Technical Proofs and Additional Results

Appendix A: Limitations of DP for Finding Markovian Stopping Rules in Non-Markovian Optimal Stopping Problems

In the context of non-Markovian optimal stopping problems, a subtle but important challenge is that the exercise policies which define the best Markovian stopping rule cannot be found in general using backwards recursion. Intuitively, this problem arises because Bellman's dynamic programming equations no longer hold when the stochastic process is non-Markovian. This fact is illustrated numerically in §5.1 and motivates the development of methods which optimize over the exercise policies in all time periods simultaneously. For the sake of completeness, we provide the following example in which there is a Markovian stopping rule that is optimal for the non-Markovian stopping problem but is not obtained using backwards recursion.

Example EC.1. Consider a three-period optimal stopping problem with a one-dimensional non-Markovian stopping process that obeys the following probability distribution:

$$\mathbb{P}(x_1 = 3, x_2 = 2, x_3 = 1) = \frac{2}{3}; \quad \mathbb{P}(x_1 = 1, x_2 = 2, x_3 = 3) = \frac{1}{3}.$$

Let the reward of stopping on each period t be equal to the current state x_t, and recall that our goal is to find a stopping rule which maximizes the expected reward. We observe that there is an optimal stopping rule for this non-Markovian stopping problem that is a Markovian stopping rule, defined by exercise policies $\mu_1^*(x_1) = \textsc{Stop}$ if and only if $x_1 = 3$, $\mu_2^*(x_2) = \textsc{Continue}$ for all x_2, and $\mu_3^*(x_3) = \textsc{Stop}$ for all x_3. Applying this stopping rule yields an expected reward of ${}^2/_3 \times 3 + {}^1/_3 \times 3 = 3$.

We now show that the best Markovian stopping rule which is obtained using backwards recursion will have a strictly lower expected reward. Indeed, assume that the reward from not stopping on any period is equal to zero. Then, starting on the last period, it is clear from the dynamic programming principle that the optimal exercise policy for the last period is $\mu_3^{\mathrm{DP}}(x_3) = \textsc{Stop}$. Hence, conditioned on $x_2 = 2$, the expected reward from stopping on the third period is $1 \times \mathbb{P}(x_3 = 1 \mid x_2 = 2) + 3 \times \mathbb{P}(x_3 = 3 \mid x_2 = 2) = 1 \times {}^2/_3 + 3 \times {}^1/_3 = {}^5/_3$. Because the reward from stopping on the second period (2) is greater than the conditional expected reward of not stopping on the second period (${}^5/_3$), the dynamic programming principle says that the exercise policy in the second period should be chosen to satisfy $\mu_2^{\mathrm{DP}}(2) = \textsc{Stop}$. Finally, unfolding to the first period, we conclude that the exercise policy obtained from dynamic programming is $\mu_1^{\mathrm{DP}}(x_1) = \textsc{Stop}$ if and only if $x_1 = 3$. All together, the Markovian stopping rule for the non-Markovian optimal stopping problem that is obtained using backwards recursion yields an expected reward of ${}^2/_3 \times 3 + {}^1/_3 \times 2 = 2.6\bar{6}$. □

Appendix B: Comparison of Robust Optimization Formulations

As described in §2.2, our formulation of the robust optimization problem (RO) deviates from that of (RO'), which followed from BSS23. In this section, we show that all of the main results from §2.4 and our characterization of optimal Markovian stopping rules in §3 also hold for formulation (RO') with only minor modifications to the proofs. We will then discuss the advantages of using formulation (RO) in the context of our algorithmic techniques that are developed in §4.

B.1. The Relationship Between (RO) and (RO')

Before proceeding further, let us develop intuition for the relationship between the two formulations (RO) and (RO') by comparing them in a simple example. Speaking informally, the following Example EC.2 shows that if the reward functions $g(1,\cdot),\ldots,g(T,\cdot)$ are well behaved functions of stochastic process, and if the radius of the uncertainty sets is small, then the two robust optimization formulations are essentially equivalent. As we will see afterwards, this intuition will extend to general classes of reward functions.

Example EC.2. Consider an optimal stopping problem in which the state space is one-dimensional and the reward functions satisfy $g(t,x)=x_t$ for all periods t. In this case, we observe that the following equalities hold:

$$\begin{aligned}
\text{(RO')} &= \sup_{\mu} \frac{1}{N}\sum_{i=1}^{N} \inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)\\
&= \sup_{\mu} \frac{1}{N}\sum_{i=1}^{N} \inf_{y\in\mathcal{U}^i} \sum_{t=1}^{T} y_t \mathbb{I}\{\tau_\mu(y)=t\}\\
&= \sup_{\mu} \frac{1}{N}\sum_{i=1}^{N} \inf_{y\in\mathcal{U}^i} \sum_{t=1}^{T} \left((y_t-x_t^i)+x_t^i\right)\mathbb{I}\{\tau_\mu(y)=t\}. \qquad \text{(EC.1)}
\end{aligned}$$

Indeed, the first equality follows from the definition of (RO'), the second equality follows from the fact that $g(t,x)=x_t$ for all periods t, and the third equality follows from algebra.

For notational convenience, let $\widehat{J}_{N,\epsilon}(\mu)$ denote the objective value of the robust optimization formulation (RO) corresponding to exercise policies μ. Moreover, we recall from the definition of the uncertainty sets in §2.2 that the inequality $|y_t-x_t^i|\le\epsilon$ is satisfied for all sample paths $i\in\{1,\ldots,N\}$, periods $t\in\{1,\ldots,T\}$, and $y\equiv(y_1,\ldots,y_T)\in\mathcal{U}^i$. Therefore, it follows from line (EC.1) and algebra that

$$\begin{aligned}
\text{(RO')} &\le \sup_{\mu} \frac{1}{N}\sum_{i=1}^{N} \inf_{y\in\mathcal{U}^i} \sum_{t=1}^{T} \left(\epsilon+x_t^i\right)\mathbb{I}\{\tau_\mu(y)=t\} = T\epsilon+\sup_{\mu}\widehat{J}_{N,\epsilon}(\mu) = T\epsilon+\text{(RO)}, \text{ and}\\
\text{(RO')} &\ge \sup_{\mu} \frac{1}{N}\sum_{i=1}^{N} \inf_{y\in\mathcal{U}^i} \sum_{t=1}^{T} \left(-\epsilon+x_t^i\right)\mathbb{I}\{\tau_\mu(y)=t\} = -T\epsilon+\sup_{\mu}\widehat{J}_{N,\epsilon}(\mu) = -T\epsilon+\text{(RO)}.
\end{aligned}$$

We have thus shown that if the robustness parameter $\epsilon\ge 0$ is small, then the two formulations will be close to one another with respect to optimal objective value. Moreover, it follows by identical reasoning that for all exercise policies μ,

$$\left|\widehat{J}_{N,\epsilon}(\mu) - \frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)\right| \le T\epsilon.$$

Therefore, we conclude that if the robustness parameter $\epsilon\ge 0$ is small, then the objective values of the two formulations will be close to one another, uniformly over the space of all exercise policies. □

The above example is insightful because it reveals, at least for problems with linear reward functions, that the robust optimization formulations (RO) and (RO') become essentially equivalent in the asymptotic regime in which the robustness parameter converges to zero. We will see shortly that the same intuition from Example EC.2 will extend to a broad class of reward functions.

We emphasize that the above intuition, of course, does *not imply* that the two formulations (RO) and (RO') are guaranteed to have identical performance on any fixed collection of simulated sample paths. Indeed, we do not preclude the possibility that one of the two formulations (RO) and (RO') may have better finite-sample performance than the other formulation in finding Markovian stopping rules which perform well with respect to (OPT). Fortunately, in the particular setting of the present paper in which the joint probability distribution of the underlying stochastic problem is *known*, any potential differences in finite-sample performance between the two formulations can be decreased arbitrarily by simulating larger number of sample paths when constructing the robust optimization problem and then choosing a smaller robustness parameter.

B.2. Optimality Guarantees for (RO')

In view of the above intuition, we proceed to prove that the convergence guarantees from §2.4 will also hold if we opted instead to use formulation (RO'). Our convergence guarantees in the following Theorem EC.1 will be developed by extending the intuition from Example EC.2, that is, by showing that the gap between the objective values for formulations (RO') and (RO) converges to zero, almost surely, uniformly over the space of all exercise policies. We establish these convergence guarantees for formulation (RO') when the reward function in the optimal stopping problem satisfies the following assumption:

ASSUMPTION EC.1. $\lim_{\epsilon\to 0}\Omega_\epsilon(x)=0$ *almost surely, where*

$$\Omega_\epsilon(x) \triangleq \max_{t\in\{1,\dots,T\}}\left\{\sup_{y\in\mathcal{X}^T:\|y-x\|_\infty\le\epsilon} g(t,y)-g(t,x)\right\}.$$

We readily observe that above assumption, in conjunction with Assumption 1, is equivalent to requiring that the reward functions $g(1,\cdot),\dots,g(T,\cdot)$ are continuous functions of the stochastic process almost surely. Hence, Assumption EC.1 can be viewed as a mild assumption that is frequently satisfied in the applications of optimal stopping to options pricing. In the following theorem, as in §2.4, we let $\widehat{J}_{N,\epsilon}(\mu)$ denote the objective value of the robust optimization problem (RO) corresponding to exercise policies μ.

THEOREM EC.1 **(Uniform convergence for robust optimization formulations)**. *If Assumptions 4 and EC.1 hold, then the gap in objective values between formulations* (RO) *and* (RO') *converges to zero, almost surely, uniformly over the space of all exercise policies:*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{\mu}\left|\widehat{J}_{N,\epsilon}(\mu)-\frac{1}{N}\sum_{i=1}^N\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)\right|=0 \quad \text{almost surely.}$$

Proof. Consider any arbitrary sample path $x^i=(x_1^i,\dots,x_T^i)$ and robustness parameter $\epsilon\ge 0$. We first observe from algebra that

$$\begin{aligned}\left|\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)-\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),x^i)\right| &\le \sup_{y\in\mathcal{U}^i}\left|g(\tau_\mu(y),y)-g(\tau_\mu(y),x^i)\right|\\ &\le \max_{t\in\{1,\dots,T\}}\sup_{y\in\mathcal{U}^i}\left|g(t,y)-g(t,x^i)\right|\\ &\le \max_{t\in\{1,\dots,T\}}\left\{\sup_{y\in\mathcal{X}^T:\|y-x^i\|_\infty\le\epsilon} g(t,y)-g(t,x^i)\right\}\\ &=\Omega_\epsilon(x^i), \end{aligned} \tag{EC.2}$$

where the last inequality follows from the definition of the uncertainty sets. Therefore,

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{\mu}\left|\widehat{J}_{N,\epsilon}(\mu)-\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)\right|\\
&\le \limsup_{\epsilon\to 0}\limsup_{N\to\infty}\sup_{\mu}\frac{1}{N}\sum_{i=1}^{N}\left|\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)-\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),x^i)\right|\\
&\le \limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\Omega_\epsilon(x^i)\\
&= \limsup_{\epsilon\to 0}\mathbb{E}\left[\Omega_\epsilon(x)\right] \quad \text{almost surely}\\
&= 0.
\end{aligned}$$

where the first inequality follows from triangle inequality, the second inequality follows from (EC.2), the first equality follows from the strong law of large numbers, and the second equality follows from the dominated convergence theorem and Assumption EC.1. We note that the strong law of large numbers and dominated convergence theorem can be applied in both cases because of the boundedness of the reward function (Assumption 4). This concludes the proof of Theorem EC.1. □

Using standard proof techniques from stochastic programming (see, for example, Shapiro et al. (2014, §5.1.1)), Theorem EC.1 readily implies that the alternative formulation (RO') enjoys identical convergence guarantees as those in Theorems 1-3 under Assumptions 1-4 and EC.1.

B.3. Characterization of Optimal Markovian Stopping Rules for (RO')

Next, we show that our characterization of the structure of optimal Markovian stopping rules for formulation (RO) from §3 can be readily extended to the alternative formulation (RO') using similar proof techniques to those used in Theorem 4. Our following analysis in Theorem EC.2 uses the following additional but relatively mild assumption on the reward functions in the optimal stopping problem:

ASSUMPTION EC.2. *For each period t, the reward function satisfies $g(t,x)=h(t,x_t)$.*

The above assumption says that the reward function depends in each period only on the current state. This is a common assumption in the optimal stopping and options pricing literature and is often without loss of generality. With this assumption, the following theorem establishes the structure of optimal Markovian stopping rules for formulation (RO').

THEOREM EC.2. *Under Assumption EC.2, there exists $\mu \in \mathcal{M}$ that is optimal for* (RO').

Our proof of Theorem EC.2 follows the same pruning technique that was used in the proof of Theorem 4 in §3.2. Specifically, our proof of Theorem EC.2 makes use of the following four lemmas, which are essentially restatements of Lemmas 1-4 from §3.2.

LEMMA EC.1. *The optimal objective value of* (RO') *is equal to the optimal objective value of*

$$\begin{aligned}
\sup_{\mu} \quad & \frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),y)\\
\text{subject to} \quad & \text{for each } i\in\{1,\ldots,N\}, \text{ there exists } t\in\{1,\ldots,T\}\\
& \text{such that } \mu_t(y_t)=\textsc{Stop} \text{ for all } y_t\in\mathcal{U}_t^i
\end{aligned} \qquad (\text{RO}_T\text{'})$$

Proof. The proof of Lemma EC.1 is identical to the proof of Lemma 1. $\square$

LEMMA EC.2. *Let Assumption EC.2 hold, consider any $\mu \equiv (\mu_1, \ldots, \mu_T)$ that satisfies the constraints of* (RO$_T$'), *and define*

$$\sigma^i \triangleq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{STOP for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}.$$

Then the following equality holds for each $i \in \{1,\ldots,N\}$:

$$\inf_{y \in \mathcal{U}^i} g(\tau_\mu(y), x^i) = \min_{t \in \{1,\ldots,\sigma^i\}} \inf_{y_t \in \mathcal{U}_t^i} \{h(t, y_t) : \mu_t(y_t) = \text{STOP}\}.$$

Proof. Let Assumption EC.2 hold, consider any $\mu \equiv (\mu_1, \ldots, \mu_T)$ that satisfies the constraints of (RO$_T$'), and define

$$\sigma^i \triangleq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{STOP for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}.$$

It follows from identical reasoning as in the proof of Lemma 2 that the following equality holds for each sample path $i \in \{1,\ldots,N\}$:

$$\left\{\tau_\mu(y) : y \in \mathcal{U}^i\right\} = \left\{t \in \{1,\ldots,\sigma^i\} : \text{there exists } y_t \in \mathcal{U}_t^i \text{ such that } \mu_t(y_t) = \text{STOP}\right\}. \tag{EC.3}$$

We thus conclude for each sample path $i \in \{1,\ldots,N\}$ that

$$\inf_{y \in \mathcal{U}^i} g(\tau_\mu(y), y) = \min_{t \in \{1,\ldots,\sigma^i\}} \inf_{y_t \in \mathcal{U}_t^i} \{h(t, y_t) : \mu_t(y_t) = \text{STOP}\},$$

where the equality follows from Assumption EC.2 and line (EC.3). This completes our proof of Lemma EC.2. $\square$

LEMMA EC.3. *If Assumption EC.2 holds and if μ' is a pruned version of μ, then $\inf_{y \in \mathcal{U}^i} g(\tau_{\mu'}(y), y) \geq \inf_{y \in \mathcal{U}^i} g(\tau_\mu(y), y)$ for all $i \in \{1,\ldots,N\}$.*

REMARK EC.1. We observe that (RO$_T$') and (RO$_T$) have identical constraints. Thus, we use the same definition of pruning in Appendix B.3 as given by Definition 1.

Proof of Lemma EC.3. Let Assumption EC.2 hold, and let μ' be a pruned version of μ. Let $\sigma^1, \ldots, \sigma^N$ satisfy the following equalities for each $i \in \{1,\ldots,N\}$:

$$\sigma^i = \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{STOP } \forall y_t \in \mathcal{U}_t^i\right\} = \min\left\{t \in \{1,\ldots,T\} : \mu'_t(y_t) = \text{STOP } \forall y_t \in \mathcal{U}_t^i\right\}. \tag{EC.4}$$

Then it follows from the fact that μ is feasible for (RO$_T$') that $\sigma^1, \ldots, \sigma^N \in \{1,\ldots,T\}$. Therefore, for each $i \in \{1,\ldots,N\}$,

$$\begin{aligned}
\inf_{y \in \mathcal{U}^i} g(\tau_{\mu'}(y), y) &= \min_{t \in \{1,\ldots,\sigma^i\}} \inf_{y_t \in \mathcal{U}_t^i} \{h(t, y_t) : \mu'_t(y_t) = \text{STOP}\} \\
&\geq \min_{t \in \{1,\ldots,\sigma^i\}} \inf_{y_t \in \mathcal{U}_t^i} \{h(t, y_t) : \mu_t(y_t) = \text{STOP}\} \\
&= \inf_{y \in \mathcal{U}^i} g(\tau_\mu(y), y).
\end{aligned}$$

Indeed, the two equalities follow from Lemma EC.2 and line (EC.4). The inequality follows from the fact that μ' is a pruned version of μ, which implies that $\{y_t \in \mathcal{X} : \mu'_t(y_t) = \text{STOP}\} \subseteq \{y_t \in \mathcal{X} : \mu_t(y_t) = \text{STOP}\}$ for all $t \in \{1,\ldots,T\}$. Our proof of Lemma EC.3 is thus complete. $\square$

Lemma EC.4. *Consider any $\mu \equiv (\mu_1, \ldots, \mu_T)$ that satisfies the constraints of* (RO$_T$'), *and define*

$$\sigma^i \triangleq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{Stop for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}.$$

Then $\mu^{\sigma^1 \cdots \sigma^N}$ is a pruned version of μ.

Proof. The proof of Lemma EC.4 is identical to the proof of Lemma 4. □

In view of the above Lemmas EC.1-EC.4, we now present the proof of Theorem EC.2.

Proof of Theorem EC.2. Lemma EC.1 shows that the robust optimization problem (RO') is equivalent to the robust optimization problem (RO$_T$'). Moreover, for any arbitrary exercise policy μ that is feasible for (RO$_T$'), Lemmas EC.3 and EC.4 together show that there exists an exercise policy $\mu' \in \mathcal{M}$ such that the objective value associated with μ is less than or equal to the objective value associated with μ'. Since μ was chosen arbitrarily, our proof of Theorem EC.2 is complete. □

B.4. Computational Tractability of (RO')

Finally, we now discuss our primary motivation for using formulation (RO) instead of formulation (RO'). As we have shown up to this point, these two robust optimization formulations are essentially equivalent with respect to convergence guarantees and characterization of optimal Markovian stopping rules. However, as we will show momentarily, we find that formulation (RO) is significantly more amenable than (RO') from an algorithmic perspective. Our subsequent discussion on the computational tractability of formulation (RO') makes use of the following Theorem EC.3, which, analogously to Theorem 5 from §3.3, provides a reformulation of (RO') as a finite-dimensional optimization problem over integer decision variables.

Theorem EC.3. *Under Assumption EC.2, the alternative formulation* (RO') *is equivalent to*

$$\underset{\sigma^1,\ldots,\sigma^N \in \{1,\ldots,T\}}{\text{maximize}} \ \frac{1}{N}\sum_{i=1}^{N} \min_{t \in \{1,\ldots,\sigma^i\}} \ \min_{j : \sigma^j = t} \ \inf_{y_t \in \mathcal{U}_t^i \cap \mathcal{U}_t^j} h(t, y_t). \tag{IP'}$$

Proof. Our proof of Theorem EC.3 is essentially identical to the proof of Theorem 5. Indeed, let Assumption EC.2 hold. With this assumption, our proof of Theorem EC.3 is split into the following two intermediary claims:

Claim EC.1. *The optimal objective value of* (RO') *is less than or equal to the optimal objective value of* (IP').

Proof of Claim EC.1. Consider any arbitrary exercise policy $\mu \in \mathcal{M}$, and define the integers

$$\sigma^i \triangleq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\}, \quad \forall i \in \{1,\ldots,N\}. \tag{EC.5}$$

It follows from the definition of $\mathcal{M}$ that μ satisfies the constraints of (RO$_T$'), and so Lemma EC.4 implies that $\mu^{\sigma^1 \cdots \sigma^N}$ is a pruned version of μ. Therefore, it follows from Definition 1 and line (EC.5) that the following equalities hold:

$$\sigma^i = \min\left\{t \in \{1,\ldots,T\} : \mu_t^{\sigma^1 \cdots \sigma^N}(y_t) = \text{Stop for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}. \tag{EC.6}$$

Therefore, we observe that

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g\left(\tau_{\mu}(y), y\right)\\
&\le \frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g\left(\tau_{\mu^{\sigma^1\cdots\sigma^N}}(y), y\right)\\
&= \frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\inf_{y_t\in\mathcal{U}_t^i}\left\{h(t,y_t):\ \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\text{Stop}\right\}\\
&= \frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\min_{j:\sigma^j=t}\inf_{y_t\in\mathcal{U}_t^i\cap\mathcal{U}_t^j} h(t,y_t),
\end{aligned}\tag{EC.7}$$

where the first inequality follows from Lemma EC.3, the first equality follows from Lemma EC.2 and line (EC.6), and the second equality follows from the fact that $\mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\text{Stop}$ if and only if there exists a sample path j such that $y_t\in\mathcal{U}_t^j$ and $\sigma^j=t$. Because $\mu\in\mathcal{M}$ was chosen arbitrarily, we conclude from line (EC.7) and Theorem EC.2 that the optimal objective value of (RO') is less than or equal to the optimal objective value of (IP'). Our proof of Claim EC.1 is thus complete. □

Claim EC.2. *The optimal objective value of* (RO') *is greater than or equal to the optimal objective value of* (IP'). *Furthermore, for any choice of integers* $\sigma^1,\ldots,\sigma^N\in\{1,\ldots,T\}$, *the corresponding exercise policy* $\mu^{\sigma^1\cdots\sigma^N}\equiv(\mu_1^{\sigma^1\cdots\sigma^N},\ldots,\mu_T^{\sigma^1\cdots\sigma^N})$ *satisfies*

$$\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g\left(\tau_{\mu^{\sigma^1\cdots\sigma^N}}(y), y\right)\ge \frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\min_{j:\sigma^j=t}\inf_{y_t\in\mathcal{U}_t^i\cap\mathcal{U}_t^j} h(t,y_t).$$

Proof of Claim EC.2. Consider any integers $\sigma^1,\ldots,\sigma^N\in\{1,\ldots,T\}$. For each sample path $i\in\{1,\ldots,N\}$, we observe that

$$\begin{aligned}
&\min\left\{t\in\{1,\ldots,T\}:\ \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\text{Stop for all } y_t\in\mathcal{U}_t^i\right\}\\
&=\min\left\{t\in\{1,\ldots,\sigma^i\}:\ \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\text{Stop for all } y_t\in\mathcal{U}_t^i\right\}\\
&\le \sigma^i,
\end{aligned}\tag{EC.8}$$

where the equality follows from the fact that $\mu^{\sigma^1\cdots\sigma^N}\equiv(\mu_1^{\sigma^1\cdots\sigma^N},\ldots,\mu_T^{\sigma^1\cdots\sigma^N})$ by construction satisfies $\mu_{\sigma^i}^{\sigma^1\cdots\sigma^N}(y_{\sigma^i})=\text{Stop}$ for all $y_{\sigma^i}\in\mathcal{U}_{\sigma^i}^i$, and the inequality follows from algebra. Therefore,

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g\left(\tau_{\mu^{\sigma^1\cdots\sigma^N}}(y), y\right)\\
&\ge \frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\inf_{y_t\in\mathcal{U}_t^i}\left\{h(t,y_t):\ \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\text{Stop}\right\}\\
&= \frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\min_{j:\sigma^j=t}\inf_{y_t\in\mathcal{U}_t^i\cap\mathcal{U}_t^j} h(t,y_t),
\end{aligned}\tag{EC.9}$$

where the inequality follows from Lemma EC.2 and line (EC.8), and the equality follows from the fact that $\mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\text{Stop}$ if and only if there exists a sample path j such that $y_t\in\mathcal{U}_t^j$ and $\sigma^j=t$. Because $\sigma^1,\ldots,\sigma^N\in\{1,\ldots,T\}$ were chosen arbitrarily, we conclude from line (EC.9) that the optimal objective value of (RO') is greater than or equal to the optimal objective value of (IP'), which concludes our proof of Claim EC.2. □

Combining Claims EC.1 and EC.2, our proof of Theorem EC.3 is thus complete.
□

Equipped with the above Theorem EC.3, we now explain why formulation (RO) is more amenable from an tractability perspective than (RO'). Indeed, we recall from Theorem 5 in §3.3 that the robust optimization problem (RO) is equivalent to

$$\underset{\sigma^1,\ldots,\sigma^N \in \{1,\ldots,T\}}{\text{maximize}} \; \frac{1}{N}\sum_{i=1}^{N} \underbrace{\min_{t \in \{1,\ldots,\sigma^i\}} \left\{ g(t,x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\}}_{\nu_i(\sigma)}, \tag{IP}$$

where we observe for each sample path $i \in \{1,\ldots,N\}$ that the quantity $\nu_i(\sigma)$ satisfies

$$\nu_i(\sigma) \in \bigcup_{t=1}^{T} \{g(t,x^i)\}.$$

In contrast, we showed in the above Theorem EC.3 that (RO') is equivalent to

$$\underset{\sigma^1,\ldots,\sigma^N \in \{1,\ldots,T\}}{\text{maximize}} \; \frac{1}{N}\sum_{i=1}^{N} \underbrace{\min_{t \in \{1,\ldots,\sigma^i\}} \min_{j:\sigma^j = t} \inf_{y_t \in \mathcal{U}_t^i \cap \mathcal{U}_t^j} h(t,y_t)}_{\nu_i'(\sigma)}, \tag{IP'}$$

where we observe for each sample path $i \in \{1,\ldots,N\}$ that the quantity $\nu_i'(\sigma)$ satisfies

$$\nu_i'(\sigma) \in \bigcup_{t=1}^{T} \bigcup_{j:\mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset} \left\{ \inf_{y_t \in \mathcal{U}_t^i \cap \mathcal{U}_t^j} h(t,y_t) \right\}.$$

Hence, for each sample path $i \in \{1,\ldots,N\}$, we observe that the number of possible values for $\nu_i'(\sigma)$ can be a factor of $\mathcal{O}(N)$ greater than the number of possible values for $\nu_i(\sigma)$. This demonstrates that (RO) is more amenable to compact reformulations than (RO') and concludes our motivation for using formulation (RO) instead of formulation (RO') throughout the paper.

