Appendix.

A. Vertex structure

To aid in considering the properties of the optimizers of the formulations from Section 3.8, we will first consider the vertex structure of the relaxed feasible set $\mathbb{S}'$ and of several special cases. For $\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{g} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \in \mathbb{R}^{4T}$, the system (18) defines a (possibly empty) polytope $\mathbb{S}'$ whose extreme points are its vertices [20]. Each such vertex $\begin{bmatrix} \widehat{\boldsymbol{p}} \\ \widehat{\boldsymbol{g}} \\ \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{v}} \end{bmatrix} \in \mathbb{R}^{4T}$ is a "basic feasible solution;" that is, there is a $4T$ element subset of the rows of (18) such that:

- the corresponding rows of the coefficient matrix are linearly independent and,
- if we treat these rows as equalities, then $\begin{bmatrix} \widehat{\boldsymbol{p}} \\ \widehat{\boldsymbol{g}} \\ \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{v}} \end{bmatrix}$ is the unique solution of this set of linear equalities.

The development in Section 4 seeks conditions under which vertices of the system (18) have integer-valued entries for $\boldsymbol{u}$ and $\boldsymbol{v}$ at solutions of the optimization formulations from Section 3.8. Generically, the coefficient matrix and right-hand side of (18) has some non-integer values. To understand more clearly the values of $\boldsymbol{u}$ and $\boldsymbol{v}$ at the vertices of the system (18), it is convenient to first scale the coefficient matrix. Under Assumptions 2 and 3, the rows of the coefficient matrix can be scaled to have mostly integer values. In particular, re-arranging (18) and noting that, by Assumption 2, $\check{s} = \alpha\overline{p} = \beta\overline{g}$, we can obtain:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \boldsymbol{L} & -\boldsymbol{L}+\mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{L}+\mathbf{I} & \boldsymbol{L} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}/\overline{p} \\ \boldsymbol{g}/\overline{g} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1}(\overline{s}-s_0)/\check{s} \\ \mathbf{1}(s_0-\underline{s})/\check{s} \end{bmatrix}. \tag{50}$$

In the rest of this section, we will assume that Assumptions 2 and 3 hold and consider:

1. the vertices of $\mathbb{S}'$,
2. the vertices of a special case where SOC limits are not enforced, and
3. the vertices of special cases where mutual exclusivity constraints are not enforced.

We will see that the special cases have more favorable vertex structures than $\mathbb{S}'$, highlighting that it is the combination of SOC limits and mutual exclusivity constraints in an integer linear programming formulation that present difficulties.

A.1. Vertices of $\mathbb{S}'$

Even under Assumptions 2 and 3, we will find that some of the vertices of $\mathbb{S}'$ do not have integer values for $\boldsymbol{u}$ and $\boldsymbol{v}$. To see why this would be the case, we first observe that the result in [22, Appendix] regarding integrality is not applicable to $\mathbb{S}'$ because of the constraint structure. In particular, there are too many constraints involving the variables $\boldsymbol{p}$ and $\boldsymbol{g}$ in (50) to apply that result.

As a second observation, consider the property of total unimodularity, which applies to matrices with entries that are $0, \pm 1$. In the special case that:

- $\underline{p} = 0$ or $\underline{p} = \overline{p}$, and
- $\underline{g} = 0$ or $\underline{g} = \overline{g}$,

then the coefficient matrix of (50) consists entirely of $0, \pm 1$ entries. Moreover, by Assumption 3, the right-hand side of (50) is integral. So, if the coefficient matrix *were* totally unimodular then the vertices of the system (50) would all have binary values for $\boldsymbol{u}$ and $\boldsymbol{v}$. Unfortunately, even under these various assumptions, the coefficient matrix of (50) is not totally unimodular since some of the determinants of sub-matrices of the coefficient matrix of (50) are not equal to $0, \pm 1$. To see this, we will perform elementary row operations on the coefficient matrix, and identify a sub-matrix of the result that has determinant equal to 2.

Consider the coefficient matrix of (50) and subtract $\boldsymbol{L}$ times the third block row and subtract $(\boldsymbol{L} - \mathbf{I})$ times the fifth row from the eighth row. This results in the following coefficient matrix:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{L} & -\boldsymbol{L}+\mathbf{I} \\ -\boldsymbol{L}+\mathbf{I} & \boldsymbol{L} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Consider the following rows of this matrix:

1. the second row of the second block row,
2. the first row of the seventh block row,
3. the second row of the seventh block row, and
4. the first row of the eighth block row.

We re-order the coefficient matrix to make these the first four rows of the re-arranged matrix. Moreover, consider the following columns of this matrix:

1. the second column of the fourth block row (corresponding to v_2),
2. the second column of the third block row (corresponding to u_2),

3. the first column of the fourth block row (corresponding to v_1), and
4. the first column of the third block row (corresponding to u_1).

We re-order the coefficient matrix to make these the first four columns of the re-arranged matrix. This creates a lower block-triangular matrix of the form:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix},$$

where $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$ is:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Direct calculation shows that the determinant of $\boldsymbol{A}$ is equal to -2 (and the determinant of sub-matrices of $\boldsymbol{A}$ are $0, \pm 1, \pm 2$). That is, $\boldsymbol{A}$ is not totally unimodular and *a fortiori* the full coefficient matrix of (50) is not totally unimodular. The structure of $\boldsymbol{A}$ is repeated throughout the coefficient matrix.

The existence of sub-matrices with determinant not equal to $0, \pm 1$ is due to the combination of mutual exclusivity constraints, binary variables to represent the pumping and generating operating regions, and the opposite sign of the contribution of pumping and generating to the SOC. This is consistent with the observation in Section 2 that these features of the problem pose a challenge for efficient computation. To summarize, we are not able to use general results for totally unimodular coefficient matrices and generalizations such as [22, Appendix] to analyze the vertices of the system (18); however, we will see that special cases that omit either the SOC limits or the mutual exclusivity constraints do have coefficient matrices that are totally unimodular.

