) = \max\{\mathcal{L}(x_1;C), l(x_1;C)\}$. As both $\mathcal{L}(x_1;C)$ and $l(x_1;C)$ are increasing continuous in C, we have $\widetilde{l}(x_1;C)$ is increasing and continuous in C.

G.10. Lemma 17 and its Proof

LEMMA 17. *Define $p_b(x)$ as a function which balances the compatible ratio of two points $(x, \underline{h}(x))$ and $(x, \bar{h}(x))$ for any $x \in [\underline{x}, \bar{x}]$; that is,*

$$CP_o(p_b(x); (x, \underline{h}(x))) = CP_u(p_b(x); (x, \bar{h}(x)))\,.$$

Then, $p_b(x)$ exists for any $x \in [\underline{x}, \bar{x}]$, and

$$CP_o(p_b(x); (x, \underline{h}(x))) = CP_u(p_b(x); (x, \bar{h}(x))) \geqslant C^\star(\mathcal{R})\,,$$

where $C^\star(\mathcal{R})$ is the maximum consistent ratio among all PLAs given that the ML advice $\mathcal{R}$.

Proof of Lemma 17. Fix any $x \in [\underline{x}, \bar{x}]$, define $f(p) = \mathrm{CP}_o(p; (x, \underline{h}(x))) - \mathrm{CP}_u(p; (x, \bar{h}(x)))$. As $\mathrm{CP}_o(p; (x, \underline{h}(x)))$ and $\mathrm{CP}_u(p; (x, \bar{h}(x)))$ are both continuous in p, we have $f(p)$ is continuous in p. Next, take $p = \min\{m, \underline{h}(x)\}$, then $\mathrm{CP}_o(p; (x, \underline{h}(x))) = 1$ and $\mathrm{CP}_u(p; (x, \bar{h}(x))) \leqslant 1$, and hence we have $f(\min\{m, \underline{h}(x)\}) \geqslant 0$. Take $p = \min\{m, \bar{h}(x)\}$, then $\mathrm{CP}_u(p; (x, \bar{h}(x))) = 1$ and $\mathrm{CP}_o(p; (x, \underline{h}(x))) \leqslant 1$, and hence we have $f(\min\{m, \bar{h}(x)\}) \leqslant 0$. Therefore, by mean value theorem, there must exist $p \in [\min\{m, \underline{h}(x)\}, \min\{m, \bar{h}(x)\}]$ such that $f(p) = 0$, i.e. $\mathrm{CP}_o(p_b(x); (x, \underline{h}(x))) = \mathrm{CP}_u(p_b(x); (x, \bar{h}(x)))$.

Next, we prove that $\mathrm{CP}_o(p_b(x); (x, \underline{h}(x))) = \mathrm{CP}_u(p_b(x); (x, \bar{h}(x))) \geqslant C^\star(\mathcal{R})$ for any $x \in [\underline{x}, \bar{x}]$. We prove by contradiction. Suppose that there exists $x_1 \in [\underline{x}, \bar{x}]$ such that $\mathrm{CP}_o(p_b(x); (x, \underline{h}(x))) = \mathrm{CP}_u(p_b(x); (x, \bar{h}(x))) < C^\star(\mathcal{R})$, then, if we set any $p < p_b(x)$, by Lemma 9, we have

$$\mathrm{CP}_u(p; (x, \bar{h}(x))) < \mathrm{CP}_u(p_b(x); (x, \bar{h}(x))) < C^\star(\mathcal{R})\,.$$

Similarly, if we set any $p > p_b(x)$, by Lemma 9, we have

$$\mathrm{CP}_o(p; (x, \underline{h}(x))) < \mathrm{CP}_o(p_b(x); (x, \underline{h}(x))) < C^\star(\mathcal{R})\,.$$

Therefore, there does not exist a PL function such that the consistent ratio is $C^\star(\mathcal{R})$, which is a contradiction. This is because here $C^\star(\mathcal{R})$ in is defined as an upper bound on the consistent ratio of any PLA.

G.11. Lemma 18 and its Proof

LEMMA 18. *For any fixed PL function p, we have $CP_o(p; (x, 0))$ is a decreasing function in x. That is, for any $x_1 \leqslant x_2$, we have*

$$CP_o(p; (x_1, 0)) \geqslant CP_o(p; (x_2, 0)).$$

Proof of Lemma 18 If $x_1 \leqslant x_2 \leqslant m$, by Equation (6), we have

$$\mathrm{CP}_o(p; A = (x, 0)) = \frac{0 \cdot r_h + \min\{x, m-p\} r_\ell}{0 \cdot r_h + \min\{x, m-0\} r_\ell} = \frac{\min\{x, m-p\}}{x}.$$

