Behavior-Aware Queueing: The Finite-Buffer Setting with Many Strategic Servers: Technical Appendix

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In this technical appendix, we provide proofs for the results stated in the main body of the manuscript titled: "Behavior-Aware Queueing: The Finite-Buffer Setting with Many Strategic Servers". The proofs of these results are in the order in which they appear in the main body. Throughout, we use the notation $\rho = \frac{\lambda}{\mu}$.

EC.1. Preliminaries

Our analysis requires knowledge of the Erlang B and C formulae, which are arguably the most fundamental formulae for studying queueing systems (Cooper (1981)). The Erlang B formula represents the steady-state blocking probability in the $M/M/N/N$ queue, given by

$$ErlB(N,\rho) = \frac{\rho^N/N!}{\sum_{i=0}^{N} \rho^i/i!}, \quad \rho > 0. \tag{EC.1}$$

For a constant $\rho > 0$, $ErlB(N,\rho)$ satisfies the recursion

$$ErlB(N,\rho) = \frac{\rho ErlB(N-1,\rho)}{N + \rho ErlB(N-1,\rho)}, \tag{EC.2}$$

where $ErlB(0,\rho) \equiv 1$; see, e.g., pp.82 of Cooper (1981). The Erlang C formula represents the steady-state probability of delay in the $M/M/N$ queue, given by

$$ErlC(N,\rho) = \frac{\rho^N \frac{1}{N!} \frac{N}{N-\rho}}{\sum_{i=0}^{N-1} \rho^i \frac{1}{i!} + \rho^N \frac{1}{N!} \frac{N}{N-\rho}}, \quad \rho > 0. \tag{EC.3}$$

Lemma EC.1 **(Monotonicity of Erlang B and C)**. *The following hold:*

(a) $ErlB(N,\rho)$ *is strictly decreasing in* N *and strictly increasing in* ρ*;*

(b) $ErlC(N,\rho)$ *is strictly decreasing in* N *and strictly increasing in* ρ*.*

Lemma EC.2 **(More Properties of Erlang B and C)**. *The following hold:*

(a) $ErlC(N,\rho) = N\left(\frac{N-\rho}{ErlB(N,\rho)} + \rho\right)^{-1}$.

(b) $ErlC(N,\rho)\begin{cases} <1, & \rho < N \\ =1, & \rho = N \\ >1, & \rho > N \end{cases}$. *Moreover,* $\lim_{\rho \downarrow 0} ErlC(N,\rho) = 0$ *and* $\lim_{\rho\to\infty} ErlC(N,\rho)/\rho = 1$.

(c) $\frac{1-ErlC(N,\rho)}{N-\rho} \in (0,1)$, $\forall \rho > 0$.

Lemma EC.3 **(Derivative of Erlang C)**. $\frac{\partial ErlC(N,\rho)}{\partial\rho} = ErlC(N,\rho)\left(\frac{1-ErlC(N,\rho)}{N-\rho} + \frac{N-\rho}{\rho}\right)$.

EC.1.1. Proofs of Lemmas EC.1-EC.3

The proofs can be found in Zhong et al. (2023) with consistent numbering.

EC.2. Proofs from Section 1.3

EC.2.1. Preliminaries

LEMMA EC.4 **(Properties of Idle Time)**. *$I_i(\boldsymbol{\mu};\lambda,k,N)$ satisfies the following monotonicity properties.*

(a) $I_i(\boldsymbol{\mu};\lambda,k,N)$ is a strictly increasing function of μ_j, $1\le j\le N$.

(b) $I_i(\boldsymbol{\mu};\lambda,k,N)$ is a strictly decreasing function of k.

(c) $I_i(\boldsymbol{\mu};\lambda,k,N)$ is a strictly decreasing function of λ.

EC.2.1.1. Proof of Lemma EC.4 We begin by considering an $M/M/1/k$ queueing system with arrival rate λ and service rate μ. The birth-death process is shown in Figure EC.1. Let P_ℓ denote the steady-state probability of ℓ jobs in the system. Then, the balance equations are given by

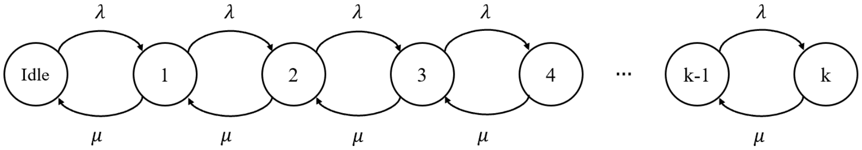


Figure EC.1 Birth-death process for the $M/M/1/k$ system

$$\begin{aligned}&-\lambda P_0+\mu P_1=0,\\&-(\lambda+\mu)P_i+\lambda P_{i-1}+\mu P_{i+1}=0,\quad 1\le i\le k-1\\&-\mu P_k+\lambda P_{k-1}=0.\end{aligned}$$

Solving the above system of equations yields the steady-state probabilities:

$$P_i=\left(\frac{\lambda}{\mu}\right)^i P_0,\quad 1\le i\le k,\quad \text{where}\quad P_0=\frac{1}{\sum_{i=1}^k\left(\frac{\lambda}{\mu}\right)^i}. \tag{EC.4}$$

Here, P_0 is the idle time, i.e., the probability that the server is idle. It is straightforward that P_0 is strictly increasing in μ, strictly decreasing in λ, and strictly decreasing in k.

Now, we extend the analysis to an $M/M/N/k$ system with arrival rate λ and heterogeneous service rates μ_i ($i=1,2,\ldots,N$). The birth-death process is demonstrated in Figure EC.2. Note that all the states framed in the red dashed square represent some servers are idle. Thus, it is plausible to regard states $\{0,1,2,\ldots,N-1\}$ as one "super" idle state and obtain an equivalent birth-death process, as shown in Figure EC.3. The equivalent birth-death process for the $M/M/N/k$ system can

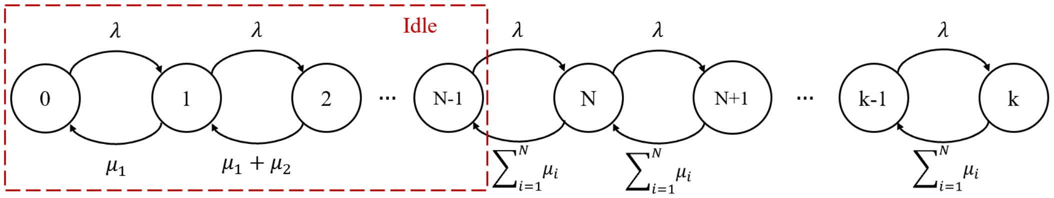


Figure EC.2 Birth-death process for the $M/M/N/k$ system

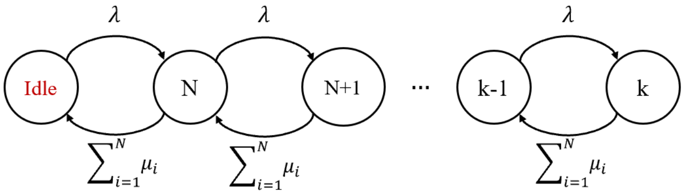


Figure EC.3 Equivalent birth-death process for the $M/M/N/k$ system

be viewed as an $M/M/1/k$ system with arrival rate λ, service rate $\sum_{i=1}^{N}\mu_i$ and a state truncation at N. Thus, following the stationary result (EC.4) in the $M/M/1/k$ system, the steady-state probabilities of this equivalent birth-death process can be written as

$$P'_i = \left(\frac{\lambda}{\sum_{i=1}^{N}\mu_i}\right)^{i-N+1} P'_0, \quad N \le i \le k, \quad \text{where} \quad P'_0 = \frac{1}{\sum_{i=1}^{k-N+1}\left(\frac{\lambda}{\sum_{i=1}^{N}\mu_i}\right)^i}, \tag{EC.5}$$

and P'_0 is the probability of the "super" idle state. Denote the probably that all servers are busy with no jobs waiting in queue by P'_N, then it follows that

$$P'_N = P'_0 \frac{\lambda}{\sum_{i=1}^{N}\mu_i} = \frac{1}{\sum_{i=0}^{k-N}\left(\frac{\lambda}{\sum_{i=1}^{N}\mu_i}\right)^i}. \tag{EC.6}$$

Next, we examine the idle probability of each individual server. Let $\mathcal{I} \subseteq \{1,2,\ldots,N\}$ be the set of idle servers ($\mathcal{I} = \emptyset$ when all servers are busy), and state $\boldsymbol{a} = (a_1, a_2, \ldots, a_{|\mathcal{I}|})$ be the ordered vector of idle servers where server j became idle before server k whenever $1 \le j < k \le |\mathcal{I}|$. Denote by $P(\boldsymbol{a})$ the steady-state probability of state $\boldsymbol{a}$. From Gopalakrishnan et al. (2016), all idle-time-order-based routing policies have the same steady-state probabilities as random routing, and satisfy

$$P(\boldsymbol{a}) = P'_N \prod_{\ell=1}^{|\mathcal{I}(\boldsymbol{a})|} \frac{\mu_\ell}{\lambda}, \quad \text{for all } \boldsymbol{a} = (a_1, a_2, \ldots, a_{|\mathcal{I}(\boldsymbol{a})|}) \text{ with } |\mathcal{I}(\boldsymbol{a})| > 0,$$

where $\mathcal{I}(\boldsymbol{a})$ denote the set of idle servers in state $\boldsymbol{a}$. Then, the steady-state probability that server i is idle (i.e., idleness fraction), denoted by I_i, can be written as

$$I_i = \sum_{\boldsymbol{a}:\{i\}\in\mathcal{I}(\boldsymbol{a})} P(\boldsymbol{a}) = \sum_{\boldsymbol{a}:\{i\}\in\mathcal{I}(\boldsymbol{a})} P'_N \prod_{\ell=1}^{|\mathcal{I}(\boldsymbol{a})|} \frac{\mu_\ell}{\lambda}, \quad \text{for all } i \in \{1,2,\ldots,N\}. \tag{EC.7}$$

From (EC.6), it is straightforward to observe that

- P'_N is strictly increasing in μ_j for all $j \in \{1,2,\ldots,N\}$;
- P'_N is strictly decreasing in k.
- P'_N is strictly decreasing in λ;

(a): As μ_j $(j \in \{1,2,\ldots,N\})$ increases, P'_N increases, and from (EC.7), I_i is also increasing in all μ_j. Thus, I_i is increasing in μ_j $(j \in \{1,2,\ldots,N\})$.

(b): As k increases, P'_N decreases, and from (EC.7), k influences I_i only through P'_N. Thus, it is clear that I_i is decreasing in k.

(c): As λ increases, P'_N decreases, and from (EC.7), I_i is also decreasing in λ. Thus, I_i is decreasing in λ.

■

EC.2.2. Proof of Lemma 1

We first observe that the steady-state probabilities when at least one server is idle have an identical form to those for the infinite-buffer system given in (21) in Gopalakrishnan et al. (2016). Then, it is sufficient to verify the detailed balance equations of the corresponding Markov chain. Identical to the proof of Theorem 9 in Gopalakrishnan et al. (2016), one can show that, for any state $\boldsymbol{a} = (a_1, a_2, \ldots, a_{|\mathcal{I}|})$, where $\mathcal{I}$ is the set of idle servers,

(i) Rate into state $\boldsymbol{a}$ due to an arrival = Rate out of state $\boldsymbol{a}$ due to a departure; and
(ii) Rate into state $\boldsymbol{a}$ due to a departure = Rate out of state $\boldsymbol{a}$ due to an arrival.

■

EC.2.3. Proof of Lemma 2

The proof is closely related to the proof of Theorem 1 in Gopalakrishnan et al. (2016), which relies on Gumbel (1960), and utilizes the same techniques. The difference is that their results are for the infinite buffer system ($M/M/N$). For the finite-buffer system ($M/M/N/k$), the Markov chain is the same as that for the infinite-buffer system except that it is truncated at state k. As a result, the balance equations for all the states except that for state k remain the same, and the normalization constant is different.

We let $(a_1, a_2, \ldots, a_\ell)$ denote the state of the $M/M/N/k$ system when there are ℓ jobs in the system $(0 < \ell < N)$ and the busy servers are $\{a_1, a_2, \ldots, a_\ell\}$, where $1 \le a_1 < a_2 < \cdots < a_\ell \le N$. Let $P(a_1, a_2, \ldots, a_\ell)$ denote the steady-state probability of the $M/M/N/k$ system being in state $(a_1, a_2, \ldots, a_\ell)$. Also let P_ℓ denote the steady-state probability of ℓ jobs in the $M/M/N/k$ system. We first note that Equations (EC.1) and (EC.6) in Gopalakrishnan et al. (2016) continue to hold for the finite-buffer $M/M/N/k$ system, and so

- When there are $\ell \in \{1,2,\ldots,N-1\}$ jobs in the system, and the tagged server (server 1) is idle,
$$P(a_1,a_2,\ldots,a_\ell)=\frac{(N-\ell)!P_0\rho^\ell}{N!},\quad 2\le a_1\le\ldots\le a_\ell\le N. \tag{EC.8}$$
- When there are $\ell \in \{1,2,\ldots,N\}$ jobs in the system, and the tagged server (server 1) is busy,
$$P(1,a_2,\ldots,a_\ell)=\frac{(N-\ell)!P_0\rho_1\rho^{\ell-1}}{N!},\quad 2\le a_2\le\ldots\le a_\ell\le N. \tag{EC.9}$$

Combining (EC.8) and (EC.9), we can write down the steady-state probability of $\ell \in \{1,2,\ldots,N-1\}$ jobs in the system as

$$\begin{aligned}P_\ell&=\sum_{2\le a_1\le\ldots\le a_\ell\le N}P(a_1,a_2,\ldots,a_\ell)+\sum_{2\le a_2\le\ldots\le a_\ell\le N}P(1,a_2,\ldots,a_\ell)\\&=P_0\left(\sum_{2\le a_1\le\ldots\le a_\ell\le N}\frac{(N-\ell)!\rho^\ell}{N!}+\sum_{2\le a_2\le\ldots\le a_\ell\le N}\frac{(N-\ell)!\rho_1\rho^{\ell-1}}{N!}\right).\end{aligned} \tag{EC.10}$$

Letting $\ell=N$ in (EC.9) implies

$$P_N=P(1,2,\ldots,N)=\frac{P_0\rho_1\rho^{N-1}}{N!}=P_0\frac{\rho_1}{\rho}\frac{\rho^N}{N!}.$$

When there are $\ell\ge N$ jobs in the system, the system behaves as a single-server queue with service rate $(N-1)\mu+\mu_1$, and so, from the balance equations,

$$P_\ell=P_N\left(\frac{\lambda}{(N-1)\mu+\mu_1}\right)^{\ell-N}=P_0\left(\frac{\rho_1}{\rho}\frac{\rho^N}{N!}\left(\frac{\rho}{N-\left(1-\frac{\rho}{\rho_1}\right)}\right)^{\ell-N}\right),\quad N\le\ell\le k. \tag{EC.11}$$

Thus, we can obtain the expression for P_0 using the normalization constraint, which yields

$$P_0=\left(1+\sum_{\ell=1}^{k}\frac{P_\ell}{P_0}\right)^{-1}.$$

Note that the ratio $\frac{P_\ell}{P_0}$ is independent of k, in both (EC.10) and (EC.11). Thus, letting $k\to\infty$ in the above display implies that the steady-state probability of an empty infinite-buffer $M/M/N$ system, denoted by $P_0^{M/M/N}$, satisfies

$$P_0^{M/M/N}=\left(1+\sum_{\ell=1}^{\infty}\frac{P_\ell}{P_0}\right)^{-1}=\left(1+\sum_{\ell=1}^{k}\frac{P_\ell}{P_0}+\sum_{\ell=k+1}^{\infty}\frac{P_\ell}{P_0}\right)^{-1}=\left((P_0)^{-1}+\sum_{\ell=k+1}^{\infty}\frac{P_\ell}{P_0}\right)^{-1}.$$

Hence, we can express P_0 in terms of $P_0^{M/M/N}$ as

$$P_0=\left(\left(P_0^{M/M/N}\right)^{-1}-\sum_{\ell=k+1}^{\infty}\frac{P_\ell}{P_0}\right)^{-1}.$$

Using the expression for $P_0^{M/M/N}$, as shown in the display following (EC.8) in Gopalakrishnan et al. (2016), and substituting that equivalence and (EC.11) into the above expression:

$$\begin{aligned}P_0&=\left(\left(1-\frac{\rho}{N}\left(1-\frac{\rho_1}{\rho}\right)\right)\sum_{\ell=0}^{N-1}\frac{\rho^\ell}{\ell!}+\frac{\rho^N}{N!}\left(1+\frac{\rho_1}{(N-\rho)-\left(1-\frac{\rho}{\rho_1}\right)}\right)-\frac{\rho_1}{\rho}\frac{\rho^N}{N!}\sum_{\ell=k+1}^{\infty}\left(\frac{\rho}{N-\left(1-\frac{\rho}{\rho_1}\right)}\right)^{\ell-N}\right)^{-1}\\&=\left(\left(1-\frac{\rho}{N}\left(1-\frac{\rho_1}{\rho}\right)\right)\sum_{\ell=0}^{N-1}\frac{\rho^\ell}{\ell!}+\frac{\rho^N}{N!}\left(1+\frac{\rho_1}{(N-\rho)-\left(1-\frac{\rho}{\rho_1}\right)}\left(1-\left(\frac{\rho}{N-\left(1-\frac{\rho}{\rho_1}\right)}\right)^{k-N}\right)\right)\right)^{-1}.\end{aligned}$$

Note that the term in red appearing in P_0 vanishes when $k \to \infty$, recovering the expression for $P_0^{M/M/N}$ in the infinite-buffer system, as given in (4) in Gopalakrishnan et al. (2016). (This is a sanity check.)

Next, we use the same manoeuver used to obtain (EC.9) in Gopalakrishnan et al. (2016) to express P_0 in terms of $ErlC(N,\rho)$; i.e., we add and subtract the term $\frac{N}{N-\rho}\frac{\rho^N}{N!}$, and follow similar algebraic simplifications to obtain

$$\begin{aligned}P_0 = &\left(\sum_{\ell=0}^{N-1}\frac{\rho^\ell}{\ell!} + \frac{N}{N-\rho}\frac{\rho^N}{N!}\right)^{-1} \\ &\cdot\left(1-\frac{\rho}{N}\left(1-\frac{\rho_1}{\rho}+\left(1-\frac{\rho_1}{\rho}+\frac{N\rho_1}{\rho}\left(1-\frac{\rho}{N}\right)\left(\frac{\rho}{N-\left(1-\frac{\rho}{\rho_1}\right)}\right)^{k-N}\right)\frac{ErlC(N,\rho)}{(N-\rho)-\left(1-\frac{\rho}{\rho_1}\right)}\right)\right)^{-1}.\end{aligned}$$

Identical to Equation (EC.5) in Gopalakrishnan et al. (2016), the formula for the tagged server's idle time is

$$I(\mu_1,\mu;\lambda,k,N) = P_0 + \sum_{\ell=1}^{N-1}\sum_{2\le a_1\le\cdots\le a_\ell\le N} P(a_1,a_2,\ldots,a_\ell).$$

Following similar final steps as those in Gopalakrishnan et al. (2016), we find:

$$\begin{aligned}&I(\mu_1,\mu;\lambda,k,N)\\ &=\left(1-\frac{\rho}{N}\right)\left(\sum_{\ell=0}^{N-1}\frac{\rho^\ell}{\ell!}+\frac{N}{N-\rho}\frac{\rho^N}{N!}\right)P_0\\ &=\left(1-\frac{\rho}{N}\right)\left(1-\frac{\rho}{N}\left(1-\frac{\mu}{\mu_1}+\left(1-\frac{\mu}{\mu_1}+\frac{N\mu}{\mu_1}\left(1-\frac{\rho}{N}\right)\left(\frac{\rho}{N-\left(1-\frac{\mu_1}{\mu}\right)}\right)^{k-N}\right)\frac{ErlC(N,\rho)}{(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right)}\right)\right)^{-1}\\ &=\left(\frac{N}{N-\rho}-\rho\left(\left(1-\frac{\mu}{\mu_1}\right)\left(1+\frac{ErlC(N,\rho)}{(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right)}\right)\frac{1}{N-\rho}+\frac{\mu}{\mu_1}\left(\frac{\rho}{N-\left(1-\frac{\mu_1}{\mu}\right)}\right)^{k-N}\frac{ErlC(N,\rho)}{(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right)}\right)\right)^{-1}\\ &=\left(1+\rho\frac{\mu}{\mu_1}\left(\frac{1-ErlC(N,\rho)}{N-\rho}+\left(1-\left(\frac{\rho}{N-\left(1-\frac{\mu_1}{\mu}\right)}\right)^{k-N}\right)\frac{ErlC(N,\rho)}{(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right)}\right)\right)^{-1},\end{aligned}$$

which establishes (1).

Finally, let $x = \frac{\rho}{N-\left(1-\frac{\mu_1}{\mu}\right)}$. Note that $(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right) = \left[N-\left(1-\frac{\mu_1}{\mu}\right)\right]-\rho = \left[N-\left(1-\frac{\mu_1}{\mu}\right)\right]-x\left[N-\left(1-\frac{\mu_1}{\mu}\right)\right] = (1-x)\left[N-\left(1-\frac{\mu_1}{\mu}\right)\right]$. Then,

$$\begin{aligned}&\left(1-\left(\frac{\rho}{N-\left(1-\frac{\mu_1}{\mu}\right)}\right)^{k-N}\right)\frac{ErlC(N,\rho)}{(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right)} = \left(1-x^{k-N}\right)\frac{ErlC(N,\rho)}{(1-x)\left[N-\left(1-\frac{\mu_1}{\mu}\right)\right]}\\ =&(1-x)\left(\sum_{i=0}^{k-N-1}x^i\right)\frac{ErlC(N,\rho)}{(1-x)\left[N-\left(1-\frac{\mu_1}{\mu}\right)\right]} = \left(\sum_{i=0}^{k-N-1}x^i\right)\frac{ErlC(N,\rho)}{N-\left(1-\frac{\mu_1}{\mu}\right)}.\end{aligned}$$

When $\lambda \ge (N-1)\mu+\mu_1$, we have $\rho \ge N-\left(1-\frac{\mu_1}{\mu}\right)$, i.e., $x\ge 1$. Thus, $\sum_{i=0}^{k-N-1}x^i \ge k-N\to\infty$, as $k\to\infty$. Hence, from (1) and the above display, $I(\mu_1,\mu;\lambda,\infty,N) := \lim_{k\to\infty} I(\mu_1,\mu;\lambda,k,N) = 0$. ∎

EC.3. Proofs from Section 2

EC.3.1. Proof of Proposition 1

From (1), $\lim_{\mu_1\downarrow 0} I(\mu_1,\mu)=0$ for all $\mu>0$. Together with $c(0)=0$, this implies that $\lim_{\mu_1\downarrow 0} U(\mu_1,\mu)=0$ for all $\mu>0$. Thus, any equilibrium $\mu^\star>0$ satisfies $U(\mu^\star,\mu^\star)\geq 0$. ■

EC.3.2. Proof of Lemma 3

We first note that the cost function c is strictly increasing with $c(0)=0$, and is therefore invertible in $[0,\infty)$. From Proposition 1, $U(\mu^\star,\mu^\star)=p\mu^\star+(v-p\mu^\star)I(\mu^\star,\mu^\star)-c(\mu^\star)\geq 0$. Using this inequality and the trivial bound $I(\mu^\star,\mu^\star)<1$, we show the next claim, whose proof will appear at the end.

Claim EC.1. *If $c(\mu^\star)>p\mu^\star$, then $\mu^\star<\frac{v}{p}$ and $\mu^\star<(c')^{-1}(v)$.*

Case (I): If $c'(0)\geq p$, because c is strictly convex, we have $c(\mu^\star)>c'(0)(\mu^\star-0)\geq p\mu^\star$. Then, it follows from Claim EC.1 that $\mu^\star\leq\min\left\{\frac{v}{p},(c')^{-1}(v)\right\}$.

Case (II): If $c'(0)<p$, because c is strictly increasing and strictly convex with $c(0)=0$, and $p\mu$ is linear with zero intercept, there exists a unique $\mu_0>0$ such that

$$c(\mu_0)=p\mu_0 \quad\text{and}\quad c(\mu)<p\mu \;\Leftrightarrow\; \mu<\mu_0. \tag{EC.12}$$

Next, we discuss two cases.

Case (II-1): If $c(\frac{v}{p})<v$, which can be equivalently written as $c(\frac{v}{p})<p\cdot\frac{v}{p}$, then it follows from (EC.12) that $\frac{v}{p}<\mu_0$. We show that $\mu^\star\leq\mu_0$ by contradiction. Suppose $\mu^\star>\mu_0$, then $c(\mu^\star)>p\mu^\star$ using (EC.12), which implies $\mu^\star<\frac{v}{p}$ from Claim EC.1. This contradicts $\mu^\star>\mu_0>\frac{v}{p}$.

Case (II-2): If $c(\frac{v}{p})\geq v$, which can be equivalently written as

$$\frac{v}{p}\geq c^{-1}(v) \;\Leftrightarrow\; v\geq p\cdot c^{-1}(v) \;\Leftrightarrow\; c\left(c^{-1}(v)\right)\geq p\cdot c^{-1}(v).$$

Then, (EC.12) implies that $c^{-1}(v)\geq\mu_0$. Thus,

$$\mu_0\leq c^{-1}(v)\leq\frac{v}{p}. \tag{EC.13}$$

- If $\mu^\star\leq\mu_0$, then it follows from (EC.13) that $\mu^\star\leq\mu_0\leq c^{-1}(v)$.
- If $\mu^\star>\mu_0$, then (EC.12) implies that $c(\mu^\star)>p\mu^\star$. Then, it follows from Claim EC.1 that $\mu^\star\leq c^{-1}(v)$, recalling from (EC.13) that $c^{-1}(v)\leq\frac{v}{p}$.

Combining all the cases above,

$$\mu^\star\leq\mu^\star_{\max}(p,v)=\begin{cases}\mu_0, & c'(0)<p \text{ and } c(\frac{v}{p})<v,\\ c^{-1}(v), & c'(0)<p \text{ and } c(\frac{v}{p})\geq v,\\ \min\left\{\frac{v}{p},c^{-1}(v)\right\}, & c'(0)\geq p.\end{cases}$$

■

Proof of Claim EC.1: Since $c(\mu^\star) > p\mu^\star$, $U(\mu^\star,\mu^\star) = p\mu^\star + (v - p\mu^\star)I(\mu^\star,\mu^\star) - c(\mu^\star) \geq 0$ holds only if $(v - p\mu^\star)I(\mu^\star,\mu^\star) > 0$, implying that $v - p\mu^\star > 0$ (since $I(\mu^\star,\mu^\star) > 0$), i.e., $\mu^\star < \frac{v}{p}$.

In addition, since $I(\mu^\star,\mu^\star) < 1$, $U(\mu^\star,\mu^\star) = p\mu^\star + (v - p\mu^\star)I(\mu^\star,\mu^\star) - c(\mu^\star) < v - c(\mu^\star)$ (recalling that $v - p\mu^\star > 0$). Thus, $U(\mu^\star,\mu^\star) = p\mu^\star + (v - p\mu^\star)I(\mu^\star,\mu^\star) - c(\mu^\star) \geq 0$ holds only if its strict upper bound is strictly positive; that is, $v - c(\mu^\star) > 0$, i.e., $\mu^\star < c^{-1}(v)$.

■

EC.4. Proofs from Section 3

EC.4.1. Preliminaries

We start by providing closed-form expressions for the first two partial derivatives of the idle time with respect to μ_1.

Lemma EC.5 **(Expressions for Derivatives of Idle Time)**. *In an M/M/N/k system where server 1 operates at rate $\mu_1 > 0$ and the other $N-1$ servers operate at rate $\mu > 0$, the first two partial derivatives of $I(\mu_1,\mu;\lambda,k,N)$, from (1), with respect to μ_1 are given by*

$$\frac{\partial I(\mu_1,\mu)}{\partial \mu_1} = \frac{1}{\mu_1}I(1-I) + \frac{I^2}{\mu_1}\left(\frac{C}{d_2}\left[\left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{\rho}{d_2} - (k-N)\left(\frac{\rho}{d_1}\right)^{k-N}\right] + \frac{k-N}{d_1}\left(\frac{\rho}{d_1}\right)^{k-N} C\right), \tag{EC.14}$$

$$\frac{\partial^2 I(\mu_1,\mu)}{\partial \mu_1^2} = \frac{I^2}{\mu_1^2}\left\{-2(1-I) + \frac{2\rho C}{d_2^2}\left[1-\left(\frac{\rho}{d_1}\right)^{k-N}\right]\left[(1-2I) - \frac{\frac{\mu_1}{\mu} + \rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I\right] - \frac{\rho C}{d_2}\frac{k-N}{d_1}\left(\frac{\rho}{d_1}\right)^{k-N}\right.$$
$$\left.\left[2(1-2I) + \left(\frac{2}{d_2} - \frac{1}{d_1}\right)\frac{\mu_1}{\mu} - 4\frac{\frac{\mu_1}{\mu} + \rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I\right] - \rho C\left(\frac{k-N}{d_1}\right)^2\left(\frac{\rho}{d_1}\right)^{k-N}\left[\frac{1}{d_2}\frac{\mu_1}{\mu} - 2\frac{\frac{\mu_1}{\mu} + \rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I - \frac{2\rho C}{d_2^2}I\right]\right\}, \tag{EC.15}$$

where

$$C := ErlC(N,\rho),\ d_1 := N - \left(1 - \frac{\mu_1}{\mu}\right) \text{ and } d_2 := d_1 - \rho.$$

Corollary EC.1 **($\mu_1 = \mu$ in Lemmas 2 and EC.5)**.

$$I(\mu,\mu) = \left(1 + \rho\left(\frac{1 - ErlC(N,\rho)}{N-\rho}\right) + ErlC(N,\rho)\sum_{i=1}^{k-N}\left(\frac{\rho}{N}\right)^i\right)^{-1} \tag{EC.16}$$

$$= \left(\left(1 + \sum_{i=0}^{N-1}\frac{N!}{i!}\left(\frac{\mu}{\lambda}\right)^{N-i} + \sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right) ErlC\left(N,\frac{\lambda}{\mu}\right)\right)^{-1} \tag{EC.17}$$

$$= \left(1 - \frac{\rho}{N}\right)\left(1 - ErlC(N,\rho)\left(\frac{\rho}{N}\right)^{k-N+1}\right)^{-1}, \tag{EC.18}$$

$$\left.\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} = \frac{1}{\mu}I(\mu,\mu)(1 - I(\mu,\mu)) + I(\mu,\mu)^2\frac{ErlC(N,\rho)}{N\mu}\sum_{i=1}^{k-N} i\left(\frac{\rho}{N}\right)^i \tag{EC.19}$$

$$= \frac{\rho}{N\mu}\frac{1 - \frac{\rho}{N} + ErlC(N,\rho)\left(\frac{1}{N} - \left(\frac{\rho}{N}\right)^{k-N}\left(\frac{1}{N} + \frac{k}{N}\left(1 - \frac{\rho}{N}\right)\right)\right)}{\left(1 - \left(\frac{\rho}{N}\right)^{k-N+1} ErlC(N,\rho)\right)^2}. \tag{EC.20}$$

Next, we present upper bounds on the derivative of idle time, which can help simplify proof.

Lemma EC.6. *The following hold for all $\lambda > 0$ and $k \geq N \geq 2$:*

(a)

$$I(\mu,\mu)^2\, ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i} < 2\sqrt{N}, \quad \forall \mu > 0.$$

(b)

$$I(\mu,\mu)^2\, ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1} < 2\sqrt{N}, \quad \forall \mu > 0.$$

COROLLARY EC.2. *The following hold for all $\lambda > 0$ and $k \ge N \ge 2$:*

(a)

$$\mu\, \frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} < I(\mu,\mu)\left(1 - I(\mu,\mu)\right) + \frac{2}{\sqrt{N}}, \quad \forall \mu > 0.$$

(b)

$$\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} < \frac{I(\mu,\mu)}{\mu} + \frac{2\sqrt{N}}{\lambda}, \quad \forall \mu > 0.$$

The next result expresses the derivative of idle time in terms of the partial derivative of idle time, which is useful for some proof.

LEMMA EC.7.

$$\frac{dI(\mu,\mu)}{d\mu} = N\cdot \frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} - \frac{I(\mu,\mu)}{\mu}\frac{\lambda}{N\mu}\left(N - \frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{1-\frac{\lambda}{N\mu}}\right).$$

Finally, we provide a useful monotonicity property of the idle time.

LEMMA EC.8. *The difference between the idle time in a finite-buffer $M/M/N/k$ system and that in an infinite-buffer $M/M/N$ system, $I(\mu,\mu;\lambda,k,N) - \left(1-\frac{\lambda}{N\mu}\right)$, is strictly decreasing in μ for $\mu \in (0,\infty)$, for all $\lambda > 0$ and $k \ge N \ge 2$, and satisfies $I(\mu,\mu;\lambda,k,N) - \left(1-\frac{\lambda}{N\mu}\right) \ge 0$, with equality holding only when $k = \infty$.*

EC.4.1.1. Proof of Lemma EC.5

First-order derivative:

This is useful to first observe that taking the reciprocal of (1) shows

$$\left(\frac{1}{I}-1\right)\frac{\mu_1}{\mu}\frac{1}{\rho} = \frac{1-C}{N-\rho} + \frac{C}{d_2} - \frac{C}{d_2}\left(\frac{\rho}{d_1}\right)^{k-N} \tag{EC.21}$$

$$\Leftrightarrow \quad \frac{C(N-\rho)}{d_2}\left(\frac{\rho}{d_1}\right)^{k-N} = 1 + \frac{N-\rho}{\rho}\left(1-\frac{1}{I}\right)\frac{\mu_1}{\mu} + \left(1-\frac{\mu_1}{\mu}\right)\frac{C}{d_2}. \tag{EC.22}$$

Differentiating $I(\mu_1,\mu)$ using (1)with respect to μ_1 yields

$$\begin{aligned}
\frac{\partial I}{\partial \mu_1} =& -I^2\cdot\rho\mu\left\{-\frac{1}{\mu_1^2}\left[\frac{1-C}{N-\rho} + \left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{C}{d_2}\right]\right. \\
&\left. + \frac{1}{\mu_1}\left[\left((k-N)\left(\frac{\rho}{d_1}\right)^{k-N-1}\frac{\rho}{d_1^2\mu}\right)\frac{C}{d_2} - \left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{C}{d_2^2\mu}\right]\right\} \\
=& \frac{I^2}{\mu_1^2}\frac{\lambda}{N-\rho}\left\{1 + C\frac{(N-\rho)-\left(1-\frac{\mu_1}{\mu}\right)^2}{d_2^2} - \frac{C(N-\rho)}{d_2}\left(\frac{\rho}{d_1}\right)^{k-N}\left(1+\frac{\mu_1}{\mu}\frac{1}{d_2}+\frac{\mu_1}{\mu}\frac{k-N}{d_1}\right)\right\}.
\end{aligned}$$

Substitution for $\frac{C(N-\rho)}{d_2}\left(\frac{\rho}{d_1}\right)^{k-N}$ using (EC.22) and additional algebra shows

$$\frac{\partial I}{\partial \mu_1} = \frac{I^2}{\mu_1}\left\{\left(1+\frac{\mu_1}{\mu}\frac{1}{d_2}\right)\left(\frac{1}{I}-1\right) - \frac{\rho\left(\frac{1-C}{N-\rho}\right)}{d_2} + \frac{k-N}{d_1}\left(\frac{\mu_1}{\mu}\left(\frac{1}{I}-1\right) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)\right)\right\}.$$

Thus,

$$\begin{aligned}\frac{\mu_1}{I^2}\frac{\partial I}{\partial \mu_1} &= \left(1+\frac{\mu_1}{\mu}\frac{1}{d_2}\right)\left(\frac{1}{I}-1\right) - \frac{\rho\left(\frac{1-C}{N-\rho}\right)}{d_2} + \frac{k-N}{d_1}\left(\frac{\mu_1}{\mu}\left(\frac{1}{I}-1\right) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)\right) \quad \text{(EC.23)}\\ &= \left(\frac{1}{I}-1\right) + \left(\frac{1}{d_2}+\frac{k-N}{d_1}\right)\frac{\mu_1}{\mu}\left(\frac{1}{I}-1\right) - \frac{\rho\left(\frac{1-C}{N-\rho}\right)}{d_2} - \frac{k-N}{d_1}\left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right).\end{aligned}$$

Substitution for the second $\left(\frac{1}{I}-1\right)$ using (EC.21) yields

$$\begin{aligned}\frac{\mu_1}{I^2}\frac{\partial I}{\partial \mu_1} &= \left(\frac{1}{I}-1\right) + \left(\frac{1}{d_2}+\frac{k-N}{d_1}\right)\frac{\mu_1}{\mu}\cdot\rho\frac{\mu}{\mu_1}\left(\frac{1-C}{N-\rho} + \left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{C}{d_2}\right) - \frac{\rho\left(\frac{1-C}{N-\rho}\right)}{d_2}\\ &\qquad - \frac{k-N}{d_1}\left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)\\ &= \left(\frac{1}{I}-1\right) + \frac{C}{d_2}\left[\left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{\rho}{d_2} - (k-N)\left(\frac{\rho}{d_1}\right)^{k-N}\left(\frac{\rho}{d_1}-1+1\right)\right]\\ &= \left(\frac{1}{I}-1\right) + \frac{C}{d_2}\left[\left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{\rho}{d_2} - (k-N)\left(\frac{\rho}{d_1}\right)^{k-N}\right] - \frac{C}{d_2}\frac{k-N}{d_1}d_1\left(\frac{\rho}{d_1}\right)^{k-N}\left(\frac{\rho}{d_1}-1\right).\end{aligned}$$

Recall that $d_1-\rho=d_2$, which implies that $\frac{d_1}{d_2}\left(\frac{\rho}{d_1}-1\right)=-1$, and so

$$\frac{\mu_1}{I^2}\frac{\partial I}{\partial \mu_1} = \left(\frac{1}{I}-1\right) + \frac{C}{d_2}\left[\left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)\frac{\rho}{d_2} - (k-N)\left(\frac{\rho}{d_1}\right)^{k-N}\right] + \frac{k-N}{d_1}\left(\frac{\rho}{d_1}\right)^{k-N}C,$$

which establishes (EC.14).