Indeed, there are vertices of (18) that have non-integer-valued entries for $\boldsymbol{u}$ and $\boldsymbol{v}$. For example, consider the parameter values $T = 2$, $\overline{s} = 2$, $s_0 = 1$, $\underline{s} = 0$, $\alpha\overline{p} = \beta\overline{g} = 1$, $\underline{p} = \underline{g} = 0$, and the specific values $\widehat{u}_1 = \widehat{v}_1 = 0.5, \widehat{p}_1 = 0.5/\alpha, \widehat{g}_1 = 0.5/\beta, \widehat{u}_2 = 1, \widehat{p}_2 = 1/\alpha$, and $\widehat{v}_2 = \widehat{g}_2 = 0$. There is a set of $4T = 8$ linearly independent constraints of the system (18) that are binding at the feasible point $\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{\boldsymbol{p}} \\ \widehat{\boldsymbol{g}} \\ \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{v}} \end{bmatrix} \in \mathbb{R}^8$, namely:

$$\begin{aligned} \widehat{p}_1 &= 0.5/\alpha, \\ &= (1/\alpha) \times 0.5, \\ &= \overline{p}\,\widehat{u}_1, \\ \widehat{g}_1 &= 0.5/\beta, \\ &= (1/\beta) \times 0.5, \end{aligned}$$

$$
\begin{aligned}
&= \overline{g}\,\widehat{v}_1, \\
\widehat{u}_1 + \widehat{v}_1 &= 0.5 + 0.5, \\
&= 1, \\
\widehat{v}_2 &= 0, \\
\widehat{p}_2 &= 1/\alpha, \\
&= (1/\alpha) \times 1, \\
&= \overline{p}\,\widehat{u}_2, \\
\widehat{g}_2 &= 0, \\
&= (1/\beta) \times 0, \\
&= \overline{g}\,\widehat{v}_2, \\
\widehat{u}_2 + \widehat{v}_2 &= 1 + 0, \\
&= 1, \\
s_0 + \alpha\widehat{p}_1 - \beta\widehat{g}_1 + \alpha\widehat{p}_2 &= 1 + 0.5 - 0.5 + 1, \\
&= \overline{s}.
\end{aligned}
$$

These are the specific constraints considered in the construction above of the sub-matrix A. It is interesting to note that the pumping and generation lower limits in this example are both zero, so the existence of this vertex is not due to the non-convexity of non-zero lower limits on pumping and generation.

Consequently, conditions such that $\boldsymbol{u}$ and $\boldsymbol{v}$ are binary-valued must be based not only the structure of the coefficient matrix, but also on characteristics of the right-hand side and the form of the objective. In particular, Lemma 1 shows that if the value of electricity production is strictly positive, then this vertex $\widehat{\boldsymbol{x}}$ cannot be a solution of a continuous relaxation of an electricity trading problem if Assumptions 1 and 2 hold.

Moreover, the tigtened SOC formulation (13)–(14) eliminates some of the non-binary-valued entries for $\boldsymbol{u}$ and $\boldsymbol{v}$ in the vertices of (18) compared to the standard formulation utilizing (12). For example, consider the same limits as in the example above but with $T = 1$, $s_0 = 2$, and the specific values: $\tilde{u}_1 = \tilde{v}_1 = \alpha\tilde{p}_1 = \beta\tilde{g}_1 = 0.5$. This point $\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{p}} \\ \tilde{\boldsymbol{g}} \\ \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} \in \mathbb{R}^4$ is not feasible for (18), since:

$$
\begin{aligned}
s_0 + \alpha\tilde{p}_1 &= 2 + 0.5, \\
&\nleq 2, \\
&= \overline{s}.
\end{aligned}
$$

However, $\tilde{\boldsymbol{x}}$ *is* a vertex of the modification of the system (18) where (13)–(14) are replaced by the standard formulation of SOC limits (12). In particular, there is a set of $4T = 4$ linearly independent constraints of the system (18) with (13)–(14) replaced by (12) that are binding at $\tilde{\boldsymbol{x}}$, namely:

$$\begin{aligned}
\tilde{p}_1 &= 0.5/\alpha, \\
&= (1/\alpha) \times 0.5, \\
&= \overline{p}\,\tilde{u}_1, \\
\tilde{g}_1 &= 0.5/\beta, \\
&= (1/\beta) \times 0.5, \\
&= \overline{g}\,\tilde{v}_1, \\
\tilde{u}_1 + \tilde{v}_1 &= 0.5 + 0.5, \\
&= 1, \\
s_0 + \alpha\tilde{p}_1 - \beta\tilde{g}_1 &= 2 + 0.5 - 0.5, \\
&= \overline{s}.
\end{aligned}$$

The specific form of (13)–(14) in the Tightened SOC model eliminates $\tilde{\boldsymbol{x}}$ from being a vertex of the system (18), and so is tighter than the Standard SOC formulation in the literature. To summarize, although (18) does not characterize the convex hull, it is a tighter representation than the standard formulation using (12), consistent with the discussion in Section 3.2.