Take any $x_1 \leqslant x_2$, if $\min\{x_2, m-p\} = x_2$, then $\min\{x_1, m-p\} = x_1$, and we have

$$\frac{\min\{x_1, m-p\}}{x_1} = \frac{\min\{x_2, m-p\}}{x_2} = 1.$$

If $\min\{x_2, m-p\} = m-p$ and $\min\{x_1, m-p\} = m-p$, we have

$$\frac{\min\{x_2, m-p\}}{x_2} = \frac{m-p}{x_2} \leqslant \frac{m-p}{x_1} = \frac{\min\{x_1, m-p\}}{x_1}.$$

If $\min\{x_2, m-p\} = m-p$ and $\min\{x_1, m-p\} = x_1$, we have

$$\frac{\min\{x_2, m-p\}}{x_2} = \frac{m-p}{x_2} \leqslant 1 = \frac{x_1}{x_1} = \frac{\min\{x_1, m-p\}}{x_1}.$$

Therefore, we have for any $x_1 \leqslant x_2 \leqslant m$, we have

$$\mathrm{CP}_o(p; (x_1, 0)) \geqslant \mathrm{CP}_o(p; (x_2, 0)).$$

Next, for $x_1 \leqslant m \leqslant x_2$, by Lemma 10, we have

$$\mathrm{CP}_o(p; (x_1, 0)) \geqslant \mathrm{CP}_o(p; (m, 0)) = \mathrm{CP}_o(p; (x_2, 0)).$$

For $m \leqslant x_1 \leqslant x_2$, again, by Lemma 10, we have

$$\mathrm{CP}_o(p; (m, 0)) = \mathrm{CP}_o(p; (x_1, 0)) = \mathrm{CP}_o(p; (x_2, 0)).$$

G.12. Lemma 19 and its Proof

LEMMA 19. *For any $C \in (0,1)$. Let $\bar{x}_u$ be defined in Equation (14), we have*

$$\min\{\bar{x}_u, m - u(\bar{x}_u; C)\} = m - u(\bar{x}_u; C).$$

Recall that H is the point in set $\bar{\mathcal{R}}$ that has the highest low-reward demand, where $\bar{\mathcal{R}} = \{(x, y) \in \mathcal{R} : y = \sup_{(x', y') \in \mathcal{R}} \min\{y', m\}\}$ Further, we have

$$\min\{x_H, m - l(x_H; C^\star(\mathcal{R}))\} = m - l(x_H; C^\star(\mathcal{R})).$$

Let $x_{-1} = \sup\{x \in [x_H, \bar{x}] : \frac{\partial l(x^-; C)}{\partial x} \leqslant -1\}$. We have

$$x_{-1} \geqslant m - l(x_{-1}; C)$$

.

Proof of Lemma 19 We prove by contradiction. If $\min\{\bar{x}_u, m-u(\bar{x}_u;C)\} = \bar{x}_u$, then by Equation (6), we have

$$\mathrm{CP}_o(u(\bar{x}_u;C);(\bar{x}_u,\underline{h}(\bar{x}_u))) = \frac{\min\{\underline{h}(\bar{x}_u),m\}r_h + \min\{m-u(\bar{x}_u;C),\bar{x}_u,\}r_\ell}{\min\{\underline{h}(\bar{x}_u),m\}r_h + \min\{(m-\underline{h}(\bar{x}_u))^+,\bar{x}_u\}r_\ell}.$$

Given that $\min\{\bar{x}_u, m-u(\bar{x}_u;C)\} = \bar{x}_u$, as $u(\bar{x}_u;C) \geqslant \min\{\underline{h}(\bar{x}_u),m\}$, we have $\min\{(m-\underline{h}(\bar{x}_u))^+,\bar{x}_u\} = \bar{x}_u$, and we have

$$\mathrm{CP}_o(u(\bar{x}_u;C);(\bar{x}_u,\underline{h}(\bar{x}_u))) = \frac{\min\{\underline{h}(\bar{x}_u),m\}r_h + \bar{x}_u r_\ell}{\min\{\underline{h}(\bar{x}_u),m\}r_h + \bar{x}_u r_\ell} = 1 > C,$$

which contradicts to the definition of $u(\cdot;C)$.