Appendix C: Proofs of Theorems 1, 2, and 3

Establishing the convergence guarantees from §2.4 for the robust optimization problem (RO) can be organized into two high-level steps. The first high-level step, comprised of Theorem 3, consists of showing that the objective function of the robust optimization problem (RO) converges almost surely to a (conservative) lower bound approximation of the objective function of the stochastic optimal stopping problem (OPT), uniformly over the space of all exercise policies. The second high-level step, comprised of Theorems 1 and 2, consists of showing that the conservativeness of the robust optimization problem (RO) disappears almost surely as the robustness parameter tends to zero and the number of sample paths tends to infinity. We present the first high-level step in Appendix C.1, and the second high-level step can be found in Appendix C.2.

Let us reflect on the novelty of the results in the present Appendix C. The first high-level step in Appendix C.1 (i.e., the proof of Theorem 3) is not novel and follows immediately from a uniform convergence result established by BSS23. Rather, the significant novelty of Appendix C is found in the second high-level step in Appendix C.2 (i.e., the proofs of Theorems 1 and 2). Stated succinctly, the second high-level step is challenging to establish because it requires showing that (a) the limit of the optimal objective value of the robust optimization problem (RO) exists almost surely as $\epsilon \to \infty$ and $N \to \infty$, and showing that (b) there always exists an arbitrarily near-optimal Markovian stopping rule μ for the stochastic optimal stopping problem (OPT) that almost surely satisfies $\tau_\mu(y) = \tau_\mu(x)$ for all sufficiently close realizations of $y \equiv (y_1, \ldots, y_T)$ to the stochastic process $x \equiv (x_1, \ldots, x_T)$.

For the general classes of stochastic dynamic optimization problems considered by BSS23 and Sturt (2020), the authors were unable to identify simple and verifiable conditions under which the aforementioned properties (a) and (b) are guaranteed to hold. Consequently, the authors only presented upper bounds on the gap between the optimal objective value of the robust optimization problem and the optimal objective value of the stochastic dynamic optimization (BSS23, Theorem 1). In certain cases, the authors alternatively assumed that the stochastic dynamic optimization problem happened to have arbitrarily near-optimal control policies with convenient structure (Sturt 2020, Assumption 15). The authors of BSS23 reflect on the weaknesses of these results in the literature at the end of their §4.3, saying that "future work may identify subclasses of [stochastic dynamic optimization problems] where the equality of the bounds can be ensured."

In Appendix C.2 of this paper, we resolve this gap in the literature by showing that the aforementioned properties (a) and (b) are guaranteed to hold for the specific class of stochastic dynamic optimization problems (OPT) under the mild assumptions that the stochastic problem has a bounded objective function (Assumption 4) and that the random variables in the stochastic problem have a continuous joint probability distribution (Assumption 3). Specifically, we establish in Lemma EC.6 of Appendix C.2 that property (a) holds by combining Assumption 4 with McDiarmid's inequality, and we establish in Lemma EC.7 of Appendix C.2 that property (b) holds by combining Assumptions 3 and 4 with elementary techniques from topology and measure theory. These lemmas allow us to establish the proofs of Theorems 1 and 2 at the end of Appendix C.2.

C.1. Proof of Theorem 3

In Appendix C.1, we establish the proof of Theorem 3 by using a uniform convergence result from BSS23. We begin by presenting some preliminary notation. Recall that $x \equiv (x_1, \ldots, x_T)$, $x^1 \equiv (x^1_1, \ldots, x^1_T)$, $x^2 \equiv (x^2_1, \ldots, x^2_T), \ldots \in \mathcal{X}^T \equiv \mathbb{R}^{Td}$ are sample paths drawn independently from an identical joint probability distribution. We will make use of the following additional notation:

$$\delta_N \triangleq N^{-\frac{1}{\max\{3, Td+1\}}}; \qquad \mathcal{U}^i(\delta_N) \triangleq \left\{ y \in \mathcal{X}^T : \|y - x^i\|_\infty \le \delta_N \right\};$$

$$\mathcal{U}^i \triangleq \left\{ y \in \mathcal{X}^T : \|y - x^i\|_\infty \le \epsilon \right\}; \qquad M_N \triangleq N^{-\frac{1}{(Td+1)(Td+2)}} \log N.$$

We now state the uniform convergence result from BSS23, which has been adapted to the notation of the present paper.

Lemma EC.5 **(Theorem 2 of BSS23)**. *Let Assumption 2 hold. Then there exists a finite $\bar{N} \in \mathbb{N}$, almost surely, such that the following inequality holds for all $N \ge \bar{N}$ and all measurable functions $f : \mathbb{R}^{Td} \to \mathbb{R}$:*

$$\mathbb{E}\left[f(x) \mathbb{I}\left\{ x \in \cup_{i=1}^N \mathcal{U}^i(\delta_N) \right\} \right] \ge \frac{1}{N} \sum_{i=1}^N \inf_{y \in \mathcal{U}^i(\delta_N)} f(y) - M_N \sup_{y \in \cup_{i=1}^N \mathcal{U}^i(\delta_N)} |f(y)|.$$

In view of the above notation and lemma, we are now ready to present the proof of Theorem 3.

Proof of Theorem 3. Consider any arbitrary choice of the robustness parameter $\epsilon > 0$, and recall that the reward function satisfies $g(\infty, y) = 0$ (see §2) and $0 \le g(1,y),\dots,g(T,y) \le U$ for all trajectories $y \in \mathcal{X}^T$ (Assumption 4). With this notation, we observe that

$$\begin{aligned}
&\liminf_{N\to\infty} \inf_{\mu} \left\{ J^*(\mu) - \widehat{J}_{N,\epsilon}(\mu) \right\} \\
&= \liminf_{N\to\infty} \inf_{\mu} \left\{ \mathbb{E}\left[g(\tau_\mu(x), x)\right] - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.10)}\\
&\ge \liminf_{N\to\infty} \inf_{\mu} \left\{ \mathbb{E}\left[g(\tau_\mu(x), x)\mathbb{I}\left\{x \in \cup_{i=1}^N \mathcal{U}^i(\delta_N)\right\}\right] - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.11)}\\
&\ge \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{U}^i(\delta_N)} g(\tau_\mu(y), y) - M_N U - \widehat{J}_{N,\epsilon}(\mu) \right\} \quad \text{almost surely} && \text{(EC.12)}\\
&= \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{U}^i(\delta_N)} g(\tau_\mu(y), y) - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.13)}\\
&\ge \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{U}^i} g(\tau_\mu(y), y) - \widehat{J}_{N,\epsilon}(\mu) \right\}. && \text{(EC.14)}
\end{aligned}$$

(EC.10) follows from applying the definition of $J^*(\mu)$; (EC.11) holds because the reward function is nonnegative; (EC.12) follows from Lemma EC.5 and the boundedness of the reward function; (EC.13) holds because $M_N \to 0$; (EC.14) holds because $\delta_N \to 0$ implies, for any arbitrary $\epsilon > 0$, that the inequality $\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y), y) \le \inf_{y\in\mathcal{U}^i(\delta_N)} g(\tau_\mu(y), y)$ is satisfied for all $i \in \mathbb{N}$, for all μ, and for all large $N \in \mathbb{N}$. Moreover:

$$\begin{aligned}
\text{(EC.14)} &= \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{U}^i} \left\{ g(\tau_\mu(y), x^i) + g(\tau_\mu(y), y) - g(\tau_\mu(y), x^i)\right\} - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.15)}\\
&\ge \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \left(\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y), x^i) + \inf_{y\in\mathcal{U}^i} \left\{ g(\tau_\mu(y), y) - g(\tau_\mu(y), x^i)\right\}\right) - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.16)}\\
&\ge \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \left(\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y), x^i) + \min_{t\in\{1,\dots,T\}} \left\{ \inf_{y\in\mathcal{U}^i} g(t, y) - g(t, x^i)\right\}\right) - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.17)}\\
&\ge \liminf_{N\to\infty} \inf_{\mu} \left\{ \frac{1}{N}\sum_{i=1}^N \left(\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y), x^i) + \Delta_\epsilon(x^i)\right) - \widehat{J}_{N,\epsilon}(\mu) \right\} && \text{(EC.18)}\\
&= \liminf_{N\to\infty} \inf_{\mu} \left\{ \widehat{J}_{N,\epsilon}(\mu) - \widehat{J}_{N,\epsilon}(\mu) \right\} + \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \Delta_\epsilon(x^i) && \text{(EC.19)}\\
&= \mathbb{E}\left[\Delta_\epsilon(x)\right] \quad \text{almost surely.} && \text{(EC.20)}
\end{aligned}$$

(EC.15), (EC.16), and (EC.17) follow from algebra; (EC.18) follows from the definition of $\Delta_\epsilon(x^i)$ (see Assumption 1); (EC.19) follows from the definition of $\widehat{J}_{N,\epsilon}(\mu)$; (EC.20) follows from the strong law of large numbers.

Since $\epsilon > 0$ was chosen arbitrarily, we have shown that

$$\lim_{\epsilon\to 0} \liminf_{N\to\infty} \inf_{\mu} \left\{ J^*(\mu) - \widehat{J}_{N,\epsilon}(\mu) \right\} \ge \lim_{\epsilon\to 0} \mathbb{E}\left[\Delta_\epsilon(x)\right] \ge 0 \quad \text{almost surely,}$$

where the first inequality holds almost surely from lines (EC.10)-(EC.20), and the second inequality holds almost surely due to the dominated convergence theorem and Assumption 1. Note that the above limits exist because $\epsilon \mapsto \liminf_{N\to\infty} \inf_\mu \{J^*(\mu) - \widehat{J}_{N,\epsilon}(\mu)\}$ and $\epsilon \mapsto \mathbb{E}[\Delta_\epsilon(x)]$ are monotonic functions. This concludes the proof of Theorem 3. $\square$

C.2. Proofs of Theorems 1 and 2

We begin Appendix C.2 by presenting the two novel intermediary results, Lemmas EC.6 and EC.7, that were discussed at the beginning of Appendix C. In the first novel intermediary result, denoted below by Lemma EC.6, we prove that the limit of the optimal objective value of the robust optimization problem (RO) exists almost surely as $\epsilon \to \infty$ and $N \to \infty$. The proof of the following lemma is based on McDiarmid's inequality.

LEMMA EC.6. *Let Assumption 4 hold. Then for all $\epsilon > 0$,*

$$\liminf_{N\to\infty} \sup_\mu \widehat{J}_{N,\epsilon}(\mu) = \limsup_{N\to\infty} \sup_\mu \widehat{J}_{N,\epsilon}(\mu) \quad \text{almost surely.}$$

Proof. Consider any fixed $\epsilon > 0$, and, for notational convenience, define the following function:

$$h_\epsilon(x^1,\ldots,x^N) \triangleq \sup_\mu \widehat{J}_{N,\epsilon}(\mu).$$

We will utilize the following intermediary claim:

CLAIM EC.3. *The function $h_\epsilon : \mathcal{X}^T \times \cdots \times \mathcal{X}^T \to \mathbb{R}$ has the following 'bounded differences' property: for all $\breve{x}^1,\ldots,\breve{x}^N \in \mathcal{X}^T$ and $\bar{x}^1,\ldots,\bar{x}^N \in \mathcal{X}^T$ that differ only on the jth coordinate ($\breve{x}^i = \bar{x}^i$ for $i \neq j$),*

$$\left| h_\epsilon(\bar{x}^1,\ldots,\bar{x}^N) - h_\epsilon(\breve{x}^1,\ldots,\breve{x}^N) \right| \le \frac{U}{N}.$$

Proof of Claim EC.3. For any arbitrary $\eta > 0$, let the exercise policies $\breve{\mu}^\eta$ be chosen to satisfy

$$\frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{X}^T:\, \|y-\breve{x}^i\|_\infty \le \epsilon} g(\tau_{\breve{\mu}^\eta}(y), \breve{x}^i) \ge h_\epsilon(\breve{x}^1,\ldots,\breve{x}^N) - \eta. \tag{EC.21}$$

We observe that

$$\begin{aligned}
&h_\epsilon(\breve{x}^1,\ldots,\breve{x}^N) - h_\epsilon(\bar{x}^1,\ldots,\bar{x}^N)\\
&\le \left(\frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{X}^T:\, \|y-\breve{x}^i\|_\infty \le \epsilon} g(\tau_{\breve{\mu}^\eta}(y), \breve{x}^i) + \eta\right) - \left(\frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{X}^T:\, \|y-\bar{x}^i\|_\infty \le \epsilon} g(\tau_{\breve{\mu}^\eta}(y), \bar{x}^i)\right)\\
&= \frac{1}{N} \inf_{y\in\mathcal{X}^T:\, \|y-\breve{x}^j\|_\infty \le \epsilon} g(\tau_{\breve{\mu}^\eta}(y), \breve{x}^j) - \frac{1}{N} \inf_{y\in\mathcal{X}^T:\, \|y-\bar{x}^j\|_\infty \le \epsilon} g(\tau_{\breve{\mu}^\eta}(y), \bar{x}^j) + \eta\\
&\le \frac{U}{N} + \eta.
\end{aligned}$$

Indeed, the first inequality holds because of line (EC.21) and because $\breve{\mu}^\eta$ is a feasible but possibly suboptimal solution to the optimization problem $\sup_\mu \frac{1}{N}\sum_{i=1}^N \inf_{y\in\mathcal{X}^T:\, \|y-\bar{x}^i\|_\infty\le\epsilon} g(\tau_\mu(y), \bar{x}^i) \equiv h_\epsilon(\bar{x}^1,\ldots,\bar{x}^N)$, and the second inequality follows from Assumption 4. Because $\eta > 0$ was chosen arbitrarily, we have shown that

$$h_\epsilon(\breve{x}^1,\ldots,\breve{x}^N) - h_\epsilon(\bar{x}^1,\ldots,\bar{x}^N) \le \frac{U}{N}.$$

It follows from symmetry that

$$h_\epsilon(\bar{x}^1,\ldots,\bar{x}^N) - h_\epsilon(\breve{x}^1,\ldots,\breve{x}^N) \le \frac{U}{N},$$

which concludes our proof of Claim EC.3. $\square$

Because the above Claim EC.3 holds, it follows from McDiarmid's inequality that

$$\mathbb{P}\left(\left|h_\epsilon(x^1,\ldots,x^N)-\mathbb{E}\left[h_\epsilon(x^1,\ldots,x^N)\right]\right|>\eta\right)\le 2\exp\left(-\frac{2\eta^2 N}{U^2}\right)\quad \forall \eta>0.$$

It follows from the above line that

$$\sum_{N=1}^{\infty}\mathbb{P}\left(\left|h_\epsilon(x^1,\ldots,x^N)-\mathbb{E}\left[h_\epsilon(x^1,\ldots,x^N)\right]\right|>\eta\right)<\infty\quad \forall \eta>0,$$

and so the Borel-Cantelli lemma implies that

$$\lim_{N\to\infty}\left|h_\epsilon(x^1,\ldots,x^N)-\mathbb{E}\left[h_\epsilon(x^1,\ldots,x^N)\right]\right|=0\quad \text{almost surely.} \tag{EC.22}$$

We observe from identical reasoning as in the proof of Shapiro et al. (2014, Proposition 5.6) that $\mathbb{E}[h_\epsilon(x^1,\ldots,x^N)]$ is monotonically decreasing with respect to $N\in\mathbb{N}$. Since the random variables $h_\epsilon(x^1,\ldots,x^N)$ are also contained in the interval $[0,U]$ for all $N\in\mathbb{N}$, we conclude that $\lim_{N\to\infty}\mathbb{E}[h_\epsilon(x^1,\ldots,x^N)]$ exists, and thus it follows from line (EC.22) that $\lim_{N\to\infty}h_\epsilon(x^1,\ldots,x^N)$ exists almost surely. This concludes the proof of Lemma EC.6. □

In the second novel intermediary result, denoted below by Lemma EC.7, we show that there always exists an arbitrarily near-optimal Markovian stopping rule μ for the stochastic optimal stopping problem (OPT) that almost surely satisfies $\tau_\mu(y)=\tau_\mu(x)$ for all sufficiently close realizations of $y\equiv(y_1,\ldots,y_T)$ to the stochastic process $x\equiv(x_1,\ldots,x_T)$. The proof of the following lemma is based on elementary techniques from topology and measure theory.

LEMMA EC.7. *Under Assumptions 3 and 4,*

$$\sup_\mu \mathbb{E}\left[g(\tau_\mu(x),x)\right]=\sup_\mu \mathbb{E}\left[\liminf_{y\to x} g(\tau_\mu(y),x)\right].$$

Proof. Our proof of Lemma EC.7 is organized into the following steps. In our first step, we will show that any feasible exercise policy to the stochastic optimal stopping problem (OPT) can be approximated to arbitrary accuracy by an exercise policy with exercise regions that are open sets. In our second step, we will show that these exercise policies from the previous step can be approximated to arbitrary accuracy by an exercise policy with exercise regions that are the union of finitely many open balls. Finally, we will then invoke Assumption 3 to conclude that the probability of a random state lying a strictly positive distance from the boundary of finitely many open balls is equal to one, which implies that the corresponding Markovian stopping rule almost surely satisfies $\tau_\mu(y)=\tau_\mu(x)$ for all sufficiently close realizations of $y\equiv(y_1,\ldots,y_T)$ to the stochastic process $x\equiv(x_1,\ldots,x_T)$. From this, the desired lemma will follow readily.

In view of the above organization, we now present our proof. Choose any arbitrary exercise policy $\bar{\mu}\equiv(\bar{\mu}_1,\ldots,\bar{\mu}_T)$ which is feasible for the stochastic optimal stopping problem (OPT), and choose any arbitrary constant $\eta>0$. In our first claim, we show that this exercise policy can be approximated by an exercise policy with exercise regions that are open sets.

CLAIM EC.4. *There exists an exercise policy $\tilde{\mu}^\eta\equiv(\tilde{\mu}_1^\eta,\ldots,\tilde{\mu}_T^\eta)$ such that:*
(a) $\mathbb{E}[g(\tau_{\tilde{\mu}^\eta}(x),x)]\ge\mathbb{E}[g(\tau_{\bar{\mu}}(x),x)]-\eta$;
(b) For each period $t\in\{1,\ldots,T\}$, $A_t^\eta\triangleq\{y_t\in\mathcal{X}:\tilde{\mu}^\eta(y_t)=\text{STOP}\}$ is an open set.

Proof of Claim EC.4. Because $\bar{\mu}_1,\ldots,\bar{\mu}_T:\mathbb{R}^d\to\{\text{STOP},\text{CONTINUE}\}$ are feasible for the optimization problem (OPT), it follows from §2.1 that $\bar{\mu}_1,\ldots,\bar{\mu}_T:\mathbb{R}^d\to\{\text{STOP},\text{CONTINUE}\}$ are measurable functions. Thus, it follows for each period $t\in\{1,\ldots,T\}$ that the set $A_t\triangleq\left\{y_t\in\mathbb{R}^d:\mu_t(y_t)=\text{STOP}\right\}$ is a Borel set in $\mathbb{R}^d$. Now, for each period $t\in\{1,\ldots,T\}$, let $\mathbb{P}_{x,t}(\cdot)\triangleq$

$\mathbb{P}(x_t \in \cdot)$ denote the marginal probability law of the stochastic process on period t. Since A_t is a Borel set, it is a well known result from measure theory (Rudin 1964, Remark 11.11(b)) that there exists an open set $A_t^\eta \subseteq \mathbb{R}^d$ which satisfies $A_t \subseteq A_t^\eta$ and $\mathbb{P}_{x,t}(A_t^\eta \setminus A_t) \le \frac{\eta}{T}$. Using these sets, we define an exercise policy on each period t by

$$\tilde{\mu}_t^\eta(y_t) \triangleq \begin{cases} \textsc{Stop}, & \text{if } y_t \in A_t^\eta, \\ \textsc{Continue}, & \text{otherwise.} \end{cases}$$

It follows from the above construction that the new exercise policy $\tilde{\mu}^\eta \equiv (\tilde{\mu}_1^\eta, \ldots, \tilde{\mu}_T^\eta)$ satisfies property (b) of Claim EC.4. To show that property (a) of Claim EC.4 holds, let us first define the following set for notational convenience:

$$B \triangleq \{y \equiv (y_1, \ldots, y_T) : y_t \notin A_t^\eta \setminus A_t \text{ for all } t \in \{1, \ldots, T\}\}$$

We observe that B is a Borel set in $\mathbb{R}^{Td}$, since B is comprised of a finite number of complements and intersections of Borel sets. Then,

$$\begin{aligned} &\mathbb{E}[|g(\tau_{\tilde{\mu}^\eta}(x), x) - g(\tau_{\bar{\mu}}(x), x)|] \\ &= \mathbb{E}\left[|g(\tau_{\tilde{\mu}^\eta}(x), x) - g(\tau_{\bar{\mu}}(x), x)| \mathbb{I}\{x \notin B\}\right] + \mathbb{E}\left[|g(\tau_{\tilde{\mu}^\eta}(x), x) - g(\tau_{\bar{\mu}}(x), x)| \mathbb{I}\{x \in B\}\right] \\ &= \mathbb{E}\left[|g(\tau_{\tilde{\mu}^\eta}(x), x) - g(\tau_{\bar{\mu}}(x), x)| \mathbb{I}\{x \notin B\}\right] + 0 \\ &\le U \mathbb{P}(x \notin B) \\ &\le \sum_{t=1}^T \mathbb{P}(x_t \in A_t^\eta \setminus A_t) \\ &\le \eta. \end{aligned}$$

The first equality follows from the law of total expectation. The second equality holds because $g(\tau_{\tilde{\mu}^\eta}(x), x) = g(\tau_{\bar{\mu}}(x), x)$ when $x \in B$. The first inequality follows from Assumption 4. The second inequality follows from the union bound and the definition of the set B. The third and final inequality holds because $\mathbb{P}(x_t \in A_t^\eta \setminus A_t) = \mathbb{P}_{x,t}(A_t^\eta \setminus A_t) \le \frac{\eta}{T}$. This concludes our proof of Claim EC.4. □

In the above Claim EC.4, we showed that we can construct a new exercise policy $\tilde{\mu}^\eta \equiv (\tilde{\mu}_1^\eta, \ldots, \tilde{\mu}_T^\eta)$ which is close to the original exercise policy $\bar{\mu} \equiv (\bar{\mu}_1, \ldots, \bar{\mu}_T)$ with respect to expected reward. However, the new exercise policy is comprised of exercise regions in each period which are open sets. We now use this open set property, along with Assumption 3, to show the following second claim.