Second-order derivative:

From (EC.23),

$$\frac{\mu_1}{I}\frac{\partial I}{\partial \mu_1} = \left(1+\frac{\mu_1}{\mu}\frac{1}{d_2}\right)(1-I) - \frac{\rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I + \frac{k-N}{d_1}\left(\frac{\mu_1}{\mu}(1-I) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I\right).$$

Set $LHS(\mu_1,\mu)$ and $RHS(\mu_1,\mu)$ equal to the left-hand and right-hand sides of the above equation, respectively. Then,

$$\frac{\partial LHS(\mu_1,\mu)}{\partial \mu_1} = \frac{\mu_1}{I}\frac{\partial^2 I}{\partial \mu_1^2} - \frac{1}{\mu_1}\left[\left(\frac{\mu_1}{I}\frac{\partial I}{\partial \mu_1}\right)^2 - \left(\frac{\mu_1}{\mu}\frac{\partial I}{\partial \mu_1}\right)\right],$$

and

$$\begin{aligned}\frac{\partial RHS(\mu_1,\mu)}{\partial \mu_1} &= \frac{N-\rho-1}{d_2^2}\frac{1}{\mu} - \frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2^2}\frac{I}{\mu} - \left(2-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2}\right)\frac{\partial I}{\partial \mu_1}\\ &\quad + (k-N)\left[\frac{N-1}{d_1^2}\frac{1}{\mu} - \left(\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1^2} - \frac{C}{d_2^2}\right)\frac{I}{\mu} - \left(1-\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1}+\frac{C}{d_2}\right)\frac{\partial I}{\partial \mu_1}\right]\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{\mu_1}\left[\frac{N-\rho-1}{d_2^2}\frac{\mu_1}{\mu}-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2^2}\frac{\mu_1}{\mu}I-\left(2-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2}\right)I\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right.\\
&\left.+(k-N)\left(\frac{N-1}{d_1^2}\frac{\mu_1}{\mu}-\left(\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1^2}-\frac{C}{d_2^2}\right)\frac{\mu_1}{\mu}I-\left(1-\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1}+\frac{C}{d_2}\right)I\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)\right]\\
=&\frac{1}{\mu_1}\left[\left\{\frac{N-\rho-1}{d_2}-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2}I+(k-N)\left(\frac{N-1}{d_1}-\left(\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1}-\frac{C}{d_2}\right)I\right)\right.\right.\\
&\left.-(k-N)\left(\frac{\rho(N-1)}{d_1^2}-\frac{\rho N\left(1-\frac{1-C}{N-\rho}\right)}{d_1^2}I\right)\right\}\cdot\left(\frac{1}{d_2}\frac{\mu_1}{\mu}\right)-\left\{\left(2-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2}\right)I\right.\\
&\left.\left.+(k-N)\left(1-\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1}+\frac{C}{d_2}\right)I\right\}\cdot\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)\right].
\end{aligned}
\tag{EC.24}
$$

Additionally, (EC.23) can be alternatively written as

$$
\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}=2-\frac{N-\rho-1}{d_2}-\left(2-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2}\right)I+(k-N)\left(1-\frac{N-1}{d_1}-\left(1-\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1}+\frac{C}{d_2}\right)I\right),
$$

which implies

$$
\begin{aligned}
\frac{N-\rho-1}{d_2}-\frac{(N-1)-\rho\left(1+\frac{1-C}{N-\rho}\right)}{d_2}I+(k-N)\left(\frac{N-1}{d_1}-\left(\frac{N\left(1-\frac{1-C}{N-\rho}\right)}{d_1}-\frac{C}{d_2}\right)I\right)&\\
=(k-N+2)(1-I)-\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)&.
\end{aligned}
$$

Substitution into (EC.24) yields

$$
\begin{aligned}
\frac{\partial RHS(\mu_1,\mu)}{\partial\mu_1}=&\frac{1}{\mu_1}\left[\left\{(k-N+2)(1-I)-\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)-(k-N)\left(\frac{\rho(N-1)}{d_1^2}-\frac{\rho N\left(1-\frac{1-C}{N-\rho}\right)}{d_1^2}I\right)\right\}\cdot\left(\frac{1}{d_2}\frac{\mu_1}{\mu}\right)\right.\\
&\left.-\left\{2-\frac{N-\rho-1}{d_2}+(k-N)\left(1-\frac{N-1}{d_1}\right)-\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right\}\cdot\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)\right]\\
=&\frac{1}{\mu_1}\left[\frac{2(1-I)}{d_2}\frac{\mu_1}{\mu}+(k-N)\left(1-I-\frac{\rho}{d_1^2}\left(N-1-N\left(1-\frac{1-C}{N-\rho}\right)I\right)\right)\frac{1}{d_2}\frac{\mu_1}{\mu}\right.\\
&\left.-\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)\left(\left(\frac{2}{d_2}+\frac{k-N}{d_1}\right)\frac{\mu_1}{\mu}+2\right)+\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)+\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)^2\right]\\
=&\frac{1}{\mu_1}\left[\frac{2(1-I)}{d_2}\frac{\mu_1}{\mu}+\frac{k-N}{d_1}\frac{\mu_1}{\mu}\left((1-I)+\left(\frac{1}{d_2}-\frac{1}{d_1}\right)\left(\frac{\mu_1}{\mu}(1-I)-\left(\rho\left(\frac{1-C}{N-\rho}\right)-C\right)I\right)\right)\right.\\
&\left.+\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)\left(2\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)-2\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)-\frac{k-N}{d_1}\frac{\mu_1}{\mu}\right)+\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)-\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)^2\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial^2 I}{\partial\mu_1^2}=&\frac{I}{\mu_1^2}\left[\frac{2(1-I)}{d_2}\frac{\mu_1}{\mu}+\frac{k-N}{d_1}\frac{\mu_1}{\mu}\left((1-I)+\left(\frac{1}{d_2}-\frac{1}{d_1}\right)\left(\frac{\mu_1}{\mu}(1-I)-\left(\rho\left(\frac{1-C}{N-\rho}\right)-C\right)I\right)\right)\right.\\
&\left.+\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)\left(2\left(\frac{\mu_1}{I}\frac{\partial I}{\partial\mu_1}\right)-2\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right)-\frac{k-N}{d_1}\frac{\mu_1}{\mu}\right)\right].
\end{aligned}
$$

Substituting for $\frac{\mu_1}{I}\frac{\partial I}{\partial \mu_1}$ from (EC.23), and after additional algebra, we obtain

$$\begin{aligned}\frac{\partial^2 I}{\partial \mu_1^2} = \frac{I}{\mu_1^2}\Bigg\{ &-2I(1-I) + \frac{2}{d_2}\left[(1-2I) - \frac{\frac{\mu_1}{\mu}+\rho\left(\frac{1-C}{N-\rho}\right)}{d_2} I\right]\left[\frac{\mu_1}{\mu}(1-I) - \rho\left(\frac{1-C}{N-\rho}\right)I\right] \\ &+ \frac{k-N}{d_1}\left[2(1-2I) + \left(\frac{2}{d_2}-\frac{1}{d_1}\right)\frac{\mu_1}{\mu} - 4\frac{\frac{\mu_1}{\mu}+\rho\left(\frac{1-C}{N-\rho}\right)}{d_2} I\right]\left[\frac{\mu_1}{\mu}(1-I) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I\right] \\ &+ \left(\frac{k-N}{d_1}\right)^2\left[\frac{\mu_1}{\mu} - 2\left(\frac{\mu_1}{\mu}+\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I\right]\left[\frac{\mu_1}{\mu}(1-I) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I\right]\Bigg\}. \end{aligned} \tag{EC.25}$$

We further simplify the two terms $\left[\frac{\mu_1}{\mu}(1-I) - \rho\left(\frac{1-C}{N-\rho}\right)I\right]$ and $\left[\frac{\mu_1}{\mu}(1-I) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I\right]$ as follows. Recall from (EC.22):

$$\frac{C(N-\rho)}{d_2}\left(\frac{\rho}{d_1}\right)^{k-N} = 1 + \frac{N-\rho}{\rho}\left(1-\frac{1}{I}\right)\frac{\mu_1}{\mu} + \left(1-\frac{\mu_1}{\mu}\right)\frac{C}{d_2},$$

which implies

$$\frac{\mu_1}{\mu}(1-I) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I = -\frac{\rho C}{d_2}\left(\frac{\rho}{d_1}\right)^{k-N} I, \tag{EC.26}$$

and furthermore,

$$\frac{\mu_1}{\mu}(1-I) - \rho\left(\frac{1-C}{N-\rho}\right)I = \frac{\rho C}{d_2}\left(1-\left(\frac{\rho}{d_1}\right)^{k-N}\right)I. \tag{EC.27}$$

Substituting for $\left[\frac{\mu_1}{\mu}(1-I) - \rho\left(\frac{1-C}{N-\rho}\right)I\right]$ and $\left[\frac{\mu_1}{\mu}(1-I) - \left(\rho\left(\frac{1-C}{N-\rho}\right)+\frac{\rho C}{d_2}\right)I\right]$ in (EC.25) using (EC.26) and (EC.27) respectively:

$$\begin{aligned}\frac{\partial^2 I}{\partial \mu_1^2} = \frac{I^2}{\mu_1^2}\Bigg\{ &-2(1-I) + \frac{2\rho C}{d_2^2}\left[1-\left(\frac{\rho}{d_1}\right)^{k-N}\right]\left[(1-2I) - \frac{\frac{\mu_1}{\mu}+\rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I\right] - \frac{\rho C}{d_2}\frac{k-N}{d_1}\left(\frac{\rho}{d_1}\right)^{k-N} \\ &\left[2(1-2I) + \left(\frac{2}{d_2}-\frac{1}{d_1}\right)\frac{\mu_1}{\mu} - 4\frac{\frac{\mu_1}{\mu}+\rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I\right] - \rho C\left(\frac{k-N}{d_1}\right)^2\left(\frac{\rho}{d_1}\right)^{k-N}\left[\frac{1}{d_2}\frac{\mu_1}{\mu} - 2\frac{\frac{\mu_1}{\mu}+\rho\left(\frac{1-C}{N-\rho}\right)}{d_2}I - \frac{2\rho C}{d_2^2}I\right]\Bigg\}, \end{aligned}$$

which establishes (EC.15).

∎

EC.4.1.2. Proof of Corollary EC.1

Idle time (EC.16): From Lemma 2, substituting $\mu_1 = \mu$ into (1) yields

$$\begin{aligned} I(\mu,\mu) &= \left(1+\rho\left(\frac{1-ErlC\,(N,\rho)}{N-\rho} + \left(1-\left(\frac{\rho}{N}\right)^{k-N}\right)\frac{ErlC\,(N,\rho)}{N-\rho}\right)\right)^{-1} \\ &= \left(1+\rho\left(\frac{1-ErlC\,(N,\rho)}{N-\rho}\right) + ErlC\,(N,\rho)\,\frac{\rho\left(1-\left(\frac{\rho}{N}\right)^{k-N}\right)}{N-\rho}\right)^{-1} \\ &= \left(1+\rho\left(\frac{1-ErlC\,(N,\rho)}{N-\rho}\right) + ErlC\,(N,\rho)\sum_{i=1}^{k-N}\left(\frac{\rho}{N}\right)^{i}\right)^{-1}. \end{aligned} \tag{EC.28}$$

Idle time (EC.17): Using the relationship between $ErlB$ and $ErlC$ (from Lemma EC.2 (a)), note that

$$
\begin{aligned}
1+\rho\left(\frac{1-ErlC(N,\rho)}{N-\rho}\right) &= 1+\rho\frac{1-\left(\frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}\right)}{N-\rho} = 1+\rho\frac{\frac{1}{ErlB(N,\rho)}-1}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}\\
=&1+\rho\frac{1-ErlB(N,\rho)}{N-(1-ErlB(N,\rho))\rho} = \frac{N}{N-(1-ErlB(N,\rho))\rho} = \frac{N}{\frac{N-\rho}{ErlB(N,\rho)}+\rho}\frac{1}{ErlB(N,\rho)} = \frac{ErlC(N,\rho)}{ErlB(N,\rho)}.
\end{aligned}
$$

Substitution into (EC.28) using the above display and by definition of $ErlB(N,\rho)$ from (EC.1):

$$
\begin{aligned}
I(\mu,\mu) &= \left(\left(\frac{1}{ErlB(N,\rho)}+\sum_{i=1}^{k-N}\left(\frac{\rho}{N}\right)^i\right)ErlC(N,\rho)\right)^{-1}\\
&= \left(\left(1+\sum_{i=0}^{N-1}\frac{N!}{i!}\left(\frac{\mu}{\lambda}\right)^{N-i}+\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)ErlC\left(N,\frac{\lambda}{\mu}\right)\right)^{-1}.
\end{aligned}
$$

Idle time (EC.18): From Lemma 2, substituting $\mu_1=\mu$ into (1) yields

$$
\begin{aligned}
I(\mu,\mu) &= \left(1+\rho\left(\frac{1-ErlC\,(N,\rho)}{N-\rho}+\left(1-\left(\frac{\rho}{N}\right)^{k-N}\right)\frac{ErlC\,(N,\rho)}{N-\rho}\right)\right)^{-1}\\
&= \left(1+\rho\left(\frac{1}{N-\rho}-\frac{ErlC\,(N,\rho)\left(\frac{\rho}{N}\right)^{k-N}}{N-\rho}\right)\right)^{-1}\\
&= \left(1-\frac{\rho}{N}\right)\left(1-ErlC(N,\rho)\left(\frac{\rho}{N}\right)^{k-N+1}\right)^{-1}.
\end{aligned}
$$

First-order derivative (EC.19): From Lemma EC.5, substituting $\mu_1=\mu$ into (EC.14) yields

$$
\begin{aligned}
\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu} =& \frac{I(\mu,\mu)^2}{\mu}\left(\frac{1}{I(\mu,\mu)}-1+\frac{ErlC(N,\rho)}{N-\rho}\left(\left(1-\left(\frac{\rho}{N}\right)^{k-N}\right)\frac{\rho}{N-\rho}-(k-N)\left(\frac{\rho}{N}\right)^{k-N}\right)\right.\\
&\left.+\frac{k-N}{N}\left(\frac{\rho}{N}\right)^{k-N}ErlC(N,\rho)\right)\\
=&\frac{I(\mu,\mu)}{\mu}\left(1-I(\mu,\mu)+I(\mu,\mu)\cdot ErlC(N,\rho)\left(\frac{\rho}{(N-\rho)^2}-\frac{\rho}{(N-\rho)^2}\left(\frac{\rho}{N}\right)^{k-N}-(k-N)\left(\frac{\rho}{N}\right)^{k-N}\frac{\rho}{N(N-\rho)}\right)\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{i=1}^{k-N}i\left(\frac{\rho}{N}\right)^i =& \frac{\rho}{N}\frac{(k-N)\left(\frac{\rho}{N}\right)^{k-N+1}-(k-N+1)\left(\frac{\rho}{N}\right)^{k-N}+1}{\left(\frac{\rho}{N}-1\right)^2}\\
=&N\left(\frac{\rho}{(N-\rho)^2}-\frac{\rho}{(N-\rho)^2}\left(\frac{\rho}{N}\right)^{k-N}-\frac{k-N}{N-\rho}\left(\frac{\rho}{N}\right)^{k-N+1}\right),
\end{aligned}
$$

and thus $\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}$ can be equivalently written as

$$
\begin{aligned}
\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu} =& \frac{I(\mu,\mu)}{\mu}\left(1-I(\mu,\mu)+I(\mu,\mu)\cdot ErlC(N,\rho)\cdot\frac{1}{N}\sum_{i=1}^{k-N}i\left(\frac{\rho}{N}\right)^i\right)\\
=&\frac{1}{\mu}I(\mu,\mu)\left(1-I(\mu,\mu)\right)+I(\mu,\mu)^2\frac{ErlC\,(N,\rho)}{N\mu}\sum_{i=1}^{k-N}i\left(\frac{\rho}{N}\right)^i.
\end{aligned}
$$

First-order derivative (EC.20): From (EC.14) in Lemma EC.5, letting $\mu_1=\mu$ and substituting for $1-I(\mu,\mu)$ using (EC.16) in Corollary EC.1:

$$
\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu} = \frac{I^2(\mu,\mu)}{\mu}\left[\rho\frac{1-ErlC(N,\rho)}{N-\rho}+\left(1-\left(\frac{\rho}{N}\right)^{k-N}\right)\frac{\rho ErlC(N,\rho)}{N-\rho}\right]
$$

$$
\begin{aligned}
&+\frac{I^2(\mu,\mu)}{\mu}\left[\frac{ErlC(N,\rho)}{N-\rho}\left(1-\left(\frac{\rho}{N}\right)^{k-N}\right)\frac{\rho}{N-\rho}-\frac{ErlC(N,\rho)}{N-\rho}(k-N)\left(\frac{\rho}{N}\right)^{k-N}\right.\\
&\left.+\frac{k-N}{N}\left(\frac{\rho}{N}\right)^{k-N}ErlC(N,\rho)\right]\\
=&\frac{I^2(\mu,\mu)}{\mu}\left[\frac{\rho\left(1-ErlC(N,\rho)\right)}{N-\rho}+\frac{\rho ErlC(N,\rho)}{N-\rho}-\frac{\rho ErlC(N,\rho)}{N-\rho}\left(\frac{\rho}{N}\right)^{k-N}+\frac{\rho ErlC(N,\rho)}{(N-\rho)^2}\right.\\
&\left.-\frac{\rho ErlC(N,\rho)}{(N-\rho)^2}\left(\frac{\rho}{N}\right)^{k-N}-\frac{ErlC(N,\rho)}{N-\rho}(k-N)\left(\frac{\rho}{N}\right)^{k-N}+\frac{k-N}{N}\left(\frac{\rho}{N}\right)^{k-N}ErlC(N,\rho)\right]\\
=&\frac{I^2(\mu,\mu)}{\mu}\left[\frac{\rho}{N-\rho}+\frac{\rho ErlC(N,\rho)}{(N-\rho)^2}-\frac{\rho ErlC(N,\rho)}{N-\rho}\left(\frac{\rho}{N}\right)^{k-N}\left(1+\frac{1}{N-\rho}+\frac{k-N}{N}\right)\right]\\
=&\frac{\rho}{(N-\rho)^2}\frac{I^2(\mu,\mu)}{\mu}\left[N-\rho+ErlC(N,\rho)\left(1-\left(\frac{\rho}{N}\right)^{k-N}\left(1+\frac{k}{N}(N-\rho)\right)\right)\right].
\end{aligned}
$$

Substituting for $I(\mu,\mu)$ using (EC.18) in the above display yields

$$
\begin{aligned}
\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}=&\frac{\rho}{(N-\rho)^2}\frac{1}{\mu}\frac{\left(1-\frac{\rho}{N}\right)^2}{\left(1-\left(\frac{\rho}{N}\right)^{k-N+1}ErlC(N,\rho)\right)^2}\left[N-\rho+ErlC(N,\rho)\left(1-\left(\frac{\rho}{N}\right)^{k-N}\left(1+\frac{k}{N}(N-\rho)\right)\right)\right]\\
=&\frac{\rho}{N\mu}\frac{1-\frac{\rho}{N}+ErlC(N,\rho)\left(\frac{1}{N}-\left(\frac{\rho}{N}\right)^{k-N}\left(\frac{1}{N}+\frac{k}{N}\left(1-\frac{\rho}{N}\right)\right)\right)}{\left(1-\left(\frac{\rho}{N}\right)^{k-N+1}ErlC(N,\rho)\right)^2}.
\end{aligned}
$$

■

EC.4.1.3. Proof of Lemma EC.6

(a): From (EC.18) in Corollary EC.1,

$$
\begin{aligned}
I(\mu,\mu)=&\frac{1-\frac{\lambda}{N\mu}}{1-ErlC\left(N,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N+1}}\\
\overset{(i)}{=}&\frac{1-\frac{\lambda}{N\mu}}{1-ErlC\left(N,\frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{N\mu}\right)-\left(1-\frac{\lambda}{N\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right]}\\
=&\frac{ErlB\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\lambda}{N\mu}}{1-\frac{\lambda}{N\mu}}+ErlB\left(N,\frac{\lambda}{\mu}\right)ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i}\\
\overset{(ii)}{=}&\frac{ErlB\left(N,\frac{\lambda}{\mu}\right)}{ErlC\left(N,\frac{\lambda}{\mu}\right)\left(1+ErlB\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)},
\end{aligned}
$$

where (i) follows from the finite summation formula, and (ii) follows from the relationship between $ErlB$ and $ErlC$ (from Lemma EC.2 (a)). Substitution for $I(\mu,\mu)$ using the above display yields

$$
\begin{aligned}
&I(\mu,\mu)^2ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^i\\
=&\left(\frac{ErlB\left(N,\frac{\lambda}{\mu}\right)}{ErlC\left(N,\frac{\lambda}{\mu}\right)\left(1+ErlB\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)}\right)^2ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^i\\
=&\frac{ErlB\left(N,\frac{\lambda}{\mu}\right)^2}{ErlC\left(N,\frac{\lambda}{\mu}\right)}\frac{\sum_{i=0}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^i}{\left(1-ErlB\left(N,\frac{\lambda}{\mu}\right)+ErlB\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)^2}
\end{aligned}
$$

$$\overset{(iii)}{\le} \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\right)^{2}} \tag{EC.29}$$

$$\overset{(iv)}{<} \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}, \tag{EC.30}$$

where (iii) follows by noting that $ErlB\left(N,\frac{\lambda}{\mu}\right) \le 1$ and (iv) follows because

$$\begin{aligned}
&\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\right)^{2}} = \frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\sum_{i=0}^{k-N}\sum_{j=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i+j}} = \frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\sum_{\ell=0}^{2(k-N)}\sum_{i=\max\{0,\ell-(k-N)\}}^{\min\{\ell,k-N\}}\left(\frac{\lambda}{N\mu}\right)^{\ell}}\\
=&\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\sum_{\ell=0}^{k-N}\sum_{i=0}^{\ell}\left(\frac{\lambda}{N\mu}\right)^{\ell}+\sum_{\ell=k-N+1}^{2(k-N)}\sum_{i=\ell-(k-N)}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{\ell}}\\
=&\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\sum_{\ell=0}^{k-N}(\ell+1)\left(\frac{\lambda}{N\mu}\right)^{\ell}+\sum_{\ell=k-N+1}^{2(k-N)}(2(k-N)-\ell)\left(\frac{\lambda}{N\mu}\right)^{\ell}} \qquad \text{(EC.31)}\\
<&1,
\end{aligned}$$

noting that $\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i} < \sum_{\ell=0}^{k-N}(\ell+1)\left(\frac{\lambda}{N\mu}\right)^{\ell}$.

Case (I): When $0<\mu\le\frac{\lambda}{N}$, i.e., $\frac{\lambda}{N\mu}\ge 1$, we have $ErlC\left(N,\frac{\lambda}{\mu}\right)\ge 1$ (from Lemma EC.2 (b)). Then, from (EC.29),

$$\begin{aligned}
I(\mu,\mu)^{2}ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i} \le& \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\right)^{2}}\\
\le&\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\right)^{2}}\\
\overset{(*)}{\le}&\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\frac{\lambda}{N}}\right)^{i}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\frac{\lambda}{N}}\right)^{i}\right)^{2}}\\
=&\frac{1}{2}\frac{k-N}{k-N+1}<\frac{1}{2}<2\sqrt{N}, \quad \forall k\ge N\ge 2,
\end{aligned}$$

where $(*)$ follows from the next claim, whose proof appears at the end.

Claim EC.2. *For any $\lambda>0$ and $k\ge N\ge 2$, $\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\right)^{2}}$ is increasing in μ for $\mu\in(0,\frac{\lambda}{N}]$.*

Case (II): When $\mu>\frac{\lambda}{N}$, i.e., $0<\frac{\lambda}{N\mu}<1$, we have $0<ErlC\left(N,\frac{\lambda}{\mu}\right)<1$ (from Lemma EC.2 (b)). Using the finite summation formula,

$$I(\mu,\mu)^{2}ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i}$$

$$
\begin{aligned}
&= I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\left[1-\left(\frac{\lambda}{N\mu}\right)^{k-N}\left(1+(k-N)\left(1-\frac{\lambda}{N\mu}\right)\right)\right]\\
&\le ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2},
\end{aligned}
\tag{EC.32}
$$

where the last inequality follows because $I(\mu,\mu)\le 1$ and $1-\left(\frac{\lambda}{N\mu}\right)^{k-N}\left(1+(k-N)\left(1-\frac{\lambda}{N\mu}\right)\right)<1$.

Combining (EC.30) and (EC.32) yields

$$
\begin{aligned}
I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^i &\le \min\left\{\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}, ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\right\}\\
&= 2\sqrt{N}\cdot\min\left\{\frac{1}{2\sqrt{N}ErlC\left(N,\frac{\lambda}{\mu}\right)}, \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{2\sqrt{N}}\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\right\}\\
&\overset{(*)}{<} 2\sqrt{N},
\end{aligned}
$$

where $(*)$ follows from the next claim, whose proof appears at the end.

Claim EC.3. *The following holds for all* λ, $k\ge N\ge 2$ *and* $\mu>\frac{\lambda}{N}$*:*

$$
\min\left\{\frac{1}{2\sqrt{N}ErlC\left(N,\frac{\lambda}{\mu}\right)}, \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{2\sqrt{N}}\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\right\}<1.
$$

Therefore, together the above two cases establish that

$$
I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^i<2\sqrt{N}.
$$

(b): From (EC.29),

$$
\begin{aligned}
I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1} &\le \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)^2}\\
&\overset{(*)}{<}\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)},
\end{aligned}
\tag{EC.33}
$$

where $(*)$ follows by noting, from (EC.31), that

$$
\begin{aligned}
\frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}}{\left(\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)^2} &= \frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}}{\sum_{\ell=0}^{k-N}(\ell+1)\left(\frac{\lambda}{N\mu}\right)^{\ell}+\sum_{\ell=k-N+1}^{2(k-N)}\left(2(k-N)-\ell\right)\left(\frac{\lambda}{N\mu}\right)^{\ell}}\\
&= \frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}}{\sum_{\ell=0}^{k-N}(\ell+1)\left(\frac{\lambda}{N\mu}\right)^{\ell}+\left(\frac{\lambda}{N\mu}\right)^{k-N+1}\sum_{\ell=0}^{k-N}(k-N-\ell)\left(\frac{\lambda}{N\mu}\right)^{\ell}}\\
&= \frac{\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}}{1+\sum_{i=0}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}+2\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}+\left(\frac{\lambda}{N\mu}\right)^{k-N+1}\sum_{i=1}^{k-N}(k-N-i)\left(\frac{\lambda}{N\mu}\right)^{i}}<1.
\end{aligned}
$$

Case (I): When $0 < \mu \le \frac{\lambda}{N}$, i.e., $\frac{\lambda}{N\mu} \ge 1$, we have $ErlC\left(N, \frac{\lambda}{\mu}\right) \ge 1$ (from Lemma EC.2 (b)). Then, from (EC.33),

$$I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1} \le \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \le 1 < 2\sqrt{N}, \quad \forall k \ge N \ge 2,$$

Case (II): When $\mu > \frac{\lambda}{N}$, i.e., $0 < \frac{\lambda}{N\mu} < 1$, we have $0 < ErlC\left(N, \frac{\lambda}{\mu}\right) < 1$ (from Lemma EC.2 (b)). Using the finite summation formula,

$$\begin{aligned}
&I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1}\\
=&I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\frac{\lambda}{N\mu}\left[1-\left(\frac{\lambda}{N\mu}\right)^{k-N}\left(1+(k-N)\left(1-\frac{\lambda}{N\mu}\right)\right)\right]\\
\le&ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2},
\end{aligned} \tag{EC.34}$$

where the last inequality follows because $I(\mu,\mu) \le 1$, $\frac{\lambda}{N\mu} < 1$ and $1 - \left(\frac{\lambda}{N\mu}\right)^{k-N}\left(1+(k-N)\left(1-\frac{\lambda}{N\mu}\right)\right) < 1$.

Combining (EC.33) and (EC.34) yields

$$\begin{aligned}
I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i+1} \le& \min\left\{\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}, ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\right\}\\
=&2\sqrt{N}\cdot\min\left\{\frac{1}{2\sqrt{N}ErlC\left(N,\frac{\lambda}{\mu}\right)}, \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{2\sqrt{N}}\frac{\frac{\lambda}{N\mu}}{\left(1-\frac{\lambda}{N\mu}\right)^2}\right\}\\
\overset{(*)}{<}&2\sqrt{N},
\end{aligned}$$

where $(*)$ follows from Claim EC.3. ∎

Proof of Claim EC.2: Let $x := \frac{\lambda}{N\mu}$ and define $f(x) := \frac{\sum_{i=0}^{k} i x^i}{\left(\sum_{i=0}^{k} x^i\right)^2}$ for $x \ge 0$. It suffices to show $f(x)$ is decreasing in $x \in [1,\infty)$. Using the finite summation formula in $f(x)$ yields

$$f(x) = \frac{x + kx^{k+2} - (k+1)x^{k+1}}{(1-x^{k+1})^2}.$$

Consider the first-order derivative of the function $f(x)$:

$$f'(x) = \frac{1-(k+1)^2x^k + (k^2+4k+1)x^{k+1} - (k+1)^2x^{2k+1} + k^2x^{2(k+1)}}{(1-x^{k+1})^3} =: \frac{g(x)}{(1-x^{k+1})^3}.$$

We first evaluate $f'(x)$ at $x=1$ by applying L'Hôpital's rule for three times and algebra:

$$\begin{aligned}
f'(1) =& \left.\frac{\left[1-(k+1)^2x^k + (k^2+4k+1)x^{k+1} - (k+1)^2x^{2k+1} + k^2x^{2(k+1)}\right]'''}{\left[(1-x^{k+1})^3\right]'''}\right|_{x=1}\\
=&\frac{k(k+1)^2(k-1)}{-k(k+1)^3} = -\frac{k(k-1)}{k(k+1)} \le 0, \quad \forall k \ge 1.
\end{aligned}$$

This means that $f(x)$ is decreasing at $x=1$. In what follows, we focus on $x\in(1,\infty)$. Consider the first-order derivative of the function $g(x)$:

$$\begin{aligned} g'(x) =&(k+1)x^{k-1}\left[(2k^2x^{k+1}+k(k+1))(x-1)-(3k+1)x(x^k-1)\right] \\ =&(k+1)x^{k-1}(x-1)\left[2k^2x^{k+1}-(3k+1)\sum_{i=1}^{k}x^i+k(k+1)\right] \\ =:&\,(k+1)x^{k-1}(x-1)\cdot h(x), \end{aligned} \tag{EC.35}$$

where

$$h(x):=\begin{cases} 2x^2-4x+2, & \text{if } k=1, \\ 2k^2x^{k+1}-(3k+1)\sum_{i=1}^{k}x^i+k(k+1), & \text{if } k\geq 2. \end{cases}$$

If $k=1$, then it is clear that $h(x)>0$ for $x>1$. If $k\geq 2$, then consider the first-order derivative of the function $h(x)$:

$$h'(x)=2k^2(k+1)x^k-(3k+1)\left(1+\sum_{i=2}^{k}ix^{i-1}\right),$$

which changes the sign of the coefficients exactly once. Descartes' rule of signs for polynomials implies that $h'(x)$ has at most one positive real root. Note that $h'(0)=-(3k+1)<0$ and $h'(1)=\frac{k(k-1)(k+1)}{2}>0$ for all $k\geq 2$, then the intermediate value theorem implies that $h'(x)$ has at least one root in $(0,1)$. Therefore, $h'(x)$ has one unique root, denoted by $x_0\in(0,1)$, with $h'(x)<0$ for all $x\in(0,x_0)$ and $h'(x)>0$ for all $x\in(x_0,\infty)$. This implies that $h(x)$ is strictly increasing in x for $x\in(1,\infty)$. Note that $h(1)=0$, thus $h(x)>h(1)=0$ for all $x>1$.

Then, from (EC.35), $h(x)>0$ for all $x>1$ implies that $g'(x)>0$ for all $x>1$, which means that the function $g(x)$ is strictly increasing in x for $(1,\infty)$. Note that $g(1)=0$, thus $g(x)>g(0)=0$ for all $x>1$, implying that $f'(x)<0$ for all $x>1$ (since $1-x^{k+1}<0$ for $x>1$); that is, $f(x)$ is strictly decreasing in x for $x\in(1,\infty)$.

Therefore, we conclude that $f(x)$ is decreasing in $[1,\infty)$. ▮

Proof of Claim EC.3: Let $f(x):=\frac{1}{ErlC(N,Nx)}$ and $g(x):=\frac{x\cdot ErlC(N,Nx)}{(1-x)^2}$ for $x\in(0,1)$. From Lemma EC.1 (b) and Lemma EC.2 (b), it is clear that $f(x)$ is strictly decreasing in x with $\lim_{x\to 0}f(x)=\infty$ and $\lim_{x\to 1}f(x)=1$; and $g(x)$ is strictly increasing in x with $\lim_{x\to 0}g(x)=0$ and $\lim_{x\to 1}g(x)=\infty$. This implies that $f(x)$ and $g(x)$ intersect once in $(0,1)$. We denote the unique solution to $f(x)=g(x)$ by $x^\star\in(0,1)$, i.e.,

$$ErlC(N,Nx^\star)=\frac{1}{\sqrt{x^\star}}-\sqrt{x^\star}, \tag{EC.36}$$

and it is straightforward that $\min\{f(x),g(x):x\in(0,1)\}\leq f(x^\star)=g(x^\star)$. Thus, to prove Claim EC.3, it suffices to show that

$$\frac{f(x^\star)}{2\sqrt{N}}<1,\ \text{or, equivalently, } 2\sqrt{N}ErlC(N,Nx^\star)>1.$$

Substituting for $ErlC(N, Nx^\star)$ using (EC.36) into the above display shows

$$\frac{1}{\sqrt{x^\star}} - \sqrt{x^\star} > \frac{1}{2\sqrt{N}},$$

which after algebra implies

$$\sqrt{x^\star} < \sqrt{1 + \frac{1}{16N}} - \frac{1}{4\sqrt{N}} =: \sqrt{\bar{x}}. \tag{EC.37}$$

To show (EC.37) is true, we introduce an auxiliary function $h(x) := ErlC(N, Nx) - \frac{1}{\sqrt{x}} + \sqrt{x}$ for $x \in (0,1)$. It is clear that $h(x)$ is strictly increasing in x (from Lemma EC.1 (b)). Thus, showing $\bar{x} > x^\star$ (i.e., (EC.37)) is equivalent to showing $h(\bar{x}) > h(x^\star)$. Note that, by definition of $x^\star$ in (EC.36), $h(x^\star) = 0$, it suffices to show $h(\bar{x}) > 0$, i.e.,

$$ErlC(N, N\bar{x}) - \frac{1}{\sqrt{\bar{x}}} + \sqrt{\bar{x}} > 0,$$

or, equivalently,

$$ErlC(N, N\bar{x}) > \frac{1}{\sqrt{\bar{x}}} - \sqrt{\bar{x}} \overset{(*)}{=} \frac{1}{2\sqrt{N}}, \tag{EC.38}$$

where $(*)$ follows by definition of $\bar{x}$. The remainder of the proof is devoted to verifying (EC.38). To show (EC.38), we consider $N = 1$, $N = 2$ and $N \geq 3$ separately.

- When $N = 1$, $\bar{x} = 0.7808$ and (EC.38) becomes to $ErlC(1, 0.7808) > \frac{1}{2}$, which is true.
- When $N = 2$, $\bar{x} = 0.8387$ and (EC.38) becomes to $ErlC(2, 2 \times 0.8387) > \frac{1}{2\sqrt{2}}$, which is true.
- When $N \geq 3$, from Proposition 2 in Harel (2010),

  $$ErlC(N, N\bar{x}) > 1 - (1 - \bar{x}^2)\sqrt{\frac{\pi N}{8}} > 1 - (1 - \bar{x}^2)\sqrt{\frac{N}{2}}.$$

  To verify (EC.38) stands, it is sufficient to show

  $$1 - (1 - \bar{x}^2)\sqrt{\frac{N}{2}} > \frac{1}{2\sqrt{N}},$$

  which after algebra implies

  $$\bar{x}^2 > 1 - \sqrt{\frac{2}{N}} + \frac{1}{\sqrt{2}N}.$$

  Plugging in the expression for $\bar{x}$ from (EC.37):

  $$\left(\sqrt{1 + \frac{1}{16N}} - \frac{1}{4\sqrt{N}}\right)^2 > 1 - \sqrt{\frac{2}{N}} + \frac{1}{\sqrt{2}N},$$

  which can be equivalently written as

  $$\sqrt{N}\left(\sqrt{2} - \frac{1}{2}\sqrt{1 + \frac{1}{16N}}\right) > \frac{1}{\sqrt{2}} - \frac{1}{8}. \tag{EC.39}$$

  It is clear that the left-hand side of the above display is strictly increasing in N and the right-hand side of the above display is a constant. Note that, when $N = 3$, the left-hand side evaluates to 1.5745, which is greater than the right-hand side $\frac{1}{\sqrt{2}} - \frac{1}{8} = 0.5821$. Hence, (EC.39) is true for all $N \geq 3$.

Therefore, combining the analysis in the above three bullet points establishes that (EC.38) holds for all $N \geq 1$, and thus the Claim is proved.