Next, to show the second statement, We still prove by contradiction. If $\min\{x_H, m-l(x_H;C^\star(\mathcal{R}))\} = x_H$, then by Equation (7), we have

$$\mathrm{CP}_u(l(x_H;C^\star(\mathcal{R}));H) = \frac{\max\{l(x_H;C^\star(\mathcal{R})),\min\{y_H,(m-x_H)^+\}\}r_h + \min\{x_H,m-l(x_H;C^\star(\mathcal{R}))\}r_\ell}{y_H r_h + \min\{x_H,m-y_H\}r_\ell}.$$

Given that $\min\{x_H, m-l(x_H;C^\star(\mathcal{R}))\} = x_H$, we have

$$\mathrm{CP}_u(l(x_H;C^\star(\mathcal{R}));H) = \frac{\min\{y_H,(m-x_H)^+\}r_h + x_H r_\ell}{y_H r_h + \min\{x_H,m-y_H\}r_\ell} = \mathrm{CP}_u(0;H),$$

which implies that $x_H \geqslant \bar{x}_\ell$, and this contradicts to the definition of $\bar{x}_\ell$ since $x_H \leqslant x_{-1} < \bar{x}_\ell$.

Finally, we show that $x_{-1} \geqslant m - l(x_{-1};C)$. We still prove by contradiction. Suppose that $x_{-1} < m - l(x_{-1};C)$. By Equation (7), we have

$$\mathrm{CP}_u(l(x_{-1};C);(x_{-1},\bar{h}(x_{-1}))) = \frac{\max\{l(x_{-1};C),\min\{\bar{h}(x_{-1}),(m-x_{-1})^+\}\}r_h + \min\{x_{-1},m-l(x_{-1};C)\}r_\ell}{\min\{\bar{h}(x_{-1}),m\}r_h + \min\{x_{-1},(m-\bar{h}(x_{-1}))^+\}r_\ell}$$

If we have $x_{-1} < m - l(x_{-1};C)$, then we have

$$C = \mathrm{CP}_u(l(x_{-1};C);(x_{-1},\bar{h}(x_{-1}))) = \frac{\min\{\bar{h}(x_{-1}),(m-x_{-1})^+\}r_h + x_{-1}r_\ell}{\min\{\bar{h}(x_{-1}),m\}r_h + \min\{x_{-1},(m-\bar{h}(x_{-1}))^+\}r_\ell} = \mathrm{CP}_u(0;(x_{-1},\bar{h}(x_-$$

which implies that $x_{-1} \geqslant \bar{x}_\ell$, which is a contradiction.

G.13. Lemma 20 and its Proof

LEMMA 20. *Let $C^\star(\mathcal{R})$ be the optimal consistent ratio of $\mathcal{R}$. Take any $C > C^\star(\mathcal{R})$, then $u(\hat{x};C) = u(\hat{x};C^\star(\mathcal{R}))$ only if both of them are equal to m.*

Proof of Lemma 20 If $u(\hat{x};C^\star(\mathcal{R}))<m$, then by definition of $\underline{x}_u$, we have $\hat{x}\in[\underline{x}_u,\bar{x}]$. For $\hat{x}\in[\underline{x}_u,\bar{x}_u]$, by Equation (13), we have $u(\hat{x};C^\star(\mathcal{R}))=\sup\left\{p\in[0,m]:\mathrm{CP}_o(p;(\hat{x},\underline{h}(\hat{x})))=C^\star(\mathcal{R})\right\}$. For $C>C^\star(\mathcal{R})$, if $u(\hat{x};C)=u(\hat{x};C^\star(\mathcal{R}))$, then we have

$$\begin{aligned}\sup\left\{p\in[0,m]:\mathrm{CP}_o(p;(\hat{x},\underline{h}(\hat{x})))=C\right\}&=u(\hat{x};C)=u(\hat{x};C^\star(\mathcal{R}))\\&=\sup\left\{p\in[0,m]:\mathrm{CP}_o(p;(\hat{x},\underline{h}(\hat{x})))=C^\star(\mathcal{R})\right\},\end{aligned}$$

which implies that $\mathrm{CP}_o(u(\hat{x};C);(x,\underline{h}(\hat{x})))=C$ and $\mathrm{CP}_o(u(\hat{x};C);(x,\underline{h}(\hat{x})))=C^\star(\mathcal{R})$, which is a contradiction. For $\hat{x}\in[\bar{x}_u,\bar{x}]$, we have $u(\hat{x};C^\star(\mathcal{R}))=u(\bar{x}_u;C^\star(\mathcal{R}))$ and $u(\hat{x};C)=u(\bar{x}_u;C)$. By the similar statement above, we have

$$\begin{aligned}\sup\left\{p\in[0,m]:\mathrm{CP}_o(p;(\hat{x},\underline{h}(\hat{x})))=C\right\}&=u(\bar{x}_u;C)=u(\bar{x}_u;C^\star(\mathcal{R}))\\&=\sup\left\{p\in[0,m]:\mathrm{CP}_o(p;(\hat{x},\underline{h}(\hat{x})))=C^\star(\mathcal{R})\right\},\end{aligned}$$

which implies that $\mathrm{CP}_o(u(\bar{x}_u;C);(\hat{x},\underline{h}(\hat{x})))=C$ and $\mathrm{CP}_o(u(\bar{x}_u;C);(\hat{x},\underline{h}(\hat{x})))=C^\star(\mathcal{R})$, which is a contradiction.