Claim EC.5. *There exists an exercise policy $\breve{\mu}^\eta \equiv (\breve{\mu}_1^\eta, \ldots, \breve{\mu}_T^\eta)$ such that:*
(a) $\mathbb{E}[g(\tau_{\breve{\mu}^\eta}(x), x)] \ge \mathbb{E}[g(\tau_{\tilde{\mu}^\eta}(x), x)] - \eta$;
(b) $\lim_{y \to x} \tau_{\breve{\mu}^\eta}(y) = \tau_{\breve{\mu}^\eta}(x)$ *almost surely.*

Proof of Claim EC.5. For each period $t \in \{1, \ldots, T\}$, consider the exercise region $A_t^\eta \triangleq \{y_t \in \mathcal{X} : \tilde{\mu}^\eta(y_t) = \textsc{Stop}\}$ corresponding to the exercise policy $\tilde{\mu}_t^\eta$. Since A_t^η is an open set, it is a well-known result from measure theory (Rudin 1964, Remark 11.11(a)) that A_t^η is the union of a countable collection of open balls. That is, there exists a countable set of tuples $\{(y_t^{\eta,\ell}, \epsilon_t^{\eta,\ell})\}_{\ell \in \mathbb{N}}$ such that $\epsilon_t^{\eta,\ell} > 0$ for all $\ell \in \mathbb{N}$ and the following equality is satisfied:

$$A_t^\eta = \lim_{k \to \infty} \underbrace{\bigcup_{\ell=1}^k \left\{y_t \in \mathbb{R}^d : \|y_t - y_t^{\eta,\ell}\|_\infty \le \epsilon_t^{\eta,\ell}\right\}}_{C_t^{\eta,k}}.$$

We observe from the above construction that $C_t^{\eta,k}$, $k \in \mathbb{N}$, is an increasing sequence of sets which converges to A_t^η. Consequently, we have $\lim_{k\to\infty} \mathbb{P}\left(x_t \in A_t^\eta \setminus C_t^{\eta,k}\right) = 0$, and so it follows from the definition of a limit that there exists a $k_t^\eta \in \mathbb{N}$ such that

$$\mathbb{P}\left(x_t \in A_t^\eta \setminus C_t^{\eta,k_t^\eta}\right) \le \frac{\eta}{T}. \tag{EC.23}$$

Using these sets, we define an exercise policy on each period t by

$$\breve{\mu}_t^\eta(y_t) \triangleq \begin{cases} \textsc{Stop}, & \text{if } y_t \in C_t^{\eta,k_t^\eta}, \\ \textsc{Continue}, & \text{otherwise.} \end{cases}$$

Since the inequality (EC.23) holds for each period t, it follows from identical reasoning as the proof of property (a) of Claim EC.4 that the new exercise policy $\breve{\mu}^\eta \equiv (\breve{\mu}_1^\eta, \ldots, \breve{\mu}_T^\eta)$ satisfies property (a) of Claim EC.5.

Moreover, since the exercise regions C_t^{η,k_t^η} are unions of finite numbers of open balls, and since the Lebesgue measure of the boundaries of a finite number of open balls is equal to zero, it follows from Assumption 3 that the random state in each period $x_t \in \mathbb{R}^d$ will be a strictly positive distance from the boundary of C_t^{η,k_t^η} with probability one. This concludes our proof of property (b) of Claim EC.5, and thus concludes the proof of Claim EC.5. □

We now combine Claims EC.4 and EC.5 to conclude our proof of Lemma EC.7. Indeed, it follows from these claims that

$$\mathbb{E}\left[g(\tau_{\bar{\mu}}(x), x)\right] \le \mathbb{E}\left[g(\tau_{\tilde{\mu}^\eta}(x), x)\right] + \eta \le \mathbb{E}\left[g(\tau_{\breve{\mu}^\eta}(x), x)\right] + 2\eta = \mathbb{E}\left[\liminf_{y\to x} g(\tau_{\breve{\mu}^\eta}(y), x)\right] + 2\eta,$$

where the first inequality follows from property (a) of Claim EC.4, the second inequality follows from property (a) of Claim EC.5, and the third inequality follows from property (b) of Claim EC.5. Since the exercise policy $\bar{\mu} \equiv (\bar{\mu}_1, \ldots, \bar{\mu}_T)$ and constant $\eta > 0$ were chosen arbitrarily, we have proven that

$$\sup_\mu \mathbb{E}\left[g(\tau_{\bar{\mu}}(x), x)\right] \le \sup_\mu \mathbb{E}\left[\liminf_{y\to x} g(\tau_\mu(y), x)\right].$$

The other direction of the inequality obviously holds, and so our proof of Lemma EC.7 is complete. □

We now combine the above novel intermediary lemmas to establish our proofs of Theorems 1 and 2.

Proof of Theorem 1. We first show that the optimal objective value of (RO) is an asymptotic lower bound on the optimal objective value of (OPT). Indeed,

$$0 \ge \lim_{\epsilon\to 0} \limsup_{N\to\infty} \sup_\mu \left\{\widehat{J}_{N,\epsilon}(\mu) - J^*(\mu)\right\} \quad \text{almost surely} \tag{EC.24}$$

$$\ge \lim_{\epsilon\to 0} \limsup_{N\to\infty} \left(\sup_\mu \widehat{J}_{N,\epsilon}(\mu) - \sup_\mu J^*(\mu)\right) \tag{EC.25}$$

$$= -\sup_\mu J^*(\mu) + \lim_{\epsilon\to 0} \limsup_{N\to\infty} \sup_\mu \widehat{J}_{N,\epsilon}(\mu) \tag{EC.26}$$

$$= -\sup_\mu J^*(\mu) + \lim_{\epsilon\to 0} \lim_{N\to\infty} \sup_\mu \widehat{J}_{N,\epsilon}(\mu) \quad \text{almost surely,} \tag{EC.27}$$

where (EC.24) follows from Theorem 3, (EC.25) and (EC.26) follow from algebra, and (EC.27) follows from Lemma EC.6. Note that all of the above limits exist because $\epsilon \mapsto \limsup_{N\to\infty} \sup_\mu \{\widehat{J}_{N,\epsilon}(\mu) - J^*(\mu)\}$ and $\epsilon \mapsto \limsup_{N\to\infty} \sup_\mu \widehat{J}_{N,\epsilon}(\mu)$ are monotonic functions.

We next show that the optimal objective value of (RO) provides an asymptotic upper bound on the optimal objective value of (OPT). Indeed, we observe that

$$\begin{aligned}
\lim_{\epsilon\to 0}\lim_{N\to\infty}\sup_{\mu}\widehat{J}_{N,\epsilon}(\mu) &= \lim_{\epsilon\to 0}\liminf_{N\to\infty}\sup_{\mu}\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),x^i) \\
&\ge \lim_{\epsilon\to 0}\sup_{\mu}\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),x^i) && \text{(EC.28)}\\
&= \lim_{\epsilon\to 0}\sup_{\mu}\mathbb{E}\left[\inf_{y\in\mathcal{X}^T:\,\|y-x\|\le\epsilon} g(\tau_\mu(y),x)\right] \quad \text{almost surely} && \text{(EC.29)}\\
&= \sup_{\mu}\mathbb{E}\left[\liminf_{y\to x} g(\tau_\mu(y),x)\right] \quad \text{almost surely.} && \text{(EC.30)}\\
&= \sup_{\mu}\mathbb{E}\left[g(\tau_\mu(x),x)\right] && \text{(EC.31)}
\end{aligned}$$

(EC.28) follows from algebra; (EC.29) follows from the strong law of large numbers; (EC.30) follows from the dominated convergence theorem. We note that the strong law of large numbers and dominated convergence theorem can both be applied because of Assumption 4. Finally, (EC.31) follows from Lemma EC.7. Combining the above, our proof of Theorem 1 is complete. □

Proof of Theorem 2. We observe from Theorems 1 and 3 that for every arbitrary $\eta > 0$, there exists a finite $\bar{\epsilon}(\eta) > 0$ almost surely such that the following statements hold for all $0 < \epsilon < \bar{\epsilon}(\eta)$:

$$\left|\lim_{N\to\infty}\widehat{J}_{N,\epsilon}(\hat{\mu}_{N,\epsilon}) - \sup_{\mu} J^*(\mu)\right| \le \eta \quad \text{almost surely;} \tag{EC.32}$$

$$\liminf_{N\to\infty}\left(J^*(\hat{\mu}_{N,\epsilon}) - \widehat{J}_{N,\epsilon}(\hat{\mu}_{N,\epsilon})\right) \ge -\eta \quad \text{almost surely.} \tag{EC.33}$$

Therefore,

$$\sup_{\mu} J^*(\mu) \ge \limsup_{N\to\infty} J^*(\hat{\mu}_{N,\epsilon}) \ge \liminf_{N\to\infty} J^*(\hat{\mu}_{N,\epsilon}) \ge \liminf_{N\to\infty}\widehat{J}_{N,\epsilon}(\hat{\mu}_{N,\epsilon}) - \eta \ge \sup_{\mu} J^*(\mu) - 2\eta,$$

where the first inequality holds because each $\hat{\mu}_{N,\epsilon}$ is a feasible but possibly suboptimal solution to (OPT), the second inequality is obvious, the third inequality follows from (EC.33), and the final inequality follows from (EC.32). Rearranging the above line, we have shown that the following statements hold for all $0 < \epsilon < \bar{\epsilon}(\eta)$:

$$\left|\limsup_{N\to\infty} J^*(\hat{\mu}_{N,\epsilon}) - \sup_{\mu} J^*(\mu)\right| \le 2\eta \quad \text{almost surely;}$$

$$\left|\liminf_{N\to\infty} J^*(\hat{\mu}_{N,\epsilon}) - \sup_{\mu} J^*(\mu)\right| \le 2\eta \quad \text{almost surely.}$$

Since $\eta > 0$ was chosen arbitrarily, our proof of Theorem 2 is complete. □

Appendix D: Proofs from §3

Proof of Lemma 1. We recall from §2.1 that a Markovian stopping rule satisfies $\tau_\mu(y) = \infty$ for a trajectory $y \equiv (y_1,\ldots,y_T) \in \mathcal{X}^T$ if and only if $\mu_t(y_t) = \text{Continue}$ for each period $t \in \{1,\ldots,T\}$. Therefore, we observe the robust optimization problem (RO_T) is equivalent to the following optimization problem:

$$\begin{aligned}
\sup_{\mu} \quad & \frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g(\tau_\mu(y),x^i) \\
\text{subject to} \quad & \tau_\mu(y) < \infty \quad \text{for all } i \in \{1,\ldots,N\} \text{ and } y \in \mathcal{U}^i
\end{aligned} \tag{RO”}$$

Now consider any arbitrary exercise policy $\mu \equiv (\mu_1, \ldots, \mu_T)$, and let $\mu' \equiv (\mu', \ldots, \mu')$ be an exercise policy that is defined for each period $t \in \{1, \ldots, T\}$ and state $y_t \in \mathcal{X}$ as

$$\mu'_t(y_t) \triangleq \begin{cases} \mu_t(y_t), & \text{if } t \in \{1, \ldots, T-1\}, \\ \textsc{Stop}, & \text{if } t = T. \end{cases}$$

We readily observe that $\tau_{\mu'}(y) \le T$ for each $i \in \{1, \ldots, N\}$ and $y \in \mathcal{U}^i$, which implies that μ' satisfies the constraints of (RO"). Moreover, we observe for each $i \in \{1, \ldots, N\}$ and $y \in \mathcal{U}^i$ that

$$\tau_{\mu'}(y) = \begin{cases} \tau_\mu(y), & \text{if } \tau_\mu(y) < \infty, \\ T, & \text{if } \tau_\mu(y) = \infty. \end{cases}$$

Since the reward function satisfies $g(1, y), \ldots, g(T, y) \ge 0$ and $g(\infty, y) = 0$ for all $y \in \mathcal{X}^T$, we have shown for each $i \in \{1, \ldots, N\}$ and $y \in \mathcal{U}^i$ that

$$\begin{aligned} &g(\tau_\mu(y), x^i) = g(\tau_{\mu'}(y), x^i) && \text{if } \tau_\mu(y) < \infty, \text{ and} \\ &g(\tau_\mu(y), x^i) = 0 \le g(T, x^i) = g(\tau_{\mu'}(y), x^i) && \text{if } \tau_\mu(y) = \infty. \end{aligned}$$

We thus conclude that the objective value $\frac{1}{N}\sum_{i=1}^N \inf_{y \in \mathcal{U}^i} g(\tau_{\mu'}(y), x^i)$ associated with the new exercise policy $\mu' \equiv (\mu'_1, \ldots, \mu'_T)$ is always greater than or equal to the objective value $\frac{1}{N}\sum_{i=1}^N \inf_{y \in \mathcal{U}^i} g(\tau_\mu(y), x^i)$ associated with the original exercise policy $\mu \equiv (\mu_1, \ldots, \mu_T)$. Since $\mu \equiv (\mu_1, \ldots, \mu_T)$ was chosen arbitrarily, our proof of Lemma 1 is complete. □

Proof of Lemma 2. Consider any $\mu \equiv (\mu_1, \ldots, \mu_T)$ that satisfies the constraints of (RO_T), and define

$$\sigma^i \triangleq \min\left\{t \in \{1, \ldots, T\} : \mu_t(y_t) = \textsc{Stop} \text{ for all } y_t \in \mathcal{U}^i_t\right\} \quad \forall i \in \{1, \ldots, N\}.$$

It follows from the fact that μ is feasible for (RO_T) that $\sigma^1, \ldots, \sigma^N \in \{1, \ldots, T\}$. Moreover, for each sample path $i \in \{1, \ldots, N\}$ and trajectory $y \in \mathcal{U}^i$, we observe that

$$\tau_\mu(y) = \min\{t \in \{1, \ldots, T\} : \mu_t(y_t) = \textsc{Stop}\} \le \sigma^i,$$

where the equality is simply the definition of a Markovian stopping rule and the inequality follows from the fact that $\mu_{\sigma^i}(y_{\sigma^i}) = \textsc{Stop}$ for all $y_{\sigma^i} \in \mathcal{U}^i_{\sigma^i}$. Therefore, we observe for each sample path $i \in \{1, \ldots, N\}$ that

$$\begin{aligned} \left\{\tau_\mu(y) : y \in \mathcal{U}^i\right\} &= \left\{ t \in \{1, \ldots, \sigma^i\} : \begin{array}{l} \text{there exists } y \in \mathcal{U}^i \text{ such that } \mu_{t'}(y_{t'}) = \textsc{Continue} \\ \text{for all } t' \in \{1, \ldots, t-1\} \text{ and } \mu_t(y) = \textsc{Stop} \end{array} \right\} \\ &= \left\{t \in \{1, \ldots, \sigma^i\} : \text{there exists } y_t \in \mathcal{U}^i_t \text{ such that } \mu_t(y_t) = \textsc{Stop}\right\}. \end{aligned}$$

Indeed, the first equality follows from the definition of a Markovian stopping rule and from the fact that $\tau_\mu(y) \le \sigma^i$ for all $y \in \mathcal{U}^i$. The second equality follows from the fact that $\mathcal{U}^i_t$ is not a subset of $\{y_t \in \mathcal{X} : \mu_t(y_t) = \textsc{Stop}\}$ for each period $t \in \{1, \ldots, \sigma^i - 1\}$, which implies for each $t \in \{1, \ldots, \sigma^i - 1\}$ that there exists a $y_t \in \mathcal{U}^i_t$ that satisfies $\mu_t(y_t) = \textsc{Continue}$. We thus conclude for each sample path $i \in \{1, \ldots, N\}$ that

$$\inf_{y \in \mathcal{U}^i} g(\tau_\mu(y), x^i) = \min_{t \in \{1, \ldots, \sigma^i\}} \left\{g(t, x^i) : \text{there exists } y_t \in \mathcal{U}^i_t \text{ such that } \mu_t(y_t) = \textsc{Stop}\right\},$$

which completes our proof of Lemma 2. □

Proof of Lemma 3. Let μ' be a pruned version of μ, and let $\sigma^1,\ldots,\sigma^N$ satisfy the following equalities for each $i \in \{1,\ldots,N\}$:

$$\sigma^i = \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\} = \min\left\{t \in \{1,\ldots,T\} : \mu'_t(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\}. \tag{EC.34}$$

Then it follows from the fact that μ is feasible for (RO$_T$) that $\sigma^1,\ldots,\sigma^N \in \{1,\ldots,T\}$. Therefore, for each $i \in \{1,\ldots,N\}$,

$$\begin{aligned}\inf_{y\in\mathcal{U}^i} g(\tau_{\mu'}(y), x^i) &= \min_{t\in\{1,\ldots,\sigma^i\}} \left\{g(t,x^i) : \text{there exists } y_t \in \mathcal{U}_t^i \text{ such that } \mu'_t(y_t) = \text{Stop}\right\}\\ &\geq \min_{t\in\{1,\ldots,\sigma^i\}} \left\{g(t,x^i) : \text{there exists } y_t \in \mathcal{U}_t^i \text{ such that } \mu_t(y_t) = \text{Stop}\right\}\\ &= \inf_{y\in\mathcal{U}^i} g(\tau_{\mu}(y), x^i).\end{aligned}$$

Indeed, the two equalities follow from Lemma 2 and line (EC.34). The inequality follows from the fact that μ' is a pruned version of μ, which implies that $\{y_t \in \mathcal{X} : \mu'_t(y_t) = \text{Stop}\} \subseteq \{y_t \in \mathcal{X} : \mu_t(y_t) = \text{Stop}\}$ for all $t \in \{1,\ldots,T\}$. Our proof of Lemma 3 is thus complete. □

Proof of Lemma 4. Consider any $\mu \equiv (\mu_1,\ldots,\mu_T)$ that satisfies the constraints of (RO$_T$), and define

$$\sigma^i \triangleq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{Stop for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}.$$

Our proof that $\mu^{\sigma^1\cdots\sigma^N}$ is a pruned version of μ is split into the following two intermediary claims.

Claim EC.6. $\{y_t \in \mathcal{X} : \mu_t^{\sigma^1\cdots\sigma^N}(y_t) = \text{Stop}\} \subseteq \{y_t \in \mathcal{X} : \mu_t(y_t) = \text{Stop}\}$ *for all* $t \in \{1,\ldots,T\}$.

Proof of Claim EC.6. Indeed, we observe for each period $t \in \{1,\ldots,T\}$ that

$$\begin{aligned}&\left\{y_t \in \mathcal{X} : \mu_t^{\sigma^1\cdots\sigma^N}(y_t) = \text{Stop}\right\}\\ &= \left\{y_t \in \mathcal{X} : y_t \in \bigcup_{i:\,\sigma^i=t} \mathcal{U}_t^i\right\}\\ &\subseteq \left\{y_t \in \mathcal{X} : y_t \in \bigcup_{i:\,\sigma^i=t} \{y'_t \in \mathcal{X} : \mu_t(y'_t) = \text{Stop}\}\right\}\\ &= \begin{cases}\{y_t \in \mathcal{X} : \mu_t(y_t) = \text{Stop}\}, & \text{if there exists } i \in \{1,\ldots,N\} \text{ such that } \sigma^i = t,\\ \emptyset, & \text{otherwise}\end{cases}\\ &\subseteq \{y_t \in \mathcal{X} : \mu_t(y_t) = \text{Stop}\},\end{aligned}$$

where the first equality follows from the definition of $\mu_t^{\sigma^1\cdots\sigma^N}$, the first inclusion follows from the fact that $\mathcal{U}_t^i \subseteq \{y_t \in \mathcal{X} : \mu_t(y_t) = \text{Stop}\}$ for each sample path i that satisfies $\sigma^i = t$, the second equality follows from algebra, and the second inclusion follows from algebra. This concludes our proof of Claim EC.6. □

Claim EC.7. *For each* $i \in \{1,\ldots,N\}$,

$$\min\left\{t \in \{1,\ldots,T\} : \mu_t^{\sigma^1\cdots\sigma^N}(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\} = \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\}.$$

Proof of Claim EC.7. Consider any sample path $i \in \{1,\ldots,N\}$. We first observe that

$$\begin{aligned}\min\left\{t \in \{1,\ldots,T\} : \mu_t^{\sigma^1\cdots\sigma^N}(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\} &\leq \sigma^i\\ &= \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \text{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\},\end{aligned}$$

where the inequality follows from the fact that $\mu_{\sigma^i}^{\sigma^1 \cdots \sigma^N}(y_{\sigma^i}) = \textsc{Stop}$ for all $y_{\sigma^i} \in \mathcal{U}_{\sigma^i}^i$, and the equality follows from the definition of σ^i. Moreover, it follows immediately from Claim EC.6 that

$$\min\left\{t \in \{1,\ldots,T\} : \mu_t^{\sigma^1 \cdots \sigma^N}(y_t) = \textsc{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\} \geq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \textsc{Stop}\ \forall y_t \in \mathcal{U}_t^i\right\}.$$

Our proof of Claim EC.7 is thus complete. □

Combining Claims EC.6 and EC.7 with Definition 1, we conclude that $\mu^{\sigma^1 \cdots \sigma^N}$ is a pruned version of μ, which completes our proof of Lemma 4. □

The following proof follows the identical reasoning as discussed in §3.2, and is stated formally here for the sake of completeness.

Proof of Theorem 4. Lemma 1 shows that the robust optimization problem (RO) is equivalent to the robust optimization problem (RO_T). Moreover, for any arbitrary exercise policy μ that is feasible for (RO_T), Lemmas 3 and 4 together show that there exists an exercise policy $\mu' \in \mathcal{M}$ such that the objective value associated with μ is less than or equal to the objective value associated with μ'. Since μ was chosen arbitrarily, our proof of Theorem 4 is complete. □

Proof of Theorem 5. We split our proof into the following two intermediary claims.

Claim EC.8. *The optimal objective value of* (RO) *is less than or equal to the optimal objective value of* (IP).