A.2. No SOC limits

Consider the special case of (18), or equivalently (50), where there are no SOC limits. In practice, this would correspond to a PSH having a very large range for $(\overline{s} - \underline{s})$ and with s_0 somewhere in the middle of this range. The inequalities describing the resulting feasible region can be obtained from (50) by deleting the eighth and ninth block rows to obtain:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}/\overline{p} \\ \boldsymbol{g}/\overline{g} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}. \tag{51}$$

The no SOC limits feasible operating region is the set:

$$\mathbb{M} = \{(\boldsymbol{p}, \boldsymbol{g}, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{Z}^T \times \mathbb{Z}^T \,|\, (51)\}.$$

The corresponding continuous relaxation of $\mathbb{M}$ is:

$$\mathbb{M}' = \{(\boldsymbol{p}, \boldsymbol{g}, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \,|\, (51)\},$$

so that $\mathbb{M}'$ is obtained from $\mathbb{M}$ by relaxing the integrality constraints.

Now consider the first, second, and seventh block rows of (51). These constraints involve only the variables $\boldsymbol{u}$ and $\boldsymbol{v}$. Moreover, if we temporarily ignore the variables $\boldsymbol{p}$ and $\boldsymbol{g}$ and temporarily ignore the constraints involving $\boldsymbol{p}$ and $\boldsymbol{g}$, we note that the first, second, and seventh block rows characterize the convex hull of the region specified by:

- the requirement for $\boldsymbol{u}$ and $\boldsymbol{v}$ to be binary vectors, and
- the mutual exclusivity constraints.

Now consider all of the variables $\boldsymbol{p}, \boldsymbol{g}, \boldsymbol{u}$, and $\boldsymbol{v}$ and all of the rows of the coefficient matrix of (51). By the result in [22, Appendix], which considers the convex hull of systems with both continuous and integer variables, we can show that all the vertices of this system have binary values for the entries of $\boldsymbol{u}$ and $\boldsymbol{v}$. That is, a continuous optimization algorithm that seeks vertex solutions will automatically yield a solution that has binary values for $\boldsymbol{u}$ and $\boldsymbol{v}$ if we do not include the SOC limits. Under Assumptions 2 and 3, a formulation with mutual exclusivity constraints but without SOC limits has a more convenient structure for computation than the full formulation with both mutual exclusivity constraints and SOC limits. Consistent with this conclusion, we observe that the points $\widehat{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}$ defined in Section A.1 are not vertices of $\mathbb{M}'$ for the corresponding values of parameters because for both points there are, respectively, only 7 and 3 linearly independent constraints binding.

A.3. No mutual exclusivity constraints

In this section, we consider three special cases of the formulation where there are SOC limits but no mutual exclusivity constraints. We show that the cases with SOC limits but without mutual exclusivity constraints are also more convenient for computation than the full problem with both sets of constraints.

The three formulations we consider are:

1. where only pumping is possible, but not generation,
2. where only generation is possible, but not pumping,
3. where both pumping and generation is simultaneously possible.

In each case we will find that the convex hull of the feasible region is given by its continuous relaxation.

The case where only pumping is possible is a relevant model for energy-limited *consumption* of energy. This literally applies to the pumping of water for use in a gravity-fed water supply, but also applies, for example, to industrial loads that have some flexibility in the intensity of electricity consumption and also have a desired maximum amount of energy consumption over the scheduling horizon. In the latter example of an industrial load, we interpret "pumping" to also include consumption not necessarily literally associated with hydroelectric pumping. In this "pumping"-only case, we interpret $\overline{s} - s_0$ to be the maximum energy consumption over the scheduling horizon.

The case where only generation is possible is relevant to energy-limited reservoir hydroelectric generation, and to other energy-limited generation resources, if we interpret $s_0 - \underline{s}$ to be the maximum energy that can be used from the resource over the scheduling horizon.

The third case represents a so-called ternary PSH system [6][27, section 3.3], where there can be simultaneous charging and discharging associated with a single unit, resulting in a net injection or withdrawal of electric energy determined by the difference between pumping and generating in a single interval. That case will show that deleting the mutual exclusivity constraints in a pumping and generating model also results in a more tractable problem.

A.3.1. Pumping only Turning first to the case of pumping-only, the inequalities describing the feasible region can be obtained from (50) by deleting the second, fifth, sixth, seventh, and ninth block rows and deleting the second and fourth block columns to obtain:

$$\begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & (\underline{p}/\overline{p})\mathbf{I} \\ \boldsymbol{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}/\overline{p} \\ \boldsymbol{u} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1}(\overline{s} - s_0)/(\alpha \overline{p}) \end{bmatrix}. \tag{52}$$

The pumping-only feasible operating region is the set:

$$\mathbb{P} = \{(\boldsymbol{p}, \boldsymbol{u}) \in \mathbb{R}^T \times \mathbb{Z}^T | (52)\}.$$

The corresponding continuous relaxation of $\mathbb{P}$ is:

$$\mathbb{P}' = \{(\boldsymbol{p}, \boldsymbol{u}) \in \mathbb{R}^T \times \mathbb{R}^T | (52)\},$$

so that $\mathbb{P}'$ is obtained from $\mathbb{P}$ by relaxing the integrality constraints. We observe that if $\underline{p} = 0$ or $\underline{p} = \overline{p}$ then the coefficient matrix of (52) consists entirely of $0, \pm 1$ entries and is the concatenation of zero and identity matrices with $\boldsymbol{L}$, which is a matrix having the "consecutive ones" property, and so the coefficient matrix of (52) is totally unimodular. Therefore, since $\alpha \overline{p}$ divides evenly into $\overline{s} - s_0$ by Assumption 3, so that the right-hand side of the scaled system is integral, then the convex hull of $\mathbb{P}$ is its continuous relaxation $\mathbb{P}'$, and all vertices have integer-valued entries for $\boldsymbol{p}/\overline{p}$ and $\boldsymbol{u}$. To summarize, specialization to the case of pumping-only results in a formulation for which the convex hull can be straightforward to characterize compactly. We observe that the points $(\widehat{\boldsymbol{p}}, \widehat{\boldsymbol{u}})$ and $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{u}})$ defined in Section A.1 are not vertices of $\mathbb{P}'$ for the corresponding values of parameters.