G.14. Lemma 21 and its Proof

LEMMA 21. *Recall that $u(\cdot;C)$ is defined in Equation (13), and $\bar{x}_u$ is defined in Equation (14). We have $u(x;C)$ gets its minimum value at $x\in[\bar{x}_u,\bar{x}]$.*

Proof of Lemma 21 By the first property of Lemma 6, we have

$$\frac{\partial u(x;C)}{\partial x}=\begin{cases}((1-C)\frac{r_h}{r_\ell}+C)\underline{\mathcal{H}}'(x) & \text{if } x+\underline{\mathcal{H}}(x)\geqslant m;\\ (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x)-C & \text{if } x+\underline{\mathcal{H}}(x)<m.\end{cases}\tag{58}$$

Recall that

$$\bar{x}_u=\begin{cases}x_L & \text{if } x_L+y_L\geqslant m;\\ \sup\{x\in[x_L,\bar{x}]:(1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-)-C<0\} & \text{Otherwise.}\end{cases}\tag{59}$$

Therefore, if $x_L+y_L\geqslant m$, we have $\bar{x}_u=x_L$. As L is the lowest point, which implies that $\underline{h}'(x_L^-)<0$ and $\underline{h}'(x_L^+)>0$. Recall that $\underline{\mathcal{H}}(x)=\min\{\underline{h},m\}$, we have $\underline{\mathcal{H}}'(x_L^-)\leqslant 0$ and $\underline{\mathcal{H}}'(x_L^+)\geqslant 0$. For $x+\underline{\mathcal{H}}(x)\geqslant m$, as we have $\frac{\partial u(x;C)}{\partial x}=((1-C)\frac{r_h}{r_\ell}+C)\underline{\mathcal{H}}'(x)$, we have $\frac{\partial u(x;C)}{\partial x}<0$ for $x<x_L$ and $\frac{\partial u(x;C)}{\partial x}>0$ for $x>x_L$, which implies that $\bar{x}_u=x_L$ is the point such that $u(x;C)$ achieves its lowest value. As we force $u(x;C)=u(\bar{x}_u;C)$, we have $u(x;C)$ gets its minimum value at $x\in[\bar{x}_u,\bar{x}]$.

In other cases, by taking $\bar{x}_u=\sup\{x\in[x_L,\bar{x}]:(1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-)-C<0\}$, we have $\frac{\partial u(x;C)}{\partial x}<0$ for $x<\bar{x}_u$ and $\frac{\partial u(x;C)}{\partial x}>0$ for $x>\bar{x}_u$, which implies that $\bar{x}_u$ is the point such that $u(x;C)$ achieves its lowest value. As we force $u(x;C)=u(\bar{x}_u;C)$, we have $u(x;C)$ gets its minimum value at $x\in[\bar{x}_u,\bar{x}]$.

G.15. Lemma 22 and its Proof

LEMMA 22. *Recall that $\bar{x}_u$ is defined in Equation (14), and $\mathcal{V}$ is the x-vertices set of a polyhedron $\mathcal{R}$ plus all elements of $\mathcal{R}_0$, where $\mathcal{R}_0 = \{(x, \underline{h}(x) : x \in [\underline{x}, \bar{x}]\} \cap \{(x,y) : x + y = m\}$, then $\bar{x}_u \in \mathcal{V}$.*

Proof of Lemma 22 Recall that

$$\bar{x}_u = \begin{cases} x_L & \text{if } x_L + y_L \geqslant m; \\ \sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\} & \text{Otherwise}, \end{cases} \tag{60}$$

That is, $\bar{x}_u$ equals to either x_L or $\sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\}$. As L is a vertex of $\mathcal{R}$, we have $x_L \in \mathcal{V}$.

Then, we claim that $\bar{x}_u = \sup\{x \in [x_L, \bar{x}] : (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x^-) - C < 0\} \in \mathcal{V}$. As $\mathcal{R}$ is a polyhedron, we have $\underline{h}(\cdot)$ is a piecewise linear function. As $\underline{\mathcal{H}}(x) = \min\{\underline{h}(x), m\}$, we have $\underline{\mathcal{H}}(\cdot)$ is also a piecewise linear function. If $\bar{x}_u$ is not a x-vertex, then there exists $\epsilon > 0$ such that $\underline{\mathcal{H}}'(x^-) = \underline{\mathcal{H}}'((x+\epsilon)^-)$, and we have

$$(1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'((x+\epsilon)^-) - C < 0,$$

which contradicts the definition of $\bar{x}_u$.