■

EC.4.1.4. Proof of Corollary EC.2

From (EC.19) in Corollary EC.1,

$$\mu \frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} = I(\mu,\mu)(1-I(\mu,\mu)) + I(\mu,\mu)^2 \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{N}\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^i .$$

(a): Using the bound in Lemma EC.6 (a),

$$\mu \frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} < I(\mu,\mu)(1-I(\mu,\mu)) + \frac{2\sqrt{N}}{N} = I(\mu,\mu)(1-I(\mu,\mu)) + \frac{2}{\sqrt{N}}, \quad \forall \mu > 0.$$

(b): Using the bound in Lemma EC.6 (b),

$$\mu \frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} < I(\mu,\mu)(1-I(\mu,\mu)) + \frac{1}{N}\frac{N\mu}{\lambda}\cdot 2\sqrt{N} < I(\mu,\mu) + \frac{2\mu\sqrt{N}}{\lambda},$$

which implies that

$$\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} < \frac{I(\mu,\mu)}{\mu} + \frac{2\sqrt{N}}{\lambda}, \quad \forall \mu > 0.$$

■

EC.4.1.5. Proof of Lemma EC.7 From (EC.16) in Corollary EC.1,

$$\frac{1}{I(\mu,\mu)} = 1 + \frac{\lambda}{\mu}\left(\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\right) + ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i . \tag{EC.40}$$

Differentiating the above display on both sides yields

$$\begin{aligned}
&-\frac{1}{I(\mu,\mu)^2}\frac{dI(\mu,\mu)}{d\mu}\\
&=\left(-\frac{\lambda}{\mu^2}\right)\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}} + \frac{\lambda}{\mu}\frac{-\frac{\partial ErlC\left(N,\frac{\lambda}{\mu}\right)}{\partial\mu}\left(N-\frac{\lambda}{\mu}\right) - \left(1-ErlC\left(N,\frac{\lambda}{\mu}\right)\right)\frac{\lambda}{\mu^2}}{\left(N-\frac{\lambda}{\mu}\right)^2}\\
&\qquad + \frac{\partial ErlC\left(N,\frac{\lambda}{\mu}\right)}{\partial\mu}\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i - ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{i-1}\left(-\frac{\lambda}{N\mu^2}\right)\\
&= -\frac{\lambda}{\mu^2}\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}} - \left(1-ErlC\left(N,\frac{\lambda}{\mu}\right)\right)\frac{\frac{\lambda}{\mu}\frac{\lambda}{\mu^2}}{\left(N-\frac{\lambda}{\mu}\right)^2}\\
&\qquad + \frac{\partial ErlC(N,\rho)}{\partial\rho}\left(-\frac{\lambda}{\mu^2}\right)\left[\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i - \frac{\frac{\lambda}{\mu}}{N-\frac{\lambda}{\mu}}\right] - \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu}\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^i\\
&= -\frac{\lambda}{\mu^2}\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\frac{N}{N-\frac{\lambda}{\mu}} + \frac{\partial ErlC(N,\rho)}{\partial\rho}\left(-\frac{\lambda}{\mu^2}\right)\left[\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i - \frac{\frac{\lambda}{\mu}}{N-\frac{\lambda}{\mu}}\right] - \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu}\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^i .
\end{aligned} \tag{EC.41}$$

On the other hand, note that

$$\frac{N}{\mu}\frac{1-I(\mu,\mu)}{I(\mu,\mu)} + \frac{1}{\mu}ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^i - \frac{\lambda}{\mu^2}\frac{1}{I(\mu,\mu)}\left(-\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}\frac{\partial ErlC(N,\rho)}{\partial\rho} + \frac{N\mu}{\lambda}\right)$$

$$
\begin{aligned}
&= \frac{1}{I(\mu,\mu)} \cdot \frac{\lambda}{\mu^2} \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} - \frac{N}{\mu} + \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu} \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i \\
&\overset{(i)}{=} \left[1 + \frac{\lambda}{\mu} \frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}} + ErlC\left(N,\frac{\lambda}{\mu}\right) \sum_{i=1}^{k-N} \left(\frac{\lambda}{N\mu}\right)^i \right] \frac{\lambda}{\mu^2} \frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} \\
&\qquad\qquad - \frac{N}{\mu} + \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu} \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i \\
&= \frac{\lambda}{\mu^2} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} \left[\sum_{i=1}^{k-N} \left(\frac{\lambda}{N\mu}\right)^i - \frac{\frac{\lambda}{\mu}}{N-\frac{\lambda}{\mu}} + \frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)\left(N-\frac{\lambda}{\mu}\right)}\right] - \frac{N}{\mu} + \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu} \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i \\
&\overset{(ii)}{=} \frac{\lambda}{\mu^2} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} \left[\sum_{i=1}^{k-N} \left(\frac{\lambda}{N\mu}\right)^i - \frac{\frac{\lambda}{\mu}}{N-\frac{\lambda}{\mu}}\right] + \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu} \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i \\
&\qquad\qquad + \frac{\lambda}{\mu^2} ErlC\left(N,\frac{\lambda}{\mu}\right) \left(\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}} + \frac{N-\frac{\lambda}{\mu}}{\frac{\lambda}{\mu}}\right) \frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)\left(N-\frac{\lambda}{\mu}\right)} - \frac{N}{\mu} \\
&= \frac{\lambda}{\mu^2} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} \left[\sum_{i=1}^{k-N} \left(\frac{\lambda}{N\mu}\right)^i - \frac{\frac{\lambda}{\mu}}{N-\frac{\lambda}{\mu}}\right] + \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\mu} \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i + \frac{\lambda}{\mu^2} \frac{N}{N-\frac{\lambda}{\mu}} \frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}},
\end{aligned}
\tag{EC.42}
$$

where (i) follows from (EC.40), and (ii) follows from Lemma EC.3.

Comparing (EC.41) and (EC.42) finds that

$$
\begin{aligned}
\frac{1}{I(\mu,\mu)^2} \frac{dI(\mu,\mu)}{d\mu} = \frac{N}{\mu} \frac{1-I(\mu,\mu)}{I(\mu,\mu)} + \frac{1}{\mu} ErlC\left(N,\frac{\lambda}{\mu}\right) \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i \\
- \frac{\lambda}{\mu^2} \frac{1}{I(\mu,\mu)} \left(-\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} + \frac{N\mu}{\lambda}\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{dI(\mu,\mu)}{d\mu} &= \frac{N}{\mu} I(\mu,\mu)(1-I(\mu,\mu)) + \frac{1}{\mu} I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right) \sum_{i=1}^{k-N} i \left(\frac{\lambda}{N\mu}\right)^i \\
&\quad - \frac{\lambda}{\mu^2} I(\mu,\mu) \left(-\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} + \frac{N\mu}{\lambda}\right) \\
&\overset{(i)}{=} N \left.\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{I(\mu,\mu)}{\mu} \frac{\lambda}{\mu} \left(-\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)} \frac{\partial ErlC\left(N,\rho\right)}{\partial \rho} + \frac{N\mu}{\lambda}\right) \\
&\overset{(ii)}{=} N \left.\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{I(\mu,\mu)}{\mu} \frac{\lambda}{N\mu} \left(N - \frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{1-\frac{\lambda}{N\mu}}\right),
\end{aligned}
$$

where (i) follows from (EC.19) in Corollary EC.1 and (ii) follows from Lemma EC.3.

$\blacksquare$

EC.4.1.6. Proof of Lemma EC.8 Letting $k=\infty$ in (EC.18) in Corollary EC.1 yields

$$
I(\mu,\mu;\lambda,\infty,N) = \lim_{k\to\infty} I(\mu,\mu;\lambda,k,N) = 1 - \frac{\lambda}{N\mu}.
$$

This implies that the difference between the idle time in an $M/M/N/k$ system and that in an $M/M/N$ system is given by $I(\mu,\mu)-\left(1-\frac{\lambda}{N\mu}\right)$. For any $\mu\in(0,\infty)$,

$$
\begin{aligned}
&\mu\cdot\frac{d}{d\mu}\left(I(\mu,\mu)-\left(1-\frac{\lambda}{N\mu}\right)\right)\\
=&\mu\frac{dI(\mu,\mu)}{d\mu}-\frac{\lambda}{N\mu}\\
\overset{(i)}{=}&N\left(\mu\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}\right)-I(\mu,\mu)\frac{\lambda}{\mu}\left(1-\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\right)-\frac{\lambda}{N\mu}\\
\overset{(ii)}{=}&N\left(I(\mu,\mu)(1-I(\mu,\mu))+I(\mu,\mu)^2\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{N}\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^i\right)-I(\mu,\mu)\left(\frac{\lambda}{\mu}-\left(\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-1\right)\right)-\frac{\lambda}{N\mu}\\
\overset{(iii)}{=}&I(\mu,\mu)\Bigg\{-NI(\mu,\mu)+I(\mu,\mu)ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^i+N-\frac{\lambda}{N\mu}+\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-1\\
&\qquad\qquad-\frac{\lambda}{N\mu}\left(\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}+ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right)\Bigg\}\\
=&I(\mu,\mu)\left\{I(\mu,\mu)ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^i+N(1-I(\mu,\mu))-\frac{\lambda}{\mu}-ErlC\left(N,\frac{\lambda}{\mu}\right)\frac{\lambda}{N\mu}\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right\}\\
=&I(\mu,\mu)\Bigg\{N\left(\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-1\right)I(\mu,\mu)+I(\mu,\mu)ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}(i+N)\left(\frac{\lambda}{N\mu}\right)^i\\
&\qquad\qquad-\frac{\lambda}{\mu}-\frac{\lambda}{N\mu}ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\Bigg\}\\
=&\frac{\lambda}{N\mu}I(\mu,\mu)\Bigg\{\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{1-\frac{\lambda}{N\mu}}NI(\mu,\mu)-N+I(\mu,\mu)ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N-1}(N+i+1)\left(\frac{\lambda}{N\mu}\right)^i\\
&\qquad\qquad-ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\Bigg\}\\
\overset{(iv)}{=}&\frac{\lambda}{N\mu}I(\mu,\mu)\Bigg\{\left[N\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{1-\frac{\lambda}{N\mu}}-N-\frac{\lambda}{\mu}\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{1-\frac{\lambda}{N\mu}}-NErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right]I(\mu,\mu)\\
&\qquad\qquad+I(\mu,\mu)ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N-1}(N+i+1)\left(\frac{\lambda}{N\mu}\right)^i-ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\Bigg\}\\
=&\frac{\lambda}{N\mu}I(\mu,\mu)ErlC\left(N,\frac{\lambda}{\mu}\right)\left\{\left[\sum_{i=0}^{k-N}(i+1)\left(\frac{\lambda}{N\mu}\right)^i-(k+1)\left(\frac{\lambda}{N\mu}\right)^{k-N}\right]I(\mu,\mu)-\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right\}\\
=&\frac{\lambda}{N\mu}I(\mu,\mu)^2ErlC\left(N,\frac{\lambda}{\mu}\right)\Bigg\{\sum_{i=0}^{k-N}(i+1)\left(\frac{\lambda}{N\mu}\right)^i-(k+1)\left(\frac{\lambda}{N\mu}\right)^{k-N}\\
&\qquad\qquad-\left[N\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}+ErlC\left(N,\frac{\lambda}{\mu}\right)\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\right]\sum_{i=0}^{k-N}\left(\frac{\lambda}{N\mu}\right)^i\Bigg\}\\
=&\frac{\lambda}{N\mu}I(\mu,\mu)^2ErlC\left(N,\frac{\lambda}{\mu}\right)\Bigg\{-N(k-N+1)\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\left(\frac{\lambda}{N\mu}\right)^{k-N+1}-(k+1)\left(\frac{\lambda}{N\mu}\right)^{k-N}\\
&\qquad\qquad-ErlC\left(N,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N+1}\sum_{i=0}^{k-N}(k-N-i)\left(\frac{\lambda}{N\mu}\right)^i\Bigg\}
\end{aligned}
$$

$$\overset{(v)}{=} -\left(\frac{\lambda}{N\mu}\right)^{k-N+1} I(\mu,\mu)^2 ErlC\left(N,\frac{\lambda}{\mu}\right)\left(N+ErlC\left(N,\frac{\lambda}{\mu}\right)\left(\frac{k-N+1}{ErlB\left(N,\frac{\lambda}{\mu}\right)}+\frac{\lambda}{N\mu}\sum_{i=1}^{k-N} i\left(\frac{\lambda}{N\mu}\right)^{k-N+i}\right)\right)<0,$$

where (i) follows from Lemma EC.7, (ii) follows from (EC.19) in Corollary EC.1 and $\rho\frac{1-ErlC(N,\rho)}{N-\rho}=\frac{ErlC(N,\rho)}{ErlB(N,\rho)}-1$ (using the relationship between $ErlB$ and $ErlC$ from Lemma EC.2 (a)), (iii) follows from (EC.17) in Corollary EC.1 and the definition of $ErlB$ in (EC.1), (iv) follows from (EC.16) in Corollary EC.1, and (v) follows from $\rho\frac{1-ErlC(N,\rho)}{N-\rho}=\frac{ErlC(N,\rho)}{ErlB(N,\rho)}-1$.

Hence, $I(\mu,\mu;\lambda,k,N)-\left(1-\frac{\lambda}{N\mu}\right)$ is strictly decreasing in μ for $\mu\in(0,\infty)$. Moreover, note that $\lim_{\mu\to\infty} I(\mu,\mu;\lambda,k,N)-\left(1-\frac{\lambda}{N\mu}\right)=1-1=0$ (from Lemma EC.2 (b)). Thus, $I(\mu,\mu;\lambda,k,N)-\left(1-\frac{\lambda}{N\mu}\right)>0$ for all $\mu\in(0,\infty)$. Hence, the difference between the idle time in an $M/M/N/k$ system and that in an $M/M/N$ system satisfies $I(\mu,\mu;\lambda,k,N)-\left(1-\frac{\lambda}{N\mu}\right)\geq 0$ with equality holding only when $k=\infty$. $\blacksquare$

EC.4.2. Proof of Theorem 1

Since $\mu\neq 0$, the FOC in (6) for a symmetric equilibrium is equivalent to

$$\frac{\mu c'(\mu)}{I(\mu,\mu)}=p\left(\frac{\mu}{I(\mu,\mu)}-\mu\right)+(v-p\mu)\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)},$$

where $I'(\mu,\mu)$ denotes $\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}$. Set $LHS(\mu)$ and $RHS(\mu)$ equal to the left-hand side and the right-hand side of the above display, respectively. Since $LHS(\mu)$ and $RHS(\mu)$ are both continuous functions of $\mu\in(0,\infty)$, a sufficient condition for the FOC to admit a solution for $\mu\in(0,\infty)$ is

$$\lim_{\mu\downarrow 0} LHS(\mu)<\lim_{\mu\downarrow 0} RHS(\mu) \text{ and } \lim_{\mu\uparrow\infty}\mu\cdot LHS(\mu)>\lim_{\mu\uparrow\infty}\mu\cdot RHS(\mu). \tag{EC.43}$$

Claim EC.4. *The following hold:*

$$\lim_{\mu\downarrow 0} LHS(\mu)=\lim_{\mu\downarrow 0}\frac{c'(\mu)}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1},\quad \lim_{\mu\downarrow 0} RHS(\mu)=\lim_{\mu\downarrow 0} p\cdot\frac{1}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}+v\frac{k}{N},$$
$$\lim_{\mu\uparrow\infty}\mu\cdot LHS(\mu)=\lim_{\mu\uparrow\infty}\mu^2 c'(\mu),\quad \lim_{\mu\uparrow\infty}\mu\cdot RHS(\mu)=-\frac{1}{2}p\lambda^2+v\frac{\lambda}{N}-p\mu\lambda\left(1+\frac{1}{N}\right).$$

By Claim EC.4, (EC.43) becomes

$$\lim_{\mu\downarrow 0}\frac{c'(\mu)}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}<\lim_{\mu\downarrow 0} p\cdot\frac{1}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}+v\frac{k}{N}, \text{ and}$$
$$\lim_{\mu\uparrow\infty}\mu^2 c'(\mu)>\lim_{\mu\uparrow\infty}-\frac{1}{2}p\lambda^2+v\frac{\lambda}{N}-p\mu\lambda\left(1+\frac{1}{N}\right),$$

which can be equivalently written as

$$\lim_{\mu\downarrow 0}\frac{c'(\mu)-p}{\mu^{k-N}}<v\frac{k}{N^2}\left(\frac{N}{\lambda}\right)^{k-N+1}, \text{ and } \lim_{\mu\uparrow\infty}\mu^2 c'(\mu)+p\mu\lambda\left(1+\frac{1}{N}\right)>v\frac{\lambda}{N}-\frac{1}{2}p\lambda^2,$$

where the second condition is always true, and the first condition establishes (7).

The second part of the theorem is established by applying (7) to the Taylor expansion of $c(\mu)$ and $c'(\mu)$ at $\mu = 0$, recalling that an entire function is equal to the sum of its Taylor series everywhere. We first express $c'(\mu)$ as a Taylor series at 0:

$$c'(\mu) = \sum_{i=1}^{\infty} \frac{c^{(i)}(0)}{(i-1)!}\mu^{i-1} \quad \Leftrightarrow \quad \frac{c'(\mu)}{\mu^{k-N}} = \sum_{i=1}^{\infty} \frac{c^{(i)}(0)}{(i-1)!}\mu^{i-(k-N+1)}.$$

Then, (7) can be rewritten as

$$\lim_{\mu\downarrow 0} \sum_{i=1}^{\infty} \frac{c^{(i)}(0)}{(i-1)!}\mu^{i-(k-N+1)} < v\frac{k}{N^2}\left(\frac{N}{\lambda}\right)^{k-N+1} + \lim_{\mu\downarrow 0}\frac{p}{\mu^{k-N}}.$$

This inequality holds if and only if

$$\begin{cases} c^{(1)}(0) < p, \text{ and } c^{(i)}(0) \text{ any for } i \geq 2, & \text{if } k > N, \\ c^{(1)}(0) < \frac{v}{\lambda} + p, \text{ and } c^{(i)}(0) \text{ any for } i \geq 2, & \text{if } k = N. \end{cases} \tag{EC.44}$$

Next, we express $c(\mu)$ as a Taylor series at 0:

$$c(\mu) = c(0) + \sum_{i=1}^{\infty} \frac{c^{(i)}(0)}{i!}\mu^{i},$$

Let integer $q > 0$ be such that $c^{(i)}(0) = 0$ for all $1 \leq i < q$ and $c^{(q)}(0) \neq 0$. Then, the above display can be rewritten as

$$\begin{aligned} c(\mu) &= c(0) + \sum_{i=q}^{\infty} \frac{c^{(i)}(0)}{i!}\mu^{i} \\ &= c(0) + \frac{c^{(q)}(0)}{q!}\mu^{q}\left(1 + \sum_{i=q+1}^{\infty} \frac{c^{(i)}(0)}{c^{(q)}(0)}\frac{q!}{i!}\mu^{i-q}\right) \\ &=: b_E + c_E\mu^q \cdot h(\mu), \end{aligned}$$

where $b_E := c(0) = 0$ (recalling that $c(0) = 0$), $c_E := \frac{c^{(q)}(0)}{q!} \neq 0$, and $h(\mu) := 1 + \sum_{i=q+1}^{\infty} \frac{c^{(i)}(0)}{c^{(q)}(0)}\frac{q!}{i!}\mu^{i-q} = h(0) + \sum_{i=q+1}^{\infty} \frac{h^{(i-q)}(0)}{(i-q)!}\mu^{i-q}$ is an entire function with $h(0) = 1$.

- If $q \geq 2$, then $c^{(1)}(0) = 0$. Thus, condition (EC.44) is satisfied.
- If $q = 1$, then condition (EC.44) holds if and only if

$$c_E = \frac{c^{(1)}(0)}{1!} = c^{(1)}(0) < \begin{cases} p, & \text{if } k > N, \\ \frac{v}{\lambda} + p, & \text{if } k = N. \end{cases}$$

Therefore, an entire function c satisfies (7) if and only if it is of the form $c(\mu) = c_E\mu^q \cdot h(\mu)$, where $c_E \in \mathbb{R}$, $c_E \neq 0$, $q \in \mathbb{Z}_+$; h is an entire function with $h(0) = 1$; and, either (a) $q = 1$ and $c_E < p + \frac{v}{\lambda}\mathbb{1}\{k = N\}$, or (b) $q \geq 2$. ∎

Proof of Claim EC.4: Note that the building blocks for evaluating these limits are $\frac{\mu}{I(\mu,\mu)}$ and $\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)}$, so we first investigate their associated limits as $\mu \downarrow 0$ and as $\mu \uparrow \infty$.

From (EC.17) in Corollary EC.1,

$$\begin{aligned}
\frac{\mu}{I(\mu,\mu)} =&\mu\left(1+\sum_{i=0}^{N-1}\frac{N!}{i!}\left(\frac{\mu}{\lambda}\right)^{N-i}+\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\right) ErlC\left(N,\frac{\lambda}{\mu}\right)\\
=&\mu\left(\left(\frac{N\mu}{\lambda}\right)^{k-N}+\sum_{i=0}^{N-1}\frac{N!}{i!}\left(\frac{\mu}{\lambda}\right)^{N-i}\left(\frac{N\mu}{\lambda}\right)^{k-N}+\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\left(\frac{N\mu}{\lambda}\right)^{k-N}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N} ErlC\left(N,\frac{\lambda}{N\mu}\right)\\
=&\mu\left(1+\sum_{i=1}^{k-N}\left(\frac{N\mu}{\lambda}\right)^{i}+N^{k-N}\sum_{i=k-N+1}^{k}\frac{N!}{(k-i)!}\left(\frac{\mu}{\lambda}\right)^{i}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N} ErlC\left(N,\frac{\lambda}{\mu}\right)\\
=&\frac{1}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\frac{\lambda}{\mu}}\left(1+\sum_{i=1}^{k-N}\left(\frac{N\mu}{\lambda}\right)^{i}+N^{k-N}\sum_{i=k-N+1}^{k}\frac{N!}{(k-i)!}\left(\frac{\mu}{\lambda}\right)^{i}\right),
\end{aligned}$$

which implies that

$$\lim_{\mu\downarrow 0}\frac{\mu}{I(\mu,\mu)}=\lim_{\mu\downarrow 0}\frac{1}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\frac{\lambda}{\mu}}\overset{(*)}{=}\lim_{\mu\downarrow 0}\frac{1}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}, \tag{EC.45}$$

where $(*)$ follows from $\lim_{\mu\downarrow 0}\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\frac{\lambda}{\mu}}=1$ (Lemma EC.2 (b)), and

$$\begin{aligned}
&\lim_{\mu\uparrow\infty}\mu\cdot\frac{\mu}{I(\mu,\mu)}=\lim_{\mu\uparrow\infty}\frac{\mu}{\mu^{k-N}}N\left(\frac{\lambda}{N}\right)^{k-N+1}\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{\frac{\lambda}{\mu}}N^{k-N}N!\left(\frac{\mu}{\lambda}\right)^{k}\\
=&\lim_{\mu\uparrow\infty}\mu^{2}\left(\frac{\mu}{\lambda}\right)^{N}N!ErlC\left(N,\frac{\lambda}{\mu}\right)\overset{(**)}{=}\lim_{\mu\uparrow\infty}\mu^{2}\frac{\frac{N}{N-\lambda/\mu}}{\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^{i}\frac{1}{i!}+\left(\frac{\lambda}{\mu}\right)^{N}\frac{1}{N!}\frac{N}{N-\lambda/\mu}},
\end{aligned} \tag{EC.46}$$

where $(**)$ follows from the expression for $ErlC$ in (EC.3).

From (EC.17) and (EC.19) in Corollary EC.1,

$$\begin{aligned}
\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)}=&1-I(\mu,\mu)\left(1-\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{N}\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^{i}\right)\\
=&1-\frac{1-\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{N}\left(\sum_{i=1}^{k-N}i\left(\frac{\lambda}{N\mu}\right)^{i}\left(\frac{N\mu}{\lambda}\right)^{k-N}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N}}{\left(\left(\frac{N\mu}{\lambda}\right)^{k-N}+\sum_{i=0}^{N-1}\frac{N!}{i!}\left(\frac{\mu}{\lambda}\right)^{N-i}\left(\frac{N\mu}{\lambda}\right)^{k-N}+\sum_{i=1}^{k-N}\left(\frac{\lambda}{N\mu}\right)^{i}\left(\frac{N\mu}{\lambda}\right)^{k-N}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N}ErlC\left(N,\frac{\lambda}{\mu}\right)}\\
=&1-\frac{1-\left(k-N+\sum_{i=1}^{k-N}(k-N-i)\left(\frac{N\mu}{\lambda}\right)^{i}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N}\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{N}}{\left(1+\sum_{i=1}^{k-N}\left(\frac{N\mu}{\lambda}\right)^{i}+N^{k-N}\sum_{i=k-N+1}^{k}\frac{N!}{(k-i)!}\left(\frac{\mu}{\lambda}\right)^{i}\right)\left(\frac{\lambda}{N\mu}\right)^{k-N}ErlC\left(N,\frac{\lambda}{\mu}\right)}\\
=&\frac{\frac{k}{N}+\sum_{i=1}^{k-N-1}\frac{k-i}{N}\left(\frac{N\mu}{\lambda}\right)^{i}+N^{k-N}\sum_{i=k-N+1}^{k}\frac{(N-1)!}{(k-i)!}\left(\frac{\mu}{\lambda}\right)^{i-1}}{1+\sum_{i=1}^{k-N}\left(\frac{N\mu}{\lambda}\right)^{i}+N^{k-N}\sum_{i=k-N+1}^{k}\frac{N!}{(k-i)!}\left(\frac{\mu}{\lambda}\right)^{i}},
\end{aligned}$$

which implies that

$$\lim_{\mu\downarrow 0}\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)}=\frac{k}{N}, \tag{EC.47}$$

and

$$\lim_{\mu\uparrow\infty}\mu\cdot\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)}=\lim_{\mu\uparrow\infty}\mu\cdot\frac{N^{k-N}(N-1)!\left(\frac{\mu}{\lambda}\right)^{k-1}}{N^{k-N}N!\left(\frac{\mu}{\lambda}\right)^{k}}=\frac{\lambda}{N}. \tag{EC.48}$$

When $\lim_{\mu\downarrow 0}\frac{c'(\mu)}{\mu^{k-N}}$ exists and is finite, (EC.45) implies that

$$\lim_{\mu\downarrow 0} LHS(\mu) = \lim_{\mu\downarrow 0}\frac{c'(\mu)}{\mu^{k-N}} N\left(\frac{\lambda}{N}\right)^{k-N+1}, \tag{EC.49}$$

and (EC.46) implies that

$$\lim_{\mu\uparrow\infty}\mu\cdot LHS(\mu) = \lim_{\mu\uparrow\infty}\mu^2 c'(\mu)\frac{\frac{N}{N-\lambda/\mu}}{\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}+\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\frac{N}{N-\lambda/\mu}} \overset{(*)}{=} \lim_{\mu\uparrow\infty}\mu^2 c'(\mu), \tag{EC.50}$$

where $(*)$ follows because $\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}\to 1$, $\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\frac{N}{N-\lambda/\mu}\to 0$, and $\frac{N}{N-\frac{\lambda}{\mu}}\to 1$, as $\mu\to\infty$.

From (EC.45) and (EC.47),

$$\lim_{\mu\downarrow 0} RHS(\mu) = p\cdot\lim_{\mu\downarrow 0}\frac{\mu}{I(\mu,\mu)} + \lim_{\mu\downarrow 0}(v-p\mu)\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)} = \lim_{\mu\downarrow 0} p\cdot\frac{1}{\mu^{k-N}} N\left(\frac{\lambda}{N}\right)^{k-N+1} + v\frac{k}{N}, \tag{EC.51}$$

and from (EC.46) and (EC.48),

$$\begin{aligned}
\lim_{\mu\uparrow\infty}\mu\cdot RHS(\mu) &= \lim_{\mu\uparrow\infty} p\cdot\left(\mu\cdot\frac{\mu}{I(\mu,\mu)}-\mu^2\right) + v\cdot\lim_{\mu\uparrow\infty}\mu\cdot\frac{\mu I'(\mu,\mu)}{I(\mu,\mu)} - p\cdot\lim_{\mu\uparrow\infty}\frac{\mu^3 I'(\mu,\mu)}{I(\mu,\mu)}\\
&= \lim_{\mu\uparrow\infty} p\cdot\mu^2\frac{\frac{N}{N-\lambda/\mu}}{\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}+\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\frac{N}{N-\lambda/\mu}} - \mu^2 + v\cdot\frac{\lambda}{N} - p\cdot\frac{\lambda\mu}{N}\\
&= \lim_{\mu\uparrow\infty} p\cdot\mu^2\frac{\frac{N}{N-\frac{\lambda}{\mu}}\left(1-\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\right)-\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}}{\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}+\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\frac{N}{N-\lambda/\mu}} + (v-p\mu)\frac{\lambda}{N}\\
&= \lim_{\mu\uparrow\infty} p\cdot\frac{\frac{N}{N-\frac{\lambda}{\mu}}\left(\mu^2-\left(\frac{\lambda}{\mu}\right)^{N-2}\frac{\lambda^2}{N!}\right)-\left(\mu^2+\lambda\mu+\frac{1}{2!}\lambda^2+\frac{1}{3!}\frac{\lambda^3}{\mu}+\cdots+\frac{1}{(N-1)!}\frac{\lambda^{N-1}}{\mu^{N-3}}\right)}{\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}+\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\frac{N}{N-\lambda/\mu}} + (v-p\mu)\frac{\lambda}{N}\\
&\overset{(*)}{=} \lim_{\mu\uparrow\infty} p\cdot\left(\mu^2-\left(\mu^2+\lambda\mu+\frac{1}{2}\lambda^2\right)\right) + (v-p\mu)\frac{\lambda}{N}\\
&= \lim_{\mu\uparrow\infty} -\frac{1}{2}p\lambda^2 + v\frac{\lambda}{N} - p\mu\lambda\left(1+\frac{1}{N}\right),
\end{aligned} \tag{EC.52}$$

where $(*)$ follows by noting that $\sum_{i=0}^{N-1}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}\to 1$, $\left(\frac{\lambda}{\mu}\right)^N\frac{1}{N!}\frac{N}{N-\lambda/\mu}\to 0$, $\frac{N}{N-\frac{\lambda}{\mu}}\to 1$, $\left(\frac{\lambda}{\mu}\right)^{N-2}\frac{\lambda^2}{N!}\to 0$, and $\frac{1}{3!}\frac{\lambda^3}{\mu}+\cdots+\frac{1}{(N-1)!}\frac{\lambda^{N-1}}{\mu^{N-3}}\to 0$, as $\mu\to\infty$.

■

EC.5. Proofs from Section 4.1

EC.5.1. Preliminaries

The expressions for $I(\mu_1,\mu)$ and its derivatives simplify significantly for a loss system, i.e., when $k=N$. We provide the expressions and their properties in the following lemmas.

LEMMA EC.9. *In an M/M/N/N loss system, the steady-state probability that the tagged server is idle, and its first two partial derivatives with respect to μ_1 satisfy the following expressions:*

(a) $I(\mu_1,\mu) = \frac{1-\frac{\rho}{N}(1-ErlB(N,\rho))}{1-\left(1-\frac{\mu}{\mu_1}\right)\frac{\rho}{N}(1-ErlB(N,\rho))}$.

(b) $\frac{\partial I(\mu_1,\mu)}{\partial \mu_1} = \frac{I(\mu_1,\mu)^2}{\mu_1}\left(\frac{1}{I(\mu_1,\mu)} - 1\right) > 0, \quad \forall \mu_1, \mu > 0.$
(c) $\frac{\partial^2 I(\mu_1,\mu)}{\partial \mu_1^2} = -2\frac{I(\mu_1,\mu)^2}{\mu_1}(1 - I(\mu_1,\mu)) < 0, \quad \forall \mu_1, \mu > 0.$

Corollary EC.3 (**$\mu_1 = \mu$ in Lemma EC.9**).

(a) $I(\mu,\mu) = 1 - \frac{\rho}{N}(1 - ErlB(N,\rho))$.
(b) $\left.\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} = \frac{I(\mu,\mu)^2}{\mu}\left(\frac{1}{I(\mu,\mu)} - 1\right) > 0, \quad \forall \mu > 0.$
(c) $\left.\frac{\partial^2 I(\mu_1,\mu)}{\partial \mu_1^2}\right|_{\mu_1=\mu} = -2\frac{I(\mu,\mu)^2}{\mu}(1 - I(\mu,\mu)) < 0, \quad \forall \mu > 0.$

Lemma EC.10. *In an M/M/N/N loss system,* $\frac{I(\mu,\mu;\lambda,N)}{\mu}$ *is strictly increasing in* μ *for* $\mu \in (0, \frac{5\lambda}{4N}]$ *for all* $\lambda > 0$ *and* $N \geq 6$.

Lemma EC.11. *In an M/M/N/N loss system, for any* $x \in (0,1)$, $\mu > 0$, *and* $N > 0$, *there exists a unique* $\lambda > 0$ *such that* $I(\mu,\mu;\lambda,N) = x$. *Similarly, for any* $x \in (0,1)$, $\mu > 0$, *and* $\lambda > 0$, *there exists a unique (real-valued)* $N > 0$ *such that* $I(\mu,\mu;\lambda,N) = x$.

EC.5.1.1. Proof of Lemma EC.9

(a): From Lemma 2, substituting $k = N$ into (1) yields

$$I(\mu_1,\mu) = \left(1 + \rho\frac{\mu}{\mu_1}\left(\frac{1 - ErlC(N,\rho)}{N - \rho}\right)\right)^{-1}.$$

Using the relationship between $ErlB$ and $ErlC$ from Lemma EC.2 (a),

$$\begin{aligned} I(\mu_1,\mu) = \left(1 + \rho\frac{\mu}{\mu_1}\left(\frac{1 - \frac{N}{\frac{N-\rho}{ErlB(N,\rho)} + \rho}}{N - \rho}\right)\right)^{-1} &= \left(1 + \frac{\rho}{N}\frac{\mu}{\mu_1}\frac{1 - ErlB(N,\rho)}{1 - \frac{\rho}{N}(1 - ErlB(N,\rho))}\right)^{-1} \\ &= \frac{1 - \frac{\rho}{N}(1 - ErlB(N,\rho))}{1 - \frac{\rho}{N}(1 - ErlB(N,\rho)) + \frac{\rho}{N}\frac{\mu}{\mu_1}(1 - ErlB(N,\rho))} \\ &= \frac{1 - \frac{\rho}{N}(1 - ErlB(N,\rho))}{1 - \left(1 - \frac{\mu}{\mu_1}\right)\frac{\rho}{N}(1 - ErlB(N,\rho))}. \end{aligned}$$

(b): From Lemma EC.5, substituting $k = N$ into (EC.14) yields

$$\frac{\partial I(\mu_1,\mu)}{\partial \mu_1} = \frac{1}{\mu_1}I(\mu_1,\mu)(1 - I(\mu_1,\mu)) = \frac{I(\mu_1,\mu)^2}{\mu_1}\left(\frac{1}{I(\mu_1,\mu)} - 1\right) > 0, \quad \forall \mu_1, \mu > 0,$$

because $I(\mu_1,\mu) \in (0,1)$ by definition.

(c): From Lemma EC.5, substituting $k = N$ into (EC.15) yields

$$\frac{\partial^2 I(\mu_1,\mu)}{\partial \mu_1^2} = -2\frac{I(\mu_1,\mu)^2}{\mu_1}(1 - I(\mu_1,\mu)) < 0, \quad \forall \mu_1, \mu > 0,$$

because $I(\mu_1,\mu) \in (0,1)$ by definition.

$\blacksquare$

EC.5.1.2. Proof of Corollary EC.3 This immediately follows from Lemma EC.9 by substituting $\mu_1 = \mu$.

$\blacksquare$

EC.5.1.3. Proof of Lemma EC.10 Note that

$$\frac{d}{d\mu}\left(\frac{I(\mu,\mu)}{\mu}\right)=\frac{1}{\mu}\frac{dI(\mu,\mu)}{d\mu}-\frac{I(\mu,\mu)}{\mu^2}=\frac{I(\mu,\mu)}{\mu^2}\left(\frac{\mu}{I(\mu,\mu)}\frac{dI(\mu,\mu)}{d\mu}-1\right).$$

From Lemma EC.7,

$$\begin{aligned}
\frac{\mu}{I(\mu,\mu)}\frac{dI(\mu,\mu)}{d\mu}-1 &= \frac{N\mu}{I(\mu,\mu)}\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}-\frac{\lambda}{\mu}\left(1-\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\right)-1\\
&\overset{(i)}{=} N\mu\frac{1}{\mu}\left(1-I(\mu,\mu)\right)-\frac{\lambda}{\mu}\left(1-\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\right)-1\\
&\overset{(ii)}{=} N\frac{\lambda}{N\mu}\left(1-ErlB\left(N,\frac{\lambda}{\mu}\right)\right)-\frac{\lambda}{\mu}\left(1-\frac{1-ErlC\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}}\right)-1\\
&\overset{(iii)}{=} \frac{\lambda}{\mu}\left(1-ErlB\left(N,\frac{\lambda}{\mu}\right)\right)-\frac{\lambda}{\mu}+\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-1-1\\
&= \frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)-2,
\end{aligned}$$

where (i) follows from Corollary EC.3 (b), (ii) follows from Corollary EC.3 (a), and (iii) follows by noting that $\frac{\lambda}{\mu}\left(\frac{1-ErlC(N,\frac{\lambda}{\mu})}{N-\frac{\lambda}{\mu}}\right)=\frac{ErlC(N,\frac{\lambda}{\mu})}{ErlB(N,\frac{\lambda}{\mu})}-1$ (using the relationship between $ErlB$ and $ErlC$ from Lemma EC.2 (a)).