Proof of Claim EC.8. Consider any arbitrary $\mu \in \mathcal{M}$, and define the integers

$$\sigma^i \triangleq \min\left\{t \in \{1,\ldots,T\} : \mu_t(y_t) = \textsc{Stop} \text{ for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}. \tag{EC.35}$$

It follows from the definition of $\mathcal{M}$ that μ satisfies the constraints of (RO_T), and so Lemma 4 implies that $\mu^{\sigma^1 \cdots \sigma^N}$ is a pruned version of μ. Therefore, it follows from Definition 1 and line (EC.35) that the following equalities hold:

$$\sigma^i = \min\left\{t \in \{1,\ldots,T\} : \mu_t^{\sigma^1 \cdots \sigma^N}(y_t) = \textsc{Stop} \text{ for all } y_t \in \mathcal{U}_t^i\right\} \quad \forall i \in \{1,\ldots,N\}. \tag{EC.36}$$

Therefore, we observe that

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \inf_{y \in \mathcal{U}^i} g\left(\tau_\mu(y), x^i\right) \\
&\leq \frac{1}{N}\sum_{i=1}^N \inf_{y \in \mathcal{U}^i} g\left(\tau_{\mu^{\sigma^1 \cdots \sigma^N}}(y), x^i\right) \\
&= \frac{1}{N}\sum_{i=1}^N \min_{t \in \{1,\ldots,\sigma^i\}} \left\{g(t, x^i) : \text{there exists } y_t \in \mathcal{U}_t^i \text{ such that } \mu_t^{\sigma^1 \cdots \sigma^N}(y_t) = \textsc{Stop}\right\} \\
&= \frac{1}{N}\sum_{i=1}^N \min_{t \in \{1,\ldots,\sigma^i\}} \left\{g(t, x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t\right\},
\end{aligned} \tag{EC.37}$$

where the first inequality follows from Lemma 3, the first equality follows from Lemma 2 and line (EC.36), and the second equality follows from the fact that $\mu_t^{\sigma^1 \cdots \sigma^N}(y_t) = \textsc{Stop}$ if and only if there exists a sample path j such that $y_t \in \mathcal{U}_t^j$ and $\sigma^j = t$. Because $\mu \in \mathcal{M}$ was chosen arbitrarily, we conclude from line (EC.37) and Theorem 4 that the optimal objective value of (RO) is less than or equal to the optimal objective value of (IP). Our proof of Claim EC.8 is thus complete. □

Claim EC.9. *The optimal objective value of* (RO) *is greater than or equal to the optimal objective value of* (IP). *Furthermore, for any choice of integers* $\sigma^1,\ldots,\sigma^N \in \{1,\ldots,T\}$, *the corresponding exercise policy* $\mu^{\sigma^1\cdots\sigma^N} \equiv (\mu_1^{\sigma^1\cdots\sigma^N},\ldots,\mu_T^{\sigma^1\cdots\sigma^N})$ *satisfies*

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g\left(\tau_{\mu^{\sigma^1\cdots\sigma^N}}(y),x^i\right)\\ &\geq \frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\left\{g(t,x^i):\ \text{there exists } j \text{ such that } \mathcal{U}_t^i\cap\mathcal{U}_t^j\neq\emptyset \text{ and } \sigma^j=t\right\}.\end{aligned}$$

Proof of Claim EC.9. Consider any arbitrary integers $\sigma^1,\ldots,\sigma^N\in\{1,\ldots,T\}$. For each sample path $i\in\{1,\ldots,N\}$, we observe that

$$\begin{aligned}&\min\left\{t\in\{1,\ldots,T\}:\ \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\textsc{Stop} \text{ for all } y_t\in\mathcal{U}_t^i\right\}\\ &=\min\left\{t\in\{1,\ldots,\sigma^i\}:\ \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\textsc{Stop} \text{ for all } y_t\in\mathcal{U}_t^i\right\}\\ &\leq\sigma^i,\end{aligned}\tag{EC.38}$$

where the equality follows from the fact that $\mu^{\sigma^1\cdots\sigma^N}\equiv(\mu_1^{\sigma^1\cdots\sigma^N},\ldots,\mu_T^{\sigma^1\cdots\sigma^N})$ by construction satisfies $\mu_{\sigma^i}^{\sigma^1\cdots\sigma^N}(y_{\sigma^i})=\textsc{Stop}$ for all $y_{\sigma^i}\in\mathcal{U}_{\sigma^i}^i$, and the inequality follows from algebra. Therefore,

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\inf_{y\in\mathcal{U}^i} g\left(\tau_{\mu^{\sigma^1\cdots\sigma^N}}(y),x^i\right)\\ &\geq\frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\left\{g(t,x^i):\text{there exists } y_t\in\mathcal{U}_t^i \text{ such that } \mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\textsc{Stop}\right\}\\ &=\frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\left\{g(t,x^i):\ \text{there exists } j \text{ such that } \mathcal{U}_t^i\cap\mathcal{U}_t^j\neq\emptyset \text{ and } \sigma^j=t\right\},\end{aligned}\tag{EC.39}$$

where the inequality follows from Lemma 2 and line (EC.38), and the equality follows from the fact that $\mu_t^{\sigma^1\cdots\sigma^N}(y_t)=\textsc{Stop}$ if and only if there exists a sample path j such that $y_t\in\mathcal{U}_t^j$ and $\sigma^j=t$. Because $\sigma^1,\ldots,\sigma^N\in\{1,\ldots,T\}$ were chosen arbitrarily, we conclude from line (EC.39) and Theorem 4 that the optimal objective value of (RO) is greater than or equal to the optimal objective value of (IP), which concludes our proof of Claim EC.9 is complete. □

Combining Claims EC.8 and EC.9, our proof of Theorem 5 is thus complete. □

Appendix E: Proof of Theorem 7

Our proof of the computational complexity of (IP) consists of a reduction from MIN-2-SAT, which is shown to be strongly NP-hard by Kohli et al. (1994):

MIN-2-SAT

The optimization version of MIN-2-SAT is to compute the optimal objective value of the binary linear optimization problem

$$\begin{aligned} v^{\text{MIN-2-SAT}} \triangleq \quad & \underset{b,z}{\text{minimize}} && \sum_{k=1}^{K} z_k \\ & \text{subject to} && z_k \geq b_\ell && \forall k \in \{1,\ldots,K\}, \forall \ell \in I_k^+ \\ &&& z_k \geq 1 - b_\ell && \forall k \in \{1,\ldots,K\}, \forall \ell \in I_k^- \\ &&& b_\ell \in \{0,1\} && \forall \ell \in \{1,\ldots,L\}, \end{aligned}$$

where the given sets $I_k^+, I_k^- \subseteq \{1,\ldots,L\}$ satisfy $|I_k^+| + |I_k^-| = 2$ for each $k \in \{1,\ldots,K\}$.

Note that the following equality is obtained by replacing each decision variable z_k with $1 - z_k$:

$$\begin{aligned} v^{\text{MIN-2-SAT}} = \quad N \; - \; & \underset{b,z}{\text{maximize}} && \sum_{k=1}^{K} z_k \\ & \text{subject to} && z_k \leq 1 - b_\ell && \forall k \in \{1,\ldots,K\}, \forall \ell \in I_k^+ \\ &&& z_k \leq b_\ell && \forall k \in \{1,\ldots,K\}, \forall \ell \in I_k^- \\ &&& b_\ell \in \{0,1\} && \forall \ell \in \{1,\ldots,L\}. \end{aligned}$$

We now show that any instance of the above maximization problem can be equivalently reformulated as polynomially-size instance of (IP) with $T = 3$ periods.

Proof of Theorem 7. Consider any arbitrary instance of the binary linear optimization problem

$$\begin{aligned} & \underset{b,z}{\text{maximize}} && \sum_{k=1}^{K} z_k \\ & \text{subject to} && z_k \leq 1 - b_\ell && \forall k \in \{1,\ldots,K\}, \forall \ell \in I_k^+ \\ &&& z_k \leq b_\ell && \forall k \in \{1,\ldots,K\}, \forall \ell \in I_k^- \\ &&& b_\ell \in \{0,1\} && \forall \ell \in \{1,\ldots,L\}, \end{aligned} \tag{¬MIN-2-SAT}$$

and let $e_\ell \in \mathbb{R}^{L+1}$ denote the ℓ-th column vector of the identity matrix. We construct an instance of (IP) defined as follows:

- The number of periods is $T = 3$.
- The state space is $\mathcal{X} = \mathbb{R}^{L+1}$.
- The reward function for each period $t \in \{1,2,3\}$ is $g(t,y) = y_t \cdot e_{L+1} + K$.
- The robustness parameter in the uncertainty sets is $\epsilon = \frac{2}{3}$.
- The number of sample paths is $N \triangleq L + K$, and the sample paths are defined as follows:
  - —For each $\ell \in \{1,\ldots,L\}$, let $x_1^\ell = x_2^\ell = e_\ell$ and $x_3^\ell = -K e_{L+1}$.
  - —For each $k \in \{1,\ldots,K\}$, let $x_1^{L+k} = \frac{1}{2}\sum_{\ell \in I_k^+} e_\ell$, $x_2^{L+k} = \frac{1}{2}\sum_{\ell \in I_k^-} e_\ell$, and $x_3^{L+k} = e_{L+1}$.

In the remainder of the proof, we show that the above instance of (IP) is equivalent to (¬MIN-2-SAT). Indeed, it follows immediately from the above construction that the values of $g(t,x^i)$ for each sample path i and period t are:

$$\begin{aligned} g(1,x^\ell) &= K; & g(2,x^\ell) &= K; & g(3,x^\ell) &= 0, & \forall \ell &\in \{1,\ldots,L\}, \\ g(1,x^{L+k}) &= K; & g(2,x^{L+k}) &= K; & g(3,x^{L+k}) &= K+1, & \forall k &\in \{1,\ldots,K\}. \end{aligned}$$

We require two intermediary claims:

Claim EC.10. *There exists an optimal solution for* (IP) *which satisfies*

$$\sigma^1,\ldots,\sigma^L \in \{1,2\} \text{ and } \sigma^{L+1} = \cdots = \sigma^{L+K} = 3.$$

Proof of Claim EC.10. We observe that the optimal objective value of (IP) is greater than or equal to K, since this objective value would be achieved by setting $\sigma^1 = \cdots = \sigma^N = 1$. Now consider any solution to (IP) where $\sigma^{\ell'} = 3$ for some $\ell' \in \{1,\ldots,L\}$. For that solution,

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} \min_{t\in\{1,\ldots,\sigma^i\}} \left\{ g(t,x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\
&= \frac{1}{N} \sum_{\ell\in\{1,\ldots,L\}:\ell\neq\ell'} \min_{t\in\{1,\ldots,\sigma^\ell\}} \left\{ g(t,x^\ell) : \text{ there exists } j \text{ such that } \mathcal{U}_t^\ell \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\
&\quad + \frac{1}{N} \min_{t\in\{1,\ldots,\sigma^{\ell'}\}} \left\{ g(t,x^{\ell'}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{\ell'} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\
&\quad + \frac{1}{N}\sum_{k=1}^{K} \min_{t\in\{1,\ldots,\sigma^{L+k}\}} \left\{ g(t,x^{L+k}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{L+k} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\
&\le \frac{1}{N} \sum_{\ell\in\{1,\ldots,L\}:\ell\neq\ell'} K + 0 + \sum_{k=1}^{K} (K+1) \\
&= K.
\end{aligned}$$

Because the objective value associated with this solution is never better than the objective value obtained by the solution $\sigma^1 = \cdots = \sigma^N = 1$, we have shown that there exists an optimal solution for (IP) that satisfies $\sigma^1,\ldots,\sigma^L \in \{1,2\}$.

Consider any arbitrary solution $\sigma^1,\ldots,\sigma^N \in \{1,2,3\}$ that satisfies $\sigma^1,\ldots,\sigma^L \in \{1,2\}$, and suppose that $\sigma^{L+k} \in \{1,2\}$ for some $k \in \{1,\ldots,K\}$. To perform an exchange argument, we construct an alternative solution $\bar{\sigma}^1,\ldots,\bar{\sigma}^N \in \{1,2,3\}$ defined as

$$\bar{\sigma}^i \triangleq \begin{cases} \sigma^i, & \text{if } i \neq L+k, \\ 3, & \text{if } i = L+k. \end{cases}$$

We observe that the inclusion $\{j : \sigma^j \le \sigma^i\} \supseteq \{j : \bar{\sigma}^j \le \bar{\sigma}^i\}$ holds for all $i \in \{1,\ldots,N\} \setminus \{L+k\}$. Moreover, we observe that

$$\begin{aligned}
&\min_{t\in\{1,\ldots,\sigma^{L+k}\}} \left\{ g(t,x^{L+k}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{L+k} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} = K, \text{ and} \\
&\min_{t\in\{1,\ldots,\bar{\sigma}^{L+k}\}} \left\{ g(t,x^{L+k}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{L+k} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \bar{\sigma}^j = t \right\} \in \{K, K+1\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} \min_{t\in\{1,\ldots,\sigma^i\}} \left\{ g(t,x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\
&\le \frac{1}{N}\sum_{i=1}^{N} \min_{t\in\{1,\ldots,\bar{\sigma}^i\}} \left\{ g(t,x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \bar{\sigma}^j = t \right\}
\end{aligned}$$

Because $\sigma^1,\ldots,\sigma^N$ was chosen arbitrarily, we conclude that there exists an optimal solution for (IP) which satisfies $\sigma^1,\ldots,\sigma^L \in \{1,2\}$ and $\sigma^{L+1} = \cdots = \sigma^{L+K} = 3$. This concludes our proof of Claim EC.10. □

Claim EC.11. *If $\sigma^{L+1} = \cdots = \sigma^{L+K} = 3$, then the following equality holds for each $k \in \{1,\ldots,K\}$:*

$$\begin{aligned}&\min_{t\in\{1,\ldots,\sigma^{L+k}\}} \left\{ g(t, x^{L+k}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{L+k} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\ &= K + \mathbb{I}\left\{ \sigma^\ell = 2 \text{ for all } \ell \in I_k^+ \text{ and } \sigma^\ell = 1 \text{ for all } \ell \in I_k^- \right\}.\end{aligned}$$

Proof of Claim EC.11. For each $\ell \in \{1,\ldots,L\}$ and $k \in \{1,\ldots,K\}$, we observe that

$$\left\| x_1^\ell - x_1^{L+k} \right\|_\infty = \left\| e_\ell - \frac{1}{2} \sum_{\ell' \in I_k^+} e_{\ell'} \right\|_\infty = \begin{cases} \frac{1}{2}, & \text{if } \ell \in I_k^+, \\ 1, & \text{otherwise.} \end{cases}$$

$$\left\| x_2^\ell - x_2^{L+k} \right\|_\infty = \left\| e_\ell - \frac{1}{2} \sum_{\ell' \in I_k^-} e_{\ell'} \right\|_\infty = \begin{cases} \frac{1}{2}, & \text{if } \ell \in I_k^-, \\ 1, & \text{otherwise.} \end{cases}$$

This implies that the set

$$\mathcal{U}_1^\ell \cap \mathcal{U}_1^{L+k} = \left\{ y_1 \in \mathbb{R}^{L+1} : \|y_1 - x_1^\ell\|_\infty \le \frac{2}{3} \right\} \cap \left\{ y_1 \in \mathbb{R}^{L+1} : \|y_1 - x_1^{L+k}\|_\infty \le \frac{2}{3} \right\}$$

is nonempty if and only if $\ell \in I_k^+$, and the set

$$\mathcal{U}_2^\ell \cap \mathcal{U}_2^{L+k} = \left\{ y_2 \in \mathbb{R}^{L+1} : \|y_2 - x_2^\ell\|_\infty \le \frac{2}{3} \right\} \cap \left\{ y_2 \in \mathbb{R}^{L+1} : \|y_2 - x_2^{L+k}\|_\infty \le \frac{2}{3} \right\}$$

is nonempty if and only if $\ell \in I_k^-$. Consequently, since $\sigma^{L+1} = \cdots = \sigma^{L+K} = 3$, we conclude that the following equalities hold for all $k \in \{1,\ldots,K\}$:

$$\begin{aligned}&\min_{t\in\{1,\ldots,\sigma^{L+k}\}} \left\{ g(t, x^{L+k}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{L+k} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \\ &= \min \left\{ \min_{\ell \in I_k^+ : \sigma^\ell = 1} K, \min_{\ell \in I_k^- : \sigma^\ell = 2} K, K+1 \right\} \\ &= K + \mathbb{I}\left\{ \sigma^\ell = 2 \text{ for all } \ell \in I_k^+ \text{ and } \sigma^\ell = 1 \text{ for all } \ell \in I_k^- \right\}\end{aligned}$$

This concludes the proof of Claim EC.11. $\square$

We now combine Claims EC.10 and EC.11 to complete our proof of Theorem 7:

$$\text{(IP)} \tag{EC.40}$$

$$= \max_{\substack{\sigma^1,\ldots,\sigma^L \in \{1,2\} \\ \sigma^{L+1} = \cdots = \sigma^{L+K} = 3}} \frac{1}{L+K} \sum_{i=1}^{L+K} \min_{t\in\{1,\ldots,\sigma^i\}} \left\{ g(t, x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \tag{EC.41}$$

$$\begin{aligned}= \max_{\substack{\sigma^1,\ldots,\sigma^L \in \{1,2\} \\ \sigma^{L+1} = \cdots = \sigma^{L+K} = 3}} \Bigg\{ &\frac{1}{L+K} \sum_{\ell=1}^{L} K \\ &+ \frac{1}{L+K} \sum_{k=1}^{K} \min_{t\in\{1,\ldots,\sigma^{L+k}\}} \left\{ g(t, x^{L+k}) : \text{ there exists } j \text{ such that } \mathcal{U}_t^{L+k} \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \Bigg\}\end{aligned} \tag{EC.42}$$

$$= \max_{\sigma^1,\ldots,\sigma^L \in \{1,2\}} \left\{ \frac{LK}{L+K} + \frac{1}{L+K} \sum_{k=1}^{K} \left(K + \mathbb{I}\left\{ \sigma^\ell = 2 \text{ for all } \ell \in I_k^+ \text{ and } \sigma^\ell = 1 \text{ for all } \ell \in I_k^- \right\} \right) \right\} \tag{EC.43}$$

$$= K + \left(\frac{1}{L+K} \right) \max_{\sigma^1,\ldots,\sigma^L \in \{1,2\}} \left\{ \sum_{k=1}^{K} \mathbb{I}\left\{ \sigma^\ell = 2 \text{ for all } \ell \in I_k^+ \text{ and } \sigma^\ell = 1 \text{ for all } \ell \in I_k^- \right\} \right\}$$

$$= K + \left(\frac{1}{L+K} \right) * (\neg\text{MIN-2-SAT}). \tag{EC.44}$$

Indeed, (EC.41) follows from Claim EC.10; (EC.42) holds because $g(1,x^\ell) = g(2,x^\ell) = K$ for all $\ell \in \{1,\ldots,L\}$; (EC.43) follows from Claim EC.11; (EC.44) follows from algebra and setting $b^\ell = 0$ if and only if $\sigma^\ell = 2$. We have thus shown that any instance of MIN-2-SAT can be reduced to solving a polynomially-sized instance of (IP) with $T = 3$, which concludes our proof of Theorem 7. □

Appendix F: Reformulation of (BP) as Mixed-Integer Linear Optimization Problem

Zero-one bilinear programs can be transformed into equivalent mixed-integer linear optimization problems by introducing auxiliary decision variables (Adams and Sherali 1986). In numerical experiments in §5, we perform such a linearization of the bilinear program (BP) by introducing auxiliary continuous decision variables $f^i_{t\ell}$ which obey the constraints

$$\begin{aligned} f^i_{t\ell} &\le b^i_t && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T\},\ \ell \in \{1,\ldots,L^i_t - 1\} \\ f^i_{t\ell} &\le 1 - w^i_{t\ell} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T\},\ \ell \in \{1,\ldots,L^i_t - 1\} \end{aligned}$$

and replacing the objective function of (BP) with

$$\underset{b,w,f}{\text{maximize}} \quad \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{\ell=1}^{L^i_t - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell) f^i_{t\ell}.$$

To strengthen this linear relaxation of (BP), we also add the valid constraints:

$$\begin{aligned} & w^i_{1,1} = 0 && \text{for all } i \in \{1,\ldots,N\} \\ & b^i_t + w^i_{t,1} = w^i_{t+1,1} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T-1\} \\ & b^i_T + w^i_{T,1} = 1 && \text{for all } i \in \{1,\ldots,N\}. \end{aligned}$$

Indeed, the validity of the above constraints for (BP) follows from the fact that there is an optimal solution to this zero-one bilinear program which satisfies $\sum_{t=1}^{T} b^i_t = 1$ for each sample path i. In summary, this linearization procedure transforms (BP) into the following equivalent mixed-integer linear optimization problem:

$$
\begin{array}{lll}
\underset{b,w,f}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{\ell=1}^{L_t^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)f_{t\ell}^i & \\
\text{subject to} & f_{t\ell}^i \le b_t^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,L_t^i-1\}\\
& f_{t\ell}^i \le 1-w_{t\ell}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,L_t^i-1\}\\
& w_{t,\ell}^i \le w_{t+1,\ell}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T-1\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w_{t\ell}^i \le w_{t,\ell+1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& w_{1,1}^i = 0 & \text{for all } i\in\{1,\ldots,N\}\\
& b_t^i + w_{t,1}^i = w_{t+1,1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T-1\}\\
& b_T^i + w_{T,1}^i = 1 & \text{for all } i\in\{1,\ldots,N\}\\
& b_t^i \le w_{t+1,1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T-1\}\\
& b_t^j \le w_{t\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ and } t\in\{1,\ldots,T\} \text{ such that } g(t,x^i)=\kappa_\ell^i\\
& & \text{and } \mathcal{U}_t^i\cap\mathcal{U}_t^j\neq\emptyset\\
& b_t^i\in\{0,1\} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\}\\
& w_{t\ell}^i\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}
$$

Appendix G: Proofs of Theorem 8, Lemma 5, and Lemma 6

Proof of Lemma 5. Our proof of Lemma 5 is split into two intermediary steps.

In the first intermediary step of our proof of Lemma 5, we show that every feasible solution for (IP) can be transformed into a feasible solution for the (BP-1) with the same objective value. Indeed, consider any feasible solution $\sigma^1,\ldots,\sigma^N\in\{1,\ldots,T\}$ for the optimization problem (IP). From these integers, we can define binary variables $b_t^i\triangleq\mathbb{I}\{\sigma^i=t\}$ for each sample path $i\in\{1,\ldots,N\}$ and period $t\in\{1,\ldots,T\}$. With this definition of a binary vector b, we observe for each sample path $i\in\{1,\ldots,N\}$ that

$$
\begin{aligned}
&\psi^i(b)\\
&=\sum_{t=1}^{T}b_t^i\left(\prod_{s=1}^{t-1}(1-b_s^i)\right)\min_{s\in\{1,\ldots,t\}}\left\{g(s,x^i): \text{there exists } j \text{ such that } \mathcal{U}_s^i\cap\mathcal{U}_s^j\neq\emptyset \text{ and } b_s^j=1\right\} && \text{(EC.45)}\\
&=\min_{s\in\{1,\ldots,\sigma^i\}}\left\{g(s,x^i): \text{there exists } j \text{ such that } \mathcal{U}_s^i\cap\mathcal{U}_s^j\neq\emptyset \text{ and } b_s^j=1\right\} && \text{(EC.46)}\\
&=\min_{s\in\{1,\ldots,\sigma^i\}}\left\{g(s,x^i): \text{there exists } j \text{ such that } \mathcal{U}_s^i\cap\mathcal{U}_s^j\neq\emptyset \text{ and } \sigma^j=s\right\}, && \text{(EC.47)}
\end{aligned}
$$

where line (EC.45) follows from the definition of $\psi^i(b)$, and lines (EC.46) and (EC.47) follow from the fact that we have defined the binary variables to satisfy $b_t^j=\mathbb{I}\{\sigma^j=t\}$ for each sample path $j\in\{1,\ldots,N\}$ and period $t\in\{1,\ldots,T\}$. Because the above reasoning holds for each sample path $i\in\{1,\ldots,N\}$, we have shown that

$$
\frac{1}{N}\sum_{i=1}^{N}\psi^i(b)=\frac{1}{N}\sum_{i=1}^{N}\min_{t\in\{1,\ldots,\sigma^i\}}\left\{g(t,x^i):\ \text{there exists } j \text{ such that } \mathcal{U}_t^i\cap\mathcal{U}_t^j\neq\emptyset \text{ and } \sigma^j=t\right\}. \tag{EC.48}
$$

Therefore, we conclude that every feasible solution for (IP) can be transformed into a feasible solution for (BP-1) with the same objective value.