G.16. Lemma 23 and its Proof

LEMMA 23. *If $\tilde{l}(\hat{x}; C^\star(\mathcal{R})) = u(\hat{x}; C^\star(\mathcal{R}))$ for $x_H \leqslant \hat{x} \leqslant x_{-1}$, we have $\hat{x} \in [\underline{x}_u, \bar{x}_u]$.*

Proof of Lemma 23 As $u(\hat{x}; C^\star(\mathcal{R})) = \tilde{l}(\hat{x}; C^\star(\mathcal{R})) < m$, we have $\hat{x} > \underline{x}_u$. Then, we show that $\hat{x} \leqslant \bar{x}_u$ by contradiction. If $\hat{x} > \bar{x}_u$, as $u(\cdot; C^\star(\mathcal{R}))$ is constant between $[\bar{x}_u, \hat{x}]$ and $\tilde{l}(\cdot; C^\star(\mathcal{R}))$ is a decreasing function for $x \in [x_H, \hat{x}]$, we have there exists $\epsilon > 0$ such that $\tilde{l}(\hat{x} - \epsilon; C^\star(\mathcal{R})) > u(\hat{x} - \epsilon; C^\star(\mathcal{R}))$, which is a contradiction. Therefore, we have $\hat{x} \in [\underline{x}_u, \bar{x}_u]$,

G.17. Lemma 24 and its Proof

LEMMA 24. *Recall that $x_{-1} = \sup\{x \in [x_H, \bar{x}] : \frac{\partial l(x^-; C^\star(\mathcal{R}))}{\partial x} \leqslant -1\}$. Given a polyhedron $\mathcal{R}$, recall that $\mathcal{V}$ is the x-vertices set of $\mathcal{R}$ plus all elements in $\mathcal{R}_0$, then $x_{-1} \in \mathcal{V}$.*

Proof of Lemma 24 As $\mathcal{R}$ is a polyhedron, we have $\bar{h}(\cdot)$ is a piecewise linear function. By the second property of Lemma 6, we have $l(\cdot; C)$ is also a piecewise linear function. Then, we prove by contradiction. Suppose that $x_{-1} \notin \mathcal{V}$. Then, there exists $\epsilon > 0$ such that $\frac{\partial l(x_{-1}^-; C^\star(\mathcal{R}))}{\partial x} = \frac{\partial l((x_{-1}+\epsilon)^-; C^\star(\mathcal{R}))}{\partial x}$. Then, we have $\frac{\partial l((x_{-1}+\epsilon)^-; C^\star(\mathcal{R}))}{\partial x} \leqslant -1$, which contradicts to the definition of x_{-1}.

G.18. Lemma 25 and its Proof

LEMMA 25. *Suppose that $\mathcal{R}$ is a polyhedron. Recall that $u(\cdot;C)$ and $\widetilde{l}(\cdot;C)$ are defined in Equations (13) and (17) respectively. Then, we have $u(\cdot;C)$ and $\widetilde{l}(\cdot;C)$ are piecewise linear functions and all of their x-vertices are a subset of $\mathcal{V}$. In addition, the elements of $\mathcal{R}_0$ are x-vertices of $u(\cdot;C)$, where $\mathcal{R}_0 = \{(x, \underline{h}(x) : x \in [\underline{x}, \bar{x}]\} \cap \{(x,y) : x + y = m\}$.*

Proof of Lemma 25 As $\mathcal{R}$ is a polyhedron, we have both $\bar{h}(\cdot)$ and $\underline{h}(\cdot)$ are piecewise linear. Since $\overline{\mathcal{H}}(x) = \min\{m, \bar{h}(x)\}$ and $\underline{\mathcal{H}}(x) = \min\{m, \underline{h}(x)\}$, we have both $\overline{\mathcal{H}}(\cdot)$ and $\underline{\mathcal{H}}(\cdot)$ are piecewise linear and both $\overline{\mathcal{H}}'(\cdot)$ and $\underline{\mathcal{H}}'(\cdot)$ are piecewise constant.

Recall that by Lemma 6, we have for any $x \in (x_H, \bar{x}_l)$ and $C \in [0,1]$,

$$\frac{\partial l(x;C)}{\partial x} = C\overline{\mathcal{H}}'(x).$$

Therefore, $\frac{\partial l(x;C)}{\partial x}$ is piecewise constant and $l(x;C)$ is piecewise linear. In addition, we can find that the x-vertices of $l(\cdot;C)$ are also ones of $\bar{h}(\cdot)$. By Equation (17) and Lemma 24, we have $\widetilde{l}(\cdot;C)$ is also piecewise linear with x-vertices belong to ones of $\bar{h}(\cdot)$.