Note that we can use Lemma EC.2 (a) to evaluate

$$\begin{aligned}
&\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)-2\\
&\quad=\frac{\frac{N}{\frac{N-\frac{\lambda}{\mu}}{ErlB\left(N,\frac{\lambda}{\mu}\right)}+\frac{\lambda}{\mu}}}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)-2\\
&\quad=\frac{N}{N-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)-2\\
&\quad=\frac{\frac{\lambda}{\mu}\left(2+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)\right)\left(1-ErlB\left(N,\frac{\lambda}{\mu}\right)\right)-N\left(1+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)\right)}{N-\frac{\lambda}{\mu}\left(1-ErlB\left(N,\frac{\lambda}{\mu}\right)\right)},
\end{aligned}$$

where the denominator is strictly positive because, from Lemma EC.2 (a), $ErlC\left(N,\frac{\lambda}{\mu}\right)=\frac{NErlB\left(N,\frac{\lambda}{\mu}\right)}{N-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)}>0$ implies $N-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)>0$. Thus, the above display implies that, to show $\frac{ErlC\left(N,\frac{\lambda}{\mu}\right)}{ErlB\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)-2>0$, it suffices to show

$$\begin{aligned}
&\frac{\lambda}{\mu}\left(2+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)\right)(1-ErlB\left(N,\frac{\lambda}{\mu}\right))-N\left(1+\frac{\lambda}{\mu}ErlB\left(N,\frac{\lambda}{\mu}\right)\right)>0\\
\Leftrightarrow\quad&\left(\frac{2\lambda}{\mu}-N\right)\left(\frac{\mu}{\lambda ErlB\left(N,\frac{\lambda}{\mu}\right)}\right)^2+\left(\frac{\lambda}{\mu}-N-2\right)\left(\frac{\mu}{\lambda ErlB\left(N,\frac{\lambda}{\mu}\right)}\right)-1>0
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad \left(\frac{\mu}{\lambda ErlB\left(N,\frac{\lambda}{\mu}\right)}\right)^2+\frac{\frac{\lambda}{\mu}-N-2}{\frac{2\lambda}{\mu}-N}\left(\frac{\mu}{\lambda ErlB\left(N,\frac{\lambda}{\mu}\right)}\right)-\frac{1}{\frac{2\lambda}{\mu}-N}>0\\
&\Leftrightarrow \quad \left(\frac{\mu}{\lambda ErlB\left(N,\frac{\lambda}{\mu}\right)}+\frac{\frac{\lambda}{\mu}-N-2}{2\left(\frac{2\lambda}{\mu}-N\right)}\right)^2>\frac{\left(\frac{\lambda}{\mu}-N\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}{4\left(\frac{2\lambda}{\mu}-N\right)^2}\\
&\Leftrightarrow \quad \frac{\mu}{\lambda ErlB\left(N,\frac{\lambda}{\mu}\right)}+\frac{\frac{\lambda}{\mu}-N-2}{2\left(\frac{2\lambda}{\mu}-N\right)}>\frac{\sqrt{\left(\frac{\lambda}{\mu}-N\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}}{2\left(\frac{2\lambda}{\mu}-N\right)}\\
&\Leftrightarrow \quad ErlB\left(N,\frac{\lambda}{\mu}\right)<\frac{\sqrt{\left(\frac{\lambda}{\mu}-N\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}+\left(\frac{\lambda}{\mu}-N-2\right)}{\frac{2\lambda}{\mu}}. \qquad \text{(EC.53)}
\end{aligned}
$$

We prove (EC.53) by induction. Suppose (EC.53) holds, then we want to show that it holds with N replaced by $N+1$; that is,

$$
\begin{aligned}
&ErlB\left(N+1,\frac{\lambda}{\mu}\right)<\frac{\sqrt{\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}+\left(\frac{\lambda}{\mu}-(N+1)-2\right)}{\frac{2\lambda}{\mu}}\\
\overset{(*)}{\Leftrightarrow}\quad &\left(1+\frac{(N+1)\mu}{\lambda}\frac{1}{ErlB\left(N,\frac{\lambda}{\mu}\right)}\right)^{-1}<\frac{\sqrt{\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}+\left(\frac{\lambda}{\mu}-(N+1)-2\right)}{\frac{2\lambda}{\mu}}\\
\Leftrightarrow\quad &1+\frac{(N+1)\mu}{\lambda}\frac{1}{ErlB\left(N,\frac{\lambda}{\mu}\right)}>\frac{\frac{\lambda}{\mu}\sqrt{\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-(N+1)-2\right)}{2\left(\frac{2\lambda}{\mu}-(N+1)\right)}\\
\Leftrightarrow\quad &ErlB\left(N,\frac{\lambda}{\mu}\right)<\frac{\frac{\lambda}{\mu}\sqrt{\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}+\left(\frac{\lambda}{\mu}-(N+1)\right)\left(2+\frac{\lambda}{\mu}\right)}{\frac{2\lambda}{\mu}\left(\frac{\lambda}{\mu}+1\right)}, \qquad \text{(EC.54)}
\end{aligned}
$$

where $(*)$ follows from (EC.2). Suppose we can establish that, for all $\lambda>0$, $N\geq 6$ and $\mu\in(0,\frac{5\lambda}{4N}]$,

$$
\frac{\sqrt{\left(\frac{\lambda}{\mu}-N\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}+\left(\frac{\lambda}{\mu}-N-2\right)}{\frac{2\lambda}{\mu}}<\frac{\frac{\lambda}{\mu}\sqrt{\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}+\left(\frac{\lambda}{\mu}-(N+1)\right)\left(2+\frac{\lambda}{\mu}\right)}{\frac{2\lambda}{\mu}\left(\frac{\lambda}{\mu}+1\right)}, \qquad \text{(EC.55)}
$$

then the induction hypothesis (EC.53) would imply that (EC.54) holds, as desired. To see (EC.55), it is equivalent to show

$$
\left(\frac{\lambda}{\mu}+1\right)\sqrt{\left(\frac{\lambda}{\mu}-N\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}\sqrt{\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)}<\frac{2\lambda}{\mu}-N. \qquad \text{(EC.56)}
$$

Note that, when $\mu\in(0,\frac{5\lambda}{4N}]$, the right-hand side of (EC.56) satisfies

$$
\frac{2\lambda}{\mu}-N\geq\frac{2\lambda}{\frac{5\lambda}{4N}}-N=\frac{3N}{5}>0.
$$

- If the left-hand side of (EC.56) is non-positive, then (EC.56) trivially holds.

- If the left-hand side of (EC.56) is strictly positive, then squaring both sides of (EC.56) and algebra yields

$$\left(\frac{\lambda}{\mu}-N\right)\left(\frac{\lambda}{\mu}-N-1\right)+4\left(\frac{\lambda}{\mu}+1\right)<\sqrt{\left[\left(\frac{\lambda}{\mu}-N\right)^2+4\left(1+\frac{\lambda}{\mu}\right)\right]\left[\left(\frac{\lambda}{\mu}-(N+1)\right)^2+4\left(1+\frac{\lambda}{\mu}\right)\right]}. \tag{EC.57}$$

Note that the left-hand side of (EC.57) satisfies

$$\left(\frac{\lambda}{\mu}-N\right)\left(\frac{\lambda}{\mu}-N-1\right)+4\left(\frac{\lambda}{\mu}+1\right)=\left(\frac{\lambda}{\mu}-N\right)^2+N+3\frac{\lambda}{\mu}+4>0.$$

Then, squaring both sides of (EC.57) and algebra implies

$$\begin{aligned}&2\left(\frac{\lambda}{\mu}-N\right)\left(\frac{\lambda}{\mu}-N-1\right)<\left(\frac{\lambda}{\mu}-N\right)^2+\left(\frac{\lambda}{\mu}-N\right)^2+\left(\frac{\lambda}{\mu}-N-1\right)^2\\ \Leftrightarrow\quad&\left(\frac{\lambda}{\mu}-N-1\right)^2-1<\left(\frac{\lambda}{\mu}-N-1\right)^2,\end{aligned}$$

which is clearly true. This concludes the induction step. Hence, (EC.53) holds as desired.

We conclude that $\frac{I(\mu,\mu;\lambda,N)}{\mu}$ is strictly increasing in μ for $\mu\in(0,\frac{5\lambda}{4N}]$ for all $\lambda>0$ and $N\geq 6$.

∎

EC.5.1.4. Proof of Lemma EC.11

From Corollary EC.3 (a), $I(\mu,\mu;\lambda,N)=1-\frac{\lambda}{N\mu}(1-ErlB(N,\frac{\lambda}{\mu}))$, and from Lemma EC.1 (c), $I(\mu,\mu;\lambda,N)$ is strictly decreasing in λ. In particular,

$$\lim_{\lambda\downarrow 0}I(\mu,\mu;\lambda,N)=1,$$

because $ErlB(N,\frac{\lambda}{\mu})\overset{(i)}{=}\frac{N-\frac{\lambda}{\mu}}{\frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}}\overset{(ii)}{\to}0$ as $\lambda\to 0$ (where (i) follows from the relationship between $ErlB$ and $ErlC$ in Lemma EC.2 (a), and (ii) follows from Lemma EC.2 (b))), and

$$\lim_{\lambda\uparrow\infty}I(\mu,\mu;\lambda,N)=0,$$

because $\frac{\lambda}{N\mu}(1-ErlB\left(N,\frac{\lambda}{\mu}\right))\overset{(iii)}{=}\frac{\lambda}{N\mu}\frac{\frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-N}{\frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}}\overset{(iv)}{\to}1$ as $\lambda\to\infty$ (where (iii) follows from the relationship between $ErlB$ and $ErlC$ in Lemma EC.2 (a), and (iv) follows from Lemma EC.2 (b)). Since $I(\mu,\mu;\lambda,N)$ is a monotonic and continuous function of λ, it follows that for any $x\in(0,1)$, there exists a unique $\lambda>0$ such that $I(\mu,\mu;\lambda,N)=x$.

Using the continuous extension of the Erlang B formula (Jagerman (1974)), the definition of $I(\mu,\mu;\lambda,N)$ can be extended to all real-valued $N>0$. Then, using the same technique as above,

$$\lim_{N\downarrow 0}I(\mu,\mu;\lambda,N)=0,$$

because $\frac{\lambda}{N\mu}(1-ErlB\left(N,\frac{\lambda}{\mu}\right))=\frac{\lambda}{N\mu}\frac{\frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-N}{\frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}}=\frac{\lambda}{\mu}\frac{\frac{1}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-1}{\frac{N}{ErlC\left(N,\frac{\lambda}{\mu}\right)}-\frac{\lambda}{\mu}}\to 1$ as $N\to 0$ (noting that $ErlC\left(N,\frac{\lambda}{\mu}\right)\to\infty$ as $N\to 0$, from Problem 2 in Whitt 2002, p.8), and

$$\lim_{N\uparrow\infty}I(\mu,\mu;\lambda,N)=1,$$

because $ErlB\left(N,\frac{\lambda}{\mu}\right)=\left[\frac{\lambda}{\mu}\int_0^\infty e^{-\frac{\lambda}{\mu}y}(1+y)^N dy\right]^{-1}\to 0$ as $N\to\infty$ (from (1.5) and (1.16) in Whitt (2002)). Since (the extended) $I(\mu,\mu;\lambda,N)$ is a monotonic and continuous function of N, it follows that for any $x\in(0,1)$, there exists a unique (real-valued) $N>0$ such that $I(\mu,\mu;\lambda,N)=x$.

■

EC.5.2. Proof of Theorem 2

Using Corollary EC.3 (b), the FOC (6) simplifies to

$$c'(\mu)=p\left(1-I(\mu,\mu;\lambda,N)\right)+(v-p\mu)\frac{I(\mu,\mu;\lambda,N)}{\mu}\left(1-I(\mu,\mu;\lambda,N)\right), \tag{EC.58}$$

which can be equivalently written as

$$c'(\mu)=\left(p\big(1-I(\mu,\mu;\lambda,N)\big)+\frac{v}{\mu}I(\mu,\mu;\lambda,N)\right)(1-I(\mu,\mu;\lambda,N)),$$

which establishes (10).

Next, note that by definition, a solution $\mu^\star>0$ to the symmetric FOC (10) is a symmetric equilibrium if and only if $\mu_1=\mu^\star$ is a global maximizer of the utility function $U(\mu_1,\mu^\star)$ (obtained by setting $k=N$ in (4)). To establish the latter, we show that for the loss system, $U(\mu_1,\mu^\star)$ is strictly concave; that is, $\frac{\partial^2 U(\mu_1,\mu)}{\partial\mu_1^2}<0$, for all $\mu_1,\mu>0$. Note that

$$\begin{aligned}
\frac{\partial^2 U(\mu_1,\mu)}{\partial\mu_1^2}=&\frac{\partial^2}{\partial\mu_1^2}\left(p\mu_1+(v-p\mu_1)I(\mu_1,\mu)-c(\mu_1)\right)\\
=&\frac{\partial}{\partial\mu_1}\left(p\left(1-I(\mu_1,\mu)\right)-p\mu_1\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}+v\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}-c'(\mu_1)\right)\\
\overset{(i)}{=}&\frac{\partial}{\partial\mu_1}\left(p\left(1-I(\mu_1,\mu)\right)-p\mu_1\frac{I(\mu_1,\mu)}{\mu_1}\left(1-I(\mu_1,\mu)\right)+v\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}-c'(\mu_1)\right)\\
=&\frac{\partial}{\partial\mu_1}\left(p\left(1-I(\mu_1,\mu)\right)^2+v\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}-c'(\mu_1)\right)\\
=&-2p\big(1-I(\mu_1,\mu)\big)\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}+v\frac{\partial^2 I(\mu_1,\mu)}{\partial\mu_1^2}-c''(\mu_1)\\
\overset{(ii)}{<}&0,\quad\forall\mu_1,\mu>0,
\end{aligned}$$

where (i) follows from Lemma EC.9 (b), and (ii) follows because $\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}>0$ for all $\mu_1,\mu>0$ (from Lemma EC.9 (b)), $\frac{\partial^2 I(\mu_1,\mu)}{\partial\mu_1^2}<0$ for all $\mu_1,\mu>0$ (from Lemma EC.9 (c)), and $c''>0$ (recalling that c is strictly convex). Hence, the utility function for the loss system is concave, as desired.

Finally, when $k=N$, the sufficient condition (7) of Theorem 1 that guarantees rge existence of a solution to the symmetric FOC (10) simplifies to $c'(0)<p+v/\lambda$.

■

EC.5.3. Proof of Proposition 2

Suppose $\mu_1^\star$, $\mu_2^\star$ are symmetric equilibrium service rates such that $\mu_1^\star>\mu_2^\star>0$. Setting $\mu_1=\mu$ in (4) yields the server utility at any symmetric service rate μ. Therefore,

$$U(\mu,\mu)=p\mu\left(1-I(\mu,\mu)\right)+vI(\mu,\mu)-c(\mu),\ \text{for }\mu=\mu_1^\star,\mu_2^\star. \tag{EC.59}$$

The service rates $\mu_1^\star$ and $\mu_2^\star$ satisfy the symmetric FOC (10) from Theorem 2, that is,

$$\mu c'(\mu) = \big(U(\mu,\mu) + c(\mu)\big)\,(1 - I(\mu,\mu))\,, \text{ for } \mu = \mu_1^\star, \mu_2^\star. \tag{EC.60}$$

Moreover, strict convexity of c implies that

$$c(\mu_1^\star) - c(\mu_2^\star) < (\mu_1^\star - \mu_2^\star)c'(\mu_1^\star) < \mu_1^\star c'(\mu_1^\star) - \mu_2^\star c'(\mu_2^\star),$$

where the second inequality follows from $c'(\mu_1^\star) > c'(\mu_2^\star)$ for $\mu_1^\star > \mu_2^\star$. Thus,

$$c(\mu_1^\star) - c(\mu_2^\star) < \mu_1^\star c'(\mu_1^\star) - \mu_2^\star c'(\mu_2^\star). \tag{EC.61}$$

Substituting for $\mu_1^\star c'(\mu_1^\star)$ and $\mu_2^\star c'(\mu_2^\star)$ using (EC.60), (EC.61) becomes

$$c(\mu_1^\star) - c(\mu_2^\star) < \big(U(\mu_1^\star,\mu_1^\star) + c(\mu_1^\star)\big)\,(1 - I(\mu_1^\star,\mu_1^\star)) - \big(U(\mu_2^\star,\mu_2^\star) + c(\mu_2^\star)\big)\,(1 - I(\mu_2^\star,\mu_2^\star))\,,$$

which, after algebra, can be rewritten as

$$U(\mu_1^\star,\mu_1^\star) - U(\mu_2^\star,\mu_2^\star) > (U(\mu_1^\star,\mu_1^\star) + c(\mu_1^\star))\,I(\mu_1^\star,\mu_1^\star) - (U(\mu_2^\star,\mu_2^\star) + c(\mu_2^\star))\,I(\mu_2^\star,\mu_2^\star). \tag{EC.62}$$

Finally, we show that $(U(\mu,\mu) + c(\mu))\,I(\mu,\mu)$ is a strictly increasing function of $\mu > 0$, so that (EC.62) implies $U(\mu_1^\star,\mu_1^\star) - U(\mu_2^\star,\mu_2^\star) > 0$ for $\mu_1^\star > \mu_2^\star$.

Let $f(\mu) := (U(\mu,\mu) + c(\mu))\,I(\mu,\mu)$ for $\mu > 0$. From (EC.59) and Corollary EC.3 (a),

$$f(\mu) = (p\mu(1 - I(\mu,\mu)) + vI(\mu,\mu))\,I(\mu,\mu) = \left(p\frac{\lambda}{N}\left(1 - ErlB\left(N, \frac{\lambda}{\mu}\right)\right) + vI(\mu,\mu)\right) I(\mu,\mu),$$

where $ErlB(N,\rho)$ is strictly increasing in ρ (from Lemma EC.1 (a)), and thus is strictly decreasing in μ (since $\rho = \lambda/\mu$), and $I(\mu,\mu)$ is strictly increasing in μ (either by directly calculating $\frac{dI(\mu,\mu)}{d\mu}$ using Corollary EC.3 (a) or by applying Lemma EC.4 (a) twice). Hence, using the fact that the product of two strictly positive and strictly increasing functions is strictly increasing, $f(\mu)$ is strictly increasing in μ, as desired.

■

EC.5.4. Proof of Proposition 3

From Theorem 2, obtaining $\mu_{\max}(p,v)$ (defined in (5)) for loss systems is equivalent to maximizing a solution to the FOC (10) over all λ, N. Let $x = I(\mu,\mu;\lambda,N)$. For $\mu \neq 0$, the FOC (10), after multiplying by μ on both sides, can be written as

$$\mu c'(\mu) = \big(p\mu(1-x) + vx\big)\cdot(1-x). \tag{EC.63}$$

Given any $x \in (0,1)$, $v > 0$, and $p \geq 0$, we observe that (i) the right-hand side of (EC.63) is linear in μ with a strictly positive y-intercept, $vx(1-x)$, and a non-negative slope, $p(1-x)^2$, and (ii) the left-hand side of (EC.63) is a strictly increasing and strictly convex function (from Assumption 1) with zero y-intercept, thereby resulting in the existence of a unique solution $\mu(x;p,v) > 0$. From Lemma EC.11, for any $\mu > 0$ and $x \in (0,1)$, there exists a pair (λ, N) such that

$x = I(\mu,\mu;\lambda,N)$. As a result, since we are optimizing over all (λ,N), (5) becomes equivalent to $\mu^\star_{\max}(p,v) = \sup_{x\in(0,1)}\mu(x;p,v)$.

We study $\mu'(x)$ and $\mu''(x)$ by differentiating (EC.63) twice with respect to x, and obtain the following observations: (i) $\mu'(x) < 0$ when $\mu'(x) = 0$, and (ii) $\mu'(x) = 0$ has at most one solution in $(0,\frac{1}{2}]$ and no solutions in $(\frac{1}{2},1)$. To see this, we first differentiate (EC.63) with respect to x and after algebra we obtain

$$(\mu c''(\mu) + c'(\mu))\,\mu'(x) = v(1-2x) - 2p\mu(1-x) + p(1-x)^2\mu'(x),$$

which implies

$$\mu'(x) = \frac{v(1-2x) - 2p\mu(1-x)}{\mu c''(\mu) + c'(\mu) - p(1-x)^2} \overset{(*)}{=} \frac{v(1-2x) - 2p\mu(1-x)}{\mu c''(\mu) + \frac{v}{\mu}x(1-x)},$$

where $(*)$ follows from (EC.63). Note that the above display can be rewritten as

$$\frac{1}{2}\mu'(x) = \frac{v\left(\frac{\frac{1}{2}-x}{1-x}\right) - p\mu(x)}{\frac{\mu(x)c''(\mu(x))}{1-x} + v\frac{x}{\mu(x)}}, \tag{EC.64}$$

which is strictly negative when $x > \frac{1}{2}$. Thus, $\mu'(x) = 0$ has no solution in $(\frac{1}{2},1)$ and, if it has a solution in $(0,1)$, then that solution must lie in $(0,\frac{1}{2}]$. To see this, we take the derivative of (EC.64):

$$\frac{1}{2}\mu''(x) = \frac{\left(\frac{\mu(x)c''(\mu(x))}{1-x} + v\frac{x}{\mu(x)}\right)\left(-\frac{v}{2(1-x)^2} - p\mu'(x)\right) - \left(v\left(\frac{\frac{1}{2}-x}{1-x}\right) - p\mu(x)\right)\frac{d}{dx}\left(\frac{\mu(x)c''(\mu(x))}{1-x} + v\frac{x}{\mu(x)}\right)}{\left(\frac{\mu(x)c''(\mu(x))}{1-x} + v\frac{x}{\mu(x)}\right)^2},$$

which is strictly negative when $\mu'(x) = 0$, by noting that $v\left(\frac{\frac{1}{2}-x}{1-x}\right) - p\mu(x) = \frac{1}{2(1-x)}\left(v(1-2x) - 2p\mu(1-x)\right) = 0$. Hence, $\mu'(x) = 0$ has at most one solution in $(0,\frac{1}{2}]$ because $\mu''(x)$ evaluated at two consecutive solutions to $\mu'(x) = 0$ must have opposite signs, using the fact that the derivative of a continuous function evaluated at its two consecutive roots have opposite signs. From the above, the two properties (i) and (ii) are proved.

Based on (i) and (ii), we can argue that if a solution to $\mu'(x) = 0$ exists, denoted by $x^\star \in (0,\frac{1}{2}]$, then $\mu^\star_{\max} = \mu(x^\star)$; if not, then $\mu(x)$ is strictly decreasing in $x \in (0,1)$, implying that $\mu^\star_{\max} = \lim_{x\downarrow 0}\mu(x)$. Substituting $x = 0$ into (EC.63), we define

$$\mu(0) := \lim_{x\downarrow 0}\mu(x) = \begin{cases} 0, & \text{if } 0 \le p \le c'(0), \\ (c')^{-1}(p), & \text{otherwise .} \end{cases} \tag{EC.65}$$

Substituting $x = 0$ into (EC.64), we define

$$\mu'(0) := \lim_{x\downarrow 0}\mu'(x) = \frac{v - 2p\mu(0)}{\mu(0)c''(\mu(0)) + v\lim_{x\downarrow 0}\frac{x}{\mu(x)}}$$

Dividing (EC.63) by μ on both sides and substituting for $x = 0$ yields $\lim_{x\downarrow 0}\frac{x}{\mu(x)} = \frac{c'(\mu(0)) - p}{v}$. Then, substitution into the above display using this and (EC.65) yields

$$\mu'(0) = \frac{v - 2p\mu(0)}{\mu(0)c''(\mu(0)) + c'(\mu(0)) - p} = \begin{cases} \frac{v}{c'(0)-p}, & \text{if } 0 \le p < c'(0), \\ +\infty, & \text{if } p = c'(0), \\ \frac{v - 2p(c')^{-1}(p)}{(c')^{-1}(p)c''((c')^{-1}(p))}, & \text{if } p > c'(0). \end{cases} \tag{EC.66}$$

Next, the intermediate value theorem implies that $\mu'(x)=0$ has a unique solution $x^\star \in (0, \frac{1}{2}]$ if and only if $\mu'(0)>0$, since $\mu'(\frac{1}{2})<0$ from (EC.64). This condition, from (EC.66), is equivalent to

$$\frac{v}{2p} > (c')^{-1}(p). \tag{EC.67}$$

Given $v>0$, let $p^\ddagger(v)$ be the unique solution for $p>c'(0)$ to $\frac{v}{2p}=(c')^{-1}(p)$, or, equivalently, $c'(\frac{v}{2p})=p$. Note that $\frac{v}{2p}$ is a strictly decreasing function of p for $p>c'(0)$, with value $\frac{v}{2c'(0)}$ at $p=c'(0)$ and value 0 at $p=\infty$; $(c')^{-1}(p)$ is a strictly increasing function of p for $p>c'(0)$, with value 0 at $p=c'(0)$ and ∞ at $p=\infty$. This implies that $p^\ddagger(v)$ must exist and is unique. Then, (EC.67) is equivalent to $0\le p<p^\ddagger(v)$. This allows us to characterize $\mu^\star_{\max}$ and $I(\mu^\star_{\max},\mu^\star_{\max})$ by considering the two cases $0\le p<p^\ddagger(v)$ and $p\ge p^\ddagger(v)$ separately.

Case (I): If $0\le p<p^\ddagger(v)$, then $\mu'(x)=0$ has a unique solution $x^\star\in(0,\frac{1}{2}]$. From (EC.64),

$$v\left(\frac{\frac{1}{2}-x^\star}{1-x^\star}\right)=p\mu(x^\star).$$

Note that $\frac{\frac{1}{2}-x^\star}{1-x^\star}$ is a strictly decreasing function of $x^\star\in(0,\frac{1}{2}]$, with value $\frac{1}{2}$ at $x^\star=0$ and value 0 at $x^\star=\frac{1}{2}$; that is, $\frac{\frac{1}{2}-x^\star}{1-x^\star}<\frac{1}{2}$ for all $x^\star\in(0,\frac{1}{2}]$. Thus, the above equation implies that $\mu^\star_{\max}=\mu(x^\star)<\frac{v}{2p}$. Additionally, the above equation implies

$$I(\mu^\star_{\max},\mu^\star_{\max})=x^\star=\frac{v-2p\mu^\star_{\max}}{2v-2p\mu^\star_{\max}},$$

recalling that $\mu^\star_{\max}=\mu(x^\star)$ and $x^\star=I(\mu^\star_{\max},\mu^\star_{\max})$. Substituting the expression for $x^\star$ into (EC.63), we obtain

$$\mu^\star_{\max}c'(\mu^\star_{\max})=\frac{v^2}{4\left(v-p\mu^\star_{\max}\right)}, \tag{EC.68}$$

equivalently, $\mu^\star_{\max}$ solves

$$(v-p\mu)\mu c'(\mu)=\frac{v^2}{4}. \tag{EC.69}$$

In what follows, we verify that (EC.69) has a unique solution in $(0,\frac{v}{2p})$. Note that the right-hand side of (EC.69) is a constant, and the left-hand side of (EC.69), denoted by $LHS(\mu)$, is a strictly increasing function of μ for $\mu\in(0,\frac{v}{2p})$. To see this, differentiating $LHS(\mu)$ with respect to μ:

$$\begin{aligned}LHS'(\mu)=&(v-p\mu)(\mu c''(\mu)+c'(\mu))-p\mu c'(\mu)\\=&(v-p\mu)\mu c''(\mu)+(v-2p\mu)c'(\mu)>0,\end{aligned}$$

when $\mu\in(0,\frac{v}{2p})$, noting that $c'>0$ and $c''>0$. Moreover, note that $LHS(0)=0<\frac{v^2}{4}$, and $LHS(\frac{v}{2p})=\frac{v^2}{4}\frac{1}{p}c'(\frac{v}{2p})>\frac{v^2}{4}$ because $p<p^\ddagger(v)$; therefore, it follows that (EC.69) admits a unique solution in $(0,\frac{v}{2p})$; that is, there exists a unique $\mu^\star_{\max}$ in $(0,\frac{v}{2p})$.

When p increases, the left-hand side of (EC.69) (which is a strictly increasing function of μ) decreases for all μ, and the right-hand side of (EC.69) remains unchanged for each μ, so $\mu^\star_{\max}$ (which is the solution to (EC.69)) is strictly increasing in p.

Case (II): If $p \geq p^\ddagger(v)$, (EC.65) implies that $\mu^\star_{\max} = \mu(0) = (c')^{-1}(p)$ (since $p \geq c'(\frac{v}{2p}) > c'(0)$ where the first inequality follows from $p \geq p^\ddagger(v)$), which is strictly increasing in p (since c' is strictly increasing by strict convexity of c, and the inverse of a strictly monotone function is strictly monotone). Hence, $c'(\mu^\star_{\max}) = p$ and $I(\mu^\star_{\max}, \mu^\star_{\max}) = 0$.

■

EC.6. Proofs From Section 4.2

From (11) and (12), the server's utility function is given by

$$U(\mu;\lambda,k,p,v) = \frac{v - p\mu}{1 + \sum_{i=1}^{k} \left(\frac{\lambda}{\mu}\right)^i} + p\mu - c(\mu). \tag{EC.70}$$

EC.6.1. Proof of Lemma 4

The server's utility function (EC.70), after algebra, can be equivalently written as

$$U(\mu;\lambda,k,p,v) = \frac{v}{1 + \sum_{i=1}^{k} \left(\frac{\lambda}{\mu}\right)^i} + \frac{p\lambda}{\frac{\lambda}{\mu} + \left(1 + \sum_{i=1}^{k-1} \left(\frac{\lambda}{\mu}\right)^i\right)^{-1}} - c(\mu).$$

Then, it is clear that $\lim_{\mu \downarrow 0} U(\mu;\lambda,k,p,v) = 0$ and $\lim_{\mu \to \infty} U(\mu;\lambda,k,p,v) = -\infty$. Since $U(\mu;\lambda,k,p,v)$ is a continuous function of $\mu > 0$, if $U(\mu;\lambda,k,p,v)$ is non-negative for some μ, it must attain a global maximum for some $\mu \in (0,\infty)$, i.e., there exists an equilibrium.

■

EC.6.2. Proof of Proposition 4

When $p \leq c'(0)$, $c'(\mu) > c'(0) \geq p$ for all $\mu > 0$, because c' is a strictly increasing function (recalling strict convexity of c). Thus, $c'(\mu^\star) > p$ for any equilibrium $\mu^\star > 0$. By definition, any equilibrium $\mu^\star > 0$ satisfies the FOC (13), where $c'(\mu^\star) > p$ ensures that the left-hand side of (13) is strictly positive, implying that the right-hand side of (13) is also strictly positive. Hence, $\mu^\star < \frac{v}{p}$.

Suppose that $\mu^\star(k) > 0$ is a server equilibrium for an $M/M/1/k$ queueing system for some $k \in \mathbb{Z}_+ \cup \{\infty\}$. Then, $U(\mu^\star(k);\lambda,k,p,v) \geq 0$ (from Proposition 1). Note that, when $\mu < \frac{v}{p}$, the server's utility function (EC.70) is strictly decreasing in k for all $k \in \mathbb{Z}_+ \cup \{\infty\}$; in particular, $\lim_{k\to\infty} U(\mu;\lambda,k,p,v) < U(\mu;\lambda,k,p,v)$ for all $k \in \mathbb{Z}_+$ when $\mu < \frac{v}{p}$. Thus, for any $1 \leq k' \leq k$, $U(\mu^\star(k);\lambda,k',p,v) \geq U(\mu^\star(k);\lambda,k,p,v) \geq 0$ (since $\mu^\star(k) < \frac{v}{p}$). Therefore, there exists some $\mu > 0$ (namely, $\mu^\star(k)$) for which $U(\mu;\lambda,k',p,v) \geq 0$. From Lemma 4, we conclude that there exists an equilibrium $\mu^\star(k') > 0$ in the $M/M/1/k'$ system.

■

EC.6.3. Proof of Theorem 3

Differentiating $U(\mu;\lambda,k,p,v)$ in (EC.70) with respect to μ yields

$$U'(\mu;\lambda,k,p,v)=\left(\frac{v}{\mu}-p\right)\frac{\sum_{i=0}^{k} i\left(\frac{\lambda}{\mu}\right)^i}{\left(\sum_{i=0}^{k}\left(\frac{\lambda}{\mu}\right)^i\right)^2}-\left(c'(\mu)-p+\frac{p}{\sum_{i=0}^{k}\left(\frac{\lambda}{\mu}\right)^i}\right) \tag{EC.71}$$

$$=\left(\frac{v-p\mu}{\lambda}\right)\frac{k+\sum_{i=1}^{k}(k-i)\left(\frac{\mu}{\lambda}\right)^i}{\left(1+\sum_{i=1}^{k}\left(\frac{\mu}{\lambda}\right)^i\right)^2}\left(\frac{\mu}{\lambda}\right)^{k-1}-\frac{p}{1+\sum_{i=1}^{k}\left(\frac{\lambda}{\mu}\right)^i}+p-c'(\mu). \tag{EC.72}$$

(a): When $k=1$, from (EC.71),

$$U'(\mu;\lambda,1,p,v)=\left(\frac{v}{\mu}-p\right)\frac{\frac{\lambda}{\mu}}{\left(1+\frac{\lambda}{\mu}\right)^2}-\left(c'(\mu)-p+\frac{p}{1+\frac{\lambda}{\mu}}\right)=\left(\frac{\lambda}{\lambda+\mu}\right)^2\left(\frac{v}{\lambda}+p\right)-c'(\mu),$$

which is strictly decreasing in $\mu\in(0,\infty)$, with value $\frac{v}{\lambda}+p-c'(0)$ when $\mu\to 0$ and value $-\infty$ when $\mu\to\infty$. Therefore, $\mu^\star(1)$ exists if and only if $U'(\mu;\lambda,1,p,v)$ at $\mu=0$ is strictly positive; that is, $p>c'(0)-\frac{v}{\lambda}$. Hence, when $p\le c'(0)-\frac{v}{\lambda}$, $\mu^\star(1)>0$ does not exist. Then, Proposition 4 implies that $\mu^\star(k)>0$ does not exist for any $k\in\mathbb{Z}_+\cup\{\infty\}$, when $p\le c'(0)-\frac{v}{\lambda}$.

(b): When $c'(0)-\frac{v}{\lambda}<p\le c'(0)$, we first note that $\mu^\star(1)$ exists from the proof of (a) since $c'(0)-\frac{v}{\lambda}<p$. Then, it suffices to derive the existence conditions for an $M/M/1/k$ system when $k\to\infty$, i.e., an infinite-buffer $M/M/1$ system. Then, Proposition 4 guarantees that an equilibrium exists for all $k\in\mathbb{Z}_+\cup\{\infty\}$.

We begin by evaluating the derivative of $\lim_{k\to\infty}U(\mu;\lambda,k,p,v)$ with respect to μ, denoted by $(\lim_{k\to\infty}U(\mu;\lambda,k,p,v))'$. From (EC.70),

$$\lim_{k\to\infty}U(\mu;\lambda,k,p,v)=\begin{cases}p\mu-c(\mu), & \mu\le\lambda,\\ v+p\lambda-\frac{v\lambda}{\mu}-c(\mu), & \mu>\lambda.\end{cases}$$

Differentiating the above display with respect to μ yields

$$\left(\lim_{k\to\infty}U(\mu;\lambda,k,p,v)\right)'=\begin{cases}p-c'(\mu), & \mu\le\lambda,\\ \frac{v\lambda}{\mu^2}-c'(\mu), & \mu>\lambda.\end{cases}$$

It is clear that

- $(\lim_{k\to\infty}U(\mu;\lambda,k,p,v))'$ is strictly decreasing in $\mu\in(0,\lambda]$, with value $p-c'(0)$ at $\mu=0$ and $p-c'(\lambda)$ at $\mu=\lambda$;
- $(\lim_{k\to\infty}U(\mu;\lambda,k,p,v))'$ is strictly decreasing in $\mu\in(\lambda,\infty)$, with value $\frac{v}{\lambda}-c'(\lambda)$ as $\mu\to\lambda+$ and $-\infty$ as $\mu\to\infty$;
- $\lim_{k\to\infty}U(\mu;\lambda,k,p,v)$ is not differentiable at $\mu=\lambda$ when $p\neq\frac{v}{\lambda}$.

When $p\le c'(0)$, $p-c'(\mu)<p-c'(0)\le 0$, i.e., $(\lim_{k\to\infty}U(\mu;\lambda,k,p,v))'$ is strictly negative for all $\mu\in(0,\lambda]$. This implies that $\lim_{k\to\infty}U(\mu;\lambda,k,p,v)$ is strictly negative for all $\mu\in(0,\lambda]$ (recalling that $\lim_{k\to\infty}U(0;\lambda,k,p,v)=0$). Therefore, an equilibrium, if exists, must be under-loaded. Recall that $(\lim_{k\to\infty}U(\mu;\lambda,k,p,v))'$ is strictly decreasing in $\mu\in(\lambda,\infty)$, the solution to $(\lim_{k\to\infty}U(\mu;\lambda,k,p,v))'=0$ in (λ,∞), if exists, is unique, and a local maximizer. Hence, when $p\le c'(0)$, there exists an equilibrium $\mu^\star(k)>\lambda$ as $k\to\infty$ if and only if

(i) $(\lim_{k\to\infty} U(\mu;\lambda,k,p,v))' > 0$ when $\mu \to \lambda+$ (given that $\lim_{k\to\infty} U(\mu;\lambda,k,p,v)$ is continuous at $\mu = \lambda$), and

(ii) The solution $\mu^{\star,?} \in (\lambda,\infty)$ that satisfies $\left(\lim_{k\to\infty} U(\mu^{\star,?};\lambda,k,p,v)\right)' = 0$ also satisfies $\lim_{k\to\infty} U(\mu^{\star,?};\lambda,k,p,v) \geq \lim_{k\to\infty} U(0;\lambda,k,p,v)$.

Condition (i) can be equivalently written as

$$\frac{v}{\lambda} - c'(\lambda) > 0 \quad \Leftrightarrow \quad v > \lambda c'(\lambda).$$

For condition (ii), note that the solution $\mu^{\star,?} \in (\lambda,\infty)$ to $\left(\lim_{k\to\infty} U(\mu^{\star,?};\lambda,k,p,v)\right)' = 0$ satisfies

$$\frac{v\lambda}{(\mu^{\star,?})^2} - c'(\mu^{\star,?}) = 0. \tag{EC.73}$$

Then, $\lim_{k\to\infty} U(\mu^{\star,?};\lambda,k,p,v) \geq \lim_{k\to\infty} U(0;\lambda,k,p,v) = 0$ can be equivalently written as

$$\frac{v\lambda}{\mu^{\star,?}} + c(\mu^{\star,?}) \leq v + p\lambda.$$

Substitution using (EC.73) yields

$$\mu^{\star,?} c'(\mu^{\star,?}) + c(\mu^{\star,?}) \leq v + p\lambda.$$

(c): When $p > c'(0)$. From (EC.72), note that

$$\begin{aligned}
\lim_{\mu\downarrow 0} U'(\mu;\lambda,k,p,v) &= \lim_{\mu\downarrow 0}\left[\left(\frac{v-p\mu}{\lambda}\right)\frac{k+\sum_{i=1}^{k}(k-i)\left(\frac{\mu}{\lambda}\right)^i}{\left(1+\sum_{i=1}^{k}\left(\frac{\mu}{\lambda}\right)^i\right)^2}\left(\frac{\mu}{\lambda}\right)^{k-1} - \frac{p}{1+\sum_{i=1}^{k}\left(\frac{\lambda}{\mu}\right)^i} + p - c'(\mu)\right] \\
&= \frac{v}{\lambda}\cdot\lim_{\mu\downarrow 0}\frac{k}{\left(1-\left(\frac{\mu}{\lambda}\right)^{k+1}\right)^2}\cdot\lim_{\mu\downarrow 0}\left(\frac{\mu}{\lambda}\right)^{k-1} + p - c'(0) \\
&= \frac{vk}{\lambda}\cdot\lim_{\mu\downarrow 0}\left(\frac{\mu}{\lambda}\right)^{k-1} + p - c'(0) > 0, \quad \text{for all } k \geq 1,
\end{aligned}$$

noting that $\lim_{\mu\downarrow 0}\left(\frac{\mu}{\lambda}\right)^{k-1} \geq 0$, and $p - c'(0) > 0$ by assumption. Recalling that $\lim_{\mu\downarrow 0} U(\mu;\lambda,k,p,v) = 0$, $\lim_{\mu\downarrow 0} U'(\mu;\lambda,k,p,v) > 0$ implies that there must exist some $\mu > 0$ for which $U(\mu;\lambda,k,p,v) > 0$. Lemma 4 then guarantees an equilibrium.

$\blacksquare$

EC.7. Proofs From Section 5

EC.7.1. Preliminaries A: Asymptotic Properties of Erlang Formulae Under Linear Staffing

We present the following asymptotic properties of the Erlang B and Erlang C Formulae, which are useful for the proofs from Section 5.

LEMMA EC.12 **(Asymptotic Properties of Erlang Formulae)**. *The following hold under linear staffing (14).*

(a)

$$\lim_{\lambda\to\infty} ErlB\left(N^\lambda, \frac{\lambda}{\mu}\right) = \left(1 - \frac{\mu}{a}\right)^+ = \begin{cases} 1 - \frac{\mu}{a}, & \mu < a, \\ 0, & \mu \geq a. \end{cases}$$

(b)

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases} \infty, & \mu < a, \\ 0, & \mu > a, \end{cases}$$

and, when $\mu = a$,

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{a}\right) = \begin{cases} \infty, & 0 < \frac{\lambda}{a} - N^\lambda \in \omega(\sqrt{\lambda}) \cap o(\lambda), \\ \left(1 - \frac{z\Phi^c(z)}{\phi(z)}\right)^{-1} \in (1,\infty), & 0 < \frac{\lambda}{a} - N^\lambda \in \Theta(\sqrt{\lambda}), \\ 1, & |N^\lambda - \frac{\lambda}{a}| \in o(\sqrt{\lambda}), \\ \left(1 - \frac{z\Phi^c(z)}{\phi(z)}\right)^{-1} \in (0,1), & 0 < N^\lambda - \frac{\lambda}{a} \in \Theta(\sqrt{\lambda}), \\ 0, & 0 < N^\lambda - \frac{\lambda}{a} \in \omega(\sqrt{\lambda}) \cap o(\lambda), \end{cases}$$

where $z = \lim_{\lambda\to\infty} \frac{\frac{\lambda}{a} - N^\lambda}{\sqrt{N^\lambda}}$, $\Phi^c(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{t^2}{2}} dt$ *and* $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.