In the second intermediary step of the proof of Lemma 5, we show that if b is a feasible solution for (BP-1) and if $\sigma^i \triangleq \min\{\min\{t \in \{1,\dots,T\} : b_t^i = 1\}, T\}$ for each $i \in \{1,\dots,N\}$, then

$$\frac{1}{N}\sum_{i=1}^N \min_{t\in\{1,\dots,\sigma^i\}} \left\{ g(t,x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\} \geq \frac{1}{N}\sum_{i=1}^N \psi^i(b). \tag{EC.49}$$

Indeed, consider any feasible solution b for (BP-1), and let us define a new binary vector $\bar{b}$ as

$$\bar{b}_t^i \triangleq \begin{cases} b_t^i, & \text{if } t \in \{1,\dots,T-1\},\\ 1, & \text{if } t = T.\end{cases}$$

It follows immediately from the above definition of $\bar{b}$ and from the definition of the function $\psi^i(\cdot)$ for each sample path $i \in \{1,\dots,N\}$ that

$$\begin{aligned}
&\psi^i\left(\bar{b}\right) - \psi^i\left(b\right)\\
&= \begin{cases} \min\limits_{s\in\{1,\dots,T\}} \{g(s,x^i) : \text{there exists } j \text{ such that } \mathcal{U}_s^i \cap \mathcal{U}_s^j \neq \emptyset \text{ and } b_s^j = 1\} - 0, & \text{if } b_1^i = \cdots = b_T^i = 0,\\ 0, & \text{otherwise}\end{cases}\\
&\geq 0.
\end{aligned}$$

where the inequality holds because the reward function is nonnegative, that is, $g(t,x^i) \geq 0$ for each sample path $i \in \{1,\dots,N\}$ and period $s \in \{1,\dots,T\}$ (see §2.1). Therefore, we have proven that $\bar{b}$ has the same or better objective value as b in the optimization problem (BP-1), i.e.,

$$\frac{1}{N}\sum_{i=1}^N \psi^i\left(b\right) \leq \frac{1}{N}\sum_{i=1}^N \psi^i\left(\bar{b}\right). \tag{EC.50}$$

Now, for each sample path $i \in \{1,\dots,N\}$, let us define the following integer:

$$\sigma^i \triangleq \min\left\{ \min_{t\in\{1,\dots,T\}} \left\{t : b_t^i = 1\right\}, T \right\} = \min_{t\in\{1,\dots,T\}} \left\{t : \bar{b}_t^i = 1\right\}.$$

Given the integers $\sigma^1,\dots,\sigma^N \in \{1,\dots,T\}$ defined above, it follows from identical reasoning as in lines (EC.45), (EC.46), (EC.47), and (EC.48) that

$$\frac{1}{N}\sum_{i=1}^N \psi^i\left(\bar{b}\right) = \frac{1}{N}\sum_{i=1}^N \min_{t\in\{1,\dots,\sigma^i\}} \left\{ g(t,x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \text{ and } \sigma^j = t \right\}.$$

Combining the above equality with (EC.50), we have proven that line (EC.49) holds, thereby completing our proof of the second intermediary step.

In conclusion, we have shown through the above two intermediary steps that every feasible solution for either (IP) or (BP-1) can be transformed into a feasible solution for the other problem with the same or better objective value, which implies that the optimal objective values of (IP) and (BP-1) are equal. Moreover, the inequality in Lemma 5 follows immediately from the second intermediary step. Thus, our proof of Lemma 5 is complete. $\square$

Proof of Lemma 6. Consider any sample path $i \in \{1,\dots,N\}$ and binary vector b. For the sake of convenience, let us repeat below the linear optimization problem that is found in the statement of Lemma 6 for the given sample path i and binary vector b:

$$\begin{aligned}
\underset{w^i}{\text{maximize}} \quad & \sum_{t=1}^{T} \sum_{\ell=1}^{L_t^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i) b_t^i (1 - w_{t\ell}^i) \\
\text{subject to} \quad & w_{t\ell}^i \le w_{t+1,\ell}^i \quad \text{for all } t \in \{1,\dots,T-1\},\ \ell \in \{1,\dots,|\mathcal{K}^i|\} \\
& w_{t\ell}^i \le w_{t,\ell+1}^i \quad \text{for all } t \in \{1,\dots,T\},\ \ell \in \{1,\dots,|\mathcal{K}^i|-1\} \\
& b_t^j \le w_{t\ell}^i \quad \text{for all } j \in \{1,\dots,N\} \text{ and } t \in \{1,\dots,T\} \\
& \qquad\qquad \text{such that } g(t,x^i) = \kappa_\ell^i \text{ and } \mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset \\
& b_t^i \le w_{t+1,1}^i \quad \text{for all } t \in \{1,\dots,T-1\} \\
& w_{t\ell}^i \in \mathbb{R} \quad \text{for all } t \in \{1,\dots,T\},\ \ell \in \{1,\dots,|\mathcal{K}^i|\}.
\end{aligned} \tag{3}$$

In the remainder of this proof, we show that the optimal objective value of (3) is equal to $\psi^i(b)$.

To this end, we begin by characterizing the decision variables w^i in (3) at optimality. Indeed, it follows from the construction of the constants κ_ℓ^i that each quantity $\kappa_{\ell+1}^i - \kappa_\ell^i$ is strictly positive. Therefore, since each b_t^j is binary, we readily observe that there always exists an optimal solution w^i for (3) where the following equality holds for each period $t \in \{1,\dots,T\}$ and each $\ell \in \{1,\dots,|\mathcal{K}^i|\}$:

$$w_{t\ell}^i = \begin{cases} 0, & \text{if } [b_s^j = 0 \text{ for all } j \in \{1,\dots,N\} \text{ and } s \in \{1,\dots,t\} \text{ such that } g(s,x^i) \le \kappa_\ell^i \text{ and } \mathcal{U}_s^i \cap \mathcal{U}_s^j \neq \emptyset] \\ & \text{and } [b_s^i = 0 \text{ for all } s \in \{1,\dots,t-1\}], \\ 1, & \text{otherwise.} \end{cases} \tag{EC.51}$$

Now consider any optimal solution w^i for the optimization problem (3) for which the equality in (EC.51) is satisfied for each period $t \in \{1,\dots,T\}$ and each $\ell \in \{1,\dots,|\mathcal{K}^i|\}$. For each period $t \in \{1,\dots,T\}$, let $\tilde{\ell}_t^i$ be defined as the smallest integer such that there exists a sample path $j \in \{1,\dots,N\}$ and a period $s \in \{1,\dots,t\}$ that satisfy $b_s^j = 1$, $g(s,x^i) = \kappa_{\tilde{\ell}_t^i}^i$, and $\mathcal{U}_s^i \cap \mathcal{U}_s^j \neq \emptyset$.[12] In particular, we observe from (EC.51) that the quantity $\tilde{\ell}_t^i$ is equal to the smallest integer such that $w_{t\tilde{\ell}_t^i}^i = 1$. Hence, the optimal objective value of the optimization problem (3) is equal to:

$$\sum_{t=1}^{T} \sum_{\ell=1}^{L_t^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i) b_t^i (1 - w_{t\ell}^i)$$

$$= \sum_{t \in \{1,\dots,T\}:\, b_t^i = 1} \sum_{\ell=1}^{L_t^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)(1 - w_{t\ell}^i) \tag{EC.52}$$

$$= \begin{cases} \sum_{\ell=1}^{L_t^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)(1 - w_{t\ell}^i), & \text{if } b_1^i = \cdots = b_{t-1}^i = 0 \text{ and } b_t^i = 1 \text{ for } t \in \{1,\dots,T\}, \\ 0, & \text{otherwise} \end{cases} \tag{EC.53}$$

$$= \begin{cases} \sum_{\ell=1}^{\tilde{\ell}_t^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i), & \text{if } b_1^i = \cdots = b_{t-1}^i = 0 \text{ and } b_t^i = 1 \text{ for } t \in \{1,\dots,T\}, \\ 0, & \text{otherwise} \end{cases} \tag{EC.54}$$

$$= \begin{cases} \kappa_{\tilde{\ell}_t^i}^i, & \text{if } b_1^i = \cdots = b_{t-1}^i = 0 \text{ and } b_t^i = 1 \text{ for } t \in \{1,\dots,T\}, \\ 0, & \text{otherwise} \end{cases} \tag{EC.55}$$

[12] If no such integer exists, then we assign $\tilde{\ell}_t^i$ to be equal to L_t^i.

$$= \sum_{t=1}^{T} b_t^i \left(\prod_{s=1}^{t-1} \left(1 - b_s^i\right) \right) \kappa^i_{\tilde{\ell}^i_t} \tag{EC.56}$$

$$= \sum_{t=1}^{T} b_t^i \left(\prod_{s=1}^{t-1} \left(1 - b_s^i\right) \right) \min_{s \in \{1,\ldots,t\}} \left\{ g(s, x^i) : \text{ there exists } j \text{ such that } \mathcal{U}_s^i \cap \mathcal{U}_s^j \neq \emptyset \text{ and } b_s^j = 1 \right\} \tag{EC.57}$$

$$= \psi^i(b).$$

Indeed, we observe that the optimal objective value of (3) is equal to $\sum_{t=1}^{T} \sum_{\ell=1}^{L_t^i - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell) b_t^i (1 - w^i_{t\ell})$ because w^i is an optimal solution for (3). Line (EC.52) follows from algebra. Line (EC.53) follows from (EC.51), which implies that if $b_t^i = 1$, then $w^i_{s\ell} = 1$ for all $s \in \{t+1, \ldots, T\}$ and all $\ell \in \{1, \ldots, |\mathcal{K}^i|\}$. Line (EC.54) follows from the definition of $\tilde{\ell}^i_t$ and from (EC.51). Line (EC.55) follows from the fact that $\kappa^i_1 = 0$. Line (EC.56) follows from algebra. Line (EC.57) follows from the definition of $\tilde{\ell}^i_t$ and the definition of the constants $\kappa^i_1, \ldots, \kappa^i_{|\mathcal{K}^i|}$. The final equality follows from the definition of $\psi^i(b)$. This completes our proof of Lemma 6. □

Proof of Theorem 8. The proof of Theorem 8 follows immediately from Lemmas 5 and 6. □

Appendix H: Proofs from §4.2

H.1. Proofs from §4.2.1

Proof of Proposition 1. Let $\hat{b}, \hat{w}$ denote an optimal solution for (BP), and let the set $\widehat{\mathcal{T}}^i \triangleq \{t : \hat{b}_t^i = 1\}$ be defined for each sample path i. We will henceforth consider the optimization problem (H) in the case where the linear objective function $f(b, w)$ is defined equal to the following function:

$$\hat{f}(b, w) \triangleq \frac{1}{N} \sum_{i=1}^{N} \sum_{t \in \widehat{\mathcal{T}}^i} \sum_{\ell=1}^{L_t^i - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell)(b_t^i - w^i_{t\ell}).$$

We readily observe that the function $\hat{f}(b, w)$ is linear in b and w. Moreover, it follows from algebra that $\hat{f}(b, w)$ is less than or equal to the objective function of (BP) for all feasible solutions of (BP); that is, the linear function $\hat{f}(b, w)$ satisfies the condition from line (4).[13] Thus, we conclude that solving the optimization problem (H) with objective function $f(b, w) = \hat{f}(b, w)$ will provide a lower-bound approximation of the optimization problem (BP), and any optimal solution for (H) will be a feasible solution for (BP). For notational convenience, we henceforth let J^{H} denote the optimal objective value of (H) with objective function $f(b, w) = \hat{f}(b, w)$, and we let J^{BP} denote the optimal objective value of (BP).

We first show that the optimal objective value of (H) with objective function $f(b, w) = \hat{f}(b, w)$ is equal to the optimal objective value of (BP). Indeed, it follows from the fact that (H) and (BP) have the same constraints that the optimal solution $\hat{b}, \hat{w}$ for (BP) is a feasible solution for (H). Therefore,

$$J^{\mathrm{H}} \geq \frac{1}{N} \sum_{i=1}^{N} \sum_{t \in \widehat{\mathcal{T}}^i} \sum_{\ell=1}^{L_t^i - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell) \left(\hat{b}_t^i - \hat{w}^i_{t\ell} \right)$$

[13] To see why $\hat{f}(b, w)$ is a lower bound on the objective function of (BP), consider any arbitrary vectors b, w that satisfy the constraints of (BP). Since feasibility for the optimization problem (BP) implies that b is a binary vector, we observe that the equality $b_t^i (1 - w^i_{t\ell}) = \max\left\{ b_t^i - w^i_{t\ell}, 0 \right\}$ holds for each $i \in \{1, \ldots, N\}$, $t \in \{1, \ldots, T\}$, and $\ell \in \{1, \ldots, |\mathcal{K}^i|\}$.

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{t:\hat{b}_t^i=1}\sum_{\ell=1}^{L_t^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)(1-\hat{w}_{t\ell}^i)+\frac{1}{N}\sum_{i=1}^{N}\sum_{t:\hat{b}_t^i=0}\sum_{\ell=1}^{L_t^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)0\\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{\ell=1}^{L_t^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)\hat{b}_t^i(1-\hat{w}_{t\ell}^i)\\
&= J^{\mathrm{BP}},
\end{aligned}
$$

where the first inequality holds because $\hat{b},\hat{w}$ is a feasible but possibly suboptimal solution for (H), the first equality follows from the construction of the sets $\widehat{\mathcal{T}}^1,\ldots,\widehat{\mathcal{T}}^N \subseteq \{1,\ldots,T\}$, the second equality follows from algebra, and the final equality follows from the fact that $\hat{b},\hat{w}$ is an optimal solution for (BP). Since the optimal objective value of (H) is always less than or equal to the optimal objective value of (BP), we have proved that the optimal objective value of (H) is equal to the optimal objective value of (BP).

We conclude the proof of Proposition 1 by showing that every optimal solution for (H) with objective function $f(b,w)=\hat{f}(b,w)$ is an optimal solution for (BP). Indeed, let $\bar{b},\bar{w}$ denote an optimal solution for (H). Then,

$$
J^{\mathrm{H}} = \hat{f}(\bar{b},\bar{w}) \le \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{\ell=1}^{L_t^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)\bar{b}_t^i\left(1-\bar{w}_{t\ell}^i\right) \le J^{\mathrm{BP}},
$$

where the first equality follows from the fact that $\bar{b},\bar{w}$ is an optimal solution for (H) and from the fact that the objective function of (H) is $f(b,w)=\hat{f}(b,w)$, the first inequality follows from the fact that the linear function $\hat{f}(b,w)$ satisfies the condition from line (4), and the final inequality follows from the fact that $\bar{b},\bar{w}$ is a feasible but possibly suboptimal solution for (BP). Since we have previously shown that the optimal objective value J^{H} of (H) is equal to the optimal objective value J^{BP} of (BP), we have thus proven that $\bar{b},\bar{w}$ is an optimal solution for (BP). This concludes our proof of Proposition 1. □

H.2. Proofs from §4.2.3

Proof of Lemma 7. Let $f(b,w)=\bar{f}(b,w)$, and consider any binary vector b that is optimal for the optimization problem (H). Since b is binary, we readily observe that there exists an optimal choice for the remaining decision variables w in the optimization problem (H) in which the following equality holds for each sample path $i\in\{1,\ldots,N\}$, period $t\in\{1,\ldots,T\}$, and $\ell\in\{1,\ldots,|\mathcal{K}^i|\}$:

$$
w_{t\ell}^i = \begin{cases} 0, & \text{if } [b_s^j=0 \text{ for all } j\in\{1,\ldots,N\} \text{ and } s\in\{1,\ldots,t\} \text{ such that } g(s,x^i)\le\kappa_\ell^i \text{ and } \mathcal{U}_s^i\cap\mathcal{U}_s^j\neq\emptyset]\\ & \text{and } [b_s^i=0 \text{ for all } s\in\{1,\ldots,t-1\}],\\ 1, & \text{otherwise.}\end{cases} \tag{EC.58}
$$

We will now construct a new solution for (H) that has the same or greater objective value than b,w. Indeed, let $\bar{b},\bar{w}$ be a solution for (H) defined by the following equalities for each sample path $i\in\{1,\ldots,N\}$, period $t\in\{1,\ldots,T\}$, and $\ell\in\{1,\ldots,|\mathcal{K}^i|\}$:

$$
\bar{b}_t^i \triangleq \begin{cases} 1, & \text{if } [b_t^i=1 \text{ and } t=T^i] \text{ or } [t=T],\\ 0, & \text{otherwise;}\end{cases}
$$

$$
\bar{w}_{t\ell}^i \triangleq \begin{cases} 0, & \text{if } \left[\bar{b}_s^j=0 \text{ for all } j\in\{1,\ldots,N\} \text{ and } s\in\{1,\ldots,t\} \text{ such that } g(s,x^i)\le\kappa_\ell^i \text{ and } \mathcal{U}_s^i\cap\mathcal{U}_s^j\neq\emptyset\right]\\ & \text{and } \left[\bar{b}_s^i=0 \text{ for all } s\in\{1,\ldots,t-1\}\right],\\ 1, & \text{otherwise.}\end{cases}
$$

We observe for each sample path $i \in \{1,\ldots,N\}$ that $\bar{b}^i_T = 1$ and $\bar{b}^i_t = 0$ for all $t \in \{1,\ldots,T\} \setminus \mathcal{T}^i$. Moreover, we readily observe from inspection that the solution $\bar{b},\bar{w}$ is feasible for (H). Therefore, it remains for us to prove that $\bar{b},\bar{w}$ is an optimal solution for (H).

To show that $\bar{b},\bar{w}$ is an optimal solution for (H), we observe for each sample path $i \in \{1,\ldots,N\}$, period $t \in \{1,\ldots,T\}$, and $\ell \in \{1,\ldots,L^i_t - 1\}$,

$$\begin{aligned}\bar{w}^i_{t\ell} &= \begin{cases} 0, & \text{if } \left[b^j_{T^j} = 0 \text{ for all } j \in \{1,\ldots,N\} \text{ such that } T^j \le t,\ g(T^j, x^i) \le \kappa^i_\ell, \text{ and } \mathcal{U}^i_{T^j} \cap \mathcal{U}^j_{T^j} \neq \emptyset\right] \\ & \text{and } \left[b^i_{T^i} = 0 \text{ if } T^i \in \{1,\ldots,t-1\}\right], \\ 1, & \text{otherwise} \end{cases} \\ &\le w^i_{t\ell}, \end{aligned} \tag{EC.59}$$

where the equality follows from the definition of $\bar{b},\bar{w}$, and the inequality follows from line (EC.58). Therefore,

$$\begin{aligned}\bar{f}(\bar{b},\bar{w}) &= \frac{1}{N}\sum_{i=1}^N \sum_{t\in\mathcal{T}^i} \sum_{\ell=1}^{L^i_t - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell)\left(\bar{b}^i_t - \bar{w}^i_{t\ell}\right) \\ &\ge \frac{1}{N}\sum_{i=1}^N \sum_{t\in\mathcal{T}^i} \sum_{\ell=1}^{L^i_t - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell)\left(b^i_t - w^i_{t\ell}\right) \\ &= \bar{f}(b,w).\end{aligned}$$

Indeed, the first equality follows from the definition of $\bar{f}(\cdot,\cdot)$. The inequality follows from line (EC.59), from the facts that $\bar{b}^i_{T^i} \ge b^i_{T^i}$ and $\bar{b}^i_T \ge b^i_T$, and from the fact that each quantity $\kappa^i_{\ell+1} - \kappa^i_\ell$ is strictly positive. The final equality follows from the definition of $\bar{f}(\cdot,\cdot)$. Since b,w was an optimal solution for (H), our proof of Lemma 7 is complete. □

Proof of Lemma 8. Let $f(b,w) = \bar{f}(b,w)$. In this case, we recall from Lemma 7 that there exists an optimal solution b,w for (H) that satisfies $b^1_T = \cdots = b^N_T = 1$ and $b^i_t = 0$ for each sample path $i \in \{1,\ldots,N\}$ and period $t \in \{1,\ldots,T\} \setminus \mathcal{T}^i$. Therefore, we can without loss of generality impose those equality constraints into the optimization problem (H). That is, (H) can be rewritten equivalently as

$$\begin{aligned} \underset{b,w}{\text{maximize}} \quad & \frac{1}{N}\sum_{i=1}^N \sum_{t\in\mathcal{T}^i} \sum_{\ell=1}^{L^i_t - 1} (\kappa^i_{\ell+1} - \kappa^i_\ell)\left(b^i_t - w^i_{t\ell}\right) \\ \text{subject to} \quad & w^i_{t\ell} \le w^i_{t+1,\ell} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T-1\},\ \ell \in \{1,\ldots,|\mathcal{K}^i|\} \\ & w^i_{t\ell} \le w^i_{t,\ell+1} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T\},\ \ell \in \{1,\ldots,|\mathcal{K}^i|-1\} \\ & b^i_t \le w^i_{t+1,1} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T-1\} \\ & b^j_t \le w^i_{t\ell} && \text{for all } i,j \in \{1,\ldots,N\} \text{ and } t \in \{1,\ldots,T\} \text{ such that } g(t,x^i) = \kappa^i_\ell \\ & && \text{and } \mathcal{U}^i_t \cap \mathcal{U}^j_t \neq \emptyset \\ & b^i_T = 1 && \text{for all } i \in \{1,\ldots,N\} \\ & b^i_t = 0 && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T\} \setminus \mathcal{T}^i \\ & b^i_t \in \{0,1\} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T\} \\ & w^i_{t\ell} \in \mathbb{R} && \text{for all } i \in \{1,\ldots,N\},\ t \in \{1,\ldots,T\},\ \ell \in \{1,\ldots,|\mathcal{K}^i|\}. \end{aligned} \tag{H-1}$$

After substituting out the decision variables b_t^i that have been set to zero or one, it follows from algebra that the above optimization problem can be rewritten equivalently as

$$\begin{aligned}
\underset{b,w}{\text{maximize}} \quad & \frac{1}{N}\sum_{i\in\{1,\ldots,N\}:\,T^i<T}\;\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\left(b^i_{T^i}-w^i_{T^i\ell}\right)+\frac{1}{N}\sum_{i=1}^N\sum_{\ell=1}^{L^i_T-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\left(1-w^i_{T\ell}\right)\\
\text{subject to} \quad & w^i_{t\ell}\le w^i_{t+1,\ell} && \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T-1\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w^i_{t\ell}\le w^i_{t,\ell+1} && \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b^i_{T^i}\le w^i_{T^i+1,1} && \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i<T\\
& b^j_{T^j}\le w^i_{T^j\ell} && \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } T^j<T,\ g(T^j,x^i)=\kappa^i_\ell, \text{ and } \mathcal{U}^i_{T^j}\cap\mathcal{U}^j_{T^j}\neq\emptyset\\
& w^i_{T,L^i_T}=1 && \text{for all } i\in\{1,\ldots,N\}\\
& b^i_{T^i}\in\{0,1\} && \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i<T\\
& w^i_{t\ell}\in\mathbb{R} && \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{aligned} \tag{H-2}$$

Let us make three observations about the above optimization problem. First, we observe from inspection that the constraints in the above optimization problem of the form $w^i_{T,L^i_T}=1$ for all $i\in\{1,\ldots,N\}$ can be removed from the above optimization problem without affecting its optimal objective value. Second, we observe that the decision variables $w^i_{t\ell}$ for each $t\in\{1,\ldots,T\}\setminus\mathcal{T}^i$ do not appear in the objective function of (H-2). Third, since each term $\kappa^i_{\ell+1}-\kappa^i_\ell$ is strictly positive, we observe that there exists an optimal solution for (H-2) that satisfies the following equality for each sample path $i\in\{1,\ldots,N\}$, period $t\in\mathcal{T}^i$, and $\ell\in\{1,\ldots,|\mathcal{K}^i|\}$:

$$w^i_{t\ell}=\begin{cases}0, & \text{if } \left[b^j_{T^j}=0 \text{ for all } j\in\{1,\ldots,N\} \text{ such that } T^j\le t,\ T^j<T,\ g(T^j,x^i)\le\kappa^i_\ell, \text{ and } \mathcal{U}^i_{T^j}\cap\mathcal{U}^j_{T^j}\neq\emptyset\right]\\ & \text{and } \left[\text{if } T^i<t, \text{ then } b^i_{T^i}=0\right],\\ 1, & \text{otherwise.}\end{cases}$$

It follows from the aforementioned three observations that (H-2) can be rewritten equivalently as

$$\begin{aligned}
\underset{b,w}{\text{maximize}} \quad & \frac{1}{N}\sum_{i\in\{1,\ldots,N\}:T^i<T}\;\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\left(b^i_{T^i}-w^i_{T^i\ell}\right)+\frac{1}{N}\sum_{i=1}^N\sum_{\ell=1}^{L^i_T-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\left(1-w^i_{T\ell}\right)\\
\text{subject to} \quad & w^i_{t\ell}\le w^i_{t,\ell+1} && \text{for all } i\in\{1,\ldots,N\},\ t\in\mathcal{T}^i,\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b^i_{T^i}\le w^i_{T1} && \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i<T\\
& b^j_{T^j}\le w^i_{t\ell} && \text{for all } i,j\in\{1,\ldots,N\} \text{ and } t\in\mathcal{T}^i \text{ such that } g(T^j,x^i)=\kappa^i_\ell,\\
& && T^j\le t,\ T^j<T, \text{ and } \mathcal{U}^i_{T^j}\cap\mathcal{U}^j_{T^j}\neq\emptyset\\
& b^i_{T^i}\in\{0,1\} && \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i<T\\
& w^i_{t\ell}\in\mathbb{R} && \text{for all } i\in\{1,\ldots,N\},\ t\in\mathcal{T}^i,\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{aligned} \tag{$\bar{\text{H}}$}$$

We have thus shown that the optimal objective value of ($\bar{\text{H}}$) is equal to the optimal objective value of (H) when $f(b,w)=\bar{f}(b,w)$. Moreover, it follows from the above reasoning that any optimal solution $\bar{b},\bar{w}$ for ($\bar{\text{H}}$) can be transformed into an optimal solution for (H) using the following equality:

$$b^i_t=\begin{cases}\bar{b}^i_t, & \text{if } t=T^i \text{ and } T^i<T,\\ 1, & \text{if } t=T,\\ 0, & \text{otherwise.}\end{cases}$$

This concludes our proof of Lemma 8. □

Proof of Proposition 2. We observe that ($\bar{\text{H}}$) is equivalent to a binary linear optimization problem with $\mathcal{O}(NT)$ binary decision variables and $\mathcal{O}(N^2+NT)$ constraints, with each constraint of the form $\lambda_i \geq \lambda_j$. It thus follows from Picard (1976, §3) that the optimization problem ($\bar{\text{H}}$) is equivalent to a problem of computing the maximal closure of a directed graph with $\mathcal{O}(NT)$ nodes and $\mathcal{O}(N^2+NT)$ edges. Furthermore, Picard (1976, §4) shows that any maximal closure problem can be solved by computing the maximum flow in an augmented graph of identical size. Applying the algorithm of Orlin (2013) to compute the maximum flow in this augmented graph, we obtain an $\mathcal{O}(N^2T(N+T))$ algorithm for solving ($\bar{\text{H}}$). Finally, the output of the maximal closure problem can be transformed into an optimal solution for (H) using Lemma 8. This concludes our proof of Proposition 2. □

H.3. Proofs from §4.2.4

Proof of Proposition 3. Suppose that the number of periods is $T=2$. We recall from Remark 2 that the optimal objective value of ($\bar{\text{H}}$) is less than or equal to the optimal objective value of (BP). Therefore, it remains for us to show that the optimal objective value of (BP) is less than or equal to the optimal objective value of ($\bar{\text{H}}$).