A.3.2. Generating only Similarly, in the case of generating-only, the inequalities describing the feasible region can be obtained from (50) by deleting the first, third, fourth, seventh, and eighth block rows and deleting the first and third block columns to obtain:

$$\begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & (\underline{g}/\overline{g})\mathbf{I} \\ \boldsymbol{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}/\overline{g} \\ \boldsymbol{v} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1}(s_0 - \underline{s})/(\beta\overline{g}) \end{bmatrix}, \tag{53}$$

The generating-only feasible operating region is the set:

$$\mathbb{G} = \{(\boldsymbol{g}, \boldsymbol{v}) \in \mathbb{R}^T \times \mathbb{Z}^T | (53)\}.$$

The corresponding continuous relaxation of $\mathbb{G}$ is:

$$\mathbb{G}' = \{(\boldsymbol{g}, \boldsymbol{v}) \in \mathbb{R}^T \times \mathbb{R}^T | (53)\},$$

so that $\mathbb{G}'$ is obtained from $\mathbb{G}$ by relaxing the integrality constraints. Again, we observe that under the assumption that $\underline{g} = 0$ or $\underline{g} = \overline{g}$, and since $\beta\overline{g}$ divides evenly into $s_0 - \underline{s}$ by Assumption 3, then we have that the convex hull of $\mathbb{G}$ is $\mathbb{G}'$, and all vertices have integer-valued entries for $\boldsymbol{g}/\overline{g}$ and $\boldsymbol{v}$. We observe that the points $(\hat{\boldsymbol{g}}, \hat{\boldsymbol{v}})$ and $(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{v}})$ defined in Section A.1 are not vertices of $\mathbb{G}'$ for the corresponding values of parameters.

A.3.3. Ternary PSH Again, the inequalities describing the feasible region can be obtained by modifying (50). In this case, as well as deleting the mutual exclusivity constraints, it is necessary to add upper limits to the binary variables and change the SOC limits from (13)–(14) to the standard formulation (12), since with ternary PSH it *is* feasible to simultaneously pump and generate. This modifies (50) to:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & (\underline{p}/\overline{p})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & (\underline{g}/\overline{g})\mathbf{I} \\ \boldsymbol{L} & -\boldsymbol{L} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{L} & \boldsymbol{L} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}/\overline{p} \\ \boldsymbol{g}/\overline{g} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1}(\overline{s} - s_0)/\breve{s} \\ \mathbf{1}(s_0 - \underline{s})/\breve{s} \end{bmatrix}. \tag{54}$$

The ternary PSH feasible operating region is the set:

$$\mathbb{T} = \{(\boldsymbol{p}, \boldsymbol{g}, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{Z}^T \times \mathbb{Z}^T | (54)\}.$$

The corresponding continuous relaxation of $\mathbb{T}$ is:

$$\mathbb{T}' = \{(\boldsymbol{p}, \boldsymbol{g}, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T | (54)\},$$

so that $\mathbb{T}'$ is obtained from $\mathbb{T}$ by relaxing the integrality constraints.

Similarly to the previous cases of pumping-only and generation-only, if $\underline{p}=0$ or $\underline{p}=\overline{p}$ and if $\underline{g}=0$ or $\underline{g}=\overline{p}$ then Assumptions 2 and 3 result in the coefficient matrix being totally unimodular and the right-hand side being integral, so that all vertices have integer-valued entries for $\boldsymbol{p}/\overline{p}, \boldsymbol{g}/\overline{g}, \boldsymbol{u}$, and $\boldsymbol{v}$. To summarize, deleting the mutual exclusivity constraints results in a formulation for which the convex hull can be straightforward to characterize compactly. We observe that the points $\widehat{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}$ defined in Section A.1 are not vertices of $\mathbb{T}'$ for the corresponding values of parameters.

A.4. Contrast with general model and summary

Under Assumptions 2 and 3, if there are either no SOC limits or no mutual exclusivity constraints, then the convex hull of the feasible region has a compact representation. In contrast, and as discussed in Appendix A.1, analogous results do not hold for the general PSH model with both pumping and generating, SOC limits, and mutual exclusivity constraints. That is, it appears that the convex hull of $\mathbb{S}$ does not in general have a compact representation. In particular, as demonstrated in Appendix A.1, the analogous assumptions for PSH: Assumptions 2 and 3; that $\underline{p}=0$ or $\underline{p}=\overline{p}$; and, that $\underline{g}=0$ or $\underline{g}=\overline{g}$, do not result in a convenient characterization of the convex hull of $\mathbb{S}$.

To summarize, this analysis further illustrates that the combination of mutual exclusivity constraints and the coupling over time due to the SOC limits underlies the computational challenge of PSH compared to the case of a generator without energy limits. This is consistent with the computational results in Section 5.

B. Proof of Theorem 1

Using Lemma 1 to specify some of the entries of $\boldsymbol{x}^{\star}$ To begin the specification of the entries of $\boldsymbol{x}^{\star}$, observe that by Lemma 1, for each $t=1,\dots,T$, at least one of $p_t^{\star\star}$ or $g_t^{\star\star}$ is equal to zero. For each t such that $p_t^{\star\star}=0$, define $u_t^{\star}=0$ and $p_t^{\star}=0$, and, if it is the case for this t that $g_t^{\star\star}=\overline{g}$, then also define $v_t^{\star}=1$ and $g_t^{\star}=\overline{g}$. Similarly, for each t such that $g_t^{\star\star}=0$, define $v_t^{\star}=0$ and $g_t^{\star}=0$ and, if it is the case for this t that $p_t^{\star\star}=\overline{p}$, then also define $u_t^{\star}=1$ and $p_t^{\star}=\overline{p}$. Note that, for each t, either the entries $u_t^{\star}$, $p_t^{\star}$, $v_t^{\star}$, and $g_t^{\star}$ have all been specified and satisfy (31)–(37), or only $u_t^{\star}$ and $p_t^{\star}$ remain unspecified, or only $v_t^{\star}$ and $g_t^{\star}$ remain unspecified. In the remainder of the proof, we specify any remaining unspecified values in such a way that $\boldsymbol{x}^{\star}\in\mathbb{S}$.