Lemma EC.13 **(Asymptotic Properties of Functions of Erlang Formulae)**. *The following hold under linear staffing (14).*

(a) If $\mu < a$, *then* $\lim_{\lambda\to\infty} \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{\lambda} = \frac{(a-\mu)^2}{a^2\mu}$. *That is,* $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ *converges to* ∞ *linearly fast as* $\lambda \to \infty$.

(b) If $\mu > a$, *then* $\lim_{\lambda\to\infty} P(\lambda) ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = 0$, *where* $P(\lambda)$ *represents any polynomial in* λ. *That is,* $ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)$ *converges to zero super-polynomially fast as* $\lambda \to \infty$.

(c) If $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(\sqrt{\lambda})$, *then* $\lim_{\lambda\to\infty} \sqrt{\lambda} \left(\frac{1 - ErlC\left(N^\lambda, \frac{\lambda}{a}\right)}{N^\lambda - \frac{\lambda}{a}}\right) \in (0,\infty)$ *and* $\lim_{\lambda\to\infty} \sqrt{\lambda}\ ErlB\left(N^\lambda, \frac{\lambda}{a}\right) \in (0,\infty)$.

(d) $\lim_{\lambda\to\infty} \lambda \left(\frac{1 - ErlC\left(N^\lambda, \frac{\lambda}{a}\right)}{N^\lambda - \frac{\lambda}{a}}\right) = \infty$.

Proofs of Lemmas EC.12- EC.13

The proofs can be found in Zhong et al. (2023) with consistent numbering. In particular, Lemma EC.12 (b) when $\mu = a$ requires a substantial amount of effort, which merits a stand-alone paper Gopalakrishnan and Zhong (2023).

EC.7.2. Preliminaries B: Limiting Idle Time and Derivative of Idle Time

Building on the asymptotic properties of the Erlang formulae in Section EC.7.1, under linear staffing 14, we derive the limiting values of $I^\lambda(\mu_1, \mu) := I(\mu, \mu; \lambda, k^\lambda, N^\lambda)$ and $\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}$ as $\lambda \to \infty$ for any $\mu_1 > 0$ and $\mu > 0$.

Proposition EC.1. *Fix* $\mu_1 > 0$. *Under linear staffing (14),*

$$\lim_{\lambda\to\infty} I^\lambda(\mu_1, \mu) = \frac{\left(1 - \frac{a}{\mu}\right)^+}{1 - \frac{a}{\mu} + \frac{a}{\mu_1}} = \begin{cases} 0, & \mu \le a, \\ \frac{1 - \frac{a}{\mu}}{1 - \frac{a}{\mu} + \frac{a}{\mu_1}}, & \mu > a. \end{cases}$$

Proposition EC.2. *Fix $\mu_1 > 0$. Under linear staffing (14),*

$$\lim_{\lambda\to\infty}\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}=\frac{\frac{a}{\mu_1^2}\left(1-\frac{a}{\mu}\right)^+}{\left(1-\frac{a}{\mu}+\frac{a}{\mu_1}\right)^2}=\begin{cases}0, & \mu\le a,\\ \frac{\frac{a}{\mu_1^2}\left(1-\frac{a}{\mu}\right)}{\left(1-\frac{a}{\mu}+\frac{a}{\mu_1}\right)^2}, & \mu>a.\end{cases}$$

For ease of presentation, we denote $I^\lambda(\mu_1,\mu)$ and $\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}$ simply by I^λ and $\frac{\partial I^\lambda}{\partial\mu_1}$, respectively; and let $d_1^\lambda := N^\lambda-\left(1-\frac{\mu_1}{\mu}\right)$, $d_2^\lambda := d_1^\lambda-\rho^\lambda=(N^\lambda-\rho^\lambda)-\left(1-\frac{\mu_1}{\mu}\right)$, and $C^\lambda := ErlC(N^\lambda,\rho^\lambda)$, where $\rho^\lambda=\frac{\lambda}{\mu}$.

Proof of Proposition EC.1: From (1) in Lemma 2,

$$I^\lambda=\left(1+\frac{\mu}{\mu_1}\left(\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}+\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right)\right)^{-1}. \tag{EC.74}$$

Observe that $\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\ge 0$ (from Lemma EC.2 (c)) and $\frac{1}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\ge 0$ (recalling the definition $d_2^\lambda := d_1^\lambda-\rho^\lambda$) for all λ, resulting in all the summands in (EC.74) being non-negative for all λ. Therefore, I^λ would vanish in the limit as $\lambda\to\infty$ even if one of them grows unboundedly with λ.

Case (I): If $\mu<a$, then $\lim_{\lambda\to\infty}\frac{\rho^\lambda}{N^\lambda-\rho^\lambda}=\left(\frac{\mu}{a}-1\right)^{-1}\in(-\infty,0)$ and $\lim_{\lambda\to\infty}C^\lambda=\infty$ (from Lemma EC.12 (b)). Using these facts, (EC.74) implies that $\lim_{\lambda\to\infty}I^\lambda=0$.

Case (II): If $\mu>a$, then $\lim_{\lambda\to\infty}\frac{\rho^\lambda}{N^\lambda-\rho^\lambda}=\left(\frac{\mu}{a}-1\right)^{-1}$ and $\lim_{\lambda\to\infty}C^\lambda=0$ (from Lemma EC.12 (b)). Moreover, $\lim_{\lambda\to\infty}\frac{\rho^\lambda}{d_2^\lambda}=\left(\frac{\mu}{a}-1\right)^{-1}\in(0,\infty)$ and $\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\in(0,1]$ for all large enough λ. Using these facts, (EC.74) implies that

$$\lim_{\lambda\to\infty}I^\lambda=\left(1+\frac{\mu}{\mu_1}\left(\frac{\mu}{a}-1\right)^{-1}\right)^{-1}=\frac{1-\frac{a}{\mu}}{1-\frac{a}{\mu}+\frac{a}{\mu_1}}.$$

Case (III): If $\mu=a$, then $\lim_{\lambda\to\infty}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}=\infty$ (from Lemma EC.13 (d), recalling that when $\mu=a$, $\rho^\lambda=\frac{\lambda}{a}$). Thus, (EC.74) implies that $\lim_{\lambda\to\infty}I^\lambda=0$.

∎

The proof of Proposition EC.2 will rely on the asymptotic behavior of k^λ in relation to N^λ. This requires the following setup. Consider a subsequence λ' on which $b:=\lim_{\lambda'\to\infty}d_2^{\lambda'}\frac{k^{\lambda'}-N^{\lambda'}}{d_1^{\lambda'}}\in\mathbb{R}\cup\{-\infty,\infty\}$. If $b=0$, then consider a further subsequence λ'' on which $d_2^{\lambda''}\frac{k^{\lambda''}-N^{\lambda''}}{d_1^{\lambda''}}>0$ for all large enough λ'' or $d_2^{\lambda''}\frac{k^{\lambda''}-N^{\lambda''}}{d_1^{\lambda''}}<0$ for all large enough λ''. Simply using λ rather than λ' or λ'' to denote the subsequence, it is sufficient to consider four cases depending on the asymptotic behavior of $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}$: (i) $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in\{-\infty\}\cup(-\infty,0)$ and $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in\{-\infty\}\cup(-\infty,0)$ for all large enough λ; (ii) $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(0,\infty)\cup\{\infty\}$ and $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(0,\infty)\cup\{\infty\}$ for all large enough λ; (iii) $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$ and $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(-\infty,0)$ for all large enough λ; and (iv) $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$ and $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(0,\infty)$ for all large enough λ. For identical reasons, when $\mu=a$, it is sufficient to consider four cases depending on the asymptotic behavior of d_2^λ: (i) $\lim_{\lambda\to\infty} d_2^\lambda \in \{-\infty\}\cup(-\infty,0)$ and $d_2^\lambda \in \{-\infty\}\cup(-\infty,0)$ for all large enough λ; (ii) $\lim_{\lambda\to\infty} d_2^\lambda \in (0,\infty)\cup\{\infty\}$ and $d_2^\lambda \in (0,\infty)\cup\{\infty\}$ for all large enough λ; (iii) $\lim_{\lambda\to\infty} d_2^\lambda = 0$ and $d_2^\lambda \in (-\infty,0)$ for all large enough λ; and (iv) $\lim_{\lambda\to\infty} d_2^\lambda = 0$ and $d_2^\lambda \in (0,\infty)$ for all large enough λ.

To prove Proposition EC.2, we need the following auxiliary lemmas, whose proofs will appear at the end. These lemmas will also be used later, in the proof of Lemma 6.

LEMMA EC.14. *Under linear staffing (14), if* $\mu = a$, *then (i)* $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \exp\left(-\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)$. *Furthermore, if* $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$, *then (ii)* $\lim_{\lambda\to\infty}\frac{1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}=1$.

LEMMA EC.15. *Under linear staffing (14), if* $\mu=a$ *and* $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in\mathbb{R}\cup\{-\infty\}$, *then the following holds:* $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r I^\lambda = 0$ *(i) for* $r=1$ *if* $\lim_{\lambda\to\infty} d_2^\lambda = 0$, *or, (ii) for all* $r\in\mathbb{N}$ *if* $\lim_{\lambda\to\infty} d_2^\lambda \in (\mathbb{R}\setminus\{0\})\cup\{-\infty,\infty\}$.

LEMMA EC.16. *Under linear staffing (14), if* $\mu=a$ *and* $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(\mathbb{R}\setminus\{0\})\cup\{-\infty,\infty\}$, *then* $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} I^\lambda \in(-\infty,\infty)$.

LEMMA EC.17. *Under linear staffing (14), if* $\mu=a$ *and either (i)* $0<N^\lambda-\frac{\lambda}{a}\in\omega(1)$ *or (ii)* $0<\frac{\lambda}{a}-N^\lambda\in\mathcal{O}(\sqrt{\lambda})\cap\omega(1)$, *then* $\lim_{\lambda\to\infty}\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda}=0$.

LEMMA EC.18. *Under linear staffing (14), if* $\mu=a$ *and* $\left|N^\lambda-\frac{\lambda}{a}\right|\in\mathcal{O}(1)$, *then the following holds:* $\lim_{\lambda\to\infty}(\rho^\lambda)^r I^\lambda=0$ *(i) for* $r\in\left[0,\frac{1}{2}\right)$ *if* $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$, *or (ii) for* $r\in[0,1)$ *if* $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(\mathbb{R}\setminus\{0\})\cup\{-\infty,\infty\}$.

LEMMA EC.19. *Under linear staffing (14), if* $\mu=a$ *and* $0<\frac{\lambda}{a}-N^\lambda\in\omega(\sqrt{\lambda})$, *then* $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda}\in\left[-\frac{\mu_1}{a},0\right]$.

LEMMA EC.20. *Under linear staffing (14), if* $\mu=a$, $\lim_{\lambda\to\infty} d_2^\lambda\in(\mathbb{R}\setminus\{0\})\cup\{-\infty,\infty\}$, *and* $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$, *then* $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2=0$ *for all* $r\in\mathbb{N}$.

Proof of Proposition EC.2: From (EC.14) in Lemma EC.5,

$$\begin{aligned}\mu_1\frac{\partial I^\lambda}{\partial\mu_1} =& I^\lambda(1-I^\lambda)+(I^\lambda)^2\left(\frac{C^\lambda}{d_2^\lambda}\left[\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\frac{\rho^\lambda}{d_2^\lambda}-(k^\lambda-N^\lambda)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right]+\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}C^\lambda\right)\\ \overset{(*)}{=}& \left[\left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right)I^\lambda(1-I^\lambda)\right]-\left[\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^2\right]-\left[\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right], \qquad \text{(EC.75)}\end{aligned}$$

where $(*)$ follows by recalling the definition $d_2^\lambda := d_1^\lambda-\rho^\lambda$ and noting that

$$(I^\lambda)^2\frac{C^\lambda}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\frac{\rho^\lambda}{d_2^\lambda}=\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}I^\lambda(1-I^\lambda)-\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^2, \quad \text{and then}$$

$$-\frac{C^\lambda}{d_2^\lambda}(k^\lambda-N^\lambda)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}+\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}C^\lambda=-\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}.$$

Case (I): If $\mu < a$, then $\lim_{\lambda\to\infty} \frac{1}{d_2^\lambda} = 0$ (recalling the definition $d_2^\lambda := N^\lambda - \rho^\lambda - 1 + \frac{\mu_1}{\mu}$, where $N^\lambda = \frac{\lambda}{a} + o(\lambda)$ under linear staffing (14) and $\rho^\lambda := \frac{\lambda}{\mu}$), $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1), and $\lim_{\lambda\to\infty} \frac{\rho^\lambda}{d_2^\lambda} = \left(\frac{\mu}{a} - 1\right)^{-1} \in (-\infty, 0)$. Furthermore, $\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \in [0,1]$ for all λ (from Lemma EC.2 (c)). Using these facts, the first two terms of (EC.75) vanish in the limit as $\lambda \to \infty$. It remains to be shown that the third term follows suit:

$$\lim_{\lambda\to\infty} \frac{\rho^\lambda C^\lambda}{d_2^\lambda} \frac{k^\lambda - N^\lambda}{d_1^\lambda} \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} (I^\lambda)^2 = \lim_{\lambda\to\infty} \frac{\frac{\rho^\lambda C^\lambda}{d_2^\lambda} \frac{k^\lambda - N^\lambda}{d_1^\lambda} \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}}{\left(1 + \rho^\lambda \frac{\mu}{\mu_1}\left(\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} + \left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\frac{C^\lambda}{d_2^\lambda}\right)\right)^2}$$

$$= \lim_{\lambda\to\infty} \frac{\frac{C^\lambda}{d_2^\lambda} \frac{\rho^\lambda}{d_1^\lambda} (k^\lambda - N^\lambda) \left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda}}{\left(\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} + \rho^\lambda \frac{\mu}{\mu_1}\left(\frac{1-C^\lambda}{N^\lambda - \rho^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} + \left(\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} - 1\right)\frac{C^\lambda}{d_2^\lambda}\right)\right)^2} = 0,$$

because, in addition to the earlier facts, $\lim_{\lambda\to\infty} \frac{C^\lambda}{d_2^\lambda} = \lim_{\lambda\to\infty} \frac{C^\lambda/\lambda}{d_2^\lambda/\lambda} = \frac{(a-\mu)^2}{a^2\mu} / \left(\frac{1}{a} - \frac{1}{\mu}\right) = \frac{\mu}{a} - 1 \in (-\infty, 0)$ (using Lemma EC.13 (a)), $\lim_{\lambda\to\infty} \frac{d_1^\lambda}{\rho^\lambda} = \frac{\mu}{a} < 1$ (recalling the definition $d_1^\lambda := N^\lambda - 1 + \frac{\mu_1}{\mu}$, where $N^\lambda = \frac{\lambda}{a} + o(\lambda)$ under linear staffing (14) and $\rho^\lambda := \frac{\lambda}{\mu}$), $\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} \le 1$ for all large enough λ, and $\lim_{\lambda\to\infty} (k^\lambda - N^\lambda)\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} < \infty$ (because, even if $\lim_{\lambda\to\infty} k^\lambda - N^\lambda = \infty$, exponential decay in terms of $k^\lambda - N^\lambda$ would dominate its linear growth).

Case (II): If $\mu > a$, then the second and third terms vanish in the limit as $\lambda \to \infty$, because $\lim_{\lambda\to\infty} \frac{1}{d_2^\lambda} = 0$ (recalling the definition $d_2^\lambda := N^\lambda - \rho^\lambda - 1 + \frac{\mu_1}{\mu}$ and the linear staffing rule (14)), $\lim_{\lambda\to\infty} \frac{\rho^\lambda}{N^\lambda - \rho^\lambda} = \left(\frac{\mu}{a} - 1\right)^{-1} \in (0, \infty)$, $\lim_{\lambda\to\infty} C^\lambda = 0$ (from Lemma EC.12 (b)), $I^\lambda \in [0,1]$ for all λ, $\lim_{\lambda\to\infty} \frac{1}{d_1^\lambda} = 0$ (recalling the definition $d_1^\lambda := N^\lambda - 1 + \frac{\mu_1}{\mu}$), and $\lim_{\lambda\to\infty} (k^\lambda - N^\lambda)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} < \infty$ (noting that $\lim_{\lambda\to\infty} \frac{\rho^\lambda}{d_1^\lambda} = \frac{a}{\mu} < 1 \Rightarrow \frac{\rho^\lambda}{d_1^\lambda} < 1$ for all large enough λ, and, even if $\lim_{\lambda\to\infty} k^\lambda - N^\lambda = \infty$, exponential decay in terms of $k^\lambda - N^\lambda$ would dominate its linear growth). Thus, (EC.75) yields

$$\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = \frac{1}{\mu_1} \lim_{\lambda\to\infty} \left[\left(1 + \frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right) I^\lambda \left(1 - I^\lambda\right)\right] = \frac{\frac{a}{\mu_1^2}\left(1 - \frac{a}{\mu}\right)}{\left(1 - \frac{a}{\mu} + \frac{a}{\mu_1}\right)^2},$$

recalling that $\lim_{\lambda\to\infty} \frac{1}{d_2^\lambda} = 0$ and $\lim_{\lambda\to\infty} I^\lambda = (1 - \frac{a}{\mu})/(1 - \frac{a}{\mu} + \frac{a}{\mu_1})$ (from Proposition EC.1).

Case (III): If $\mu = a$, then, recalling that $N^\lambda = \frac{\lambda}{a} + o(\lambda)$ under linear staffing (14), it turns out that $|d_2^\lambda| = \left|(N^\lambda - \frac{\lambda}{a}) - (1 - \frac{\mu_1}{a})\right| \in o(\lambda)$, unlike when $\mu \neq a$, leading to the possibility that $\lim_{\lambda\to\infty} d_2^\lambda$ could be finite. This complicates the analysis. To proceed, we need the following auxiliary claim, whose proof is delayed until the end.

CLAIM EC.5. *Under linear staffing (14), if* $\lim_{\lambda\to\infty} d_2^\lambda = 0$ *(implying that* $\mu = a$*) and* $\lim_{\lambda\to\infty} d_2^\lambda \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right) \in (\mathbb{R} \setminus \{0\}) \in \{-\infty, \infty\}$, *then* $\lim_{\lambda\to\infty} \frac{I^\lambda}{d_2^\lambda} = 0$.

We discuss three cases depending on the asymptotic behavior of $N^\lambda - \frac{\lambda}{a}$.

Case (A): If $0 < N^\lambda - \frac{\lambda}{a} \in \omega(1) \cap o(\lambda)$, then $\lim_{\lambda\to\infty} d_2^\lambda = \infty$, which implies that $d_2^\lambda > 0$ for all large enough λ. Furthermore, $\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \geq 0$ for all λ (from Lemma EC.2 (c)). Therefore, the second and third terms of (EC.75) are non-negative, which implies that

$$\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} \leq \lim_{\lambda\to\infty} \frac{1}{\mu_1} \left[\left(1 + \frac{1}{d_2^\lambda} \frac{\mu_1}{a}\right) I^\lambda \left(1 - I^\lambda\right) \right] = 0,$$

recalling that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ and $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1). Hence, $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ by non-negativity (recalling from Lemma EC.4 (a) that $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} \geq 0$).

Case (B): If $0 < \frac{\lambda}{a} - N^\lambda \in \omega(1) \cap o(\lambda)$, then $\lim_{\lambda\to\infty} d_2^\lambda = -\infty$, which implies that $d_2^\lambda < 0$ for all large enough λ. We investigate the three terms of (EC.75) separately and show that each of them converges to 0 as $\lambda \to \infty$.

- The first term converges to 0 as $\lambda \to \infty$, recalling that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ and $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1).
- For the second term, we further discuss two cases.
  - —**Case (B-1):** If $0 < \frac{\lambda}{a} - N^\lambda \in \omega(1) \cap \mathcal{O}(\sqrt{\lambda})$, then, by regrouping the terms, we can write

    $$\lim_{\lambda\to\infty} \frac{\rho^\lambda}{d_2^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda} (I^\lambda)^2 = \lim_{\lambda\to\infty} \left(\frac{\sqrt{\rho^\lambda} I^\lambda}{d_2^\lambda} \right) \left(\sqrt{\rho^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \right) I^\lambda = 0,$$

    because $\lim_{\lambda\to\infty} \frac{\sqrt{\rho^\lambda} I^\lambda}{d_2^\lambda} = 0$ (from Lemma EC.17), $\lim_{\lambda\to\infty} \sqrt{\rho^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda}$ is finite (from Lemma EC.13 (c), noting that $\rho^\lambda = \frac{\lambda}{a}$), and $I^\lambda \in [0,1]$ for all λ.
  - —**Case (B-2):** If $0 < \frac{\lambda}{a} - N^\lambda \in \omega(\sqrt{\lambda}) \cap o(\lambda)$, then, by regrouping the terms, we can write

    $$\lim_{\lambda\to\infty} \frac{\rho^\lambda}{d_2^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda} (I^\lambda)^2 = \lim_{\lambda\to\infty} \left(\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda} \right) \frac{1}{C^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda} I^\lambda = 0,$$

    because $\lim_{\lambda\to\infty} \frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda}$ is finite (from Lemma EC.19), $\lim_{\lambda\to\infty} C^\lambda = \infty$ (from Lemma EC.12 (b)); and $\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \in [0,1]$ (from Lemma EC.2 (c)) and $I^\lambda \in [0,1]$ for all λ.
- For the third term, we further discuss two cases depending on the value of $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}$.
  - —If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$, then the third term converges to 0 as $\lambda \to \infty$ by Lemma EC.20.
  - —If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in \{-\infty\} \cup (-\infty, 0)$, then the third term converges to 0 as $\lambda \to \infty$ by Lemmas EC.15 and EC.16.

Case (C): If $|N^\lambda - \frac{\lambda}{a}| \in \mathcal{O}(1)$, we further discuss three cases depending on the value of $\lim_{\lambda\to\infty} d_2^\lambda$.

— **Case (C-1):** If $\lim_{\lambda\to\infty} d_2^\lambda \in (0,\infty) \cup \{\infty\}$, the arguments are identical to those in Case (A).

— **Case (C-2):** If $\lim_{\lambda\to\infty} d_2^\lambda \in \{-\infty\} \cup (-\infty, 0)$, we investigate the three terms of (EC.75) separately and show that each of them converges to 0 as $\lambda \to \infty$.

- The first term converges to 0 as $\lambda \to \infty$, recalling that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ and $\lim_{\lambda\to\infty} I^\lambda = 0$.
- The second term satisfies

  $$\lim_{\lambda\to\infty} \frac{\rho^\lambda}{d_2^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda} (I^\lambda)^2 = \lim_{\lambda\to\infty} \left((\rho^\lambda)^{\frac{1}{4}} I^\lambda \right)^2 \left(\sqrt{\rho^\lambda} \frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \right) \frac{1}{d_2^\lambda} = 0,$$

  by applying Lemma EC.18 and Lemma EC.13 (c), and noting that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$.

- For the third term, the arguments are identical to those in Case (B).

— **Case (C-3):** If $\lim_{\lambda\to\infty} d_2^\lambda = 0$, then we further discuss three cases depending on the value of $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}$.

- If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in (0,\infty)\cup\{\infty\}$, then

$$\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} \le \lim_{\lambda\to\infty} \frac{1}{\mu_1}\left\{\left(1+\frac{1}{d_2}\frac{\mu_1}{\mu}\right) I^\lambda (1-I^\lambda)\right\} = \lim_{\lambda\to\infty} \frac{1}{\mu_1} I^\lambda(1-I^\lambda) + \frac{1}{\mu}(1-I^\lambda)\frac{I^\lambda}{d_2^\lambda} = 0,$$

  where the inequality follows for the same reasons as those in Case (A) and the last equality follows from $\lim_{\lambda\to\infty} I^\lambda = 0$ and Claim EC.5. Hence, $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ by non-negativity (recalling from Lemma EC.4 (a) that $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} \ge 0$).

- If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in \{-\infty\}\cup(-\infty,0)$, we investigate the three terms of (EC.75) separately and show that each of them converges to 0 as $\lambda\to\infty$.

  ⋆ The first term satisfies

$$\lim_{\lambda\to\infty}\left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right) I^\lambda(1-I^\lambda) = \lim_{\lambda\to\infty} I^\lambda(1-I^\lambda) + \frac{\mu_1}{\mu}(1-I^\lambda)\frac{I^\lambda}{d_2^\lambda} = 0,$$

  recalling that $\lim_{\lambda\to\infty} I^\lambda = 0$ and by Claim EC.5.

  ⋆ The second term satisfies

$$\lim_{\lambda\to\infty} \frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^2 = \lim_{\lambda\to\infty}\left(\sqrt{\rho^\lambda} I^\lambda\right)\left(\sqrt{\rho^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\frac{I^\lambda}{d_2^\lambda} = 0,$$

  by Lemma EC.18, Lemma EC.13 (c), and Claim EC.5.

  ⋆ The third term converges to 0 as $\lambda\to\infty$ by Lemmas EC.15 and EC.16.

- If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$, then, from (EC.75), when $\mu = a$,

$$\begin{aligned}
\mu_1 \frac{\partial I^\lambda}{\partial \mu_1} =& \left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right) I^\lambda(1-I^\lambda) - \frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^2 - \frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2 \\
=& I^\lambda(1-I^\lambda) + \frac{(I^\lambda)^2}{d_2^\lambda}\left\{\frac{\mu_1}{a}\left(\frac{1}{I^\lambda}-1\right) - \rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} - \rho^\lambda C^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right\} \\
\overset{(*)}{=}& I^\lambda(1-I^\lambda) + \frac{(I^\lambda)^2}{d_2^\lambda}\left\{\frac{\mu_1}{a}\left[\rho^\lambda\frac{a}{\mu_1}\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} + \left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\frac{C^\lambda}{d_2^\lambda}\right)\right]\right. \\
&\left. -\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} - \rho^\lambda C^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right\} \\
=& I^\lambda(1-I^\lambda) + \frac{(I^\lambda)^2}{d_2^\lambda}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left[1-\left(1+d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right],
\end{aligned} \tag{EC.76}$$

  where $(*)$ follows from (EC.74).

  Note that we can write

$$\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \exp\left(\left(k^\lambda-N^\lambda\right)\ln\left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)\right).$$

Using the properties (i) $-\frac{x}{1-x} \leq \ln(1-x) \leq -x$ for all $x<1$ and (ii) $\exp(x)$ is an increasing function of x for all $x \in \mathbb{R}$, we then obtain

$$
\begin{aligned}
&\exp\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda}\right) \leq \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} \leq \exp\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right) \\
&\overset{(\dagger)}{\Rightarrow} 1 - d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda} + \frac{e^{\alpha_1^\lambda}}{2!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda}\right)^2 \leq \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} \leq 1 - d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} + \frac{e^{\alpha_2^\lambda}}{2!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2 \\
&\Rightarrow 1 - \left(\frac{d_1^\lambda}{\rho^\lambda} - 1\right)\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right) - \frac{d_1^\lambda}{\rho^\lambda}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2 + \frac{e^{\alpha_1^\lambda}}{2!}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^2\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right) \\
&\quad \leq \left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} \leq 1 - \left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2 + \frac{e^{\alpha_2^\lambda}}{2!}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right) \\
&\Rightarrow \left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\left[1 - \frac{e^{\alpha_2^\lambda}}{2!}\left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\right] \leq 1 - \left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} \\
&\quad \leq (d_2^\lambda)^2 \frac{k^\lambda - N^\lambda}{d_1^\lambda \rho^\lambda} + \frac{d_1^\lambda}{\rho^\lambda}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\left[1 - \frac{e^{\alpha_1^\lambda}}{2!}\frac{d_1^\lambda}{\rho^\lambda}\left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\right] \\
&\Rightarrow \frac{(I^\lambda)^2}{d_2^\lambda}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\left[1 - \frac{e^{\alpha_2^\lambda}}{2!}\left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\right] \\
&\quad \leq \frac{(I^\lambda)^2}{d_2^\lambda}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left[1 - \left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right] \\
&\quad \leq I^\lambda C^\lambda\left(\frac{k^\lambda - N^\lambda}{d_1^\lambda} I^\lambda\right) + \frac{(I^\lambda)^2}{d_2^\lambda}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{d_1^\lambda}{\rho^\lambda}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\left[1 - \frac{e^{\alpha_1^\lambda}}{2!}\frac{d_1^\lambda}{\rho^\lambda}\left(1 + d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\right], \qquad \text{(EC.77)}
\end{aligned}
$$

where ($\dagger$) follows from Taylor's expansion for the function $\exp(x)$ at $x=0$ up to the first two terms plus the remainder, according to which $|\alpha_1^\lambda| \leq \left|d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\frac{d_1^\lambda}{\rho^\lambda}\right|$ and $|\alpha_2^\lambda| \leq \left|d_2^\lambda \frac{k^\lambda - N^\lambda}{d_2^\lambda}\right|$. Since $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$ (by assumption), it follows that either $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} > 0$ for all large enough λ or $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} < 0$ for all large enough λ. We complete the proof under the former scenario; the proof under the latter is identical, except that the inequalities in the next step are reversed.

Note that $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$ (by assumption), $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1 when $\mu = a$), $\lim_{\lambda\to\infty} C^\lambda = 1$ (from Lemma EC.12 (b), recalling that $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(1)$), $\lim_{\lambda\to\infty} \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$ (from Lemma EC.14 (i)), $\lim_{\lambda\to\infty} \frac{d_1^\lambda}{\rho^\lambda} = 1$, $0 \leq \lim_{\lambda\to\infty} |\alpha_1^\lambda| \leq 0$ and $0 \leq |\alpha_2^\lambda| \leq 0$. Moreover, from (EC.74) when $\mu = a$,

$$
\begin{aligned}
&\lim_{\lambda\to\infty} \frac{(I^\lambda)^2}{d_2^\lambda}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2 = \lim_{\lambda\to\infty} C^\lambda\left(\sqrt{\rho^\lambda}\frac{k^\lambda - N^\lambda}{d_1^\lambda} I^\lambda\right)^2 \\
&= \lim_{\lambda\to\infty} C^\lambda\left[\frac{d_1^\lambda}{\sqrt{\rho^\lambda}}\frac{1}{k^\lambda - N^\lambda} + \frac{\mu}{\mu_1}\sqrt{\rho^\lambda}\frac{d_1^\lambda}{k^\lambda - N^\lambda}\frac{1 - C^\lambda}{N^\lambda - \rho^\lambda} + \frac{a}{\mu_1} C^\lambda \frac{\sqrt{\rho^\lambda}}{d_2^\lambda}\frac{d_1^\lambda}{k^\lambda - N^\lambda}\left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\right]^{-2} = 0,
\end{aligned} \qquad \text{(EC.78)}
$$

because all three terms within the square bracket in the above display are non-negative, $\lim_{\lambda\to\infty} C^\lambda = 1$, and in particular, the third term satisfies

$$
\frac{a}{\mu_1}\lim_{\lambda\to\infty} C^\lambda \sqrt{\rho^\lambda}\left(\frac{1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}}{d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}}\right) = \infty,
$$

recalling that $\lim_{\lambda\to\infty} C^\lambda = 1$ and $\lim_{\lambda\to\infty} \frac{1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}} = 1$ (from Lemma EC.14 (ii)).

Based on the above facts, (EC.77) implies that

$$\lim_{\lambda\to\infty} \frac{(I^\lambda)^2}{d_2^\lambda}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left[1-\left(1+d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right]=0.$$

Hence, from (EC.76), $\lim_{\lambda\to\infty}\frac{\partial I^\lambda}{\partial\mu_1}=0$, recalling that $\lim_{\lambda\to\infty} I^\lambda = 0$.

■

Proof of Claim EC.5: We first note that $\lim_{\lambda\to\infty} d_2^\lambda = 0$ implies $\mu = a$ and $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in (-\infty, 1)$ (recalling $d_2^\lambda := N^\lambda - \frac{\lambda}{\mu} - (1-\frac{\mu_1}{\mu})$), in turn, implying $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(1)$. We discuss the following two cases depending on the asymptotic behavior of $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}$.

Case (I): If $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} \in (0,\infty)\cup\{\infty\}$. Then, from (EC.74), when $\mu = a$,

$$\frac{\rho^\lambda I^\lambda}{d_2^\lambda} = \left[\frac{d_2^\lambda}{\rho^\lambda} + \frac{a}{\mu_1}\frac{d_2^\lambda}{\sqrt{\rho^\lambda}}\sqrt{\rho^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} + \frac{a}{\mu_1}C^\lambda\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}.$$

Note that $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda} = 0$, $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\sqrt{\rho^\lambda}} = 0$ (since $\lim_{\lambda\to\infty} d_2^\lambda = 0$), $\lim_{\lambda\to\infty}\sqrt{\rho^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in (0,\infty)$ (from Lemma EC.13 (c) noting that $\left|N^\lambda-\frac{\lambda}{a}\right| \in \mathcal{O}(1)$), and $\lim_{\lambda\to\infty} C^\lambda = 1$ (from Lemma EC.12 (b)). Moreover, note that $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}} \in [0,1)$ (from Lemma EC.14 (i)). Thus, it follows that $\lim_{\lambda\to\infty}\frac{\rho^\lambda I^\lambda}{d_2^\lambda} \in \left[\frac{\mu_1}{a},\infty\right)$. Then, it is clear that $\lim_{\lambda\to\infty}\frac{I^\lambda}{d_2^\lambda} = \lim_{\lambda\to\infty}\frac{1}{\rho^\lambda}\frac{\rho^\lambda I^\lambda}{d_2^\lambda} = 0$.

Case (II): If $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} \in \{-\infty\}\cup(-\infty,0)$. Then, from (EC.74), when $\mu = a$,

$$\frac{I^\lambda}{d_2^\lambda} = \left[d_2^\lambda + \frac{a}{\mu_1}d_2^\lambda\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} + \frac{a}{\mu_1}C^\lambda\rho^\lambda\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}.$$

Note that the first two terms in the square bracket in the above display are negative and $C^\lambda \to 1$ (from Lemma EC.12 (b)) as $\lambda\to\infty$. Moreover, note that $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}} \in (1,\infty]$ (from Lemma EC.14 (i)), which implies that $\lim_{\lambda\to\infty}\rho^\lambda\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right) = -\infty$. Then, it is clear that $\lim_{\lambda\to\infty}\frac{I^\lambda}{d_2^\lambda} = 0$.

Combining both cases, it follows that $\lim_{\lambda\to\infty}\frac{I^\lambda}{d_2^\lambda} = 0$. ■

We now proceed to prove auxiliary Lemmas EC.14-EC.20.

Proof of Lemma EC.14:

(i): First, we recall the definition $d_2^\lambda := d_1^\lambda - \rho^\lambda$, where $d_1^\lambda = N^\lambda - 1 + \frac{\mu_1}{\mu} = \frac{\lambda}{a} + o(\lambda) - 1 + \frac{\mu_1}{\mu}$ under linear staffing (14). This implies that $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{d_1^\lambda} = 0$ (recalling that $\rho^\lambda = \frac{\lambda}{a}$ when $\mu = a$). Next, we can write

$$\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \left(\left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)^{-\frac{d_1^\lambda}{d_2^\lambda}}\right)^{-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}$$

$$\Rightarrow \lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \left(\lim_{\lambda\to\infty}\left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)^{-\frac{d_1^\lambda}{d_2^\lambda}}\right)^{-\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}} = \exp\left(-\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right).$$

(ii): First, we recall the definition $d_2^\lambda := d_1^\lambda - \rho^\lambda$, where $d_1^\lambda := N^\lambda - 1 + \frac{\mu_1}{\mu} > 0$ (since $N^\lambda \geq 1$, $\mu_1 > 0$, and $\mu > 0$) and $\rho^\lambda := \frac{\lambda}{\mu} > 0$ for all $\lambda > 0$. This implies that $\frac{d_2^\lambda}{d_1^\lambda} < 1$ for all $\lambda > 0$. Next, we can write

$$\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = \exp\left(\left(k^\lambda-N^\lambda\right)\ln\left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)\right).$$

Using the properties (i) $-\frac{x}{1-x} \leq \ln(1-x) \leq -x$ for all $x < 1$ and (ii) $\exp(x)$ is an increasing function of x for all $x \in \mathbb{R}$, we then obtain

$$\begin{aligned}
&\exp\left(-d_2^\lambda \frac{k^\lambda-N^\lambda}{\rho^\lambda}\right) \leq \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} \leq \exp\left(-d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)\\
&\overset{(*)}{\Rightarrow} 1 - d_2^\lambda \frac{k^\lambda-N^\lambda}{\rho^\lambda} + \frac{e^{\alpha_1^\lambda}}{2!}\left(-d_2^\lambda \frac{k^\lambda-N^\lambda}{\rho^\lambda}\right)^2 \leq \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} \leq 1 - d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} + \frac{e^{\alpha_2^\lambda}}{2!}\left(-d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^2\\
&\Rightarrow d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} - \frac{e^{\alpha_2^\lambda}}{2!}\left(-d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^2 \leq 1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} \leq d_2^\lambda \frac{k^\lambda-N^\lambda}{\rho^\lambda} - \frac{e^{\alpha_1^\lambda}}{2!}\left(-d_2^\lambda \frac{k^\lambda-N^\lambda}{\rho^\lambda}\right)^2\\
&\Rightarrow d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(1 - \frac{e^{\alpha_2^\lambda}}{2!}\left(d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)\right) \leq 1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} \leq d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\frac{d_1^\lambda}{\rho^\lambda}\left(1 - \frac{e^{\alpha_1^\lambda}}{2!}\left(d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\frac{d_1^\lambda}{\rho^\lambda}\right)\right),
\end{aligned} \tag{EC.79}$$

where $(*)$ follows from Taylor's expansion for the function $\exp(x)$ at $x = 0$ up to the first two terms plus the remainder, according to which $|\alpha_1^\lambda| \leq \left|d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\frac{d_1^\lambda}{\rho^\lambda}\right|$ and $|\alpha_2^\lambda| \leq \left|d_2^\lambda \frac{k^\lambda-N^\lambda}{d_2^\lambda}\right|$. Since $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} = 0$ (by assumption), it follows that either $d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} > 0$ for all large enough λ or $d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} < 0$ for all large enough λ. We complete the proof under the former scenario; the proof under the latter is identical, except that the inequalities in the next step are reversed. Dividing (EC.79) throughout by $d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}$, we obtain, for all large enough λ,

$$1 - \frac{e^{\alpha_2^\lambda}}{2!}\left(d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right) \leq \frac{1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}} \leq \frac{d_1^\lambda}{\rho^\lambda}\left(1 - \frac{d_1^\lambda}{\rho^\lambda}\frac{e^{\alpha_1^\lambda}}{2!}\left(d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)\right). \tag{EC.80}$$

Recalling that $d_1^\lambda = N^\lambda - 1 + \frac{\mu_1}{\mu} = \frac{\lambda}{a} + o(\lambda) - 1 + \frac{\mu_1}{\mu}$ under linear staffing (14) and that $\rho^\lambda = \frac{\lambda}{a}$ when $\mu = a$, it follows that $\lim_{\lambda\to\infty} \frac{d_1^\lambda}{\rho^\lambda} = 1$. In addition, recalling that $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} = 0$ (by assumption), it follows that $\lim_{\lambda\to\infty} e^{\alpha_1^\lambda} = \lim_{\lambda\to\infty} e^{\alpha_2^\lambda} = 1$. Using these facts, (EC.80) implies that

$$1 \leq \lim_{\lambda\to\infty} \frac{1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}} \leq 1 \quad \Rightarrow \quad \lim_{\lambda\to\infty} \frac{1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}} = 1.$$

∎

Proof of Lemma EC.15: By assumption, we know that $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} \in \mathbb{R} \cup \{-\infty\}$. We discuss three cases depending on the values of $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}$ and $\lim_{\lambda\to\infty} d_2^\lambda$.