Recall that by Lemma 6, we have for any $x \in (\underline{x}_u, \bar{x}_u)$ and $C \in [0,1]$,

$$\frac{\partial u(x;C)}{\partial x} = \begin{cases} ((1-C)\frac{r_h}{r_\ell} + C)\underline{\mathcal{H}}'(x) & \text{if } x + \underline{\mathcal{H}}(x) \geqslant m; \\ (1-C)\frac{r_h}{r_\ell}\underline{\mathcal{H}}'(x) - C & \text{if } x + \underline{\mathcal{H}}(x) < m. \end{cases} \tag{61}$$

Therefore, we have $\frac{\partial u(x;C)}{\partial x}$ is piecewise constant and $u(x;C)$ is piecewise linear. In addition, the x-vertices of $u(x;C)$ for $x + \underline{\mathcal{H}}(x) \geqslant m$ is a subset to x-vertices of $\overline{\mathcal{H}}(x)$. The x-vertices of $u(x;C)$ for $x + \underline{\mathcal{H}}(x) < m$ is also a subset to x-vertices of $\overline{\mathcal{H}}(x)$. Moreover, any point x such that $x + \underline{\mathcal{H}}(x) = m$, which is we defined as an element of $\mathcal{R}_0$, is also a vertex, and by our definition, $\mathcal{V}$ contains all elements of $\mathcal{R}_0$.

G.19. Geometric Lemmas

LEMMA 26. *Suppose that we have a line $\mathcal{L}(x)$ with any negative slope on an interval $\mathcal{I}$. $f(x)$ is a concave decreasing function which intersects $\mathcal{L}(x)$ at $(x_0, f(x_0))$. If there exists $x_1 < x_0$ such that $f(x_1) > f(x_0)$, then for all $x > x_0$, we have $f(x) < \mathcal{L}(x)$.*

Proof of Lemma 26 Suppose that there exists $x_2 > x_0$ such that $f(x_2) \geqslant \mathcal{L}(x_2)$. As $(x_1, f(x_1)$, $(x_2, f(x_2))$ are both above the line $\mathcal{L}$, if we connect $(x_1, f(x_1)$, $(x_2, f(x_2))$ by a line $\mathcal{L}_1(x)$, we have $\mathcal{L}_1(x) > \mathcal{L}(x)$ for $x \in [x_1, x_2]$. Therefore, $\mathcal{L}_1(x_0) > \mathcal{L}(x_0) = f(x_0)$ since $x_0 \in [x_1, x_2]$.

However, as $f(x)$ is a concave function, if we take $x_1 < x_2$ and connect $(x_1, f(x_1), (x_2, f(x_2))$ by a line $\mathcal{L}_1(x)$, we should always have $f(x) \geqslant \mathcal{L}_1(x)$, which is a contradiction to $\mathcal{L}_1(x_0) > \mathcal{L}(x_0) = f(x_0)$.

□

LEMMA 27. *Let $f(x)$ and $g(x)$ be two piecewise linear functions defined on an interval I, with $f(x) \geqslant g(x)$ for any $x \in I$. If $\{x : f(x) = g(x)\}$ is not empty, we have there exists $x_0 \in \mathcal{V}$, which is an x-vertex of either $f(x)$ or $g(x)$, such that $f(x_0) = g(x_0)$.*

Proof of Lemma 27 We prove by contradiction. Suppose that any $x_1 \in \{x : f(x) = g(x)\}$ is not an x-vertex of either $f(x)$ or $g(x)$. Then, we have $f'(x_1^+) = f'(x_1^-)$ and $g'(x_1^+) = g'(x_1^-)$. As $f(x) \geqslant g(x)$ everywhere and $f(x_1) = g(x_1)$, we have $f'(x_1^-) \leqslant g'(x_1^-)$.

If $f'(x_1^-) < g'(x_1^-)$, we have $f'(x_1^+) < g'(x_1^+)$ and by $f(x_1) = g(x_1)$, we have there exists $\epsilon > 0$ such that $f(x_1 + \epsilon) < g(x_1 + \epsilon)$, which is a contradiction to $f(x) \geqslant g(x)$ everywhere.

If $f'(x_1^-) = g'(x_1^-)$, we define x_2 as $x_2 = \inf\{x : f(x) = g(x)$ for any $x \in [x, x_1]\}$. Then, we have $f'(x_2^+) = g'(x_2^+)$. By the definition of x_2, we know that for any $\epsilon_1 > 0$, we have $f(x_2 - \epsilon_1) > g(x_2 - \epsilon_1)$, which implies that $f(x_2^-) \neq g(x_2^-)$. As $f'(x_2^+) = g'(x_2^+)$, we have either $f(x_2^-) \neq f'(x_2^+)$ or $g(x_2^-) \neq g'(x_2^+)$, which implies that x_2 is an x-vertex for f or g, which is a contradiction.