We begin by restating the optimization problem (BP) when $T=2$ for the sake of convenience:

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_1^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)b_1^i(1-w_{1\ell}^i)+\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_2^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)b_2^i(1-w_{2\ell}^i) & \\
\text{subject to} & w_{1\ell}^i \leq w_{2\ell}^i & \text{for all } i\in\{1,\ldots,N\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\} \\
& w_{t\ell}^i \leq w_{t,\ell+1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\} \\
& b_1^i \leq w_{21}^i & \text{for all } i\in\{1,\ldots,N\} \\
& b_1^j \leq w_{1\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(1,x^i)=\kappa_\ell^i \text{ and } \mathcal{U}_1^i\cap\mathcal{U}_1^j\neq\emptyset \\
& b_2^j \leq w_{2\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(2,x^i)=\kappa_\ell^i \text{ and } \mathcal{U}_2^i\cap\mathcal{U}_2^j\neq\emptyset \\
& b_t^i \in\{0,1\} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\} \\
& w_{t\ell}^i\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array} \tag{BP}$$

We readily observe from inspection of the optimization problem (BP) that there exists an optimal solution b,w for (BP) that satisfies the equalities $b_2^1=\cdots=b_2^N=1$ and $b_1^i=0$ for all $i\in\{1,\ldots,N\}$ such that $T^i=2$. Therefore, we observe that those equality constraints can be added to the optimization problem (BP) without changing its optimal objective value. That is, the optimization problem (BP) can be rewritten equivalently as

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_1^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)b_1^i(1-w_{1\ell}^i)+\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_2^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)b_2^i(1-w_{2\ell}^i) & \\
\text{subject to} & w_{1\ell}^i \leq w_{2\ell}^i & \text{for all } i\in\{1,\ldots,N\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\} \\
& w_{t\ell}^i \leq w_{t,\ell+1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\} \\
& b_1^i \leq w_{21}^i & \text{for all } i\in\{1,\ldots,N\} \\
& b_1^j \leq w_{1\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(1,x^i)=\kappa_\ell^i \text{ and } \mathcal{U}_1^i\cap\mathcal{U}_1^j\neq\emptyset \\
& b_2^j \leq w_{2\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(2,x^i)=\kappa_\ell^i \text{ and } \mathcal{U}_2^i\cap\mathcal{U}_2^j\neq\emptyset \\
& b_2^i = 1 & \text{for all } i\in\{1,\ldots,N\} \\
& b_1^i = 0 & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=2 \\
& b_t^i \in\{0,1\} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\} \\
& w_{t\ell}^i\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}$$

After eliminating the decision variables b_t^i that have been constrained to be equal to zero or one, the above optimization problem can be rewritten equivalently as

$$
\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i\in\{1,\ldots,N\}:T^i=1}\sum_{\ell=1}^{L_1^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)b_1^i(1-w_{1\ell}^i)+\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_2^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)(1-w_{2\ell}^i) & \\
\text{subject to} & w_{1\ell}^i\le w_{2\ell}^i & \text{for all } i\in\{1,\ldots,N\},\ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w_{t\ell}^i\le w_{t,\ell+1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b_1^i\le w_{21}^i & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& b_1^j\le w_{1\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(1,x^i)=\kappa_\ell^i,\ T^j=1, \text{ and } \mathcal{U}_1^i\cap\mathcal{U}_1^j\neq\emptyset\\
& w_{2,L_2^i}^i=1 & \text{for all } i\in\{1,\ldots,N\}\\
& b_1^i\in\{0,1\} & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& w_{t\ell}^i\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}
$$

Let us make two observations about the above optimization problem. First, we observe from inspection that the constraints in the above optimization problem of the form $w_{2,L_2^i}^i=1$ for all $i\in\{1,\ldots,N\}$ can be removed from the above optimization problem without affecting its optimal objective value. Second, since each term $\kappa_{\ell+1}^i-\kappa_\ell^i$ is strictly positive, we observe that there exists an optimal solution for the above optimization problem in which the equalities $w_{11}^i=\cdots=w_{1,L_1^i-1}^i=0$ are satisfied for each sample path $i\in\{1,\ldots,N\}$. Therefore, the above optimization problem can be rewritten equivalently as

$$
\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i\in\{1,\ldots,N\}:T^i=1}\sum_{\ell=1}^{L_1^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)b_1^i(1-w_{1\ell}^i)+\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_2^i-1}(\kappa_{\ell+1}^i-\kappa_\ell^i)(1-w_{2\ell}^i) & \\
\text{subject to} & w_{1\ell}^i\le w_{2\ell}^i & \text{for all } i\in\{1,\ldots,N\},\ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w_{t\ell}^i\le w_{t,\ell+1}^i & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b_1^i\le w_{21}^i & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& b_1^j\le w_{1\ell}^i & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(1,x^i)=\kappa_\ell^i,\ T^j=1, \text{ and } \mathcal{U}_1^i\cap\mathcal{U}_1^j\neq\emptyset\\
& w_{1\ell}^i=0 & \text{for all } i\in\{1,\ldots,N\} \text{ and } \ell\in\{1,\ldots,L_1^i-1\}\\
& b_1^i\in\{0,1\} & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& w_{t\ell}^i\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}
$$

Because the above optimization problem has the constraint that $w_{1\ell}^i=0$ for all $i\in\{1,\ldots,N\}$ and $\ell\in\{1,\ldots,L_1^i-1\}$, we can without loss of generality replace each term $b_1^i(1-w_{1\ell}^i)$ in the objective function of the above optimization problem with $b_1^i-w_{1\ell}^i$. That is, the above optimization problem is equivalent to

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \multicolumn{2}{l}{\dfrac{1}{N}\displaystyle\sum_{i\in\{1,\ldots,N\}:T^i=1}\sum_{\ell=1}^{L_1^i-1}(\kappa^i_{\ell+1}-\kappa^i_\ell)(b^i_1-w^i_{1\ell})+\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_2^i-1}(\kappa^i_{\ell+1}-\kappa^i_\ell)(1-w^i_{2\ell})}\\
\text{subject to} & w^i_{1\ell}\le w^i_{2\ell} & \text{for all } i\in\{1,\ldots,N\},\ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w^i_{t\ell}\le w^i_{t,\ell+1} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b^i_1\le w^i_{21} & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& b^j_1\le w^i_{1\ell} & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(1,x^i)=\kappa^i_\ell,\ T^j=1, \text{ and } \mathcal{U}^i_1\cap\mathcal{U}^j_1\neq\emptyset\\
& w^i_{1\ell}=0 & \text{for all } i\in\{1,\ldots,N\} \text{ and } \ell\in\{1,\ldots,L^i_1-1\}\\
& b^i_1\in\{0,1\} & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& w^i_{t\ell}\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}$$

Finally, we can relax the above optimization problem by removing the constraint that $w^i_{1\ell}=0$ for all $i\in\{1,\ldots,N\}$ and $\ell\in\{1,\ldots,L^i_1-1\}$. That is, the optimal objective value of the above optimization problem is less than or equal to the optimal objective value of the following optimization problem:

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \multicolumn{2}{l}{\dfrac{1}{N}\displaystyle\sum_{i\in\{1,\ldots,N\}:T^i=1}\sum_{\ell=1}^{L_1^i-1}(\kappa^i_{\ell+1}-\kappa^i_\ell)(b^i_1-w^i_{1\ell})+\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L_2^i-1}(\kappa^i_{\ell+1}-\kappa^i_\ell)(1-w^i_{2\ell})}\\
\text{subject to} & w^i_{1\ell}\le w^i_{2\ell} & \text{for all } i\in\{1,\ldots,N\},\ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w^i_{t\ell}\le w^i_{t,\ell+1} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b^i_1\le w^i_{21} & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& b^j_1\le w^i_{1\ell} & \text{for all } i,j\in\{1,\ldots,N\} \text{ such that } g(1,x^i)=\kappa^i_\ell,\ T^j=1, \text{ and } \mathcal{U}^i_1\cap\mathcal{U}^j_1\neq\emptyset\\
& b^i_1\in\{0,1\} & \text{for all } i\in\{1,\ldots,N\} \text{ such that } T^i=1\\
& w^i_{t\ell}\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,2\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}$$

We observe that the above optimization problem is equivalent to ($\bar{\text{H}}$) for the case where $T=2$. Therefore, we have shown that the optimal objective value of (BP) is less than or equal to the optimal objective value of ($\bar{\text{H}}$), which concludes our proof of Proposition 3. □

Our proofs of Propositions 4 and 5 will make use of two intermediary results, denoted below by Lemmas EC.8 and EC.9. These intermediary lemmas establish an upper bound on the optimal objective value of (BP) and a lower bound on the optimal objective value of ($\bar{\text{H}}$), respectively. Throughout the proofs, we let J^{BP} denote the optimal objective value of (BP) and $J^{\bar{\text{H}}}$ denote the optimal objective value of ($\bar{\text{H}}$).

Lemma EC.8. $J^{\text{BP}}\le\frac{1}{N}\sum_{i=1}^{N}g(T^i,x^i)$.

Proof of Lemma EC.8. By removing constraints from the optimization problem (BP), we obtain the following optimization problem:

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \multicolumn{2}{l}{\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{\ell=1}^{L_t^i-1}(\kappa^i_{\ell+1}-\kappa^i_\ell)b^i_t(1-w^i_{t\ell})}\\
\text{subject to} & w^i_{t\ell}\le w^i_{t+1,\ell} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T-1\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}\\
& w^i_{t\ell}\le w^i_{t,\ell+1} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|-1\}\\
& b^i_t\le w^i_{t+1,1} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T-1\}\\
& b^i_t\in\{0,1\} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\}\\
& w^i_{t\ell}\in\mathbb{R} & \text{for all } i\in\{1,\ldots,N\},\ t\in\{1,\ldots,T\},\ \ell\in\{1,\ldots,|\mathcal{K}^i|\}.
\end{array}$$

In greater detail, we observe that the above optimization problem is identical to (BP), with the exception that the above optimization problem does not include the constraints of the form $b_t^j \le w_{t\ell}^i$ for all $i,j \in \{1,\dots,N\}$ and $t \in \{1,\dots,T\}$ such that $g(t,x^i) = \kappa_\ell^i$ and $\mathcal{U}_t^i \cap \mathcal{U}_t^j \neq \emptyset$. Consequently, the optimal objective value of the above optimization problem is greater than or equal to the optimal objective value of (BP). Moreover, it follows from inspection and from the fact that each term $\kappa_{\ell+1}^i - \kappa_\ell^i$ is strictly positive that there exists an optimal solution for the above optimization problem that satisfies $b_t^i = \mathbb{I}\{t = T^i\}$ for each sample path $i \in \{1,\dots,N\}$. Therefore, the optimal objective value of the above optimization problem is equal to

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \sum_{t=1}^T \sum_{\ell=1}^{L_t^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)\mathbb{I}\left\{t = T^i\right\}(1 - w_{t\ell}^i) &= \frac{1}{N}\sum_{i=1}^N \sum_{\ell=1}^{L_{T^i}^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)(1 - w_{T^i\ell}^i)\\
&\le \frac{1}{N}\sum_{i=1}^N \sum_{\ell=1}^{L_{T^i}^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)\\
&= \frac{1}{N}\sum_{i=1}^N g(T^i, x^i),
\end{aligned}
$$

where the first equality follows from algebra, the inequality follows form the fact that each decision variable $w_{T^i\ell}^i$ must be greater than or equal to zero, and the final equality follows from the definitions of the constants κ_ℓ^i and L_t^i. This concludes our proof of Lemma EC.8. □

Lemma EC.9. *$J^{\bar{\mathrm{H}}}$ is greater than or equal to the optimal objective value of the following optimization problem:*

$$
\begin{aligned}
\underset{b,w}{\text{maximize}} \quad & \frac{1}{N}\sum_{i=1}^N \sum_{\ell=1}^{L_{T^i}^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)\left(b_{T^i}^i - w_{T^i\ell}^i\right)\\
\text{subject to} \quad & w_{T^i\ell}^i \le w_{T^i,\ell+1}^i \quad \text{for all } i \in \{1,\dots,N\},\ \ell \in \{1,\dots,|\mathcal{K}^i| - 1\}\\
& b_{T^j}^j \le w_{T^i\ell}^i \quad \text{for all } i,j \in \{1,\dots,N\} \text{ such that } g(T^j,x^i) = \kappa_\ell^i,\\
& \qquad \kappa_\ell^i < g(T^i,x^i),\ T^j < T^i, \text{ and } \mathcal{U}_{T^j}^i \cap \mathcal{U}_{T^j}^j \neq \emptyset\\
& b_{T^i}^i \in \{0,1\} \quad \text{for all } i \in \{1,\dots,N\}\\
& w_{T^i\ell}^i \in \mathbb{R} \quad \text{for all } i \in \{1,\dots,N\},\ \ell \in \{1,\dots,|\mathcal{K}^i| - 1\}.
\end{aligned}
\tag{$\bar{\mathrm{H}}$-LB}
$$

Proof of Lemma EC.9. We begin by rewriting ($\bar{\mathrm{H}}$) as the following equivalent optimization problem:

$$
\begin{aligned}
\underset{b,w}{\text{maximize}} \quad & \frac{1}{N}\sum_{i=1}^N \sum_{\ell=1}^{L_{T^i}^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)\left(b_{T^i}^i - w_{T^i\ell}^i\right) + \frac{1}{N}\sum_{i \in \{1,\dots,N\}: T^i < T} \sum_{\ell=1}^{L_T^i - 1} (\kappa_{\ell+1}^i - \kappa_\ell^i)\left(1 - w_{T\ell}^i\right)\\
\text{subject to} \quad & w_{t\ell}^i \le w_{t,\ell+1}^i \quad \text{for all } i \in \{1,\dots,N\},\ t \in \mathcal{T}^i,\ \ell \in \{1,\dots,|\mathcal{K}^i| - 1\}\\
& b_{T^i}^i \le w_{T1}^i \quad \text{for all } i \in \{1,\dots,N\} \text{ such that } T^i < T\\
& b_{T^j}^j \le w_{t\ell}^i \quad \text{for all } i,j \in \{1,\dots,N\} \text{ and } t \in \mathcal{T}^i \text{ such that } g(T^j,x^i) = \kappa_\ell^i,\\
& \qquad T^j \le t,\ T^j < T, \text{ and } \mathcal{U}_{T^j}^i \cap \mathcal{U}_{T^j}^j \neq \emptyset\\
& b_{T^i}^i \in \{0,1\} \quad \text{for all } i \in \{1,\dots,N\}\\
& w_{t\ell}^i \in \mathbb{R} \quad \text{for all } i \in \{1,\dots,N\},\ t \in \mathcal{T}^i,\ \ell \in \{1,\dots,|\mathcal{K}^i|\}.
\end{aligned}
\tag{$\bar{\mathrm{H}}'$}
$$

We claim that the optimal objective value of the above optimization problem ($\bar{\mathrm{H}}'$) is equal to the optimal objective value of ($\bar{\mathrm{H}}$). To see why this is true, we first observe that the above optimization problem is identical to ($\bar{\text{H}}$), except for the fact that the term $(1 - w^i_{T\ell})$ in the objective function of ($\bar{\text{H}}$) has been replaced with $(b^i_T - w^i_{T\ell})$ in the objective function of ($\bar{\text{H}}'$) for each $i \in \{1, \ldots, N\}$ such that $T^i = T$. Because the new decision variables $b^i_{T^i} \in \{0,1\}$ for each $i \in \{1, \ldots, N\}$ such that $T^i = T$ do not appear in any inequality constraints, and because each term $\kappa^i_{\ell+1} - \kappa^i_\ell$ is strictly positive, we observe that there exists an optimal solution for the above optimization problem in which $b^i_{T^i} = 1$ for each $i \in \{1, \ldots, N\}$ such that $T^i = T$. Hence, we conclude that the optimal objective value of ($\bar{\text{H}}'$) is equal to the optimal objective value of ($\bar{\text{H}}$).

We now construct a lower bound approximation of ($\bar{\text{H}}'$) by modifying its objective function and adding constraints. Indeed, we first observe for every feasible solution for ($\bar{\text{H}}'$) that the objective function of ($\bar{\text{H}}'$) satisfies

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1} - \kappa^i_\ell\right)\left(b^i_{T^i} - w^i_{T^i\ell}\right) + \frac{1}{N}\sum_{i \in \{1,\ldots,N\}: T^i < T}\sum_{\ell=1}^{L^i_T - 1}\left(\kappa^i_{\ell+1} - \kappa^i_\ell\right)\left(1 - w^i_{T\ell}\right) \\
\geq &\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1} - \kappa^i_\ell\right)\left(b^i_{T^i} - w^i_{T^i\ell}\right),
\end{aligned}$$

where the inequality holds because each term $\kappa^i_{\ell+1} - \kappa^i_\ell$ is strictly positive and because the decision variables $w^i_{T\ell}$ are always nonnegative. Therefore, a lower bound on the optimal objective value of ($\bar{\text{H}}'$) is given by the optimal objective value of the following optimization problem:

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1} - \kappa^i_\ell\right)\left(b^i_{T^i} - w^i_{T^i\ell}\right) & \\
\text{subject to} & w^i_{t\ell} \le w^i_{t,\ell+1} & \text{for all } i \in \{1,\ldots,N\},\ t \in \mathcal{T}^i,\ \ell \in \{1,\ldots,|\mathcal{K}^i| - 1\} \\
& b^i_{T^i} \le w^i_{T1} & \text{for all } i \in \{1,\ldots,N\} \text{ such that } T^i < T \\
& b^j_{T^j} \le w^i_{t\ell} & \text{for all } i,j \in \{1,\ldots,N\} \text{ and } t \in \mathcal{T}^i \text{ such that } g(T^j, x^i) = \kappa^i_\ell, \\
& & T^j \le t,\ T^j < T, \text{ and } \mathcal{U}^i_{T^j} \cap \mathcal{U}^j_{T^j} \neq \emptyset \\
& b^i_{T^i} \in \{0,1\} & \text{for all } i \in \{1,\ldots,N\} \\
& w^i_{t\ell} \in \mathbb{R} & \text{for all } i \in \{1,\ldots,N\},\ t \in \mathcal{T}^i,\ \ell \in \{1,\ldots,|\mathcal{K}^i|\}.
\end{array} \tag{$\bar{\text{H}}$-LB-1}$$

Moreover, we observe from inspection that the optimal objective value of the above optimization problem would not change if we added the constraints $w^i_{T\ell} = 1$ to the above optimization problem for each sample path $i \in \{1, \ldots, N\}$ and $\ell \in \{1, \ldots, |\mathcal{K}^i|\}$ that satisfies $T^i < T$. By eliminating these decision variables $w^i_{T\ell}$ that can be constrained without loss of generality to be equal to one, it follows from algebra that the optimal objective value of the optimization problem ($\bar{\text{H}}$-LB-1) is equal to

$$\begin{array}{lll}
\underset{b,w}{\text{maximize}} & \displaystyle\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1} - \kappa^i_\ell\right)\left(b^i_{T^i} - w^i_{T^i\ell}\right) & \\
\text{subject to} & w^i_{T^i\ell} \le w^i_{T^i,\ell+1} & \text{for all } i \in \{1,\ldots,N\},\ \ell \in \{1,\ldots,|\mathcal{K}^i| - 1\} \\
& b^j_{T^j} \le w^i_{T^i\ell} & \text{for all } i,j \in \{1,\ldots,N\} \text{ such that } g(T^j, x^i) = \kappa^i_\ell, \\
& & T^j \le T^i,\ T^j < T, \text{ and } \mathcal{U}^i_{T^j} \cap \mathcal{U}^j_{T^j} \neq \emptyset \\
& b^i_{T^i} \in \{0,1\} & \text{for all } i \in \{1,\ldots,N\} \\
& w^i_{T^i\ell} \in \mathbb{R} & \text{for all } i \in \{1,\ldots,N\},\ \ell \in \{1,\ldots,|\mathcal{K}^i|\}.
\end{array} \tag{$\bar{\text{H}}$-LB-2}$$

We further observe from inspection that the optimal objective value of the above optimization problem would not change if we added the constraints $w^i_{T^i,|\mathcal{K}^i|} = 1$ to the above optimization problem for each sample path $i \in \{1,\dots,N\}$.[14] By eliminating these decision variables $w^i_{T^i,|\mathcal{K}^i|}$ that can be constrained without loss of generality to be equal to one, and by observing that the above optimization problem has a constraint of the form $b^j_t \le w^i_{t\ell}$ if and only if $\ell = |\mathcal{K}^i|$ and $t = T^i = T^j$, we conclude that the optimal objective value of the optimization problem ($\bar{\text{H}}$-LB-2) is equal to

$$\begin{aligned}
\underset{b,w}{\text{maximize}} \quad & \frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\left(b^i_{T^i}-w^i_{T^i\ell}\right) \\
\text{subject to} \quad & w^i_{T^i\ell} \le w^i_{T^i,\ell+1} \quad \text{for all } i\in\{1,\dots,N\},\ \ell\in\{1,\dots,|\mathcal{K}^i|-1\} \\
& b^j_{T^j} \le w^i_{T^i\ell} \quad \text{for all } i,j\in\{1,\dots,N\} \text{ such that } g(T^j,x^i)=\kappa^i_\ell, \\
& \qquad\qquad \kappa^i_\ell < g(T^i,x^i),\ T^j<T^i, \text{ and } \mathcal{U}^i_{T^j}\cap\mathcal{U}^j_{T^j}\neq\emptyset \\
& b^i_{T^i}\in\{0,1\} \quad \text{for all } i\in\{1,\dots,N\} \\
& w^i_{T^i\ell}\in\mathbb{R} \quad \text{for all } i\in\{1,\dots,N\},\ \ell\in\{1,\dots,|\mathcal{K}^i|-1\}.
\end{aligned} \tag{$\bar{\text{H}}$-LB}$$

This concludes our proof of Lemma EC.9. □

In view of the above Lemmas EC.8 and EC.9, we now present the proofs of Propositions 4 and 5.