Case (I): If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} \in \mathbb{R}$ and $\lim_{\lambda\to\infty} d_2^\lambda \in (\mathbb{R} \setminus \{0\}) \cup \{-\infty, \infty\}$, then, by multiplying and dividing by $\left(d_2^\lambda\right)^r$, we can write

$$\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r I^\lambda = \frac{1}{\left(d_2^\lambda\right)^r}\left(d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r I^\lambda = 0 \quad \forall\, r \in \mathbb{N},$$

recalling that when $\mu = a$, $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1).

For the remaining two cases, we first use (EC.74) with $\mu = a$ to expand

$$\left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^r I^\lambda = \left[\left(\frac{d_1^\lambda}{k^\lambda - N^\lambda}\right)^r + \rho^\lambda \frac{a}{\mu_1}\frac{1-C^\lambda}{N^\lambda - \rho^\lambda}\left(\frac{d_1^\lambda}{k^\lambda - N^\lambda}\right)^r + \rho^\lambda \frac{a}{\mu_1}\left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\frac{C^\lambda}{d_2^\lambda}\left(\frac{d_1^\lambda}{k^\lambda - N^\lambda}\right)^r\right]^{-1}. \tag{EC.81}$$

Next, observe that $\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \geq 0$ (from Lemma EC.2 (c)) and $\frac{1}{d_2^\lambda}\left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right) \geq 0$ (recalling the definition $d_2^\lambda := d_1^\lambda - \rho^\lambda$) for all λ, resulting in all three terms within the square bracket of (EC.81) being non-negative for all λ. Therefore, when $\lambda \to \infty$, in order to show that the expression in (EC.81) vanishes, we need only show that one of these terms diverges to ∞. For the remaining two cases, we focus on the third term, which can be multiplied and divided by $\left(d_2^\lambda\right)^r$ and expressed as

$$\frac{a}{\mu_1}\rho^\lambda C^\lambda \left(d_2^\lambda\right)^{r-1}\left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^{-r}. \tag{EC.82}$$

Case (II): Suppose $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in \mathbb{R}$ and $\lim_{\lambda\to\infty} d_2^\lambda = 0$. First, when $r = 1$, (EC.82) becomes

$$\frac{a}{\mu_1}\rho^\lambda C^\lambda \left(\frac{1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}}{d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}}\right). \tag{EC.83}$$

Next, note that $\lim_{\lambda\to\infty} d_2^\lambda = 0$ implies that $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} = 1 - \frac{\mu_1}{a} \Rightarrow \left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(1)$ (recalling that when $\mu = a$, $d_2^\lambda = N^\lambda - \frac{\lambda}{a} - \left(1 - \frac{\mu_1}{a}\right)$); therefore, $\lim_{\lambda\to\infty} C^\lambda = 1$ (from Lemma EC.12 (b)). Furthermore, $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in \mathbb{R}$ implies that $\lim_{\lambda\to\infty} \frac{1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}}{d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} > 0$ (from Lemma EC.14 (i)). Therefore, as $\lambda \to \infty$, the expression in (EC.83) diverges to infinity because $\rho^\lambda \to \infty$, as desired.

Case (III): If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = -\infty$, then $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} < 0$ and $d_2^\lambda < 0$ for all large enough λ. The latter implies that either $0 < \frac{\lambda}{a} - N^\lambda \in \omega(1)$ or $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(1)$; therefore, $\lim_{\lambda\to\infty} C^\lambda \geq 1$ (from Lemma EC.12 (b)). Furthermore, $d_2^\lambda < 0$ for all large enough λ also implies that $\frac{d_2^\lambda}{d_1^\lambda} < 0$ and $\frac{\rho^\lambda}{d_1^\lambda} > 1$ for all large enough λ (recalling the definition $d_2^\lambda := d_1^\lambda - \rho^\lambda$).

Next, we can write

$$\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} = \left(1 - \frac{d_2^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} = \exp\left(\left(k^\lambda - N^\lambda\right)\ln\left(1 - \frac{d_2^\lambda}{d_1^\lambda}\right)\right).$$

Using the properties (i) $\ln(1-x) \geq -\frac{x}{1-x}$ for all $x < 1$ and (ii) $\exp(x)$ is an increasing function of x for all $x \in \mathbb{R}$, we then obtain

$$\begin{aligned}
\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} &\geq \exp\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda}\right) \overset{(*)}{=} 1 + \sum_{i=1}^{r}\frac{1}{i!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda}\right)^i + \frac{e^{\alpha^\lambda}}{(r+1)!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda}\right)^{r+1} \\
\Rightarrow \left(\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} - 1\right) &\geq \sum_{i=1}^{r}\frac{1}{i!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^i\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^i + \frac{e^{\alpha^\lambda}}{(r+1)!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^{r+1}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{r+1} \\
&\overset{(**)}{\geq} \frac{1}{(r+1)!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^{r+1}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{r+1},
\end{aligned} \tag{EC.84}$$

where $(*)$ follows from Taylor's expansion for the function $\exp(x)$ at $x=0$ up to the first $r+1$ terms plus the remainder, according to which $\alpha^\lambda \in \left(0, -d_2^\lambda \frac{k^\lambda - N^\lambda}{\rho^\lambda}\right)$; and $(**)$ follows by recalling that $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} < 0$ for all large enough λ and noting that $e^{\alpha^\lambda} \geq 1$ for all large enough λ.

Recalling that $\frac{\rho^\lambda}{d_1^\lambda} > 1$ and $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} < 0$ for all large enough λ, it follows that both sides of the inequality (EC.84) are strictly positive for all large enough λ. Using (EC.84), we can write

$$\left|\left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^{-r}\right| = \left|\left(\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} - 1\right)\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^{-r}\right|$$

$$\geq \frac{1}{(r+1)!}\left(-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{r+1}. \tag{EC.85}$$

Recalling that $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = -\infty$ (by assumption) and noting that $\lim_{\lambda\to\infty} \frac{d_1^\lambda}{\rho^\lambda} = 1$ (recalling that $d_1^\lambda = N^\lambda - 1 + \frac{\mu_1}{\mu} = \frac{\lambda}{a} + o(\lambda) - 1 + \frac{\mu_1}{\mu}$ under linear staffing (14) and that $\rho^\lambda = \frac{\lambda}{a}$ when $\mu = a$), the right-hand side of (EC.7.2) diverges to ∞ in the limit as $\lambda \to \infty$, implying that

$$\lim_{\lambda\to\infty}\left|\left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^{-r}\right| = \infty. \tag{EC.86}$$

Recalling that $\lim_{\lambda\to\infty} C^\lambda \geq 1$ and using (EC.7.2), the expression in (EC.82) diverges to ∞ (i) for $r=1$ if $\lim_{\lambda\to\infty} d_2^\lambda = 0$ and (ii) for all $r \in \mathbb{N}$ if $\lim_{\lambda\to\infty} d_2^\lambda \in (\mathbb{R}\setminus\{0\}) \cup \{-\infty, \infty\}$, as desired.

■

Proof of Lemma EC.16: From (EC.74), when $\mu = a$,

$$\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda = \left[\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} + \frac{a}{\mu_1}\frac{d_2^\lambda}{C^\lambda}\frac{1 - C^\lambda}{N^\lambda - \rho^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} + \frac{a}{\mu_1}\left(\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} - 1\right)\right]^{-1}$$

$$= \left[\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} + \frac{a}{\mu_1}\left\{\left(\frac{d_2^\lambda}{C^\lambda}\frac{1 - C^\lambda}{N^\lambda - \rho^\lambda} + 1\right)\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} - 1\right\}\right]^{-1}. \tag{EC.87}$$

We first evaluate the limit of the second term within the square bracket in (EC.87). Note that $\frac{d_2^\lambda}{C^\lambda}\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} = \frac{d_2^\lambda}{N^\lambda - \rho^\lambda}\frac{1 - C^\lambda}{C^\lambda} \to \lim_{\lambda\to\infty}\frac{1}{C^\lambda} - 1$ and $\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} \to e^{\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}}$, as $\lambda \to \infty$ (from Lemma EC.14 (i)). Then,

$$\lim_{\lambda\to\infty}\left(\frac{d_2^\lambda}{C^\lambda}\frac{1 - C^\lambda}{N^\lambda - \rho^\lambda} + 1\right)\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} - 1 = \left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} - 1. \tag{EC.88}$$

In suffices to show:

(i) $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} \neq 1$;

(ii) $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda} = 0$, or $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda - N^\lambda}$ and $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} - 1$ share the same the sign.

Recall that, by assumption, $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in (\mathbb{R}\setminus\{0\}) \cup \{-\infty, \infty\}$. We discuss the following three cases depending on the asymptotic behavior of $N^\lambda - \frac{\lambda}{a}$. These are **Case (I)** $0 < \frac{\lambda}{a} - N^\lambda \in \omega(1) \cap o(\lambda)$, **Case (II)** $0 < \left|\frac{\lambda}{a} - N^\lambda\right| \in \mathcal{O}(1)$, and **Case (III)** $0 < N^\lambda - \frac{\lambda}{a} \in \omega(1) \cap o(\lambda)$.

Case (I): If $0<\frac{\lambda}{a}-N^\lambda\in\omega(1)\cap o(\lambda)$, then $\lim_{\lambda\to\infty}C^\lambda\in[1,\infty)\cup\{\infty\}$ and $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in\{-\infty\}\cup(-\infty,0)$. Then, $\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\in[0,1]$ and $e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in[0,1)$, implying that $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in[0,1)$. Moreover, $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}<0$. Therefore, both (i) and (ii) hold.

Case (II): If $0<\left|\frac{\lambda}{a}-N^\lambda\right|\in\mathcal{O}(1)$, then $\lim_{\lambda\to\infty}C^\lambda=1$ and $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(\mathbb{R}\setminus\{0\})\cup\{-\infty,\infty\}$. Then, $\lim_{\lambda\to\infty}\frac{1}{C^\lambda}=1$ and $e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in[0,1)\cup(1,\infty)\cup\{\infty\}$, implying that $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in[0,1)\cup(1,\infty)\cup\{\infty\}$. Hence, (i) holds.

- If $\lim_{\lambda\to\infty}d_2^\lambda\in\{-\infty\}\cup(-\infty,0]$, then $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}\leq 0$, and $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in[0,1)$. Therefore, (ii) holds.
- If $\lim_{\lambda\to\infty}d_2^\lambda\in(0,\infty)\cup\{\infty\}$, then $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}>0$, and $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in(1,\infty)\cup\{\infty\}$. Therefore, (ii) holds.

Case (III): If $0<N^\lambda-\frac{\lambda}{a}\in\omega(1)\cap o(\lambda)$, then $\lim_{\lambda\to\infty}C^\lambda\in[0,1]$ and $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\in(0,\infty)\cup\{\infty\}$. Then, $\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\in[1,\infty)\cup\{\infty\}$ and $e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in(1,\infty)\cup\{\infty\}$, implying that $\left(\lim_{\lambda\to\infty}\frac{1}{C^\lambda}\right)e^{\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in(1,\infty)\cup\{\infty\}$. Moreover, $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda C^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}>0$. Therefore, both (i) and (ii) hold. ■

Proof of Lemma EC.17: From (EC.74),

$$\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda}=\left[\frac{d_2^\lambda}{\sqrt{\rho^\lambda}}+\frac{\mu}{\mu_1}\sqrt{\rho^\lambda}d_2^\lambda\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)+\frac{\mu}{\mu_1}C^\lambda\sqrt{\rho^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}. \tag{EC.89}$$

(i): If $0<N^\lambda-\frac{\lambda}{a}\in\omega(1)$, then $d_2^\lambda>0$ for all large enough λ. We rewrite (EC.89) as

$$\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda}=\frac{1}{\sqrt{\rho^\lambda}}\left[\frac{d_2^\lambda}{\rho^\lambda}+\frac{a}{\mu_1}(1-C^\lambda)\left(\frac{d_2^\lambda}{N^\lambda-\rho^\lambda}\right)+\frac{a}{\mu_1}C^\lambda\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}.$$

Note that $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{\rho^\lambda}=0$, $\lim_{\lambda\to\infty}\frac{d_2^\lambda}{N^\lambda-\rho^\lambda}=1$, $\lim_{\lambda\to\infty}C^\lambda\in[0,1]$ (from Lemma EC.12 (b)), and $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}=e^{-\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\in[0,1]$ (from Lemma EC.14 (i)). Thus, the terms within the square bracket in the above display are all finite, which implies that $\lim_{\lambda\to\infty}\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda}=0$.

(ii): If $0<\frac{\lambda}{a}-N^\lambda\in\mathcal{O}(\sqrt{\lambda})\cap\omega(1)$, then $d_2^\lambda<0$ for all large enough λ. Since $d_2^\lambda=d_1^\lambda-\rho^\lambda$, all three terms in (EC.89) are non-positive for all large enough λ. Furthermore, note that $\lim_{\lambda\to\infty}\sqrt{\rho^\lambda}\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\in(0,\infty)$ (from Lemma EC.13 (c)), implying that the second term diverges to $-\infty$ as $\lambda\to\infty$ (since $|d_2^\lambda|\in\omega(1)$). Therefore, $\lim_{\lambda\to\infty}\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda}=0$. ■

Proof of Lemma EC.18: From (EC.74), when $\mu = a$,

$$(\rho^\lambda)^r I^\lambda = \left[\frac{1}{(\rho^\lambda)^r} + \frac{a}{\mu_1}(\rho^\lambda)^{1-r}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} + \frac{a}{\mu_1}(\rho^\lambda)^{1-r}\frac{C^\lambda}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}. \tag{EC.90}$$

Note that $\lim_{\lambda\to\infty} C^\lambda = 1$ (from Lemma EC.12 (b)) and $|d_2^\lambda| \in \mathcal{O}(1)$, implying that $\lim_{\lambda\to\infty}\frac{C^\lambda}{d_2^\lambda} \neq 0$. Note that $\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in [0,1]$ for all λ (from Lemma EC.2 (c)) and, since $d_2^\lambda = d_1^\lambda - \rho^\lambda$, $\frac{1}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right) \geq 0$ for all λ. As a result, all three terms within (EC.90) are non-negative for all λ.

(i): If $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} = 0$, then, note that

$$\lim_{\lambda\to\infty}(\rho^\lambda)^{1-r}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} = \lim_{\lambda\to\infty}\left(\sqrt{\rho^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)(\rho^\lambda)^{\frac{1}{2}-r} = \infty, \quad \forall r \in \left[0,\frac{1}{2}\right),$$

since $\sqrt{\rho^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in (0,\infty)$ (from Lemma EC.13 (c)). Thus, the second term of (EC.90) diverges to ∞ as $\lambda\to\infty$, which implies that $\lim_{\lambda\to\infty}(\rho^\lambda)^r I^\lambda = 0$ for all $r\in[0,\frac{1}{2})$.

(ii): If $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} \in (\mathbb{R}\setminus\{0\})\cup\{-\infty,\infty\}$, then, note that

$$\lim_{\lambda\to\infty}(\rho^\lambda)^{1-r}\frac{C^\lambda}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right) = \infty, \quad \forall r\in[0,1),$$

recalling that $\lim_{\lambda\to\infty}\frac{C^\lambda}{d_2^\lambda}\neq 0$ and $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}} \neq 1$ (where the equality follows from Lemma EC.14 (i)). Thus, the third term of (EC.90) diverges to ∞ as $\lambda\to\infty$, which implies that $\lim_{\lambda\to\infty}(\rho^\lambda)^r I^\lambda = 0$ for all $r\in[0,1)$. ∎

Proof of Lemma EC.19: From (EC.74), when $\mu = a$,

$$\begin{aligned}\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda} &= \left[\frac{d_2^\lambda}{\rho^\lambda}\frac{1}{C^\lambda} + \frac{a}{\mu_1}\frac{d_2^\lambda}{N^\lambda-\rho^\lambda}\left(\frac{1}{C^\lambda}-1\right) + \frac{a}{\mu_1}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\right]^{-1}\\ &= \left[\frac{d_2^\lambda}{\rho^\lambda}\frac{1}{C^\lambda} + \frac{a}{\mu_1}\left(\frac{d_2^\lambda}{N^\lambda-\rho^\lambda}\left(\frac{1}{C^\lambda}-1\right)+1\right) - \frac{a}{\mu_1}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right]^{-1}.\end{aligned}$$

Note that $\frac{d_2^\lambda}{\rho^\lambda}\to 0$, $C^\lambda\to\infty$ (from Lemma EC.12 (b) when $0<\frac{\lambda}{a}-N^\lambda\in\omega(\sqrt{\lambda})$ (by assumption)), $\frac{d_2^\lambda}{N^\lambda-\rho^\lambda}\to 1$ (since $0<\frac{\lambda}{a}-N^\lambda\in\omega(\sqrt{\lambda})$ (by assumption)), and $\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\to e^{-\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}} \in [1,\infty]$, as $\lambda\to\infty$ (from Lemma EC.14 (i)). Hence, $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda}\in\left[-\frac{\mu_1}{a},0\right]$. ∎

Proof of Lemma EC.20: By multiplying and dividing by $(d_2^\lambda)^{r-1}$ and regrouping the terms, we can write

$$\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2 = \frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\left(d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^{r-1}\frac{1}{(d_2^\lambda)^{r-1}}.$$

Note that $\lim_{\lambda\to\infty}\left(d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^{r-1}\in\{0,1\}$ (because $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$ by assumption and $r\in\mathbb{N}$), and $\lim_{\lambda\to\infty}\frac{1}{(d_2^\lambda)^{r-1}}\in(-\infty,\infty)$ (because $\lim_{\lambda\to\infty} d_2^\lambda\neq 0$ and $r\in\mathbb{N}$). Therefore, to complete the proof, it suffices to show $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2=0$.

Using (EC.74) and after algebra, when $\mu=a$,

$$
\begin{aligned}
&\left[\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right]^{-1}\\
&=\frac{(d_2^\lambda)^2}{\rho^\lambda C^\lambda}\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)^2+\left(\frac{a}{\mu_1}\right)^2\rho^\lambda C^\lambda\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)^2\\
&+2\frac{a}{\mu_1}d_2^\lambda\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right).
\end{aligned}
\tag{EC.91}
$$

Note that each of the three terms in the above display either has the same sign as $d_2^\lambda=d_1^\lambda-\rho^\lambda$ or is 0 for all λ. Furthermore, when examining the third term, we note that

$$
\begin{aligned}
&\lim_{\lambda\to\infty}\left|d_2^\lambda\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\right|\\
&=\lim_{\lambda\to\infty}\left|d_2^\lambda\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}\left(\frac{1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}}{d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\right|\overset{(i)}{=}\lim_{\lambda\to\infty}\left|d_2^\lambda\left(1+\frac{1}{\mu_1}\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\right|\overset{(ii)}{=}\infty,
\end{aligned}
$$

where (i) follows by noting that $\lim_{\lambda\to\infty}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}=\exp\left(\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)=1$ (because $\lim_{\lambda\to\infty}d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}=0$ by assumption), from Lemma EC.14, and by recalling that $\rho^\lambda=\frac{\lambda}{a}$; and (ii) follows because $\lim_{\lambda\to\infty}d_2^\lambda\neq 0$ (by assumption) and from Lemma EC.13 (d). Therefore, from (EC.91), it follows that $[\lim_{\lambda\to\infty}\left|\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2]^{-1}\right|=\infty$, or, equivalently, $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2=0$. ■

EC.7.3. Preliminaries C: Properties of the Limiting FOC

Recall the limiting FOC (16) is

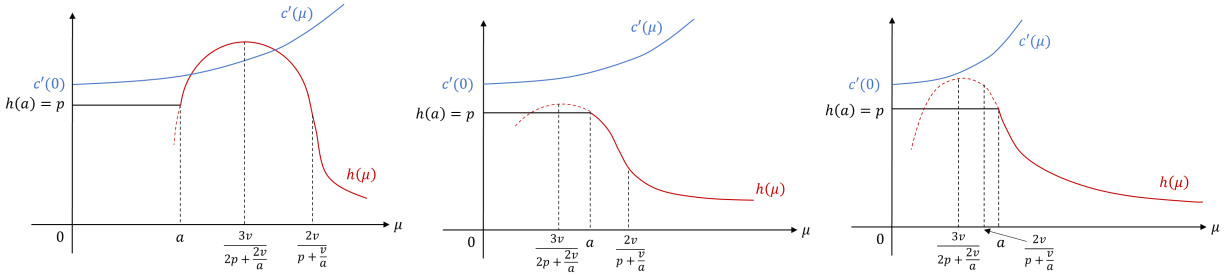
$$
c'(\mu)=p\left(1-\left[1-\frac{a^2}{\mu^2}\right]^+\right)+v\frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^+.
\tag{EC.92}
$$

Let

$$
h(\mu;a,p,v):=\frac{a^2}{\mu^2}\left(p+\frac{v}{a}-\frac{v}{\mu}\right),\quad \mu>0.
\tag{EC.93}
$$

Then, (EC.92) can be written as

$$c'(\mu) = \begin{cases} p, & \mu < a, \\ h(\mu; a, p, v), & \mu \geq a. \end{cases} \tag{EC.94}$$



(I) $p < \frac{v}{2a}$ (II) $\frac{v}{2a} \leq p < \frac{v}{a}$ (III) $p \geq \frac{v}{a}$

Figure EC.4 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.94), suppose $p \leq c'(0)$.

The next lemma provides properties of $h(\mu; a, p, v)$. We might suppress the dependence of $h(\mu; a, p, v)$ on a, p and v henceforth, when the context is clear.

Lemma EC.21. *The function $h(\mu; a, p, v)$, defined in (EC.93), satisfies the following properties:*

(a) $h(\mu)$ is strictly increasing in μ for $\mu \in \left(0, \frac{3v}{2p+\frac{2v}{a}}\right)$, and strictly decreasing in μ for $\mu \in \left(\frac{3v}{2p+\frac{2v}{a}}, \infty\right)$.

(b) $h(\mu)$ is strictly concave in μ for $\mu \in \left(0, \frac{2v}{p+\frac{v}{a}}\right)$, and strictly convex in μ for $\mu \in \left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$.

(c) $h(\mu; a, p, v)$ is strictly decreasing in $a > 0$ when $\mu \in \left(0, \frac{2v}{2p+\frac{v}{a}}\right)$, and strictly increasing in $a > 0$ when $\mu \in \left(\frac{2v}{2p+\frac{v}{a}}, \infty\right)$.

(d) $h(\mu; a, p, v)$ is strictly increasing in $p \geq 0$ for all $\mu \in (0, \infty)$.

(e) $h(\mu; a, p, v)$ is strictly decreasing in $v > 0$ when $\mu \in (0, a)$, and strictly increasing in $v > 0$ when $\mu \in (a, \infty)$.

Proof of Lemma EC.21:

(a): Differentiating $h(\mu)$ yields

$$h'(\mu) = -\frac{2a^2}{\mu^3}\left(p + \frac{v}{a} - \frac{v}{\mu}\right) + \frac{a^2}{\mu^2}\frac{v}{\mu^2} = \frac{a^2}{\mu^3}\left(-2p - \frac{2v}{a} + \frac{3v}{\mu}\right),$$

which is strictly positive when $\mu < \frac{3v}{2p+\frac{2v}{a}}$, and strictly negative when $\mu > \frac{3v}{2p+\frac{2v}{a}}$. This implies that $h(\mu)$ is strictly increasing in μ for $\mu \in \left(0, \frac{3v}{2p+\frac{2v}{a}}\right)$ and strictly decreasing in μ for $\mu \in \left(\frac{3v}{2p+\frac{2v}{a}}, \infty\right)$.

(b): Differentiating $h(\mu)$ twice yields

$$h''(\mu) = -\frac{3a^2}{\mu^4}\left(-2p - \frac{2v}{a} + \frac{3v}{\mu}\right) + \frac{a^2}{\mu^3}\left(-\frac{3v}{\mu^2}\right) = \frac{6a^2}{\mu^4}\left(p + \frac{v}{a} - \frac{2v}{\mu}\right),$$

which is strictly negative when $\mu < \frac{2v}{p+\frac{v}{a}}$, and strictly positive when $\mu > \frac{2v}{p+\frac{v}{a}}$. This implies that $h(\mu)$ is strictly concave in μ for $\mu \in \left(0, \frac{2v}{p+\frac{v}{a}}\right)$ and strictly convex in μ for $\mu \in \left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$.

(c): Differentiating $h(\mu;a,p,v)$ with respect to a yields

$$\frac{\partial h}{\partial a} = \frac{2a}{\mu^2}\left(p - \frac{v}{\mu}\right) + \frac{v}{\mu^2} = \frac{1}{\mu^2}\left(2ap + v - \frac{2av}{\mu}\right),$$

which is strictly negative when $\mu \in \left(0, \frac{2v}{2p+\frac{v}{a}}\right)$, and strictly positive when $\mu \in \left(\frac{2v}{2p+\frac{v}{a}}, \infty\right)$. This implies that $h(\mu;a,p,v)$ is strictly decreasing in $a > 0$ when $\mu \in \left(0, \frac{2v}{2p+\frac{v}{a}}\right)$, strictly increasing in $a > 0$ when $\mu \in \left[\frac{2v}{2p+\frac{v}{a}}, \infty\right)$.

(d): Differentiating $h(\mu;a,p,v)$ with respect to p yields

$$\frac{\partial h}{\partial p} = \frac{a^2}{\mu^2} > 0,$$

for all $\mu \in (0,\infty)$, which implies that $h(\mu;a,p,v)$ is strictly increasing in $p \geq 0$ for all $\mu \in (0,\infty)$.

(e): Differentiating $h(\mu;a,p,v)$ with respect to v yields

$$\frac{\partial h}{\partial v} = \frac{a^2}{\mu^2}\left(\frac{1}{a} - \frac{1}{\mu}\right),$$

which is strictly negative when $\mu \in (0,a)$, and strictly positive when $\mu \in (a,\infty)$. This implies that $h(\mu;a,p,v)$ is strictly decreasing in $v > 0$ when $\mu \in (0,a)$, and strictly increasing in $v > 0$ when $\mu \in (a,\infty)$.

■

EC.7.4. Preliminaries D: Auxiliary Definitions

In this section, we introduce auxiliary results that will be heavily used in the proofs of all the results from Section 5.2-5.3, except that of Proposition 6. Noting that they all rely on Assumption 1, throughout this section, we implicitly operate under this assumption whenever it is necessary.

We begin with $\mu^\dagger(a,p)$, which is the unique value of $\mu \in \left[a, \frac{3}{2}a\right)$ that simultaneously satisfies $c'(\mu) = h(\mu)$ (the limiting FOC (EC.94) for underloaded or critically loaded equilibria) and $c''(\mu) = h'(\mu)$, for a given $a > 0$ and $p \in [0, c'(a)]$. In other words, $c'(\mu)$ and $h(\mu)$ are tangent at $\mu^\dagger(a,p)$, as illustrated in Figure EC.5 (I)(II). $v^\dagger(a,p)$ is the unique value of v that induces $\mu^\dagger(a,p)$.

Definition EC.1.

(a) For any $a > 0$ and $p \in [0, c'(a)]$, $\mu^\dagger(a,p) \in \left[a, \frac{3}{2}a\right)$ is the unique solution for $\mu \in [a,\infty)$ that solves

$$\frac{\mu^2}{a^3}\left[(3a - 2\mu)\,c'(\mu) - (\mu - a)\mu c''(\mu)\right] = p. \tag{EC.95}$$

(b) For any $a>0$ and $p\in[0,c'(a)]$,

$$v^\dagger(a,p):=\frac{(\mu^\dagger(a,p))^3}{a^2}\left(2c'(\mu^\dagger(a,p))+\mu^\dagger(a,p)c''(\mu^\dagger(a,p))\right). \tag{EC.96}$$

Remark EC.1. For any $a>0$, when $p=c'(a)$, $\mu=a$ solves (EC.95); therefore, by uniqueness, $\mu^\dagger(a,c'(a))=a$. Then, (EC.96) yields $v^\dagger(a,c'(a))=2ac'(a)+a^2c''(a)$.

Remark EC.2. For any $a>0$ and $p\in[0,c'(a)]$, $v^\dagger(a,p)>2ap$.

Lemma EC.22 **(Validating Definition EC.1)**. *For any $a>0$, the two equations $h(\mu;a,p,v)=c'(\mu)$ and $h'(\mu;a,p,v)=c''(\mu)$ are simultaneously satisfied (i.e., $h(\mu;a,p,v)$ and $c'(\mu)$ are tangent) for some $p\geq 0$, $v>0$, and $\mu\geq a$ if and only if $p\leq c'(a)$, $v=v^\dagger(a,p)$, and $\mu=\mu^\dagger(a,p)\in\left[a,\frac{3}{2}a\right)$.*

Lemma EC.23. *For any $a>0$ and $p\in[0,c'(a)]$, $a\leq\mu^\dagger(a,p)<\frac{3v^\dagger(a,p)}{2p+\frac{2v^\dagger(a,p)}{a}}\leq\frac{3}{2}a$.*

Lemma EC.24. *For any $a>0$ and $p\in[0,c'(a)]$, $c'(\mu)\geq h(\mu;a,p,v^\dagger(a,p))$ for all $\mu\in[a,\infty)$, with equality holding only at $\mu=\mu^\dagger(a,p)$.*

Next, we revisit the piece-rate payment thresholds $p^\dagger(v)$ and $p^\ddagger(v)$, introduced in Theorem 4 and Proposition 3 (and used in Proposition 7), respectively.

Definition EC.2.

(a) For any $v>0$, $p^\dagger(v)$ is the unique solution for $p\in(c'(0),\infty)$ that solves

$$p(c')^{-1}(p)+\frac{1}{2}((c')^{-1}(p))^2c''((c')^{-1}(p))=\frac{v}{2}. \tag{EC.97}$$

This is identical to the definition in Theorem 4.

(b) For any $v>0$, $p^\ddagger(v)$ is the unique solution for $p\in(p^\dagger(v),\infty)$ that solves

$$p(c')^{-1}(p)=\frac{v}{2}. \tag{EC.98}$$

This is equivalent to the definition in Proposition 3.

Remark EC.3. When $v=0$, (EC.97) is satisfied only when $p=c'(0)$, due to the strict convexity of c. As a result, $\lim_{v\downarrow 0}p^\dagger(v)=c'(0)$.

Remark EC.4. $p^\dagger\left(v^\dagger(a,c'(a))\right)=c'(a)$, or, equivalently, $v^\dagger\left((c')^{-1}(p^\dagger(v)),p^\dagger(v)\right)=v$.

Lemma EC.25 **(Validating Definition EC.2)**. *For any $v>0$,*

(a) there exists a unique solution for $p\in(c'(0),\infty)$ that solves (EC.97).

(b) there exists a unique solution for $p\in(p^\dagger(v),\infty)$ that solves (EC.98).

The following result provides some useful monotonicity properties of the quantities defined in Definitions EC.1 and EC.2.

LEMMA EC.26.

(a) $\mu^\dagger(a,p)$ *is strictly increasing in* $a > 0$ *and strictly decreasing in* $p \in [0, c'(a)]$.

(b) $v^\dagger(a,p)$ *is strictly increasing in* $a > 0$ *and strictly decreasing in* $p \in [0, c'(a)]$.

(c) $p^\dagger(v)$ *is strictly increasing in* $v > 0$.

(d) $p^\ddagger(v)$ *is strictly increasing in* $v > 0$.

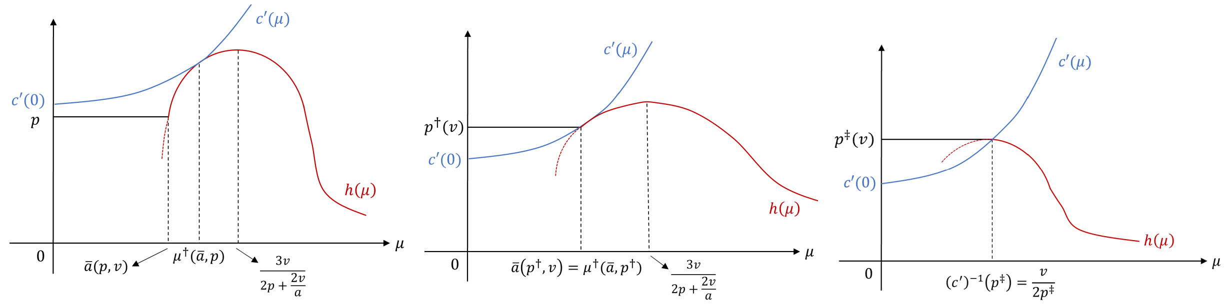
DEFINITION EC.3. For any $v > 0$ and $p \in [0, p^\dagger(v)]$, $\overline{a}(p,v)$ is the unique $a > 0$ such that $v^\dagger(a,p) = v$.

REMARK EC.5. For any $v > 0$ and $p \in (0, p^\dagger(v)]$, it follows, from Definition EC.3 and Remark EC.2, that $v > 2\overline{a}(p,v)$, or, equivalently, $\overline{a}(p,v) < \frac{v}{2p}$.

LEMMA EC.27. *Fix* $p \geq 0$. *Let* $A(p) := \{a > 0 : c'(a) \geq p\}$ *and* $V(p) := \{v > 0 : p^\dagger(v) \geq p\}$. *Then,* $v^\dagger(a,p) : A(p) \to V(p)$, *given by* (EC.96), *is an invertible function.*

COROLLARY EC.4 **(Validating Definition EC.3)**. *For any* $v > 0$ *and* $p \in [0, p^\dagger(v)]$, *there exists unique* $a > 0$ *such that* $v^\dagger(a,p) = v$.

COROLLARY EC.5. *For any* $v > 0$ *and* $p \in [0, p^\dagger(v)]$, $v < v^\dagger(a,p)$ *if* $a > \overline{a}(p,v)$; $v = v^\dagger(a,p)$ *if* $a = \overline{a}(p,v)$; *and* $v > v^\dagger(a,p)$ *if* $a < \overline{a}(p,v)$.



(I) $p < p^\dagger(v)$ (II) $p = p^\dagger(v)$ (III) $p = p^\ddagger(v)$

Figure EC.5 Illustration of $\mu^\dagger(a,p)$, $p^\dagger(v)$, $p^\ddagger(v)$ and $\overline{a}(p,v)$. Note that $v = v^\dagger(\overline{a}(p,v),p)$ in (I) and (II).