□

Appendix H: Supplementary Materials for Section 7

H.1. Stochastic Arrivals

Tables 6 and 7 present the results under different forms of ML advice, using the demand models defined in Equations (30) and (31), as well as under the stochastic order assumption. Consistent with our earlier findings, the average compatible ratio remains relatively stable as n increases, while the worst-case ratio shows more noticeable improvement. Furthermore, we observe that the best performance—both in terms of average and worst-case ratios—is achieved when using either the polyhedron advice with $\epsilon = 0.3$ or the box/ellipsoid advice.

H.2. Additional Details on the Neural Network Method

This appendix presents the figures associated with the NN-based ML advice. Following the approach of Pearce et al. (2018), the NN is trained to produce a prediction interval for each customer type. Figure 8 displays results for four training-sample sizes: $n = 10, 100, 1000,$ and 2000. Each plot reports the training loss and shows a histogram of 500 test samples, together with dashed lines indicating the learned prediction interval (PI) for the high-reward type.

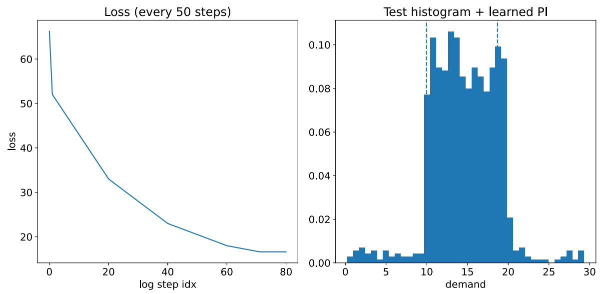
Table 6 Results under ML advice (using the demand model in Equation (30)) and stochastic order. The standard error of all reported values is less than 0.003. In each column, the top two values of average CP are highlighted in very light gray, and the top two values of worst-case CP are highlighted in light gray. Additionally, the highest average and worst-case CP across all settings for each value of n are shown in bold.

		$n=10$			$n=100$		
Algorithm	Metric	$C^\star(\mathcal{R})$	$0.9\cdot C^\star(\mathcal{R})$	$0.8\cdot C^\star(\mathcal{R})$	$C^\star(\mathcal{R})$	$0.9\cdot C^\star(\mathcal{R})$	$0.8\cdot C^\star(\mathcal{R})$
Alg. 1 (Polyhedron, $\epsilon=0.1,\theta=10$)	Avg. CP	0.852	0.840	0.831	0.855	0.842	0.836
	Worst CP	0.639	0.683	0.687	0.646	0.688	0.693
Alg. 1 (Polyhedron, $\epsilon=0.1,\theta=30$)	Avg. CP	0.855	0.841	0.832	0.854	0.844	0.838
	Worst CP	0.637	0.683	0.686	0.647	0.687	0.691
Alg. 1 (Polyhedron, $\epsilon=0.3,\theta=10$)	Avg. CP	0.941	0.927	0.900	0.949	0.935	0.902
	Worst CP	0.674	0.730	0.737	0.681	0.736	0.744
Alg. 1 (Polyhedron, $\epsilon=0.3,\theta=30$)	Avg. CP	0.940	0.928	0.900	0.949	0.937	0.903
	Worst CP	0.672	0.729	0.737	0.682	0.735	0.745
Alg. 1 (Box Advice, $z=90\%$)	Avg. CP	**0.943**	0.929	0.901	**0.953**	0.939	0.905
	Worst CP	0.676	0.733	**0.741**	0.685	0.744	**0.746**
Alg. 1 (Box Advice, $z=80\%$)	Avg. CP	0.936	0.922	0.893	0.950	0.943	0.907
	Worst CP	0.667	0.726	0.732	0.687	0.738	0.744
Point ML Advice	Avg. CP	0.910	0.853	0.814	0.918	0.860	0.819
	Worst CP	0.555	0.584	0.673	0.559	0.590	0.676
BQ Benchmark	Avg. CP	0.823	-	-	0.823	-	-
	Worst CP	0.674	-	-	0.671	-	-
PR Benchmark ($z=90\%$)	Avg. CP	0.933	-	-	0.941	-	-
	Worst CP	0.666	-	-	0.688	-	-
PR Benchmark ($z=80\%$)	Avg. CP	0.926	-	-	0.935	-	-
	Worst CP	0.658	-	-	0.668	-	-

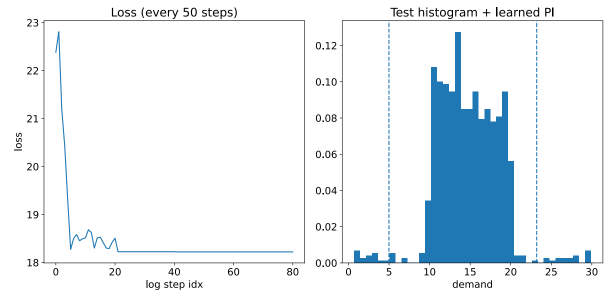
The figure highlights a clear trend: when the training set is very small ($n<1000$), the prediction intervals fluctuate substantially, reflecting underfitting and sensitivity to sampling noise. As n increases, the intervals become more stable; for $n\geqslant 2000$, we observe that the endpoints converge to a consistent range. These visual patterns align with the quantitative results in Table 4 and help explain why larger sample sizes are important for reliable NN-based ML advice.