Proof of Proposition 4. In the above Lemma EC.8, we showed that $J^{\text{BP}} \le \frac{1}{N}\sum_{i=1}^N g(T^i,x^i)$. Therefore, it remains for us to show that $J^{\bar{\text{H}}} \ge \frac{1}{T}(\frac{1}{N}\sum_{i=1}^N g(T^i,x^i))$. Indeed, let

$$t^* \triangleq \underset{t\in\{1,\dots,T\}}{\arg\max} \sum_{i\in\{1,\dots,N\}:T^i=t} g(T^i,x^i)$$

be defined as any period that maximizes the sum of the rewards $g(T^i,x^i)$ over all of the sample paths $i\in\{1,\dots,N\}$ that satisfy $T^i=t^*$. We observe from algebra and from our construction of t^* that the following inequality must hold:

$$\frac{1}{N}\sum_{i\in\{1,\dots,N\}:T^i=t^*} g(T^i,x^i) \ge \frac{1}{T}\left(\frac{1}{N}\sum_{i=1}^N g(T^i,x^i)\right). \tag{EC.60}$$

Moreover, we recall from the above Lemma EC.9 that $J^{\bar{\text{H}}}$ is greater than or equal to the optimal objective value of the optimization problem ($\bar{\text{H}}$-LB). Consider the solution b,w for the optimization problem ($\bar{\text{H}}$-LB) that satisfies the following equalities for each sample path $i\in\{1,\dots,N\}$, period $t\in\{1,\dots,T\}$, and $\ell\in\{1,\dots,|\mathcal{K}^i|-1\}$:

$$b^i_{T^i} = \begin{cases} 1, & \text{if } T^i\in\{t^*,\dots,T\}, \\ 0, & \text{if } T^i\in\{1,\dots,t^*-1\}; \end{cases} \qquad w^i_{T^i\ell} = \begin{cases} 1, & \text{if } T^i\in\{t^*+1,\dots,T\}, \\ 0, & \text{if } T^i\in\{1,\dots,t^*\}. \end{cases}$$

We observe from inspection that the solution b,w defined by the above equalities is a feasible solution for the optimization problem ($\bar{\text{H}}$-LB). Therefore, the optimal objective value of ($\bar{\text{H}}$-LB) is greater than or equal to

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\left(\mathbb{I}\left\{T^i\in\{t^*,\dots,T\}\right\}-\mathbb{I}\left\{T^i\in\{t^*+1,\dots,T\}\right\}\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{L^i_{T^i}-1}\left(\kappa^i_{\ell+1}-\kappa^i_\ell\right)\mathbb{I}\left\{T^i=t^*\right\}
\end{aligned}$$

[14] We note that the equality $L^i_{T^i} = |\mathcal{K}^i|$ holds for each sample path $i\in\{1,\dots,N\}$. This equality can be verified by simply applying the definitions of L^i_t, $\mathcal{K}^i$, and T^i.

$$= \frac{1}{N} \sum_{i \in \{1,\dots,N\}: T^i = t^*} \sum_{\ell=1}^{L^i_{T^i}-1} (\kappa^i_{\ell+1} - \kappa^i_\ell)$$
$$= \frac{1}{N} \sum_{i \in \{1,\dots,N\}: T^i = t^*} g(T^i, x^i),$$

where the last equality follows from the definition of the constants κ^i_ℓ. Combining the above equalities with line (EC.60), our proof of Proposition 4 is complete. □

Proof of Proposition 5. Let Assumption 5 hold. Our proof of Proposition 5 is split into the following two intermediary claims.

Claim EC.12. *Assume without loss of generality that $g(T^1, x^1) \ge \cdots \ge g(T^N, x^N)$. Then,*

$$J^{\bar{\mathrm{H}}} \ge \max_{i \in \{1,\dots,N\}} \frac{i}{N} g(T^i, x^i) - 2\epsilon L,$$

where ϵ is the radius of the uncertainty sets (see §2.2) and L is the constant from Assumption 5.

Proof of Claim EC.12. Assume without loss of generality that $g(T^1, x^1) \ge \cdots \ge g(T^N, x^N)$, and consider any arbitrary $i^* \in \{1,\dots,N\}$. In the remainder of the proof, we will show that

$$J^{\bar{\mathrm{H}}} \ge \frac{i^*}{N} g(T^{i^*}, x^{i^*}) - 2\epsilon L.$$

To begin, we recall from the above Lemma EC.9 that $J^{\bar{\mathrm{H}}}$ is greater than or equal to the optimal objective value of the optimization problem ($\bar{\mathrm{H}}$-LB). Next, consider the solution b, w for the optimization problem ($\bar{\mathrm{H}}$-LB) that satisfies the following equalities for each sample path $i \in \{1,\dots,N\}$, period $t \in \{1,\dots,T\}$, and $\ell \in \{1,\dots,|\mathcal{K}^i|-1\}$:

$$b^i_{T^i} = \begin{cases} 1, & \text{if } i \in \{1,\dots,i^*\}, \\ 0, & \text{if } i \in \{i^*+1,\dots,N\}; \end{cases} \qquad w^i_{T^i\ell} = \begin{cases} 1, & \text{if } \kappa^i_\ell \ge g(T^{i^*}, x^{i^*}) - 2L\epsilon, \\ 0, & \text{if } T^i \in \{1,\dots,t^*\}. \end{cases}$$

We will now prove that the solution b, w defined by the above equalities is a feasible solution for the optimization problem ($\bar{\mathrm{H}}$-LB). Indeed, we readily observe from inspection that the solution b, w defined by the above equalities satisfies the first set of constraints in the optimization problem ($\bar{\mathrm{H}}$-LB), i.e.,

$$w^i_{T^i\ell} \le w^i_{T^i,\ell+1} \quad \text{for all } i \in \{1,\dots,N\},\ \ell \in \{1,\dots,|\mathcal{K}^i|-1\}.$$

To show that the solution b, w satisfies the second set of constraints in the optimization problem ($\bar{\mathrm{H}}$-LB), consider any sample paths $i, j \in \{1,\dots,N\}$ such that $g(T^j, x^i) = \kappa^i_\ell$, $\kappa^i_\ell < g(T^i, x^i)$, $T^j < T^i$, and $\mathcal{U}^i_{T^j} \cap \mathcal{U}^j_{T^j} \neq \emptyset$. We have two cases to consider. First, suppose that the sample path j satisfies $j \in \{i^*+1,\dots,N\}$. In this case, it follows from the fact that $b^j_{T^j} = 0$ that the inequality $b^j_{T^j} \le w^i_{T^i\ell}$ is satisfied. Second, suppose that sample path j satisfies $j \in \{1,\dots,i^*\}$. In this case, we observe that

$$g(T^j, x^i) = \kappa^i_\ell \ge g(T^j, x^j) - 2\epsilon L \ge g(T^{i^*}, x^{i^*}) - 2\epsilon L,$$

where the first inequality follows from Assumption 5 and from the fact that $\mathcal{U}^i_{T^j} \cap \mathcal{U}^j_{T^j} \neq \emptyset$, and the second inequality follows from the fact that $j \in \{1,\dots,i^*\}$. Therefore, we conclude that from our construction of the solution b, w that $w^i_{T^i\ell} = 1$, which implies that the inequality $b^j_{T^j} \le w^i_{T^i\ell}$ is satisfied. We have thus shown that the solution b, w satisfies the second set of constraints in the optimization problem ($\bar{\text{H}}$-LB), i.e.,

$$b^j_{T^j} \le w^i_{T^i \ell} \quad \text{for all } i,j \in \{1,\dots,N\} \text{ such that } g(T^j, x^i) = \kappa^i_\ell, \\ \kappa^i_\ell < g(T^i, x^i),\ T^j < T^i, \text{ and } \mathcal{U}^i_{T^j} \cap \mathcal{U}^j_{T^j} \neq \emptyset.$$

This concludes our proof that the solution b, w is a feasible solution for the optimization problem ($\bar{\text{H}}$-LB).

Since b, w is a feasible solution for the optimization problem ($\bar{\text{H}}$-LB), we have the following:

$$J^{\bar{\text{H}}} \ge \frac{1}{N} \sum_{i=1}^N \sum_{\ell=1}^{L^i_{T^i}-1} (\kappa^i_{\ell+1} - \kappa^i_\ell) \left(\mathbb{I}\{i \in \{1,\dots,i^*\}\} - \mathbb{I}\left\{ \kappa^i_\ell \ge g(T^{i^*}, x^{i^*}) - 2L\epsilon \right\} \right) \tag{EC.61}$$

$$= \frac{1}{N} \sum_{i=1}^{i^*} \sum_{\ell=1}^{L^i_{T^i}-1} (\kappa^i_{\ell+1} - \kappa^i_\ell) - \frac{1}{N} \sum_{i=1}^N \sum_{\ell=1}^{L^i_{T^i}-1} \mathbb{I}\left\{ \kappa^i_\ell \ge g(T^{i^*}, x^{i^*}) - 2L\epsilon \right\} (\kappa^i_{\ell+1} - \kappa^i_\ell) \tag{EC.62}$$

$$= \frac{1}{N} \sum_{i=1}^{i^*} g(T^i, x^i) - \frac{1}{N} \sum_{i=1}^N \sum_{\ell=1}^{L^i_{T^i}-1} \mathbb{I}\left\{ \kappa^i_\ell \ge g(T^{i^*}, x^{i^*}) - 2L\epsilon \right\} (\kappa^i_{\ell+1} - \kappa^i_\ell) \tag{EC.63}$$

$$\ge \frac{1}{N} \sum_{i=1}^{i^*} g(T^i, x^i) \\ - \frac{1}{N} \sum_{i=1}^N \mathbb{I}\left\{ \exists \ell \in \{1,\dots,L^i_{T^i}-1\} \text{ such that } \kappa^i_\ell \ge g(T^{i^*}, x^{i^*}) - 2L\epsilon \right\} \left(g(T^i, x^i) - \left(g(T^{i^*}, x^{i^*}) - 2L\epsilon \right) \right) \tag{EC.64}$$

$$\ge \frac{1}{N} \sum_{i=1}^{i^*} g(T^i, x^i) - \frac{1}{N} \sum_{i=1}^N \mathbb{I}\left\{ g(T^i, x^i) \ge g(T^{i^*}, x^{i^*}) - 2L\epsilon \right\} \left(g(T^i, x^i) - g(T^{i^*}, x^{i^*}) + 2L\epsilon \right) \tag{EC.65}$$

$$= \frac{1}{N} \sum_{i=1}^{i^*} \left(g(T^{i^*}, x^{i^*}) - 2L\epsilon \right) - \frac{1}{N} \sum_{i \in \{i^*+1,\dots,N\}:\, g(T^i,x^i) \ge g(T^{i^*},x^{i^*}) - 2L\epsilon} \left(g(T^i, x^i) - g(T^{i^*}, x^{i^*}) + 2L\epsilon \right) \tag{EC.66}$$

$$\ge \frac{1}{N} \sum_{i=1}^{i^*} \left(g(T^{i^*}, x^{i^*}) - 2L\epsilon \right) - \frac{1}{N} \sum_{i \in \{i^*+1,\dots,N\}:\, g(T^i,x^i) \ge g(T^{i^*},x^{i^*}) - 2L\epsilon} 2L\epsilon \tag{EC.67}$$

$$\ge \frac{1}{N} \sum_{i=1}^{i^*} \left(g(T^{i^*}, x^{i^*}) - 2L\epsilon \right) - \frac{N - i^*}{N} 2L\epsilon \tag{EC.68}$$

$$= \frac{i^*}{N} g(T^{i^*}, x^{i^*}) - 2\epsilon L, \tag{EC.69}$$

Indeed, (EC.61) holds because $J^{\bar{\text{H}}}$ is greater than or equal to the optimal objective value of the optimization problem ($\bar{\text{H}}$-LB) and because b, w is a feasible but possibly suboptimal solution for the optimization problem ($\bar{\text{H}}$-LB). Lines (EC.62), (EC.63), and (EC.64) follow from algebra. Line (EC.65) follows from the fact that $\kappa^i_0 < \cdots < \kappa^i_{L^i_{T^i}}$. Line (EC.66) follows from rearranging terms. Line (EC.67) follows from the fact that $g(T^i, x^i) - g(T^{i^*}, x^{i^*}) \le 0$ for all $i \in \{i^*+1,\dots,N\}$. Lines (EC.68) and (EC.69) follow from algebra.

Since $i^* \in \{1,\dots,N\}$ was chosen arbitrarily, our proof of Claim EC.12 is complete. □

Claim EC.13. $J^{\bar{\mathrm{H}}} \ge \frac{1}{\log N+1}\left(\frac{1}{N}\sum_{i=1}^N g(T^i,x^i)\right) - 2\epsilon L.$

Proof of Claim EC.13. Assume without loss of generality that $g(T^1,x^1) \ge \cdots \ge g(T^N,x^N)$. In Claim EC.12, we showed that

$$J^{\bar{\mathrm{H}}} \ge \max_{i\in\{1,\ldots,N\}} \frac{i}{N} g(T^i,x^i) - 2\epsilon L.$$

For notational convenience, let us define $\gamma^i \triangleq \frac{i}{N} g(T^i,x^i)$ for each sample path $i \in \{1,\ldots,N\}$. With this notation, we observe from algebra that

$$\begin{aligned}
\frac{1}{\log N+1}\left(\frac{1}{N}\sum_{i=1}^N g(T^i,x^i)\right) - 2\epsilon L &= \frac{1}{\log N+1}\left(\sum_{i=1}^N \frac{1}{i}\gamma^i\right) - 2\epsilon L \\
&\le \frac{1}{\log N+1}\left(\max_{i\in\{1,\ldots,N\}} \gamma^i\right)\sum_{i=1}^N \frac{1}{i} - 2\epsilon L \\
&\le \max_{i\in\{1,\ldots,N\}} \gamma^i - 2\epsilon L \\
&= \max_{i\in\{1,\ldots,N\}} \frac{i}{N} g(T^i,x^i) - 2\epsilon L \\
&\le J^{\bar{\mathrm{H}}},
\end{aligned}$$

where the last line follows from Claim EC.12. This completes the proof of Claim EC.13. □

Combining Claim EC.13 with Lemma EC.8 completes our proof of Proposition 5. □

Appendix I: Additional Numerical Results

In this appendix, we present numerical results for additional parameter settings which were omitted from §5.2 due to length considerations.

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Table EC.1 Barrier Option (Asymmetric) - Expected Reward.

			Initial Price			
d	Method	Basis functions	$\bar{x}=90$	$\bar{x}=100$	$\bar{x}=110$	# of Sample Paths
8	RO	maxprice	**57.53 (0.16)**	**67.25 (0.27)**	**69.10 (0.42)**	10^3 training, 10^3 validation
8	LS	one, pricesKO, KOind, payoff	56.44 (0.11)	65.52 (0.16)	68.34 (0.09)	10^5
8	LS	one, pricesKO, payoff	56.54 (0.12)	65.74 (0.17)	68.31 (0.08)	10^5
8	LS	payoff, KOind, pricesKO	56.44 (0.11)	65.52 (0.16)	68.34 (0.09)	10^5
8	LS	pricesKO, payoff	56.51 (0.12)	65.75 (0.18)	68.31 (0.08)	10^5
8	LS	one, prices, payoff	55.56 (0.09)	61.92 (0.09)	60.07 (0.12)	10^5
8	LS	one	45.80 (0.06)	52.84 (0.07)	48.56 (0.08)	10^5
8	LS	one, KOind, prices	52.33 (0.09)	61.85 (0.10)	63.88 (0.08)	10^5
8	LS	one, prices	48.49 (0.11)	53.03 (0.09)	50.83 (0.06)	10^5
8	LS	one, pricesKO	51.67 (0.08)	61.49 (0.10)	63.14 (0.09)	10^5
8	LS	maxprice, KOind, pricesKO	52.63 (0.09)	61.97 (0.10)	64.14 (0.12)	10^5
8	PO	payoff, KOind, pricesKO	53.61 (0.18)	58.29 (0.36)	52.96 (0.30)	2×10^3 outer, 500 inner
8	PO	prices	53.90 (0.16)	60.66 (0.28)	58.14 (0.12)	2×10^3 outer, 500 inner
8	Tree	payoff, time	55.75 (0.13)	62.78 (0.31)	63.95 (0.52)	10^5
8	Tree	maxprice, time	55.75 (0.13)	62.78 (0.31)	63.95 (0.52)	10^5
16	RO	maxprice	**72.56 (0.15)**	**77.44 (0.29)**	69.74 (0.30)	10^3 training, 10^3 validation
16	LS	one, pricesKO, KOind, payoff	70.94 (0.08)	76.10 (0.10)	**70.59 (0.13)**	10^5
16	LS	one, pricesKO, payoff	71.06 (0.08)	76.14 (0.08)	70.05 (0.12)	10^5
16	LS	payoff, KOind, pricesKO	70.94 (0.08)	76.10 (0.10)	**70.59 (0.13)**	10^5
16	LS	pricesKO, payoff	71.06 (0.08)	76.14 (0.08)	70.04 (0.12)	10^5
16	LS	one, prices, payoff	67.57 (0.08)	67.07 (0.16)	56.36 (0.14)	10^5
16	LS	one	59.09 (0.05)	56.20 (0.11)	45.55 (0.05)	10^5
16	LS	one, KOind, prices	66.78 (0.06)	71.06 (0.14)	63.54 (0.20)	10^5
16	LS	one, prices	59.35 (0.07)	57.68 (0.08)	49.67 (0.07)	10^5
16	LS	one, pricesKO	66.51 (0.05)	70.70 (0.13)	62.53 (0.16)	10^5
16	LS	maxprice, KOind, pricesKO	67.00 (0.06)	71.27 (0.13)	64.04 (0.19)	10^5
16	PO	payoff, KOind, pricesKO	65.14 (0.31)	62.12 (0.34)	47.16 (0.37)	2×10^3 outer, 500 inner
16	PO	prices	66.69 (0.18)	66.23 (0.23)	53.84 (0.26)	2×10^3 outer, 500 inner
16	Tree	payoff, time	68.88 (0.15)	71.11 (0.19)	55.05 (0.11)	10^5
16	Tree	maxprice, time	68.88 (0.15)	71.11 (0.19)	55.05 (0.11)	10^5
32	RO	maxprice	**84.12 (0.28)**	79.14 (0.40)	60.66 (0.57)	10^3 training, 10^3 validation
32	LS	one, pricesKO, KOind, payoff	82.38 (0.09)	**79.17 (0.10)**	**62.92 (0.15)**	10^5
32	LS	one, pricesKO, payoff	82.45 (0.10)	78.91 (0.10)	62.27 (0.14)	10^5
32	LS	payoff, KOind, pricesKO	82.38 (0.09)	**79.17 (0.10)**	**62.92 (0.15)**	10^5
32	LS	pricesKO, payoff	82.45 (0.10)	78.91 (0.10)	62.27 (0.14)	10^5
32	LS	one, prices, payoff	73.46 (0.19)	62.81 (0.17)	48.79 (0.17)	10^5
32	LS	one	64.26 (0.12)	50.78 (0.09)	44.32 (0.05)	10^5
32	LS	one, KOind, prices	77.53 (0.11)	71.06 (0.12)	56.06 (0.21)	10^5
32	LS	one, prices	64.63 (0.11)	55.21 (0.12)	46.58 (0.05)	10^5
32	LS	one, pricesKO	77.36 (0.10)	70.44 (0.16)	55.11 (0.24)	10^5
32	LS	maxprice, KOind, pricesKO	77.67 (0.11)	71.53 (0.13)	56.30 (0.22)	10^5
32	PO	payoff, KOind, pricesKO	70.79 (0.49)	55.67 (0.31)	35.36 (1.21)	2×10^3 outer, 500 inner
32	PO	prices	73.86 (0.32)	62.09 (0.22)	44.23 (0.29)	2×10^3 outer, 500 inner
32	Tree	payoff, time	77.64 (0.44)	63.91 (0.13)	50.95 (0.04)	10^5
32	Tree	maxprice, time	77.64 (0.44)	63.91 (0.13)	50.95 (0.04)	10^5

Optimal is indicated in bold for each number of assets $d\in\{8,16,32\}$ and initial price $\bar{x}\in\{90,100,110\}$. Problem parameters are $T=54$, $Y=3$, $r=0.05$, $K=100$, $B_0=150$, $\delta=0.25$, $\sigma_a=0.1+s\frac{a\times d}{5}$, $\rho_{a,a'}=0$ for all $a\neq a'$.

Table EC.2 Barrier Option (Asymmetric) - Computation Times.