Proof of Remark EC.2: Recalling that c is strictly convex and $\mu^\dagger(a,p) \geq a$, from (EC.96),

$$\begin{aligned} v^\dagger(a,p) &= \frac{2(\mu^\dagger(a,p))^3}{a^2} c'(\mu^\dagger(a,p)) + \frac{(\mu^\dagger(a,p))^4}{a^2} c''(\mu^\dagger(a,p)) \\ &\geq 2ac'(a) + a^2 c''(\mu^\dagger(a,p)) > 2ac'(a) \geq 2ap, \end{aligned}$$

where the last inequality follows because $p \leq c'(a)$. ∎

Proof of Lemma EC.22: Recall, from (EC.93), that $h(\mu;a,p,v)=\frac{a^2}{\mu^2}\left(p+\frac{v}{a}-\frac{v}{\mu}\right)$ for all $\mu>0$. Therefore, the two equations $h(\mu;a,p,v)=c'(\mu)$ and $h'(\mu;a,p,v)=c''(\mu)$ are equivalent to

$$\frac{a^2}{\mu^2}\left(p+\frac{v}{a}-\frac{v}{\mu}\right)=c'(\mu) \quad \text{and} \quad \frac{a^2}{\mu^2}\left(\frac{3v}{\mu}-\frac{2v}{a}-2p\right)=\mu c''(\mu), \tag{EC.99}$$

respectively. Combining these two equations by eliminating v, we obtain

$$\frac{\mu^2}{a^3}\left[(3a-2\mu)\,c'(\mu)-(\mu-a)\mu c''(\mu)\right]=p,$$

which establishes (EC.95). In order to study the properties of solutions to (EC.95), we differentiate its left-hand side with respect to μ:

$$\begin{aligned}
\frac{d}{d\mu}&\left(\frac{\mu^2}{a^3}\left[(3a-2\mu)\,c'(\mu)-(\mu-a)\mu c''(\mu)\right]\right)\\
&=\frac{2\mu}{a^3}\left[(3a-2\mu)\,c'(\mu)-(\mu-a)\mu c''(\mu)\right]\\
&\quad+\frac{\mu^2}{a^3}\left[-2c'(\mu)+(3a-2\mu)c''(\mu)-\mu c''(\mu)-(\mu-a)(c''(\mu)+\mu c'''(\mu))\right]\\
&=-\frac{\mu(\mu-a)}{a^3}\left[6c'(\mu)+6\mu c''(\mu)+\mu^2c'''(\mu)\right],
\end{aligned}$$

which, under Assumption 1, is strictly positive for $\mu\in(0,a)$, zero at $\mu=a$, and strictly negative for $\mu\in(a,\infty)$. This implies that the left-hand side of (EC.95) attains a global maximum value of $c'(a)$ at $\mu=a$; therefore, (EC.95) is true for some $p\geq 0$ only if $p\leq c'(a)$. Next, observe that the left-hand side of (EC.95) is strictly decreasing in μ for $\mu\in[a,\infty)$, becoming negative at some $\mu\in\left(a,\frac{3}{2}a\right)$; therefore, given any $p\in[0,c'(a)]$, (EC.95) admits a unique solution for $\mu\in[a,\infty)$, denoted by $\mu^\dagger(a,p)\in\left[a,\frac{3}{2}a\right)$. Finally, combining the two equations in (EC.99) by eliminating p (instead of v), we obtain

$$\frac{\mu^3}{a^2}\left(2c'(\mu)+\mu c''(\mu)\right)=v,$$

which, after plugging in $\mu=\mu^\dagger(a,p)$, establishes (EC.96); consequently, given $p\in[0,c'(a)]$, $v^\dagger(a,p)$ inherits its uniqueness from $\mu^\dagger(a,p)$. ■

Proof of Lemma EC.23: From Lemma EC.21 (a), it follows that $h'(\mu;a,p,v)>0$ if and only if $\mu<\frac{3v}{2p+\frac{2v}{a}}$. By definition and from Lemma EC.22, given $a>0$ and $p\in[0,c'(a)]$, when $v=v^\dagger(a,p)$, $\mu^\dagger(v,p)\in[a,\frac{3}{2}a)$ satisfies $h'(\mu^\dagger(a,p);a,p,v^\dagger)=c''(\mu^\dagger)>0$ (due to the strict convexity of c); therefore, $\mu^\dagger(a,p)<\frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}$, which is naturally no greater than $\frac{3}{2}a$ for any $p\geq 0$. ■

Proof of Lemma EC.24: Given $a>0$ and $p\in[0,c'(a)]$, when $v=v^\dagger$, it follows from Lemma EC.23 that $\mu^\dagger\in\left[a,\frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}\right)$ satisfies $c'(\mu^\dagger)=h(\mu^\dagger)$ and $c''(\mu^\dagger)=h'(\mu^\dagger)$. Observe that:

- Under Assumption 1, $c'(\mu)$ is convex in μ for all $\mu\in(0,\infty)$:

$$c''(\mu_1)\leq\frac{c'(\mu_2)-c'(\mu_1)}{\mu_2-\mu_1}\leq c''(\mu_2) \quad \forall\ 0<\mu_1<\mu_2<\infty. \tag{EC.100}$$

- From Lemmas EC.21 (b) and EC.23, $h(\mu)$ is strictly concave in μ for all $\mu \in \left[a, \frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}\right]$:

$$h'(\mu_2) < \frac{h(\mu_2)-h(\mu_1)}{\mu_2-\mu_1} < h'(\mu_1) \quad \forall\, a \le \mu_1 < \mu_2 \le \frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}. \tag{EC.101}$$

First, for any $\mu \in [a, \mu^\dagger)$, we set $\mu_1 = \mu$, $\mu_2 = \mu^\dagger$, and use the fact that $c''(\mu^\dagger) = h'(\mu^\dagger)$ to combine (EC.100) and (EC.101) and obtain

$$\begin{aligned} &\frac{c'(\mu^\dagger)-c'(\mu)}{\mu^\dagger-\mu} < \frac{h(\mu^\dagger)-h(\mu)}{\mu^\dagger-\mu} \\ \overset{(*)}{\Longrightarrow}\;& c'(\mu) > h(\mu) \quad \forall\, \mu \in [a, \mu^\dagger), \end{aligned} \tag{EC.102}$$

where $(*)$ follows due to the fact that $c'(\mu^\dagger) = h(\mu^\dagger)$. Similarly, second, for any $\mu \in \left(\mu^\dagger, \frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}\right]$, we set $\mu_1 = \mu^\dagger$, $\mu_2 = \mu$, and use the fact that $c''(\mu^\dagger) = h'(\mu^\dagger)$ to combine (EC.100) and (EC.101) and obtain

$$\begin{aligned} &\frac{h(\mu)-h(\mu^\dagger)}{\mu-\mu^\dagger} < \frac{c'(\mu)-c'(\mu^\dagger)}{\mu-\mu^\dagger} \\ \overset{(*)}{\Longrightarrow}\;& c'(\mu) > h(\mu) \quad \forall\, \mu \in \left(\mu^\dagger, \frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}\right], \end{aligned} \tag{EC.103}$$

where $(*)$ follows due to the fact that $c'(\mu^\dagger) = h(\mu^\dagger)$. Finally, since $c'(\mu)$ is strictly increasing in μ, $h(\mu)$ is decreasing in μ for all $\mu \in \left[\frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}, \infty\right)$ (Lemma EC.21 (a)), and $c'\left(\frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}\right) > h\left(\frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}\right)$ (special case of (EC.103)), it follows that

$$c'(\mu) > h(\mu) \quad \forall\, \mu \in \left(\frac{3v^\dagger}{2p+\frac{2v^\dagger}{a}}, \infty\right). \tag{EC.104}$$

The proof is complete when combining (EC.102)-(EC.104) and recalling that $c'(\mu^\dagger) = h(\mu^\dagger)$. ∎

Proof of Remark EC.4: From Remark EC.1, $v^\dagger(a, c'(a)) = 2ac'(a) + a^2 c''(a) > 0$. Thus, from Definition EC.2 (a), $p^\dagger\left(v^\dagger(a, c'(a))\right)$ is the unique solution in p to

$$p(c')^{-1}(p) + \frac{1}{2}\left((c')^{-1}(p)\right)^2 c''\left((c')^{-1}(p)\right) = ac'(a) + \frac{1}{2}a^2 c''(a).$$

Note that $p = c'(a)$ solves the above equation. By uniqueness, it follows that $p^\dagger\left(v^\dagger(a, c'(a))\right) = c'(a)$. Similarly, when $p = p^\dagger(v) > c'(0)$ and $a = (c')^{-1}(p^\dagger(v))$, it follows from Remark EC.1 that $v^\dagger\left((c')^{-1}(p^\dagger(v)), p^\dagger(v)\right) = 2p^\dagger(v)(c')^{-1}(p^\dagger(v)) + \left((c')^{-1}(p^\dagger(v))\right)^2 c''\left((c')^{-1}(p^\dagger(v))\right) \overset{(*)}{=} 2 \cdot \frac{v}{2} = v$, where $(*)$ follows from Definition EC.2 (a). ∎

Proof of Lemma EC.25:

(a): Denote the left-hand side of (EC.97) by $LHS^\dagger(p)$. We first note that $LHS^\dagger(p)$ is a strictly increasing function of p (since c' is strictly increasing due to strict convexity of c, and c'' is also increasing by Assumption 1), and the right-hand side of (EC.97), $\frac{v}{2}$, does not depend on p. Moreover, note that when $LHS^{\dagger}(c'(0)) = 0 < \frac{v}{2}$ and $\lim_{p\to\infty} LHS^{\dagger}(p) = \infty > \frac{v}{2}$. Hence, it follows that (EC.97) admits a unique solution in $(c'(0), \infty)$.

(b): Denote the left-hand side of (EC.98) by $LHS^{\ddagger}(p)$. We first note that $LHS^{\ddagger}(p)$ is a strictly increasing function of p (since c' is strictly increasing due to strict convexity of c), and the right-hand side of (EC.98), $\frac{v}{2}$, does not depend on p. Moreover, note that $LHS^{\dagger}(p) > LHS^{\ddagger}(p)$ for all $p > 0$, while the right-hand sides of (EC.97) and (EC.98) are both $\frac{v}{2}$. This implies that $LHS^{\dagger}(p^{\ddagger}(v)) > LHS^{\ddagger}(p^{\ddagger}(v)) = \frac{v}{2} = LHS^{\dagger}(p^{\dagger}(v))$, and hence $p^{\ddagger}(v) > p^{\dagger}(v) > c'(0)$ (recalling that $LHS^{\dagger}(p)$ is a strictly increasing function of p). Additionally, note that $\lim_{p\to\infty} LHS^{\ddagger}(p) = \infty > \frac{v}{2}$. Hence, it follows that (EC.98) admits a unique solution in $(p^{\dagger}(v), \infty)$. ■

Proof of Lemma EC.26:

(a): By definition, $\mu^{\dagger}(a,p)$ is given by (EC.95), which can be rewritten as

$$\mu^2\left[(3a-2\mu)c'(\mu) - (\mu-a)\mu c''(\mu)\right] = a^3 p. \tag{EC.105}$$

Differentiating both sides of (EC.105) with respect to a yields

$$\begin{aligned}
&2\mu\frac{\partial\mu}{\partial a}\left[(3a-2\mu)c'(\mu) - (\mu-a)\mu c''(\mu)\right] \\
&+\mu^2\left[\left(3-2\frac{\partial\mu}{\partial a}\right)c'(\mu) + (3a-2\mu)c''(\mu)\frac{\partial\mu}{\partial a} - \left(\frac{\partial\mu}{\partial a}-1\right)\mu c''(\mu) - (\mu-a)\left(\frac{\partial\mu}{\partial a}c''(\mu) + \mu c'''(\mu)\frac{\partial\mu}{\partial a}\right)\right] = 3a^2 p.
\end{aligned}$$

After algebra,

$$\left[-6\mu(\mu-a)c'(\mu) - 6\mu^2(\mu-a)c''(\mu) - \mu^3(\mu-a)c'''(\mu)\right]\frac{\partial\mu}{\partial a} = 3a^2 p - 3\mu^2 c'(\mu) - \mu^3 c''(\mu). \tag{EC.106}$$

Thus, $\mu^{\dagger}(a,p)$ satisfies (EC.106). Note that the right-hand side of (EC.106) at $\mu = \mu^{\dagger}(a,p)$ satisfies

$$3a^2 p - 3\mu^2 c'(\mu) - \mu^3 c''(\mu) < 3a^2 p - 3\mu^2 c'(\mu) \le 0,$$

noting that $\mu^{\dagger}(a,p) \ge a$ and $c'(\mu^{\dagger}(a,p)) \ge c'(a) \ge p$ (where $c'(a) \ge p$ is by definition of $\mu^{\dagger}(a,p)$ in Definition EC.1 (a)). Thus, the left-hand side of (EC.106) is strictly negative. Moreover, the term in the square bracket on the left-hand side of (EC.106) is also negative, since $\mu^{\dagger}(a,p) \ge a$, $c' > 0$, $c'' > 0$ and $c''' \ge 0$ (Assumption 9). Thus, $\frac{\partial\mu^{\dagger}(a,p)}{\partial a} > 0$, i.e., $\mu^{\dagger}(a,p)$ is strictly increasing in $a > 0$.

Similarly, differentiating both sides of (EC.105) with respect to p yields

$$\begin{aligned}
&2\mu\frac{\partial\mu}{\partial p}\left[(3a-2\mu)c'(\mu) - (\mu-a)\mu c''(\mu)\right] \\
&+\mu^2\left[-2\frac{\partial\mu}{\partial p}c'(\mu) + (3a-2\mu)c''(\mu)\frac{\partial\mu}{\partial p} - \frac{\partial\mu}{\partial p}\mu c''(\mu) - (\mu-a)\left(\frac{\partial\mu}{\partial p}c''(\mu) + \mu c'''(\mu)\frac{\partial\mu}{\partial\mu}\right)\right] = a^3.
\end{aligned}$$

After algebra,

$$\left[-6\mu^2(\mu-a)c'(\mu) - 6\mu^2(\mu-a)c''(\mu) - \mu^3(\mu-a)c'''(\mu)\right]\frac{\partial\mu}{\partial p} = a^3.$$

Thus, $\mu^\dagger(a,p)$ satisfies the above equation. Note that the right-hand side of the above equation is strictly positive, which implies that the left-hand side of the above equation is also strictly positive. Moreover, note that the term in the square bracket on the left-hand side of the above equation is negative, because $\mu^\dagger(a,p) \geq a$, $c' > 0$, $c'' > 0$ and $c''' \geq 0$ (Assumption 9). Thus, $\frac{\partial \mu^\dagger(a,p)}{\partial p} < 0$, i.e., $\mu^\dagger(a,p)$ is strictly decreasing in $p \in [0, c'(a)]$.

(b): Differentiating both sides of (EC.96) yields

$$\begin{aligned}\frac{\partial v^\dagger(a,p)}{\partial a} =& \frac{3(\mu^\dagger)^2 \frac{\partial \mu^\dagger}{\partial a} a^2 - (\mu^\dagger)^3 \cdot 2a}{a^4} \left[2c'(\mu^\dagger) + \mu^\dagger c''(\mu^\dagger)\right] \\ &+ \frac{(\mu^\dagger)^3}{a^2} \left[2c''(\mu^\dagger)\frac{\partial \mu^\dagger}{\partial a} + \frac{\partial \mu^\dagger}{\partial a} c''(\mu^\dagger) + \mu^\dagger c'''(\mu^\dagger)\frac{\partial \mu^\dagger}{\partial a}\right] \\ =& \left[6\frac{(\mu^\dagger)^2}{a^2} c'(\mu^\dagger) + 6\frac{(\mu^\dagger)^3}{a^2} c''(\mu^\dagger) + \frac{(\mu^\dagger)^4}{a^2} c'''(\mu^\dagger)\right] \frac{\partial \mu^\dagger}{\partial a} - 4\frac{(\mu^\dagger)^4}{a^3} c'(\mu^\dagger) - 2\frac{(\mu^\dagger)^4}{a^3} c''(\mu^\dagger).\end{aligned}$$

Substituting for $\frac{\partial \mu^\dagger}{\partial a}$ from (EC.106) into the above display, and after algebra, implies

$$\begin{aligned}\frac{\partial v^\dagger(a,p)}{\partial a} =& \frac{\mu^\dagger}{a^2(\mu^\dagger - a)} \left(3(\mu^\dagger)^2 c'(\mu^\dagger) + (\mu^\dagger)^3 c''(\mu^\dagger) - 3a^2 p\right) - 4\frac{(\mu^\dagger)^3}{a^3} c'(\mu^\dagger) - 2\frac{(\mu^\dagger)^4}{a^3} c''(\mu^\dagger) \\ =& \left[\frac{3(\mu^\dagger)^3}{a^2(\mu^\dagger - a)} - 4\frac{(\mu^\dagger)^3}{a^3}\right] c'(\mu^\dagger) + \left[\frac{(\mu^\dagger)^4}{a^2(\mu^\dagger - a)} - 2\frac{(\mu^\dagger)^4}{a^3}\right] c''(\mu^\dagger) - \frac{\mu^\dagger}{a^2(\mu^\dagger - a)} 3a^2 p. \qquad \text{(EC.107)}\end{aligned}$$

Additionally, substituting for p in the last term of (EC.107) using (EC.105) yields

$$\begin{aligned}\frac{\mu^\dagger}{a^2(\mu^\dagger - a)} 3a^2 p =& \frac{3(\mu^\dagger)^3}{a^3(\mu^\dagger - a)} \left[(3a - 2\mu^\dagger) c'(\mu^\dagger) - (\mu^\dagger - a)\mu^\dagger c''(\mu^\dagger)\right] \\ =& \frac{3(\mu^\dagger)^3(3a - 2\mu^\dagger)}{a^3(\mu^\dagger - a)} c'(\mu^\dagger) - 3\frac{(\mu^\dagger)^4}{a^3} c''(\mu^\dagger).\end{aligned}$$

Substitution into (EC.107) yields

$$\begin{aligned}\frac{\partial v^\dagger(a,p)}{\partial a} =& \left[\frac{3(\mu^\dagger)^3}{a^2(\mu^\dagger - a)} - 4\frac{(\mu^\dagger)^3}{a^3} - \frac{3(\mu^\dagger)^3(3a - 2\mu^\dagger)}{a^3(\mu^\dagger - a)}\right] c'(\mu^\dagger) + \left[\frac{(\mu^\dagger)^4}{a^2(\mu^\dagger - a)} - 2\frac{(\mu^\dagger)^4}{a^3} + 3\frac{(\mu^\dagger)^4}{a^3}\right] c''(\mu^\dagger) \\ =& 2\frac{(\mu^\dagger)^3}{a^3} c'(\mu^\dagger) + \left[\frac{(\mu^\dagger)^4}{a^2(\mu^\dagger - a)} + \frac{(\mu^\dagger)^4}{a^3}\right] c''(\mu^\dagger) > 0,\end{aligned}$$

noting that $c' > 0$ and $c'' > 0$. Hence, $v^\dagger(a,p)$ is strictly increasing in $a > 0$.

Next, we investigate the monotonicity property of $v^\dagger(a,p)$ in terms of p. From (a), $\mu^\dagger(a,p)$ is strictly decreasing in $p \in [0, c'(a)]$; that is, $\mu^\dagger(a,p_1) > \mu^\dagger(a,p_2)$ for any $0 \leq p_1 < p_2 \leq c'(a)$. Recalling that c' is a strictly increasing function and c'' is also an increasing function (Assumption 1), it follows that $c'\left(\mu^\dagger(a,p_1)\right) > c'\left(\mu^\dagger(a,p_2)\right)$ and $c''\left(\mu^\dagger(a,p_1)\right) \geq c''\left(\mu^\dagger(a,p_2)\right)$. Therefore, $v^\dagger(a,p_1) > v^\dagger(a,p_2)$ for $0 \leq p_1 < p_2 \leq c'(a)$; that is, $v^\dagger(a,p)$ is strictly decreasing in $p \in [0, c'(a)]$.

(c): As v increases, it is clear that the right-hand side of (EC.97) increases. To equate the left-hand side of (EC.97) with the right-hand side of (EC.97), p should increase (because the left-hand side of (EC.97) is strictly increasing in p). Hence, $p^\dagger(v)$ is strictly increasing in $v > 0$.

(d): Similar to (c), we can conclude that $p^\ddagger(v)$ is strictly increasing in $v > 0$. ∎

Proof of Lemma EC.27: Fix $p \geq 0$. First, we show that for any $a \in A(p)$, $v^\dagger(a,p) \in V(p)$. Recall that $v^\dagger(a,p)$ is strictly decreasing in $p \in [0, c'(a)]$ (Lemma EC.26 (b)) and $p^\dagger(v)$ is strictly increasing in $v > 0$ (Lemma EC.26 (c)). Thus, $a \in A(p) \Rightarrow p \leq c'(a) \Rightarrow v^\dagger(a,p) \geq v^\dagger(a, c'(a)) \Rightarrow p^\dagger\left(v^\dagger(a,p)\right) \geq p^\dagger\left(v^\dagger(a,c'(a))\right)$. Moreover, from Remark EC.4, we know that $p^\dagger\left(v^\dagger(a,c'(a))\right) = c'(a)$; therefore, $p^\dagger\left(v^\dagger(a,p)\right) \geq p^\dagger\left(v^\dagger(a,c'(a))\right) = c'(a) \geq p \Rightarrow v^\dagger(a,p) \in V(p)$.

Next, since $v^\dagger(a,p) : A(p) \to V(p)$ is a continuous and strictly increasing function of $a > 0$ (Lemma EC.26 (b)), it follows that it is a one-to-one function. What remains to be shown is that it is also an onto function. For this, we rely on the following claim, stated below without proof:

Claim EC.6. *A function $f : [a,\infty) \to [b,\infty)$ that is continuous and strictly increasing is an onto function if and only if $f(a) = f(b)$ and $f(x)$ grows unboundedly with x.*

In order to apply this claim to $v^\dagger(a,p) : A(p) \to V(p)$, we need to understand the structure of the sets $A(p)$ and $V(p)$, which depends upon the relationship between p and $c'(0)$. If $0 \leq p \leq c'(0)$, then $A(p) = V(p) = (0,\infty)$; otherwise, $A(p) = [(c')^{-1}(p), \infty)$ and $V(p) = [(p^\dagger)^{-1}(p), \infty)$. Therefore, to complete the proof, we must show the following:

(i) $\lim_{a\uparrow\infty} v^\dagger(a,p) = \infty$ for all $p \geq 0$.

(ii) If $0 \leq p \leq c'(0)$, then $\lim_{a\downarrow 0} v^\dagger(a,p) = 0$.

(iii) If $p > c'(0)$, then $v^\dagger\left((c')^{-1}(p), p\right) = (p^\dagger)^{-1}(p)$, or, equivalently, $p^\dagger\left(v^\dagger\left((c')^{-1}(p), p\right)\right) = p$.

Proof of (i): For any $a > 0$ and $p \geq 0$, $c'(a) \geq p$ for all large enough a due to the strict convexity of c. Then, from Remark EC.2, $v^\dagger(a,p) > 2ap$ for all large enough a; therefore, $\lim_{a\uparrow\infty} v^\dagger(a,p) = \infty$.

Proof of (ii): For any $a > 0$ and $p \leq c'(0)$, $c'(a) \geq p$ for all $a > 0$. In order to evaluate $\lim_{a\downarrow 0} v^\dagger(a,p)$ using Definition EC.1 (b), we must first evaluate $\lim_{a\downarrow 0} \mu^\dagger(a,p)$ using Definition EC.1 (a), according to which $\mu^\dagger(a,p) \in \left[a, \frac{3a}{2}\right)$; therefore, $\lim_{a\downarrow 0} \mu^\dagger(a,p) = 0$. Moreover, (EC.95) of Definition EC.1 (a) can equivalently be written as

$$\left(\frac{\mu}{a}\right)^2 \left[\left(\frac{3}{2} - \frac{\mu}{a}\right) 2c'(\mu) - \left(\frac{\mu}{a} - 1\right) \mu c''(\mu)\right] = p. \tag{EC.108}$$

Letting $a \to 0$ in the above equation implies that, for any subsequence a' for which $\lim_{a'\downarrow 0} \frac{\mu^\dagger(a,p)}{a}$ exists, this limit must lie in the interval $\left[1, \frac{3}{2}\right]$. We simply use a rather than a' to denote the subsequence. Then, it would follow from Definition EC.1 (b) that

$$\lim_{a\downarrow 0} v^\dagger(a,p) = \left(\lim_{a\downarrow 0} \frac{\mu^\dagger(a,p)}{a}\right)^2 \lim_{a\downarrow 0} \mu^\dagger(a,p) \left(2c'\left(\lim_{a\downarrow 0} \mu^\dagger(a,p)\right) + \lim_{a\downarrow 0} \mu^\dagger(a,p) \cdot c''\left(\lim_{a\downarrow 0} \mu^\dagger(a,p)\right)\right) = 0.$$

Proof of (iii): This follows immediately from Remark EC.4 by substituting $a = (c')^{-1}(p)$. ■

Proof of Corollary EC.4: It immediately follows from Lemma EC.27 that for any $v > 0$ and $p \in [0, p^\dagger(v)]$, there exists a unique a such that $v^\dagger(a,p) = v$ (by the invertible function $v^\dagger(a,p)$). In particular, when $a = (c')^{-1}(p)$ and $p = p^\dagger(v)$, Definitions EC.1(b) and EC.2(a) imply that $v^\dagger(a,p) = v^\dagger\left((c')^{-1}(p^\dagger(v)), p^\dagger(v)\right) = v$. Equivalently, $p^\dagger\left(v^\dagger(a, c'(a)\right) = c'(a)$. ■

Proof of Corollary EC.5: By Definition EC.3, $v^\dagger(a,p) = v$ if $a = \overline{a}(p,v)$. Recall from Lemma EC.26 (b) that $v^\dagger(a,p)$ is strictly increasing in $a > 0$. Then, if $a > \overline{a}(p,v)$, then $v^\dagger(a,p) > v^\dagger(\overline{a}(p,v),p) = v$. Similarly, if $a < \overline{a}(p,v)$, then $v^\dagger(a,p) < v^\dagger(\overline{a}(p,v),p) = v$. ■

EC.7.5. Proof of Lemma 5

(a): Setting $\mu_1 = \mu$ in Propositions EC.1 and EC.2 yields

$$\lim_{\lambda\to\infty} I^\lambda(\mu,\mu) = \left[1-\frac{a}{\mu}\right]^+ \quad\text{and}\quad \lim_{\lambda\to\infty} \left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} = \frac{a[\mu-a]^+}{\mu^3}.$$

(b): Suppose $N^\lambda = f(\lambda) + o(f(\lambda))$ for $f(\lambda) \in o(\lambda)\cap\omega(1)$ or $f(\lambda)\in\omega(\lambda)$.

- When $f(\lambda)\in o(\lambda)\cap\omega(1)$, let $N_0^\lambda = \frac{\lambda}{a} + N^\lambda$. Then, it is clear that $N^\lambda \le N_0^\lambda$ for all $a>0$ and $\lambda>0$. From Lemma EC.1 (b), it follows that

$$\begin{aligned} &ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \ge ErlC\left(N_0^\lambda, \frac{\lambda}{\mu}\right), \quad \forall a>0,\ \forall\lambda>0,\\ \Rightarrow &\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \ge \lim_{\lambda\to\infty} ErlC\left(N_0^\lambda, \frac{\lambda}{\mu}\right), \quad \forall a>0.\\ \Rightarrow &\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \ge \sup_{a>0}\lim_{\lambda\to\infty} ErlC\left(N_0^\lambda, \frac{\lambda}{\mu}\right) = \infty, \end{aligned}$$

where the last equality follows from Lemma EC.12 (b) when $\mu < a$.

- When $f(\lambda)\in\omega(\lambda)$, let $N_0^\lambda = \frac{\lambda}{a}$ for any $a>0$, so that $N_0^\lambda \in o(N^\lambda)$. Then, there exists $\Lambda(a)>0$ such that $N^\lambda \ge N_0^\lambda$ for all $\lambda\ge\Lambda(a)$. From Lemma EC.1 (b), it follows that

$$\begin{aligned} &ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \le ErlC\left(N_0^\lambda, \frac{\lambda}{\mu}\right), \quad \forall a>0,\ \forall\lambda>0,\\ \Rightarrow &\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \le \lim_{\lambda\to\infty} ErlC\left(N_0^\lambda, \frac{\lambda}{\mu}\right), \quad \forall a>0.\\ \Rightarrow &\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) \le \inf_{a>0}\lim_{\lambda\to\infty} ErlC\left(N_0^\lambda, \frac{\lambda}{\mu}\right) = 0, \end{aligned}$$

where the last equality follows from Lemma EC.12 (b) when $\mu > a$.

Therefore, we conclude that

$$\lim_{\lambda\to\infty} ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right) = \begin{cases}\infty, & f(\lambda)\in o(\lambda)\cap\omega(1),\\ 0, & f(\lambda)\in\omega(\lambda).\end{cases} \tag{EC.109}$$

Using (EC.109) to evaluate the limiting value of $I^\lambda(\mu,\mu)$ from (EC.18) in Corollary EC.1,

$$\lim_{\lambda\to\infty} I^\lambda(\mu,\mu) = \begin{cases}0, & f(\lambda)\in o(\lambda)\cap\omega(1),\\ 1, & f(\lambda)\in\omega(\lambda).\end{cases}$$

Next, from Corollary EC.2 (a),

$$\lim_{\lambda\to\infty} \mu\left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu} \le \lim_{\lambda\to\infty} I^\lambda(\mu,\mu)\left(1-I^\lambda(\mu,\mu)\right) + \frac{2}{\sqrt{N^\lambda}} = 0,$$

because $\lim_{\lambda\to\infty} N^\lambda = \infty$, and $\lim_{\lambda\to\infty} I^\lambda(\mu,\mu)\left(1-I^\lambda(\mu,\mu)\right) = 0$ regardless of whether $I^\lambda(\mu,\mu)=0$ (when $f(\lambda)\in o(\lambda)\cap\omega(1)$) or 1 (when $f(\lambda)\in\omega(\lambda)$). Hence, $\lim_{\lambda\to\infty} \left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu} = 0$ by non-negativity (recalling from Lemma EC.4 (a) that $\lim_{\lambda\to\infty}\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1} \ge 0$ for all $\mu_1>0$ and $\mu>0$).

■

EC.7.6. Proof of Lemma 6

Recall that the tagged server's utility function in the finite system is given by (4). Using the notation established in Section 5.1, (4) becomes

$$U^{\lambda}(\mu_1,\mu)=p\mu_1+(v-p\mu_1)I^{\lambda}(\mu_1,\mu)-c(\mu_1).$$

Differentiating with respect to μ_1 yields

$$\frac{\partial U^{\lambda}(\mu_1,\mu)}{\partial \mu_1}=p(1-I^{\lambda}(\mu_1,\mu))+(v-p\mu_1)\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}-c'(\mu_1).$$

Differentiating once more with respect to μ_1 yields

$$\begin{aligned}\frac{\partial^2 U^{\lambda}(\mu_1,\mu)}{\partial \mu_1^2}=&-2p\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}+(v-p\mu_1)\frac{\partial^2 I^{\lambda}}{\partial \mu_1^2}-c''(\mu_1)\\ =&v\frac{\partial^2 I^{\lambda}(\mu_1,\mu)}{\partial \mu_1^2}-p\mu_1\left(\frac{\partial^2 I^{\lambda}(\mu_1,\mu)}{\partial \mu_1^2}+\frac{2}{\mu_1}\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right)-c''(\mu_1).\end{aligned}\tag{EC.110}$$

For the remainder of this proof, we inherit the shorthand notation and the setup outlined around the proof of Proposition EC.1 in the preliminary Section EC.7.2.

To begin, we recall, from Propositions EC.1 and EC.2, that

$$\lim_{\lambda\to\infty} I^{\lambda}=\begin{cases}0, & \mu\le a,\\ \frac{1-\frac{a}{\mu}}{1-\frac{a}{\mu}+\frac{a}{\mu_1}}, & \mu>a,\end{cases}\tag{EC.111}$$

and

$$\lim_{\lambda\to\infty}\frac{\partial I^{\lambda}}{\partial \mu_1}=\begin{cases}0, & \mu\le a,\\ \frac{\frac{a}{\mu_1^2}\left(1-\frac{a}{\mu}\right)}{\left(1-\frac{a}{\mu}+\frac{a}{\mu_1}\right)^2}, & \mu>a.\end{cases}\tag{EC.112}$$

Next, we recall, from (EC.15) in Lemma EC.5, that

$$\begin{aligned}\frac{\partial^2 I^{\lambda}}{\partial \mu_1^2}=&\frac{(I^{\lambda})^2}{\mu_1^2}\Bigg\{-2(1-I^{\lambda})+\frac{2\rho^{\lambda}C^{\lambda}}{(d_2^{\lambda})^2}\left[1-\left(\frac{\rho^{\lambda}}{d_1^{\lambda}}\right)^{k^{\lambda}-N^{\lambda}}\right]\left[(1-2I^{\lambda})-\frac{\frac{\mu_1}{\mu}+\rho^{\lambda}\left(\frac{1-C^{\lambda}}{N^{\lambda}-\rho^{\lambda}}\right)}{d_2^{\lambda}}I^{\lambda}\right]\\ &-\frac{\rho^{\lambda}C^{\lambda}}{d_2^{\lambda}}\frac{k^{\lambda}-N^{\lambda}}{d_1^{\lambda}}\left(\frac{\rho^{\lambda}}{d_1^{\lambda}}\right)^{k^{\lambda}-N^{\lambda}}\left[2(1-2I^{\lambda})+\left(\frac{2}{d_2^{\lambda}}-\frac{1}{d_1^{\lambda}}\right)\frac{\mu_1}{\mu}-4\frac{\frac{\mu_1}{\mu}+\rho^{\lambda}\left(\frac{1-C^{\lambda}}{N^{\lambda}-\rho^{\lambda}}\right)}{d_2^{\lambda}}I^{\lambda}\right]\\ &-\rho^{\lambda}C^{\lambda}\left(\frac{k^{\lambda}-N^{\lambda}}{d_1^{\lambda}}\right)^2\left(\frac{\rho^{\lambda}}{d_1^{\lambda}}\right)^{k^{\lambda}-N^{\lambda}}\left[\frac{1}{d_2^{\lambda}}\frac{\mu_1}{\mu}-2\frac{\frac{\mu_1}{\mu}+\rho^{\lambda}\left(\frac{1-C^{\lambda}}{N^{\lambda}-\rho^{\lambda}}\right)}{d_2^{\lambda}}I^{\lambda}-\frac{2\rho^{\lambda}C^{\lambda}}{(d_2^{\lambda})^2}I^{\lambda}\right]\Bigg\}\\ =:&\left(\frac{I^{\lambda}}{\mu_1}\right)^2\left\{-2(1-I^{\lambda})+t_1^{\lambda}-t_2^{\lambda}-t_3^{\lambda}\right\},\end{aligned}\tag{EC.113}$$

where

$$t_1^{\lambda}:=\frac{2\rho^{\lambda}C^{\lambda}}{(d_2^{\lambda})^2}\left[1-\left(\frac{\rho^{\lambda}}{d_1^{\lambda}}\right)^{k^{\lambda}-N^{\lambda}}\right]\left[(1-2I^{\lambda})-\frac{\frac{\mu_1}{\mu}+\rho^{\lambda}\left(\frac{1-C^{\lambda}}{N^{\lambda}-\rho^{\lambda}}\right)}{d_2^{\lambda}}I^{\lambda}\right],$$

$$t_2^{\lambda}:=\frac{\rho^{\lambda}C^{\lambda}}{d_2^{\lambda}}\frac{k^{\lambda}-N^{\lambda}}{d_1^{\lambda}}\left(\frac{\rho^{\lambda}}{d_1^{\lambda}}\right)^{k^{\lambda}-N^{\lambda}}\left[2(1-2I^{\lambda})+\left(\frac{2}{d_2^{\lambda}}-\frac{1}{d_1^{\lambda}}\right)\frac{\mu_1}{\mu}-4\frac{\frac{\mu_1}{\mu}+\rho^{\lambda}\left(\frac{1-C^{\lambda}}{N^{\lambda}-\rho^{\lambda}}\right)}{d_2^{\lambda}}I^{\lambda}\right],\text{ and}$$

$$t_3^{\lambda}:=\rho^{\lambda}C^{\lambda}\left(\frac{k^{\lambda}-N^{\lambda}}{d_1^{\lambda}}\right)^2\left(\frac{\rho^{\lambda}}{d_1^{\lambda}}\right)^{k^{\lambda}-N^{\lambda}}\left[\frac{1}{d_2^{\lambda}}\frac{\mu_1}{\mu}-2\frac{\frac{\mu_1}{\mu}+\rho^{\lambda}\left(\frac{1-C^{\lambda}}{N^{\lambda}-\rho^{\lambda}}\right)}{d_2^{\lambda}}I^{\lambda}-\frac{2\rho^{\lambda}C^{\lambda}}{(d_2^{\lambda})^2}I^{\lambda}\right].$$

From (EC.110), and recalling that $c''(\mu_1) > 0$ for all $\mu_1 > 0$ (due to the strict convexity of c), it follows that, in order to show that $\lim_{\lambda\to\infty} \frac{\partial^2 U^\lambda(\mu_1,\mu)}{\partial\mu_1^2}$ is strictly less than 0, it suffices to show that

- $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial\mu_1^2} \le 0$, and
- $\lim_{\lambda\to\infty} \left(\frac{\partial^2 I^\lambda}{\partial\mu_1^2} + \frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial\mu_1}\right) \ge 0$.

To do this, in what follows, we evaluate $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial\mu_1^2}$ under three cases: (I) $\mu < a$, (II) $\mu > a$, and (III) $\mu = a$.

Case (I): If $\mu < a$, then $\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in [0,1]$ for all λ (from Lemma EC.13 (c)), $\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda} \le 1$ for all large enough λ; and, as $\lambda \to \infty$, $\frac{\rho^\lambda}{d_1^\lambda} \to \frac{a}{\mu} \in (0,\infty)$, $\frac{\rho^\lambda}{d_2^\lambda} \to \left(\frac{\mu}{a}-1\right)^{-1} \in (-\infty,0)$, $d_1^\lambda \to \infty$, $d_2^\lambda \to -\infty$, $\frac{C^\lambda}{d_2^\lambda} = \frac{C^\lambda}{\lambda}/\frac{d_2^\lambda}{\lambda} \to \frac{\mu}{a} - 1 \in (-\infty,0)$ (from Lemma EC.13 (a)), $\frac{C^\lambda}{d_1^\lambda} = \frac{C^\lambda}{d_2^\lambda}/\frac{d_2^\lambda}{d_1^\lambda} \to \frac{\mu}{a}\left(1-\frac{a}{\mu}\right)^2$, $I^\lambda \to 0$ (from Proposition EC.1), $\lim_{\lambda\to\infty}(k^\lambda-N^\lambda)^r\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda} < \infty$ for all $r \in \mathbb{N}$ (because even if $\limsup_{\lambda\to\infty} k^\lambda - N^\lambda = \infty$, exponential decay in terms of $k^\lambda - N^\lambda$ dominates its polynomial growth), and

$$
\begin{aligned}
&\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\\
&=\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\left(1+\rho^\lambda\frac{\mu}{\mu_1}\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}+\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\frac{C^\lambda}{d_2^\lambda}\right)\right)^{-2}\\
&=\lim_{\lambda\to\infty}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}\left(\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}+\rho^\lambda\frac{\mu}{\mu_1}\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}+\left(\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}-1\right)\frac{C^\lambda}{d_2^\lambda}\right)\right)^{-2}\\
&=0,
\end{aligned}
$$

recalling that $\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda} \le 1$ for all large enough λ, $\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in [0,1]$ for all λ, and $\lim_{\lambda\to\infty} \frac{C^\lambda}{d_2^\lambda} = \frac{\mu}{a} - 1$. Based on these facts, it can be shown that

$$
\lim_{\lambda\to\infty}(I^\lambda)^2 t_1^\lambda = 2\frac{\rho^\lambda}{d_2^\lambda}\frac{C^\lambda}{d_2^\lambda}\left[(I^\lambda)^2-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right]\left[(1-2I^\lambda)-\frac{\frac{\mu_1}{\mu}+\rho^\lambda\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)}{d_2^\lambda}I^\lambda\right]=0,
$$

$$
\lim_{\lambda\to\infty}(I^\lambda)^2 t_2^\lambda = \frac{\rho^\lambda}{d_2^\lambda}\frac{C^\lambda}{d_1^\lambda}(k^\lambda-N^\lambda)\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\left[2(1-2I^\lambda)+\left(\frac{2}{d_2^\lambda}-\frac{1}{d_1^\lambda}\right)\frac{\mu_1}{\mu}-4\frac{\frac{\mu_1}{\mu}+\rho^\lambda\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)}{d_2^\lambda}I^\lambda\right]=0,
$$

$$
\lim_{\lambda\to\infty}(I^\lambda)^2 t_3^\lambda = \frac{\rho^\lambda}{d_1^\lambda}\frac{C^\lambda}{d_1^\lambda}(k^\lambda-N^\lambda)^2\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\left[\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}-2\frac{\frac{\mu_1}{\mu}+\rho^\lambda\left(\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)}{d_2^\lambda}I^\lambda-2\frac{\rho^\lambda}{d_2^\lambda}\frac{C^\lambda}{d_2^\lambda}I^\lambda\right]=0,
$$

which, from (EC.113), implies that

$$
\lim_{\lambda\to\infty}\frac{\partial^2 I^\lambda}{\partial\mu_1^2} = -\frac{2}{\mu_1^2}\left(\lim_{\lambda\to\infty} I^\lambda\right)^2\left(1-\lim_{\lambda\to\infty} I^\lambda\right)=0. \tag{EC.114}
$$

Together with (EC.112), it follows that

$$
\lim_{\lambda\to\infty}\frac{\partial^2 I^\lambda}{\partial\mu_1^2}+\frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial\mu_1}=0. \tag{EC.115}
$$

Hence, from (EC.110),

$$
\lim_{\lambda\to\infty}\frac{\partial^2 U^\lambda}{\partial\mu_1^2}=\lim_{\lambda\to\infty} v\frac{\partial^2 I^\lambda}{\partial\mu_1^2}-p\mu_1\left(\frac{\partial^2 I^\lambda}{\partial\mu_1^2}+\frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial\mu_1}\right)-c''(\mu_1)<0,
$$

recalling that $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$ (from (EC.114)), $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} + \frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from (EC.115)), and $c'' > 0$ (since c is strictly convex).