Considering the "uneven" intervals of $\boldsymbol{x}^{\star\star}$ The remaining unspecified values correspond to intervals t having either:

1. $u_t^{\star}$ and $p_t^{\star}$ unspecified, so that $0<p_t^{\star\star}<\overline{p}$, $u_t^{\star\star}>0$, and $g_t^{\star}=g_t^{\star\star}=0$, $v_t^{\star}=0$, or
2. $v_t^{\star}$ and $g_t^{\star}$ unspecified, so that $0<g_t^{\star\star}<\overline{g}$, $v_t^{\star\star}>0$, and $p_t^{\star}=p_t^{\star\star}=0$, $u_t^{\star}=0$,

where we observe that only one of these two cases can occur in any given interval. We call these the "uneven" intervals for the solution $\boldsymbol{x}^{\star\star}$ and refer to them, respectively, either as case 1 or 2 uneven intervals of $\boldsymbol{x}^{\star\star}$, or simply case 1 or 2 uneven intervals when the context is clear.

We call the rest of the intervals "even" intervals for the solution $\boldsymbol{x}^{\star\star}$ and note that the change of SOC in any even interval is either 0 or $\pm\breve{s} = \pm\alpha\overline{p} = \pm\beta\overline{g}$. Our specification of $\boldsymbol{x}^{\star}$ will ensure that all of its intervals are even and that the SOC constraints remain satisfied; that is, it will ensure that $\boldsymbol{x}^{\star} \in \mathbb{S}$.

Defining the notion of "epoch" and its properties To facilitate specifying the rest of the intervals, consider the intervals t such that the SOC s_t equals $\overline{s}$ or $\underline{s}$. Label these intervals as $e_1, e_2, ...$ Note that, in principle, s_T might or might not equal $\overline{s}$ or $\underline{s}$; however, for convenience we include $t = T$ as the last labeled interval irrespective of its value. Also define $t = 0$ as the 0-th labeled interval, $e_0 = 0$, so that the labeled intervals are $e_0 = 0, e_1, \ldots, e_L = T$. For $\ell = 1, \ldots, L$, define the ℓ-th "epoch" to be the collection of intervals $e_{\ell-1} + 1, \ldots, e_\ell$. That is: the first epoch includes intervals $1, \ldots, e_1$; the second epoch (if there is one) includes intervals $e_1 + 1, \ldots, e_2$; while the last epoch includes the intervals $e_{L-1} + 1, \ldots, e_L = T$.

Now consider a particular epoch $\ell \in \{1, \ldots, L\}$ and write t_k for the k-th uneven interval in the ℓ-th epoch, with $k \in \{1, \ldots, K\}$, and where there are K uneven intervals in total in the ℓ-th epoch. The set of all uneven intervals in epoch ℓ is $\{t_1, \ldots, t_k\}$, while the set of all even intervals in epoch ℓ is $\{e_{\ell-1} + 1, \ldots, e_\ell\} \backslash \{t_1, \ldots, t_k\}$. We define Δs to be the sum of the changes in SOC due to the *uneven* pumping and generation intervals in epoch ℓ. That is, $\Delta s = \sum_{k=1}^{K} (\alpha p_{t_k}^{\star\star} - \beta g_{t_k}^{\star\star})$ differs from $s_{e_\ell}^{\star\star} - s_{e_{\ell-1}}^{\star\star}$ by the sum of the changes in the SOC due to the *even* pumping and generation intervals in epoch ℓ. (We are slightly abusing notation by not including the index, ℓ, of the epoch in the notation for the uneven intervals and Δs in the ℓ-th epoch; however, the epoch index will always be clear from the context.)

We note three properties in each epoch that will be useful in the balance of the proof:

1. For each ℓ, except possibly for $\ell = L$, we have that $s_{e_\ell}^{\star\star} - s_{e_{\ell-1}}^{\star\star}$ is evenly divisible by $\breve{s} = \alpha\overline{p} = \beta\overline{g}$, where we define $s_{e_0}^{\star\star} = s_0$, and recall that by Assumption 3, $(\overline{s} - s_{e_0}^{\star\star}) = (\overline{s} - s_0)$ and $(s_{e_0}^{\star\star} - \underline{s}) = (s_0 - \underline{s})$ are both evenly divisible by $\breve{s} = \alpha\overline{p} = \beta\overline{g}$, so that:

$$
\begin{aligned}
(s_{e_1}^{\star\star} - s_{e_0}^{\star\star}) &= (s_{e_1}^{\star\star} - s_0), \\
&= \begin{cases} (\overline{s} - s_0), & \text{if } s_{e_1}^{\star\star} = \overline{s}, \\ (\underline{s} - s_0), & \text{if } s_{e_1}^{\star\star} = \underline{s}, \end{cases}
\end{aligned}
$$

is also evenly divisible by $\breve{s} = \alpha\overline{p} = \beta\overline{g}$.