			Initial Price			
d	Method	Basis functions	$\bar{x} = 90$	$\bar{x} = 100$	$\bar{x} = 110$	# of Sample Paths
8	RO	maxprice	7.19 (0.37)	8.62 (1.09)	18.38 (4.59)	10^3 training, 10^3 validation
8	LS	one, pricesKO, KOind, payoff	3.29 (0.2)	3.20 (0.31)	3.24 (0.15)	10^5
8	LS	one, pricesKO, payoff	3.16 (0.2)	3.05 (0.25)	3.13 (0.21)	10^5
8	LS	payoff, KOind, pricesKO	3.76 (0.63)	3.29 (0.58)	3.05 (0.44)	10^5
8	LS	pricesKO, payoff	3.03 (0.14)	2.89 (0.23)	2.87 (0.12)	10^5
8	LS	one, prices, payoff	3.31 (0.56)	2.97 (0.29)	2.90 (0.24)	10^5
8	LS	one	0.89 (0.07)	0.93 (0.06)	0.94 (0.08)	10^5
8	LS	one, KOind, prices	3.39 (0.55)	2.99 (0.26)	2.98 (0.21)	10^5
8	LS	one, prices	2.63 (0.19)	2.71 (0.22)	2.72 (0.19)	10^5
8	LS	one, pricesKO	2.78 (0.35)	2.62 (0.18)	2.53 (0.21)	10^5
8	LS	maxprice, KOind, pricesKO	3.01 (0.13)	2.85 (0.26)	2.87 (0.08)	10^5
8	PO	payoff, KOind, pricesKO	18.75 (0.71)	18.12 (0.41)	16.23 (0.71)	2×10^3 outer, 500 inner
8	PO	prices	23.31 (0.45)	23.57 (0.36)	22.69 (0.29)	2×10^3 outer, 500 inner
8	Tree	payoff, time	13.80 (0.28)	15.72 (5.51)	57.70 (10.81)	10^5
8	Tree	maxprice, time	13.47 (0.19)	15.04 (5.42)	55.45 (10.47)	10^5
16	RO	maxprice	6.80 (1)	10.15 (1.4)	51.58 (9.06)	10^3 training, 10^3 validation
16	LS	one, pricesKO, KOind, payoff	5.55 (0.61)	5.30 (0.34)	5.01 (0.42)	10^5
16	LS	one, pricesKO, payoff	5.50 (0.78)	5.23 (0.33)	5.24 (0.33)	10^5
16	LS	payoff, KOind, pricesKO	4.74 (0.72)	4.62 (0.54)	4.55 (0.41)	10^5
16	LS	pricesKO, payoff	5.29 (0.37)	5.18 (0.38)	4.87 (0.26)	10^5
16	LS	one, prices, payoff	5.57 (0.55)	4.93 (0.29)	5.10 (0.32)	10^5
16	LS	one	1.17 (0.11)	1.26 (0.08)	1.24 (0.05)	10^5
16	LS	one, KOind, prices	6.21 (1.46)	5.72 (0.76)	5.16 (0.51)	10^5
16	LS	one, prices	6.02 (1.63)	5.11 (1.25)	4.86 (0.73)	10^5
16	LS	one, pricesKO	4.61 (0.62)	4.58 (0.52)	4.67 (0.58)	10^5
16	LS	maxprice, KOind, pricesKO	5.11 (0.64)	4.90 (0.6)	4.73 (0.27)	10^5
16	PO	payoff, KOind, pricesKO	32.13 (1.23)	28.51 (0.96)	25.56 (0.48)	2×10^3 outer, 500 inner
16	PO	prices	43.73 (0.61)	43.75 (0.67)	41.84 (0.65)	2×10^3 outer, 500 inner
16	Tree	payoff, time	14.24 (0.4)	48.39 (0.94)	7.89 (0.09)	10^5
16	Tree	maxprice, time	13.35 (0.3)	47.49 (0.74)	8.31 (0.19)	10^5
32	RO	maxprice	7.88 (1.21)	22.32 (7.25)	154.12 (29.91)	10^3 training, 10^3 validation
32	LS	one, pricesKO, KOind, payoff	15.07 (2.97)	11.55 (1.37)	13.68 (2.69)	10^5
32	LS	one, pricesKO, payoff	15.29 (2.15)	12.20 (1.68)	13.04 (3.25)	10^5
32	LS	payoff, KOind, pricesKO	13.13 (2.59)	10.96 (1.81)	11.30 (1.81)	10^5
32	LS	pricesKO, payoff	15.11 (3.27)	11.39 (1.91)	11.62 (1.44)	10^5
32	LS	one, prices, payoff	16.78 (4.89)	10.68 (1.52)	11.92 (3.11)	10^5
32	LS	one	1.62 (0.6)	1.51 (0.16)	1.69 (0.74)	10^5
32	LS	one, KOind, prices	14.61 (2.44)	10.77 (0.79)	11.48 (1.94)	10^5
32	LS	one, prices	12.49 (3.23)	10.42 (0.57)	10.47 (0.84)	10^5
32	LS	one, pricesKO	13.61 (3.66)	10.55 (0.43)	11.02 (1.31)	10^5
32	LS	maxprice, KOind, pricesKO	12.63 (0.86)	10.82 (1.36)	11.60 (1.09)	10^5
32	PO	payoff, KOind, pricesKO	61.15 (1.43)	52.57 (4.38)	52.56 (1.89)	2×10^3 outer, 500 inner
32	PO	prices	108.40 (2.23)	108.40 (3.02)	97.29 (2.01)	2×10^3 outer, 500 inner
32	Tree	payoff, time	44.24 (19.71)	15.35 (0.31)	9.41 (0.32)	10^5
32	Tree	maxprice, time	43.92 (20.08)	16.22 (0.62)	10.92 (0.4)	10^5

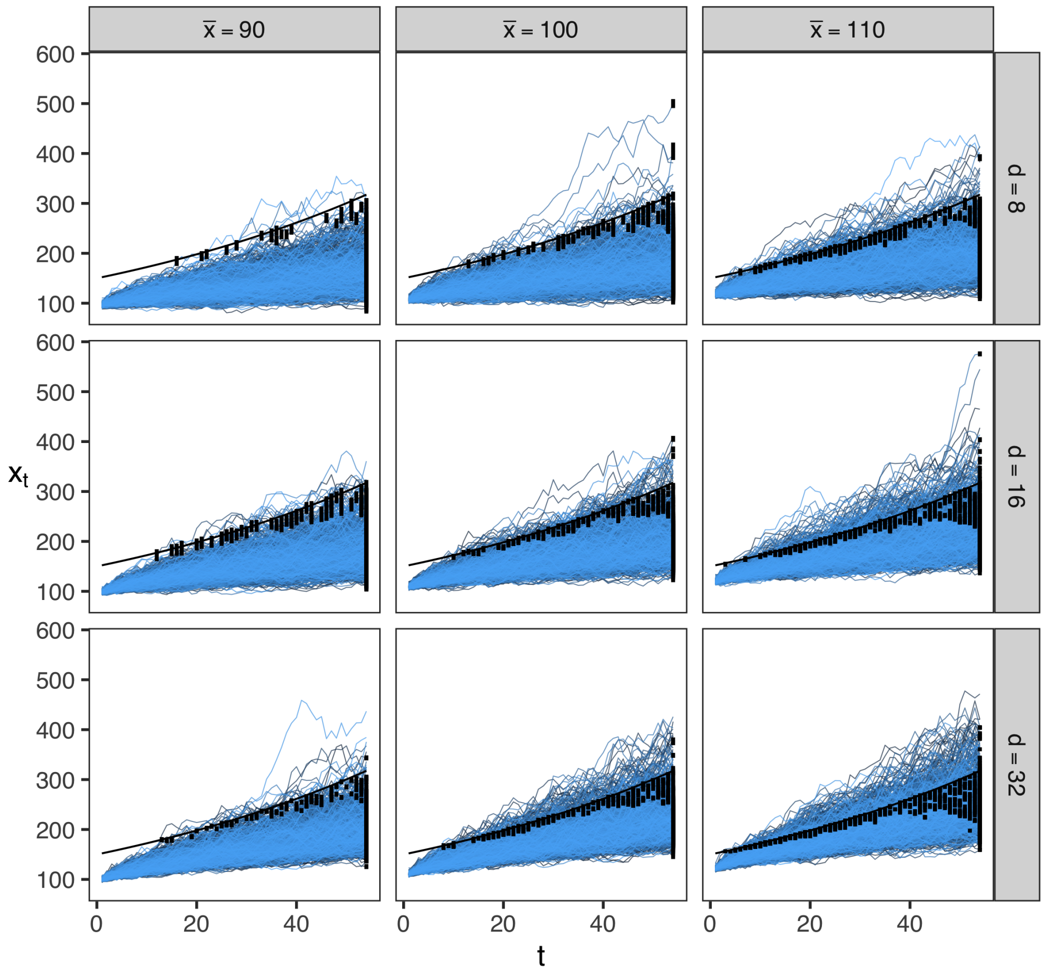
Problem parameters are $T = 54$, $Y = 3$, $r = 0.05$, $K = 100$, $B_0 = 150$, $\delta = 0.25$, $\sigma_a = 0.1 + s\frac{a \times d}{5}$, $\rho_{a,a'} = 0$ for all $a \neq a'$.

Table EC.3 Barrier Option (Asymmetric) - Best Choice of Robustness Parameter.

	Initial Price		
d	$\bar{x}=90$	$\bar{x}=100$	$\bar{x}=110$
8	8.2 (1.75)	5.9 (1.73)	4.0 (1.33)
16	6.2 (1.69)	4.6 (1.26)	3.4 (1.17)
32	5.5 (1.27)	4.0 (1.05)	2.3 (0.95)

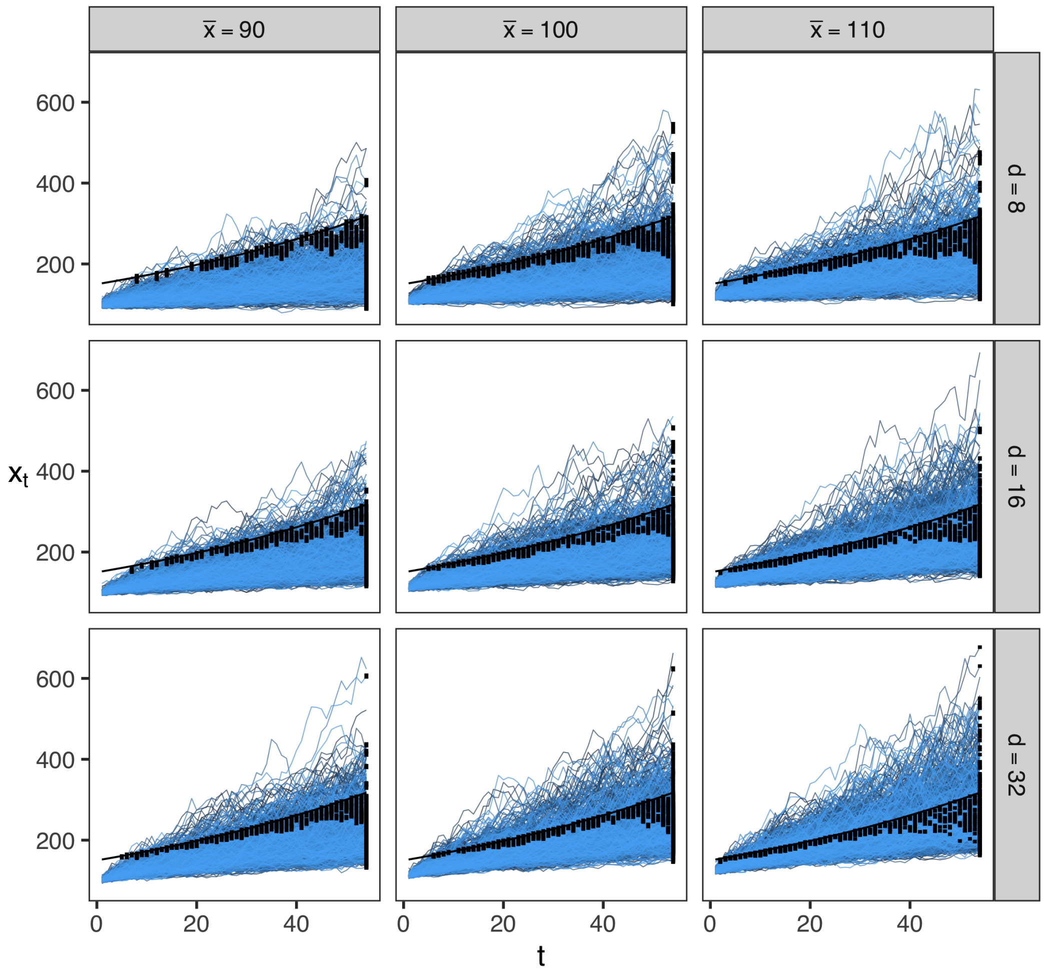
Best choice of robustness parameter ϵ found using the validation method from §2.5 in the robust optimization problems constructed from a training dataset of size $N=10^3$ and validation set of size $\bar{N}=10^3$. The remaining parameters are the same as those shown in Table EC.1.

Figure EC.1 Barrier Option (Symmetric) - Visualization of Robust Optimization Stopping Rules.



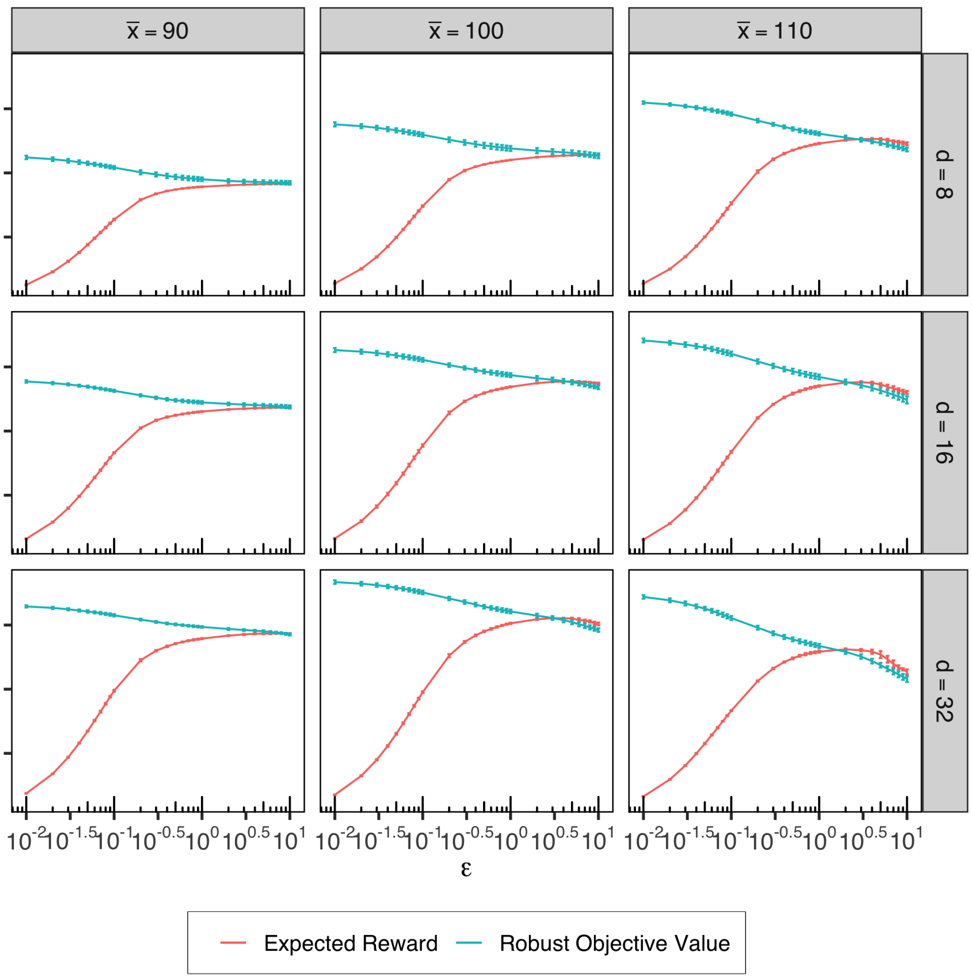
Note. Each plot shows the exercise policies obtained from solving the heuristic ($\bar{\text{H}}$) constructed from a training dataset of size $N = 10^3$ and with the robustness parameter selected using a validation set of size $\bar{N} = 10^3$. The problem parameters are the same as those shown in Table 1.

Figure EC.2 Barrier Option (Asymmetric) - Visualization of Robust Optimization Stopping Rules.



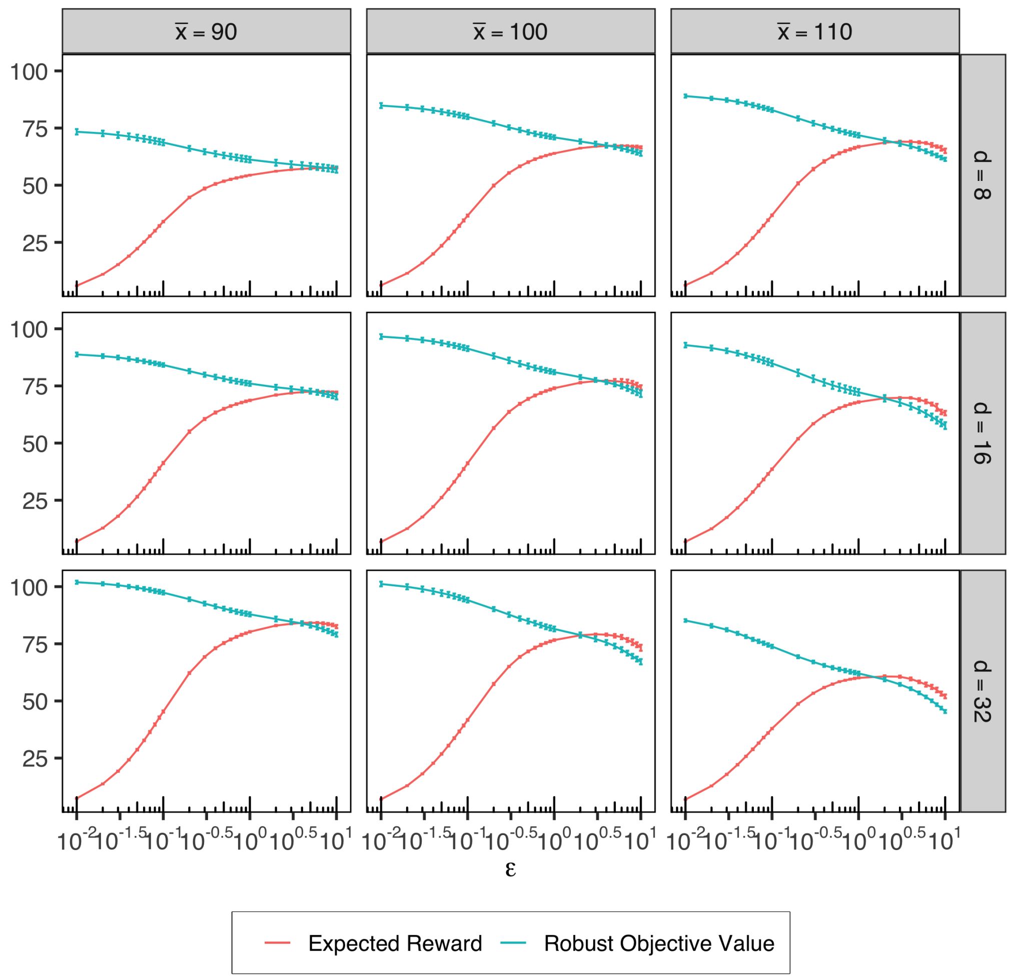
Note. Each plot shows the exercise policies obtained from solving the heuristic ($\bar{\text{H}}$) constructed from a training dataset of size $N = 10^3$ and with the robustness parameter selected using a validation set of size $\bar{N} = 10^3$. The problem parameters are the same as those shown in Table EC.1.

Figure EC.3 Barrier Option (Symmetric) - Impact of Robustness Parameter on Reward.



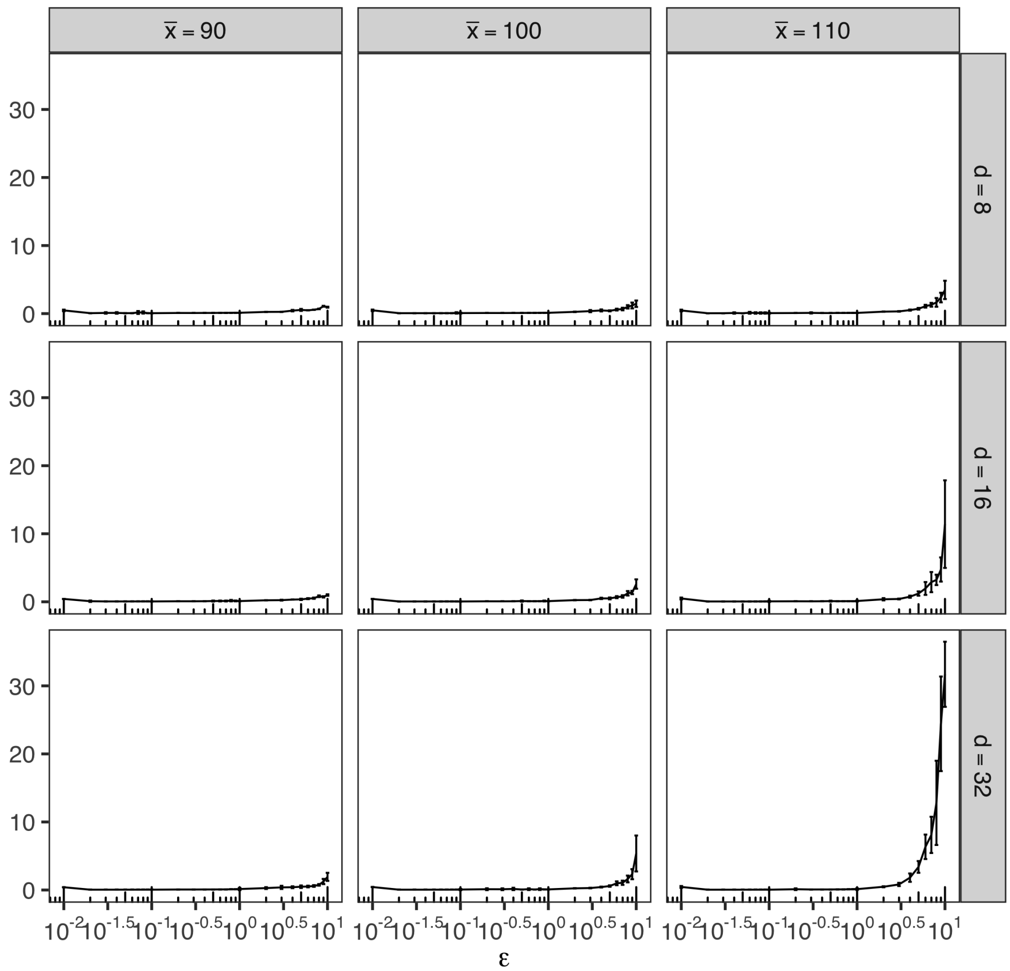
Note. Each plot shows the robust objective value and expected reward of policies obtained by solving the heuristic ($\bar{\text{H}}$) constructed from training datasets of size $N = 10^3$. The problem parameters are the same as those shown in Table 1.

Figure EC.4 Barrier Option (Asymmetric) - Impact of Robustness Parameter on Reward.



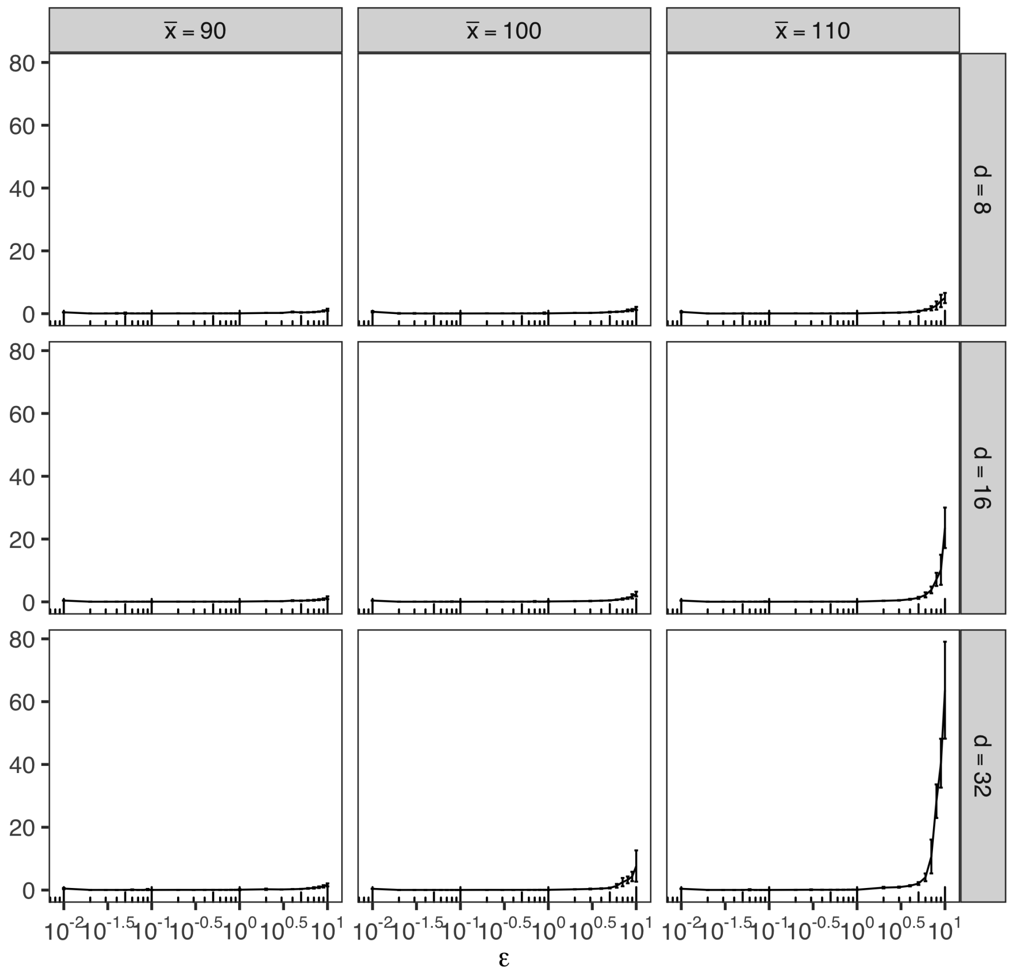
Note. Each plot shows the robust objective value and expected reward of policies obtained by solving the heuristic ($\bar{\text{H}}$) constructed from training datasets of size $N = 10^3$. The problem parameters are the same as those shown in Table EC.1.

Figure EC.5 Barrier Option (Symmetric) - Impact of Robustness Parameter on Computation Time.



Note. Each plot shows the computation times from solving the heuristic ($\bar{\text{H}}$) constructed from training datasets of size $N = 10^3$. The problem parameters are the same as those shown in Table 1.

Figure EC.6 Barrier Option (Asymmetric) - Impact of Robustness Parameter on Computation Time.



Note. Each plot shows the computation times from solving the heuristic ($\bar{\mathrm{H}}$) constructed from training datasets of size $N = 10^3$. The problem parameters are the same as those shown in Table EC.1.