Table 7 Results under ML advice (using the demand model in Equation (31)) and stochastic order. The standard error of all reported values is less than 0.005. In each column, the top two values of average CP are highlighted in very light gray, and the top two values of worst-case CP are highlighted in light gray. Additionally, the highest average and worst-case CP across all settings for each value of n are shown in bold.

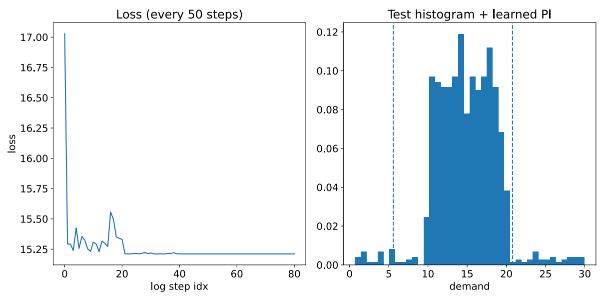
		$n=10$			$n=100$		
Algorithm	Metric	$C^\star(\mathcal{R})$	$0.9\cdot C^\star(\mathcal{R})$	$0.8\cdot C^\star(\mathcal{R})$	$C^\star(\mathcal{R})$	$0.9\cdot C^\star(\mathcal{R})$	$0.8\cdot C^\star(\mathcal{R})$
Alg. 1 (Polyhedron, $\epsilon=0.1,\theta=10$)	Avg. CP	0.858	0.850	0.842	0.867	0.857	0.848
	Worst CP	0.630	0.645	0.650	0.634	0.648	0.655
Alg. 1 (Polyhedron, $\epsilon=0.1,\theta=30$)	Avg. CP	0.859	0.852	0.842	0.868	0.859	0.849
	Worst CP	0.628	0.645	0.649	0.633	0.647	0.654
Alg. 1 (Polyhedron, $\epsilon=0.3,\theta=10$)	Avg. CP	0.939	0.910	0.883	0.942	0.915	0.890
	Worst CP	0.649	0.666	0.669	0.653	0.672	0.675
Alg. 1 (Polyhedron, $\epsilon=0.3,\theta=30$)	Avg. CP	0.938	0.908	0.882	0.941	0.915	0.889
	Worst CP	0.650	0.665	0.668	0.652	0.671	0.674
Alg. 1 (Ellipsoid Advice, $z=90\%$)	Avg. CP	**0.947**	0.922	0.891	**0.955**	0.939	0.912
	Worst CP	0.679	0.698	**0.703**	0.692	0.716	**0.724**
Alg. 1 (Ellipsoid Advice, $z=80\%$)	Avg. CP	0.934	0.918	0.880	0.952	0.931	0.895
	Worst CP	0.664	0.683	0.692	0.672	0.696	0.703
Point ML Advice	Avg. CP	0.909	0.852	0.819	0.914	0.862	0.823
	Worst CP	0.551	0.583	0.639	0.559	0.599	0.655
BQ Benchmark	Avg. CP	0.834	-	-	0.834	-	-
	Worst CP	0.670	-	-	0.670	-	-



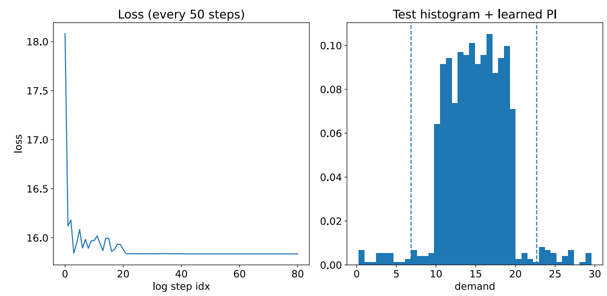
(a) $n = 10$



(b) $n = 100$



(c) $n = 1000$



(d) $n = 2000$

Figure 8 Training loss (left) and prediction intervals for the high-reward type (right), constructed using the neural network approach of Pearce et al. (2018) with training sample sizes $n \in \{10, 100, 1000, 2000\}$. The right-hand curve in each subfigure shows the learned interval (dashed lines) together with the histogram of test samples.