2. For each ℓ, except possibly for $\ell = L$, Δs is evenly divisible by $\breve{s}$, since for these epochs we have that:

$$\begin{aligned}\Delta s &= \sum_{k=1}^{K}(\alpha p^{\star\star}_{t_k} - \beta g^{\star\star}_{t_k}), \text{ by definition,}\\ &= s^{\star\star}_{e_\ell} - s^{\star\star}_{e_{\ell-1}} - \sum_{t\in\{e_{\ell-1}+1,\dots,e_\ell\}\setminus\{t_1,\dots,t_k\}} (\alpha p^{\star\star}_t - \beta g^{\star\star}_t),\\ &= \text{ (an integer multiple of } \breve{s}) - \text{ (an integer multiple of } \breve{s}),\end{aligned}$$

since $\alpha p^{\star\star}_t = \beta g^{\star\star}_t = \breve{s}$ for even intervals $t \in \{e_{\ell-1}+1,\dots,e_\ell\}\setminus\{t_1,\dots,t_k\}$.

3. For each ℓ, and for each interval $\tau \in \{e_{\ell-1}+1,\dots,e_\ell - 1\}$ (that is, for all the intervals in the range, not just the uneven intervals), the SOC $s^{\star\star}_\tau$ does not equal $\overline{s}$ nor $\underline{s}$. That is, neither (38) nor (39) are binding for these intervals, so that by complementary slackness the corresponding Lagrange multipliers satisfy $\mu^{\star\star}_{38\tau} = \mu^{\star\star}_{39\tau} = 0$ for all $\tau \in \{e_{\ell-1}+1,\dots,e_\ell - 1\}$.

Considering the coefficients of the objective for the uneven intervals in an epoch We characterize D_{t_k} for case 1 uneven intervals and characterize C_{t_k} for case 2 uneven intervals in the ℓ-th epoch. First consider case 1 uneven intervals t_k in the ℓ-th epoch. These intervals satisfy $0 < p^{\star\star}_{t_k} < \overline{p}$, $u^{\star\star}_{t_k} > 0$, and $g^{\star}_{t_k} = g^{\star\star}_{t_k} = 0$. Since $u^{\star\star}_{t_k} > 0$ and $p^{\star\star}_{t_k} > 0$, we have by complementary slackness that $\mu^{\star\star}_{31t_k} = 0$ and $\mu^{\star\star}_{34t_k} = 0$. Therefore, by (40), $\mu^{\star\star}_{33t_k} - \overline{p}\mu^{\star\star}_{35t_k} = 0$ and, so, either (33) and (35) are both binding with strictly positive values of Lagrange multipliers or $\mu^{\star\star}_{33t_k} = \mu^{\star\star}_{35t_k} = 0$.

We claim that (33) and (35) cannot be both binding with strictly positive values of Lagrange multipliers so that the only possibility is $\mu^{\star\star}_{33t_k} = \mu^{\star\star}_{35t_k} = 0$. To prove this, we consider each of the following two alternative cases for values of $v^{\star\star}_{t_k}$ and for both cases we prove by contradiction:

$v^{\star\star}_{t_k} = 0$: Suppose that (33) and (35) were both binding with strictly positive values of Lagrange multipliers. Then, since by assumption $v^{\star\star}_{t_k} = 0$ and since (33) is binding we have that $u^{\star\star}_{t_k} = 1$. Consequently, since $u^{\star\star}_{t_k} = 1$ and since (35) is binding, we also have that $p^{\star\star}_{t_k} = \overline{p}u^{\star\star}_{t_k} = \overline{p}$, contradicting that $p^{\star\star}_{t_k} < \overline{p}$. Therefore, (33) and (35) could not have been both binding with strictly positive values of Lagrange multipliers and instead we must have that $\mu^{\star\star}_{33t_k} = \mu^{\star\star}_{35t_k} = 0$.

$v^{\star\star}_{t_k} > 0$: Again suppose that (33) and (35) were both binding with strictly positive values of Lagrange multipliers. Since $g^{\star\star}_{t_k} = 0$ and $v^{\star\star}_{t_k} > 0$ we have that $0 = g^{\star\star}_{t_k} < \overline{g}v^{\star\star}_{t_k}$, so that $\mu^{\star\star}_{36t_k} > 0$ and $\mu^{\star\star}_{37t_k} = 0$. Moreover, since $v^{\star\star}_{t_k} > 0$, we have that $\mu^{\star\star}_{32t_k} = 0$. From (41),

$$\begin{aligned}\mu^{\star\star}_{33t_k} &= \mu^{\star\star}_{32t_k} + \overline{g}\mu^{\star\star}_{37t_k},\\ &= 0,\end{aligned}$$

contradicting the supposition that $\mu^{\star\star}_{33t_k} > 0$. We conclude that, in fact, $\mu^{\star\star}_{33t_k} = \mu^{\star\star}_{35t_k} = 0$.

To summarize, in both cases we have that that $\mu^{\star\star}_{33t_k} = \mu^{\star\star}_{35t_k} = 0$. Substituting into (42), and noting that:

- $\mu^{\star\star}_{34t_k} = 0$,
- $\mu^{\star\star}_{35t_k} = 0$, and
- $\mu^{\star\star}_{38\tau} = \mu^{\star\star}_{39\tau} = 0$, for $\tau \in \{e_{\ell-1}+1, \ldots, e_\ell - 1\}$,

we obtain, for each case 1 uneven interval $t_k \in \{e_{\ell-1}+1, \ldots, e_\ell - 1\}$ that:

$$\begin{aligned} D_{t_k} &= \mu^{\star\star}_{34t_k} - \mu^{\star\star}_{35t_k} - \alpha\mu^{\star\star}_{38t_k} - \alpha \sum_{\tau=t_k+1}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau}), \\ &= -\alpha \sum_{\tau=e_\ell}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau}). \end{aligned}$$

We now consider the situation where the last interval in the ℓ-th epoch, namely e_ℓ, is a case 1 uneven interval, so that $t_K = e_\ell$. By assumption 1, at most one of (38) and (39) can be binding during interval $t_K = e_\ell$. Consequently, if $t_K = e_\ell$ is a case 1 uneven interval then, since $p^{\star\star}_{t_K} > 0$, only (38) could binding, while (39) cannot be binding, and so $\mu^{\star\star}_{39t_K} = 0$. Therefore, we have:

$$\begin{aligned} D_{t_K} &= -\alpha\mu^{\star\star}_{38t_K} - \alpha \sum_{\tau=t_K+1}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau}), \\ &= -\alpha\mu^{\star\star}_{38t_K} + \beta\mu^{\star\star}_{39t_K} - \alpha \sum_{\tau=t_K+1}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau}), \\ &= -\alpha \sum_{\tau=t_K}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau}), \\ &= -\alpha \sum_{\tau=e_\ell}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau}). \end{aligned}$$

Note that this means that D_{t_k} has the same value independent of t_k for all of the case 1 uneven intervals in the ℓ-th epoch, and its common value is $-\alpha \sum_{\tau=e_\ell}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau})$. For convenience, write D for this common value. In all such case 1 uneven intervals t_k we have that $0 < p^{\star\star}_{t_k} < \overline{p}$, $u^{\star\star}_{t_k} > 0$, and $v^{\star\star}_{t_k} = 0$.

Similarly, C_{t_k} has the same value independent of t_k for all of the case 2 uneven intervals in the ℓ-th epoch, and is equal to $-\beta \sum_{\tau=e_\ell}^{T} (\mu^{\star\star}_{38\tau} - \mu^{\star\star}_{39\tau})$. For convenience, write C for this common value, and note that $C/\beta = D/\alpha$. In all such case 2 uneven intervals t_k we have that $0 < g^{\star\star}_{t_k} < \overline{g}$, $v^{\star\star}_{t_k} > 0$, and $u^{\star\star}_{t_k} = 0$.

Evaluating the contribution of uneven intervals to the objective Consider any specification of the entries of $\boldsymbol{x}^\star$ in epoch ℓ for intervals $t_1, \ldots, t_K$ such that:

- case 1 intervals are assigned zero generation in $\boldsymbol{x}^\star$, but possibly assigned non-zero pumping,
- case 2 intervals are assigned zero pumping in $\boldsymbol{x}^\star$, but possibly assigned non-zero generation, and

- the change in SOC due to the non-zero pumping and generation in intervals $t_1,\dots,t_K$ in $\boldsymbol{x}^\star$ is the same as for the intervals $t_1,\dots,t_K$ in $\boldsymbol{x}^{\star\star}$.

That is, the last condition means that the pumping and generation $p_{t_k}, g_{t_k}, k=1,\dots,K$, in the intervals $t_k, k=1,\dots,K$, are chosen to satisfy:

$$\begin{aligned}\sum_{k=1}^{K}(\alpha p^\star_{t_k} - \beta g^\star_{t_k}) &= \sum_{k=1}^{K}(\alpha p^{\star\star}_{t_k} - \beta g^{\star\star}_{t_k}),\\ &= \Delta s.\end{aligned}$$

Consider the contribution to the objective of these intervals $t_k, k=1,\dots,K$. The contribution is:

$$\begin{aligned}\sum_{k=1}^{K}(C_{t_k} g^\star_{t_k} - D_{t_k} p^\star_{t_k}) &= \sum_{k=1}^{K}(C g^\star_{t_k} - D p^\star_{t_k}),\\ &\quad\text{noting for case 1 intervals } t_k \text{ that } g^\star_{t_k}=0, \text{ and}\\ &\quad\quad\text{for case 2 intervals } t_k \text{ that } p^\star_{t_k}=0,\\ &= (C/\beta)\sum_{k=1}^{K}(\beta g^\star_{t_k} - \alpha p^\star_{t_k}),\\ &\quad\text{recalling that } C/\beta = D/\alpha\\ &= (C/\beta)\sum_{k=1}^{K}(\beta g^{\star\star}_{t_k} - \alpha p^{\star\star}_{t_k}),\\ &\quad\text{by assumption on the change in the SOC },\\ &= \sum_{k=1}^{K}(C_{t_k} g^{\star\star}_{t_k} - D_{t_k} p^{\star\star}_{t_k}),\end{aligned}$$

noting for case 1 intervals t_k that $g^{\star\star}_{t_k}=0$, and for case 2 intervals t_k that $p^\star_{t_k}=0$. That is, the contribution to the objective from the intervals $t_k, k=1,\dots,K$, in $\boldsymbol{x}^\star$ is the same as that for $\boldsymbol{x}^{\star\star}$.

Specifying the remaining entries of $\boldsymbol{x}^\star$ Using the result about the contribution to the objective, we define the remaining unspecified entries of $\boldsymbol{x}^\star$ to consolidate pumping and generation into a subset of the uneven intervals so that:

- in each case 1 uneven interval of $\boldsymbol{x}^{\star\star}$ the pumping in that interval of $\boldsymbol{x}^\star$ is either equal to 0 or $\overline{p}$,
- in each case 2 uneven interval of $\boldsymbol{x}^{\star\star}$ the generation in that interval of $\boldsymbol{x}^\star$ is either equal to 0 or $\overline{g}$,
- the sum of the changes in SOC in $\boldsymbol{s}^\star$ due to all the intervals $t_1,\dots,t_K$ is the same as the change in SOC for those intervals in $\boldsymbol{s}^{\star\star}$, and
- $\boldsymbol{s}^\star$ satisfies the SOC limits.

First consider epochs such that Δs is evenly divisible by $\breve{s}$. This applies to all epochs except possibly the last. Consider the following construction. Each specification of entries of $\boldsymbol{x}^\star$ for an uneven interval t_k will change the SOCs $s^\star_{t_k}, s^\star_{t_k+1},\dots,s^\star_T$ compared to $s^{\star\star}_{t_k}, s^{\star\star}_{t_k+1},\dots,s^{\star\star}_T$. For each successive uneven interval t_k, we choose the specification that minimizes the deviation of $s^\star_{t_k}$ from $s^{\star\star}_{t_k}$, breaking ties arbitrarily, and we claim that this will maintain satisfaction of (38)–(39). For example, if the first uneven interval in the epoch, say interval t_1, has $p^{\star\star}_{t_1} = 0.6\overline{p}$ then set $u^\star_{t_1} = 1$ and $p^\star_{t_1} = \overline{p}$. If the second uneven interval in the epoch, t_2, has $p^{\star\star}_{t_2} = 0.3\overline{p}$ then set $u^\star_{t_2} = 0$ and $p^\star_{t_2} = 0$. Note that simpler approaches, such as assigning all pumping to the earliest uneven intervals will not generally maintain feasibility with respect to the SOC limits.

More generally, consider:

$$\Delta_k = \sum_{j=1}^{k} \alpha(p^\star_{t_j} - p^{\star\star}_{t_j}) - \beta(g^\star_{t_j} - g^{\star\star}_{t_j}).$$

If t_k is an uneven case 1 interval, then choose $p^\star_{t_k}$ to be either 0 or $\overline{p}$ so as to result in the smallest value for $|\Delta_{k-1} + \alpha(p^\star_{t_k} - p^{\star\star}_{t_k})|$, where, for convenience, we define $\Delta_0 = 0$ and where ties can be broken arbitrarily. If t_k is an uneven case 2 interval, choose $g^\star_{t_k}$ to be either 0 or $\overline{g}$ so as to result in the smallest value for $|\Delta_{k-1} - \beta(p^\star_{t_k} - p^{\star\star}_{t_k})|$, again breaking ties arbitrarily. Since $\Delta_0 = 0$, we observe that, by construction, $|\Delta_k| \leqslant \breve{s}/2 = \alpha\overline{p}/2 = \beta\overline{g}/2$.

We now prove that this specification also results in the entries of the SOC $s^\star$ being within limits for all intervals in the epoch. To see this, note that if an uneven interval had state $s^{\star\star}_{t_k}$ that was within $\breve{s}$ of $\overline{s}$ then, by construction, $s^\star_{t_k}$ is either $\overline{s}$ or $\overline{s} - \breve{s}$. For subsequent even intervals t until the next uneven interval, observe that $s^{\star\star}_t$ must have been no higher than $s^{\star\star}_{t_k}$ since otherwise it would equal $s^{\star\star}_{t_k} + \breve{s}$ and the SOC limits would have been violated by $\boldsymbol{x}^{\star\star}$. Consequently, the corresponding values of $s^\star_t$ until the next uneven interval will also satisfy the SOC constraints. A similar argument applies for each uneven interval that had state $s^{\star\star}_{t_k}$ that was within $\breve{s}$ of $\underline{s}$.

Moreover, for t_K, the last uneven interval in the epoch, we must have that $\Delta_K = 0$, since

$$\begin{aligned}
\Delta_K &= \sum_{j=1}^{K} \alpha(p^\star_{t_j} - p^{\star\star}_{t_j}) - \beta(g^\star_{t_j} - g^{\star\star}_{t_j}), \\
&= -\sum_{j=1}^{K}(\alpha p^{\star\star}_{t_j} - \beta g^{\star\star}_{t_j}) + \sum_{j=1}^{K}(\alpha p^\star_{t_j} - \beta g^\star_{t_j}), \\
&= -\Delta s + \sum_{j=1}^{K}(\alpha p^\star_{t_j} - \beta g^\star_{t_j}),
\end{aligned}$$

and Δs is evenly divisible by $\breve{s}$ and the terms $(\alpha p^\star_{t_j} - \beta g^\star_{t_j})$ in the last summation are all equal to $\pm\breve{s}$ by construction. Consequently, $\Delta_K = 0, \pm\breve{s}, \pm 2\breve{s}, \ldots$. However, since $|\Delta_K| \leqslant \breve{s}/2$ by construction, this means that $\Delta_K = 0$. That is, the change in SOC due to the intervals $t_1, \ldots, t_K$ in $\boldsymbol{x}^\star$ is the same as the change in SOC due to the intervals $t_1, \ldots, t_K$ in $\boldsymbol{x}^{\star\star}$, and so the contribution to the objective is the same for these intervals.

Turning to the last epoch, $\ell = L$, note that if $s_{e_L} - s_{e_{L-1}}$ is not evenly divisible by $\breve{s}$ then for all uneven case 1 intervals t_k in the L-th epoch we must have $D_{t_k} = D = 0$ and for all uneven case 2 intervals t_k in the L-th epoch we must have $C_{t_k} = C = 0$. Otherwise, we could consider a small change either in $p^{\star\star}_{t_k}$ and $u^{\star\star}_{t_k}$, or in $g^{\star\star}_{t_k}$ and $v^{\star\star}_{t_k}$, respectively, that would improve the objective without violating any constraints, which contradicts optimality of $\boldsymbol{x}^{\star\star}$.

To summarize, for the last epoch, if $s_{e_L} - s_{e_{L-1}}$ is not evenly divisible by $\check{s}$ then because $C = D = 0$ we can arbitrarily assign the pumping for case 1 and the generation for case 2 intervals, $t_k, k = 1, \dots, K$, without affecting the objective. We can choose this assignment to maintain feasibility with respect to the SOC constraints by following the same construction as for the other epochs.