Case (II): If $\mu > a$, then $\frac{1-C^\lambda}{N^\lambda - \rho^\lambda} \in [0,1]$ for all λ (from Lemma EC.13 (c)), $I^\lambda \in [0,1]$ for all λ, $\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} \le 1$ for all large enough λ; and, as $\lambda \to \infty$, $\frac{\rho^\lambda}{d_1^\lambda} = \frac{a}{\mu} \in (0,\infty)$, $\frac{\rho^\lambda}{d_2^\lambda} \to \left(\frac{\mu}{a} - 1\right)^{-1} \in (0,\infty)$, $d_1^\lambda \to \infty$, $d_2^\lambda \to \infty$, $C^\lambda \to 0$, and $\lim_{\lambda\to\infty}(k^\lambda - N^\lambda)^r \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} < \infty$ for any $r \in \mathbb{N}$ (noting that $\frac{\rho^\lambda}{d_1^\lambda} < 1$ for all large enough λ, and, even if $\limsup_{\lambda\to\infty} k^\lambda - N^\lambda = \infty$, exponential decay in terms of $k^\lambda - N^\lambda$ dominates its polynomial growth). Based on these facts, it can be shown that

$$\lim_{\lambda\to\infty} t_1^\lambda := 2\frac{\rho^\lambda}{d_2^\lambda}\frac{C^\lambda}{d_2^\lambda}\left[1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right]\left[(1 - 2I^\lambda) - \frac{\frac{\mu_1}{\mu} + \rho^\lambda\left(\frac{1-C^\lambda}{N^\lambda - \rho^\lambda}\right)}{d_2^\lambda} I^\lambda\right] = 0,$$

$$\lim_{\lambda\to\infty} t_2^\lambda := \frac{\rho^\lambda}{d_2^\lambda} C^\lambda \frac{1}{d_1^\lambda} k^\lambda - N^\lambda \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\left[2(1 - 2I^\lambda) + \left(\frac{2}{d_2^\lambda} - \frac{1}{d_1^\lambda}\right)\frac{\mu_1}{\mu} - 4\frac{\frac{\mu_1}{\mu} + \rho^\lambda\left(\frac{1-C^\lambda}{N^\lambda - \rho^\lambda}\right)}{d_2^\lambda} I^\lambda\right] = 0,$$

$$\lim_{\lambda\to\infty} t_3^\lambda := \frac{\rho^\lambda}{d_1^\lambda} C^\lambda \frac{1}{(d_1^\lambda)^2}\left(k^\lambda - N^\lambda\right)^2 \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\left[\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu} - 2\frac{\frac{\mu_1}{\mu} + \rho^\lambda\left(\frac{1-C^\lambda}{N^\lambda - \rho^\lambda}\right)}{d_2^\lambda} I^\lambda - \frac{2\rho^\lambda C^\lambda}{(d_2^\lambda)^2} I^\lambda\right] = 0.$$

which, from (EC.113), implies that

$$\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = -\frac{2}{\mu_1^2}\left(\lim_{\lambda\to\infty} I^\lambda\right)^2\left(1 - \lim_{\lambda\to\infty} I^\lambda\right) \overset{(*)}{=} -\frac{\frac{2a}{\mu_1^3}\left(1 - \frac{a}{\mu}\right)^2}{\left(1 - \frac{a}{\mu} + \frac{a}{\mu_1}\right)^3} < 0, \tag{EC.116}$$

where $(*)$ follows from (EC.111), and the last inequality follows by noting that $1 - \frac{a}{\mu} + \frac{a}{\mu_1} > \frac{a}{\mu_1} > 0$ (when $\mu > a$). Thus, together with (EC.112),

$$\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} + \frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1} = -\frac{\frac{2a}{\mu_1^3}\left(1 - \frac{a}{\mu}\right)^2}{\left(1 - \frac{a}{\mu} + \frac{a}{\mu_1}\right)^3} + \frac{2}{\mu_1}\frac{\frac{a}{\mu_1^2}\left(1 - \frac{a}{\mu}\right)}{\left(1 - \frac{a}{\mu} + \frac{a}{\mu_1}\right)^2} = \frac{2a^2}{\mu_1^4}\frac{1 - \frac{a}{\mu}}{(1 - \frac{a}{\mu} + \frac{a}{\mu_1})^3} > 0, \tag{EC.117}$$

where the last inequality follows by noting that $1 - \frac{a}{\mu} > 0$ and $1 - \frac{a}{\mu} + \frac{a}{\mu_1} > \frac{a}{\mu_1} > 0$ (when $\mu > a$). Hence, from (EC.110),

$$\lim_{\lambda\to\infty} \frac{\partial^2 U^\lambda}{\partial \mu_1^2} = \lim_{\lambda\to\infty} v\frac{\partial^2 I^\lambda}{\partial \mu_1^2} - p\mu_1\left(\frac{\partial^2 I^\lambda}{\partial \mu_1^2} + \frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1}\right) - c''(\mu_1) < 0,$$

recalling that $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} < 0$ (from (EC.116)), $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} + \frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1} > 0$ (from (EC.117)), and $c'' > 0$ (since c is strictly convex).

Case (III): If $\mu = a$, it suffices to show that if $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in \{-\infty\} \cup [1,\infty) \cup \{\infty\}$, then $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$. Then, from (EC.110),

$$\lim_{\lambda\to\infty} \frac{\partial^2 U^\lambda}{\partial \mu_1^2} = \lim_{\lambda\to\infty} v\frac{\partial^2 I^\lambda}{\partial \mu_1^2} - p\mu_1\left(\frac{\partial^2 I^\lambda}{\partial \mu_1^2} + \frac{2}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1}\right) - c''(\mu_1) < 0,$$

recalling that $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from (EC.112)) and $c'' > 0$ (since c is strictly convex).

To show $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$ when $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in \{-\infty\} \cup [1,\infty) \cup \{\infty\}$, we need the following auxiliary claims, whose proofs are delayed until the end.

Claim EC.7. *Under linear staffing (14), if* $\mu = a$ *and* $0 < N^\lambda - \frac{\lambda}{a} \in \omega(1)$, *then* $\lim_{\lambda\to\infty} \sqrt{\rho^\lambda} C^\lambda \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^r \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda = 0$ *for all* $r \in \mathbb{N}$.

Claim EC.8. *Under linear staffing (14), if* $\mu = a$ *and* $\lim_{\lambda\to\infty} d_2^\lambda \in (\mathbb{R}\setminus\{0\}) \cup \{-\infty, \infty\}$, *then* $\lim_{\lambda\to\infty} \frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 = 0$.

Claim EC.9. *If* $\lim_{\lambda\to\infty} d_2^\lambda \in (\mathbb{R}\setminus\{0\}) \cup \{-\infty,\infty\}$ *and* $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in \mathbb{R}\cup\{\infty\}$, *then* $\lim_{\lambda\to\infty} \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^r \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} \in [0,\infty)$ *for all* $r \in \mathbb{N}$.

Now, we are ready to complete the proof by dividing the condition $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in \{-\infty\} \cup [1,\infty) \cup \{\infty\}$ into three cases. Specifically, **Case (A)** $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} = \infty$, **Case (B)** $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in [1,\infty)$, and **Case (C)** $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} = -\infty$.

Using Lemma EC.5 and after algebra, one can check that, when $\mu = a$, $\frac{\partial^2 I^\lambda}{\partial \mu_1^2}$ can be equivalently written as

$$\begin{aligned}\frac{\partial^2 I^\lambda}{\partial \mu_1^2} = & -\frac{2}{\mu_1} I^\lambda \frac{\partial I^\lambda}{\partial \mu_1} - 2\left[\frac{1}{\mu_1}\frac{\partial I^\lambda}{\partial \mu_1} - \frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 - \frac{1}{\mu_1} I^\lambda \frac{\partial I^\lambda}{\partial \mu_1} - \frac{I^\lambda}{\mu_1 a d_2^\lambda} + \frac{1}{a d_2^\lambda}\frac{\partial I^\lambda}{\partial \mu_1} + \frac{(I^\lambda)^2}{\mu_1 a d_2^\lambda}\right] \\ & + \frac{1}{\mu_1 a}\left(\frac{\rho^\lambda}{d_2^\lambda} I^\lambda\right) \frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2} C^\lambda \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda. \end{aligned} \tag{EC.118}$$

Case (A): If $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} = \infty$. Note that the last term of (EC.118) satisfies

$$\begin{aligned} & \frac{1}{\mu_1 a}\left(\frac{\rho^\lambda}{d_2^\lambda} I^\lambda\right) \frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2} C^\lambda \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda \\ = & \frac{1}{\mu_1 a}\left(\frac{\sqrt{\rho^\lambda} I^\lambda}{d_2^\lambda}\right)\left(\sqrt{\rho^\lambda} C^\lambda \frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda\right) \to 0, \quad \text{as } \lambda \to \infty, \end{aligned}$$

by Lemma EC.17 and Claim EC.7. Using this, together with Claim EC.8, and $\lim_{\lambda\to\infty} I^\lambda = 0$ and $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from Propositions EC.1 and EC.2 when $\mu = a$), one can easily check that every term in (EC.118) converges to 0 as $\lambda \to \infty$, implying that $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$.

Case (B): If $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in [1,\infty)$, we further discuss cases depending on the value of $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}$.

Case (B-1): If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$. One can easily check that every term in (EC.118) converges to 0 as $\lambda\to\infty$, using $\lim_{\lambda\to\infty} I^\lambda = 0$ and $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from Propositions EC.1 and EC.2 when $\mu = a$), together with Lemma EC.20 and Claim EC.8. Hence, $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$.

Case (B-2): If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in (0,\infty)\cup\{\infty\}$. Note that the last term of (EC.118) satisfies

$$\begin{aligned} & \frac{1}{\mu_1 a}\left(\frac{\rho^\lambda}{d_2^\lambda} I^\lambda\right) \frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2} C^\lambda \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda \\ = & \frac{1}{\mu_1 a}\frac{1}{d_2^\lambda}\left(\sqrt{\rho^\lambda} I^\lambda\right)^2 \left(\frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right) C^\lambda \to 0, \quad \text{as } \lambda\to\infty, \end{aligned}$$

by noting that $\lim_{\lambda\to\infty} C^\lambda = 1$ (from Lemma EC.12 (b)), and by Lemma EC.18 (ii) and Claim EC.9. Using this, together with Claim EC.8, and $\lim_{\lambda\to\infty} I^\lambda = 0$ and $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from Propositions EC.1 and EC.2 when $\mu = a$), one can easily check that every term in (EC.118) converges to 0 as $\lambda \to \infty$, implying that $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$.

Case (C) If $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} = -\infty$, we further discuss cases depending on the value of $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}$.

Case (C-1): If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = 0$. One can easily check that every term in (EC.118) converges to 0 as $\lambda \to \infty$, using $\lim_{\lambda\to\infty} I^\lambda = 0$ and $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from Propositions EC.1 and EC.2 when $\mu = a$), together with Lemma EC.20 and Claim EC.8.

Case (C-2): If $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \in \{-\infty\} \cup (-\infty, 0)$. Note that the term of (EC.118) satisfies

$$\begin{aligned}
&\frac{1}{\mu_1 a}\left(\frac{\rho^\lambda}{d_2^\lambda} I^\lambda\right)\frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2} C^\lambda \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda \\
=&\frac{1}{\mu_1 a}\left(\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda\right)\left(\frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2} I^\lambda\right) \to 0, \text{ as } \lambda \to \infty,
\end{aligned}$$

by Lemmas EC.15 and EC.16. Using this, together with Claim EC.8, and $\lim_{\lambda\to\infty} I^\lambda = 0$ and $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from Propositions EC.1 and EC.2 when $\mu = a$), one can easily check that every term in (EC.118) converges to 0 as $\lambda \to \infty$, implying that $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$.

Combining Cases (A)-(C), when $\mu = a$, if $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} \in \{-\infty\} \cup [1, \infty) \cup \{\infty\}$, then $\lim_{\lambda\to\infty} \frac{\partial^2 I^\lambda}{\partial \mu_1^2} = 0$. This concludes the proof of Case (III).

Together Cases (I)-(III) establish that, under the staffing rule (14), except when $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a}$ is finite and strictly less than 1, for all large enough λ, we have $\frac{\partial^2 U^\lambda}{\partial \mu_1^2} < 0$. ∎

Proof of Claim EC.7: When $0 < N^\lambda - \frac{\lambda}{\mu} \in \omega(1)$, it is clear that both d_2^λ and $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}$ are nonnegative for all large enough λ. Furthermore, $\lim_{\lambda\to\infty} d_2^\lambda = \infty$. Then, by multiplying and dividing by $(d_2^\lambda)^r$ and regrouping the terms, we can write

$$\lim_{\lambda\to\infty} \sqrt{\rho^\lambda} C^\lambda \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^r \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda = \lim_{\lambda\to\infty}\left(\frac{\sqrt{\rho^\lambda} I^\lambda}{d_2^\lambda}\right) \cdot C^\lambda \cdot \frac{1}{(d_2^\lambda)^{r-1}} \cdot \left[\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^r e^{-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}}\right] = 0,$$

because $\lim_{\lambda\to\infty} \frac{\sqrt{\rho^\lambda} I^\lambda}{d_2^\lambda} = 0$ (from Lemma EC.17), $\lim_{\lambda\to\infty} C^\lambda \in [0,1]$ (from Lemma EC.12(b)), and $\frac{1}{(d_2^\lambda)^{r-1}} \in \{0,1\}$ and $\lim_{\lambda\to\infty}\left(d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^r e^{-d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} \in [0, \infty)$ (because even if $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = \infty$, exponential decay in terms of $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}$ would dominate its polynomial growth) for all $r \in \mathbb{N}$. ∎

Proof of Claim EC.8:

We discuss two cases depending on the value of $\lim_{\lambda\to\infty} d_2^\lambda$.

Case (I): If $\lim_{\lambda\to\infty} d_2^\lambda \in (0,\infty)\cup\{\infty\}$, then $d_2^\lambda > 0$ for all large enough λ. Then, recalling that $\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} > 0$ for all λ (from Lemma EC.2 (c)), it follows from (EC.75) that

$$\begin{aligned}\frac{\partial I^\lambda}{\partial\mu_1} &\le \frac{1}{\mu_1}\left[\left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right) I^\lambda(1-I^\lambda)\right]\\ \Rightarrow \lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2 &\le \lim_{\lambda\to\infty}\frac{1}{\mu_1^2}\left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right)^2 I^\lambda\left(1-I^\lambda\right)^2 = 0,\end{aligned}$$

recalling that $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1). Hence, $\lim_{\lambda\to\infty}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2 = 0$, by non-negativity ($I^\lambda > 0$ for all λ).

Case (II): If $\lim_{\lambda\to\infty} d_2^\lambda \in \{-\infty\}\cup(-\infty,0)$, then $d_2^\lambda < 0$ for all large enough λ, which implies that (i) either $0 < \frac{\lambda}{a} - N^\lambda \in \omega(1)$ or $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(1)$ and (ii) $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \le 0$.

It follows from (EC.75) that

$$\begin{aligned}\frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2 &= \frac{1}{I^\lambda}\frac{1}{\mu_1^2}\left(\left[\left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right)I^\lambda(1-I^\lambda)\right] - \left[\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^2\right] - \left[\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^2\right]\right)^2\\ &= \frac{1}{\mu_1^2}\left(\left[\left(1+\frac{1}{d_2^\lambda}\frac{\mu_1}{\mu}\right)(I^\lambda)^{\frac{1}{2}}(1-I^\lambda)\right] - \left[\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^{\frac{3}{2}}\right] - \left[\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^{\frac{3}{2}}\right]\right)^2.\end{aligned} \tag{EC.119}$$

Note that the first term in square brackets on the right-hand side of (EC.119) vanishes in the limit as $\lambda\to\infty$, by recalling that $\lim_{\lambda\to\infty} d_2^\lambda \in \{-\infty\}\cup(-\infty,0)$ (by assumption) and $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1). To complete the proof, it suffices to show that the second and third terms in square brackets follow suit.

Second Term: We further discuss three cases depending on the asymptotic behavior of $N^\lambda - \frac{\lambda}{a}$.

- Suppose $0 < \frac{\lambda}{a} - N^\lambda \in \omega(\sqrt{\lambda})$. Then, the second term can be rearranged as follows:

  $$\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^{\frac{3}{2}} = \left(\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda}\right)\left(\frac{1}{C^\lambda}-1\right)\frac{1}{N^\lambda-\frac{\lambda}{a}}(I^\lambda)^{\frac{1}{2}},$$

  which vanishes in the limit as $\lambda\to\infty$ by noting that $\lim_{\lambda\to\infty}\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda} \in \left[-\frac{\mu_1}{a}, 0\right]$ (by Lemma EC.19) and $\lim_{\lambda\to\infty} C^\lambda = \infty$ (from Lemma EC.12 (b)) and by recalling that $\frac{\lambda}{a} - N^\lambda \in \omega(\sqrt{\lambda})$ and $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1).
- Suppose $0 < \frac{\lambda}{a} - N^\lambda \in \mathcal{O}(\sqrt{\lambda})\cap\omega(1)$. Then, recalling that $\rho^\lambda = \frac{\lambda}{a}$ when $\mu = a$, the second term can be rearranged as follows:

  $$\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^{\frac{3}{2}} = \left(\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda}\right)\frac{1}{\sqrt{a}}\left(\sqrt{\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)(I^\lambda)^{\frac{1}{2}},$$

  which vanishes in the limit as $\lambda\to\infty$ by noting that $\lim_{\lambda\to\infty}\frac{\sqrt{\rho^\lambda}I^\lambda}{d_2^\lambda} = 0$ (by Lemma EC.17) and $\lim_{\lambda\to\infty}\sqrt{\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in (0,\infty)$ (from Lemma EC.13 (c)) and by recalling that $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1).
- Suppose $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(1)$. Then, recalling that $\rho^\lambda = \frac{\lambda}{a}$ when $\mu = a$, the second term can be rearranged as follows:

  $$\frac{\rho^\lambda}{d_2^\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}(I^\lambda)^{\frac{3}{2}} = \frac{1}{d_2^\lambda}\left(\left(\rho^\lambda\right)^{\frac{1}{3}} I^\lambda\right)^{\frac{3}{2}}\frac{1}{\sqrt{a}}\left(\sqrt{\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right),$$

which vanishes in the limit as $\lambda \to \infty$ by recalling that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ (by assumption) and by noting that $\lim_{\lambda\to\infty} (\rho^\lambda)^{\frac{1}{3}} I^\lambda = 0$ (by Lemma EC.18) and $\lim_{\lambda\to\infty} \sqrt{\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in (0,\infty)$ (from Lemma EC.13 (c)).

Third Term: We further discuss three cases depending on $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda}$ and the asymptotic behavior of $N^\lambda - \frac{\lambda}{a}$.

- Suppose $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} \in \{-\infty\} \cup (-\infty, 0)$. Then, the third term can be rearranged as follows:
$$\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^{\frac{3}{2}} = \left(\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} I^\lambda\right)\left(\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^2 I^\lambda\right)^{\frac{1}{2}},$$
which vanishes in the limit as $\lambda \to \infty$ by noting that $\lim_{\lambda\to\infty} \frac{\rho^\lambda C^\lambda}{d_2^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} I^\lambda \in (-\infty,\infty)$ (by Lemma EC.16) and $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^2 I^\lambda = 0$ (by Lemma EC.15 (ii)).
- Suppose $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} = 0$ and $0 < \frac{\lambda}{a} - N^\lambda \in \omega(\sqrt{\lambda})$. Then, the third term can be rearranged as follows:
$$\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^{\frac{3}{2}} = \left(\frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda}\right)\left(\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^2 I^\lambda\right)^{\frac{1}{2}}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda},$$
which vanishes in the limit as $\lambda \to \infty$ by noting that $\lim_{\lambda\to\infty} \frac{\rho^\lambda C^\lambda I^\lambda}{d_2^\lambda} \in \left[-\frac{\mu_1}{a}, 0\right]$ (by Lemma EC.19), $\lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^2 I^\lambda = 0$ (by Lemma EC.15 (ii)), and $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} = 1$ (from Lemma EC.14 (i)).
- Suppose $\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda-N^\lambda}{d_1^\lambda} = 0$ and $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(\sqrt{\lambda})$. Then, recalling that $d_2^\lambda < 0$ for all large enough λ, the third term can be expressed as follows:
$$\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}(I^\lambda)^{\frac{3}{2}} = -\left[\left(-\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)^{\frac{4}{3}}(I^\lambda)^2\right]^{\frac{3}{4}}.$$
To complete the proof, we show that the expression within the square brackets above vanishes in the limit (by showing that its reciprocal diverges to ∞) as $\lambda \to \infty$. Using (EC.74) and after algebra, when $\mu = a$,
$$\begin{aligned}&\left[\left(-\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\frac{k^\lambda-N^\lambda}{d_1^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)^{\frac{4}{3}}(I^\lambda)^2\right]^{-1}\\ &= \left(\frac{(d_2^\lambda)^2}{\rho^\lambda C^\lambda}\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\right)^{\frac{2}{3}}\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)^2\\ &\quad + 2\left(\frac{a}{\mu_1}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\right)\left(\frac{(d_2^\lambda)^2}{\rho^\lambda C^\lambda}\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\right)^{\frac{2}{3}}\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)\\ &\quad + \left(\frac{a}{\mu_1}\frac{\rho^\lambda C^\lambda}{d_2^\lambda}\right)^2\left(\frac{(d_2^\lambda)^2}{\rho^\lambda C^\lambda}\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\right)^{\frac{2}{3}}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right)^2.\end{aligned} \tag{EC.120}$$

Observe that $\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \geq 0$ (from Lemma EC.2 (c)) and $\frac{1}{d_2^\lambda}\left(1-\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda}\right) \geq 0$ (recalling the definition $d_2^\lambda := d_1^\lambda - \rho^\lambda$) for all λ. As a result, each of the three terms on the right-hand side of (EC.120) is non-negative for all λ. Therefore, the left-hand side of (EC.120) will diverge to ∞ as $\lambda \to \infty$ even if one of these three terms grows unboundedly with λ. To complete the proof, we show that the first term on the right-hand side of (EC.120) diverges to ∞ as $\lambda \to \infty$. Recalling that $\rho^\lambda = \frac{\lambda}{a}$ when $\mu = a$, this term can be rearranged as follows:

$$\begin{aligned}
&\left(\frac{(d_2^\lambda)^2}{\rho^\lambda C^\lambda}\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\right)^{\frac{2}{3}}\left(1+\frac{a}{\mu_1}\rho^\lambda\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)^2 \\
&= \left(\frac{\sqrt{\rho^\lambda}(d_2^\lambda)^2}{C^\lambda}\left(\frac{\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda}}{-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}}\right)\right)^{\frac{2}{3}}\left(\frac{1}{\sqrt{\rho^\lambda}}+\frac{\sqrt{a}}{\mu_1}\left(\sqrt{\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda}\right)\right)^2,
\end{aligned}$$

which diverges to ∞ as $\lambda \to \infty$ by recalling that $\lim_{\lambda\to\infty}\rho^\lambda = \infty$, $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ (by assumption), and $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} = 0$; and by noting that $\lim_{\lambda\to\infty}\left(\frac{d_1^\lambda}{\rho^\lambda}\right)^{k^\lambda-N^\lambda} = 1$ (from Lemma EC.14 (i)), $\lim_{\lambda\to\infty} C^\lambda \in (0,\infty)$ (from Lemma EC.12 (b), recalling that $\left|N^\lambda - \frac{\lambda}{a}\right| \in \mathcal{O}(\sqrt{\lambda})$), and $\lim_{\lambda\to\infty}\sqrt{\lambda}\frac{1-C^\lambda}{N^\lambda-\rho^\lambda} \in (0,\infty)$ (from Lemma EC.13 (c)).

■

Proof of Claim EC.9: First, we recall the definition $d_2^\lambda := d_1^\lambda - \rho^\lambda$, where $d_1^\lambda := N^\lambda - 1 + \frac{\mu_1}{\mu} > 0$ (since $N^\lambda \geq 1$, $\mu_1 > 0$, and $\mu > 0$) and $\rho^\lambda := \frac{\lambda}{\mu} > 0$ for all $\lambda > 0$. This implies that $\frac{d_2^\lambda}{d_1^\lambda} < 1$ for all $\lambda > 0$. Next, we recall that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ (by assumption), which implies that $d_2^\lambda \neq 0$ for all large enough λ. By multiplying and dividing by $(d_2^\lambda)^r$ and regrouping the terms, we can write

$$\begin{aligned}
0 \leq \lim_{\lambda\to\infty}\left(\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} &= \lim_{\lambda\to\infty}\frac{1}{(d_2^\lambda)^r}\left(d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r\left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)^{k^\lambda-N^\lambda} \\
&= \lim_{\lambda\to\infty}\frac{1}{(d_2^\lambda)^r}\left(d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r\exp\left(\left(k^\lambda-N^\lambda\right)\ln\left(1-\frac{d_2^\lambda}{d_1^\lambda}\right)\right) \\
&\overset{(i)}{\leq} \lim_{\lambda\to\infty}\frac{1}{(d_2^\lambda)^r}\left(d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right)^r\exp\left(-d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}\right) \\
&\overset{(ii)}{<} \infty,
\end{aligned}$$

where (i) follows because $\ln(1-x) \leq -x$ for all $x < 1$ and $\exp(x)$ is an increasing function of x for all $x \in \mathbb{R}$; and (ii) follows by recalling that $\lim_{\lambda\to\infty} d_2^\lambda \neq 0$ and $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} \in \mathbb{R}\cup\{\infty\}$ (by assumption) and noting that, even if $\lim_{\lambda\to\infty} d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda} = \infty$, exponential decay in terms of $d_2^\lambda\frac{k^\lambda-N^\lambda}{d_1^\lambda}$ would dominate its polynomial growth. ■

EC.7.7. Technical Details for Endnote 7

When $\mu = a$ and $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} =: x \in (-\infty, 1)$, we exhibit a counterexample under which $\lim_{\lambda\to\infty} \frac{\partial^2 U^\lambda(\mu_1,\mu)}{\partial \mu_1^2} = \infty$ for $\mu_1 = a(1-x)$ for a range of values of the parameters a, p, and v. We inherit the shorthand notation and the setup outlined around the proof of Proposition EC.1 in the preliminary Section EC.7.2.

Consider the sequence of staffing levels $N^\lambda = \frac{\lambda}{a} - \frac{1}{\lambda^2}$ and a corresponding sequence of system sizes $k^\lambda = N^\lambda + \lambda^2$. Recalling that $\rho^\lambda = \frac{\mu_1}{a}$ when $\mu = a$, we begin by presenting the following observations which form the building blocks of our argument:

- $0 < \frac{\lambda}{a} - N^\lambda \in o(1)$, implying that $\lim_{\lambda\to\infty} C^\lambda = 1$ (from Lemma EC.12 (b)), $\lim_{\lambda\to\infty} I^\lambda = 0$ (from Proposition EC.1 when $\mu = a$), and $\lim_{\lambda\to\infty} \frac{\partial I^\lambda}{\partial \mu_1} = 0$ (from Proposition EC.2 when $\mu = a$);
- $x := \lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} = 0$;
- $\mu_1 = a(1-x) = a$;
- $d_1^\lambda = N^\lambda - \left(1 - \frac{\mu_1}{\mu}\right) = \frac{\lambda}{a} - \frac{1}{\lambda^2}$;
- $d_2^\lambda = d_1^\lambda - \frac{\lambda}{a} = -\frac{1}{\lambda^2}$;
- $d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = -\frac{1}{\frac{\lambda}{a} - \frac{1}{\lambda^2}} = -\frac{1}{\rho^\lambda - \frac{1}{\lambda^2}}$, implying that $\lim_{\lambda\to\infty} \rho^\lambda d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = -1$.

From (EC.74), when $\mu = a$,

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{I^\lambda}{d_2^\lambda} &= \lim_{\lambda\to\infty} \left[d_2^\lambda + \frac{\mu}{\mu_1}\rho^\lambda d_2^\lambda \left(\frac{1 - C^\lambda}{N^\lambda - \rho^\lambda}\right) + \frac{\mu}{\mu_1} C^\lambda \rho^\lambda \left(1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)\right]^{-1} \\
&= \lim_{\lambda\to\infty} \left[d_2^\lambda + \frac{\mu}{\mu_1}\sqrt{\rho^\lambda} d_2^\lambda \left(\sqrt{\rho^\lambda}\left(\frac{1 - C^\lambda}{N^\lambda - \rho^\lambda}\right)\right) + \frac{\mu}{\mu_1} C^\lambda \rho^\lambda d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} \left(\frac{1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}}{d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}}\right)\right]^{-1} = -1,
\end{aligned} \tag{EC.121}$$

because $\lim_{\lambda\to\infty} d_2^\lambda = 0$, $\lim_{\lambda\to\infty} \sqrt{\rho^\lambda} d_2^\lambda = 0$, $\lim_{\lambda\to\infty} \sqrt{\rho^\lambda} \frac{1 - C^\lambda}{N^\lambda - \rho^\lambda} \in (0, \infty)$ (from Lemma EC.13 (c)), $\lim_{\lambda\to\infty} C^\lambda = 1$, $\lim_{\lambda\to\infty} \rho^\lambda d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda} = -1$, and $\lim_{\lambda\to\infty} \frac{1 - \left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}}{d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} = 1$ (from Lemma EC.14 (ii)).

From (EC.76) in the proof of Proposition EC.2,

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{\mu_1^2}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial \mu_1}\right)^2 \\
&= \lim_{\lambda\to\infty} I^\lambda \left(1 - I^\lambda + \rho^\lambda C^\lambda I^\lambda \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\right)^2 && \text{(EC.122)} \\
&= \lim_{\lambda\to\infty} I^\lambda \left(1 - I^\lambda\right)^2 + 2(I^\lambda)^2\left(1 - I^\lambda\right)\rho^\lambda C^\lambda \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2 + (\rho^\lambda)^2 (C^\lambda)^2 (I^\lambda)^3 \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^4 \\
&= \lim_{\lambda\to\infty} I^\lambda \left(1 - I^\lambda\right)^2 + 2\left(1 - I^\lambda\right) C^\lambda \left(\sqrt{\rho^\lambda}\frac{k^\lambda - N^\lambda}{d_1^\lambda} I^\lambda\right)^2 + (C^\lambda)^2 \left(\left(\frac{I^\lambda}{d_2^\lambda}\right)^3 (d_2^\lambda)^3 (\rho^\lambda)^2 \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^4\right) \overset{(\dagger)}{=} a^2, && \text{(EC.123)}
\end{aligned}$$

where (‡) follows by recalling that $\lim_{\lambda\to\infty} I^\lambda = 0$, $\lim_{\lambda\to\infty} C^\lambda = 1$, $\lim_{\lambda\to\infty} \frac{I^\lambda}{d_2^\lambda} = -1$ (from (EC.121)), $\lim_{\lambda\to\infty} (d_2^\lambda)^3 (\rho^\lambda)^2 \left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^4 = \lim_{\lambda\to\infty} \left(-\frac{1}{\lambda^2}\right)^3 \left(\frac{\lambda}{a}\right)^2 \left(\frac{\lambda^2}{\frac{\lambda}{a} - \frac{1}{\lambda^2}}\right)^4 = -a^2$, and $\lim_{\lambda\to\infty} \sqrt{\rho^\lambda} \frac{k^\lambda - N^\lambda}{d_1^\lambda} I^\lambda = 0$ (from (EC.78) in the proof of Proposition EC.2).

Next, from (EC.118) and recalling that $\mu_1 = a$,

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{\partial^2 I^\lambda}{\partial\mu_1^2} &= \lim_{\lambda\to\infty}\left(-\frac{2}{\mu_1}I^\lambda\frac{\partial I^\lambda}{\partial\mu_1} - 2\left[\frac{1}{\mu_1}\frac{\partial I^\lambda}{\partial\mu_1} - \frac{1}{I^\lambda}\left(\frac{\partial I^\lambda}{\partial\mu_1}\right)^2 - \frac{1}{\mu_1}I^\lambda\frac{\partial I^\lambda}{\partial\mu_1} - \frac{I^\lambda}{\mu_1 a d_2^\lambda} + \frac{1}{a d_2^\lambda}\frac{\partial I^\lambda}{\partial\mu_1} + \frac{(I^\lambda)^2}{\mu_1 a d_2^\lambda}\right]\right.\\
&\qquad\left. + \frac{1}{\mu_1 a}\left(\frac{\rho^\lambda}{d_2^\lambda}I^\lambda\right)\frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2}C^\lambda\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda\right)\\
&= 2\left(1-\frac{1}{a^2}\right) + \frac{1}{a}\lim_{\lambda\to\infty}\left(-2\frac{I^\lambda}{d_2^\lambda}\frac{1}{I^\lambda}\frac{\partial I^\lambda}{\partial\mu_1} + \frac{1}{a}\left(\frac{\rho^\lambda}{d_2^\lambda}I^\lambda\right)\frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2}C^\lambda\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda\right)\\
&\overset{(*)}{=} 2\left(1-\frac{1}{a^2}\right) + \frac{1}{a}\lim_{\lambda\to\infty}\left(2\left[1 - I^\lambda + \frac{I^\lambda}{d_2^\lambda}C^\lambda\rho^\lambda d_2^\lambda\left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\right]\right.\\
&\qquad\left. + \frac{1}{a}\left(\frac{\rho^\lambda}{d_2^\lambda}I^\lambda\right)\frac{(k^\lambda - N^\lambda)(k^\lambda - N^\lambda + 1)}{(d_1^\lambda)^2}C^\lambda\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} I^\lambda\right)\\
&= 2\left(1-\frac{1}{a^2}\right) + \frac{1}{a^2}\lim_{\lambda\to\infty}\left(2a(1-I^\lambda) + \frac{I^\lambda}{d_2^\lambda}\left(2a + \frac{I^\lambda}{d_2^\lambda}\frac{k^\lambda - N^\lambda + 1}{k^\lambda - N^\lambda}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda}\right)C^\lambda\rho^\lambda d_2^\lambda\left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2\right)\\
&\overset{(**)}{=} 2\left(1-\frac{1}{a^2}\right) + \frac{1}{a^2}\left\{2a - (2a-1)(-\infty)\right\},
\end{aligned}$$

where $(*)$ follows from (EC.76) in the proof of Proposition EC.2 and $(**)$ follows by recalling that $\lim_{\lambda\to\infty} I^\lambda = 0$, $\lim_{\lambda\to\infty}\frac{I^\lambda}{d_2^\lambda} = -1$ (from (EC.121)), $\lim_{\lambda\to\infty}\frac{k^\lambda - N^\lambda + 1}{k^\lambda - N^\lambda} = 1$ (since $k^\lambda - N^\lambda = \lambda^2$), $\lim_{\lambda\to\infty}\left(\frac{\rho^\lambda}{d_1^\lambda}\right)^{k^\lambda - N^\lambda} = e^{-\lim_{\lambda\to\infty} d_2^\lambda \frac{k^\lambda - N^\lambda}{d_1^\lambda}} = 1$ (from Lemma EC.14 (i)), $\lim_{\lambda\to\infty} C^\lambda = 1$, and $\lim_{\lambda\to\infty}\rho^\lambda d_2^\lambda\left(\frac{k^\lambda - N^\lambda}{d_1^\lambda}\right)^2 = \lim_{\lambda\to\infty}\frac{\lambda}{a}\left(-\frac{1}{\lambda^2}\right)\left(\frac{\lambda^2}{\frac{\lambda}{a} - \frac{1}{\lambda^2}}\right)^2 = -\infty$. Hence, the above display implies that

$$\lim_{\lambda\to\infty}\frac{\partial^2 I^\lambda}{\partial\mu_1^2} = \begin{cases} -\infty, & \text{if } a < \frac{1}{2},\\ +\infty, & \text{if } a > \frac{1}{2}.\end{cases}$$

Then, from (EC.110), when $\mu_1 = a$ and recalling that $\lim_{\lambda\to\infty}\frac{\partial I^\lambda}{\partial\mu_1} = 0$, it follows that, if $p > 2ap > 2v$ or $p < 2ap < 2v$, then

$$\lim_{\lambda\to\infty}\frac{\partial^2 U^\lambda}{\partial\mu_1^2} = (v - ap)\frac{\partial^2 I^\lambda}{\partial\mu_1^2} - 2p\frac{\partial I^\lambda}{\partial\mu_1} - c''(a) = \infty.$$

∎

EC.7.8. Proof of Proposition 5

From Lemma 6, under the same restrictions on the staffing rules, the utility function $U(\mu_1,\mu)$ is a concave function of μ_1 for all $\mu > 0$, which implies that there exists a unique maximum. Thus, any solution to the FOC (6) (i.e., candidate server equilibrium) $\mu^{\star?} > 0$ is a global maximizer of $U(\mu_1,\mu^{\star?})$, that is, candidate server equilibrium $\mu^{\star?}$ is actually a sever equilibrium. Therefore, $\mu^{\star?} > 0$ is a server equilibrium if and only if it satisfies the FOC (6). ∎

EC.7.9. Proof of Theorem 4

First, we state three lemmas concerning underloaded equilibria (whose proofs appear later) that help simplify the proof of Theorem 4. Recall the definition of $p^\dagger(v)$ in Theorem 4 (also in (EC.97)).

Lemma EC.28. *If* $0 \le p < \min\{c'(a), p^\dagger(v)\}$*, then there exists a unique* $\overline{a}(p,v) > 0$ *such that* $n_u = 0$*,* 1*, or* 2 *according to whether* $a > \overline{a}(p,v)$*,* $a = \overline{a}(p,v)$*, or* $a < \overline{a}(p,v)$*, respectively.*

Lemma EC.29. *If* $p > c'(a)$ *or* $p = c'(a) < p^\dagger(v)$*, then* $n_u = 1$*.*

Lemma EC.30. *If* $p^\dagger(v) \le p \le c'(a)$*, then* $n_u = 0$*.*

Next, we note that an overloaded or critically loaded equilibrium must lie in $(0,a]$, and for this interval, the limiting FOC (16) reduces to $c'(\mu) = p$. We are now ready to prove Theorem 4.

(a) $\mathbf{p \le c'(0)}$**:** Here, $n_o = n_c = 0$ because $c'(\mu) > c'(0) \ge p$ for all $\mu \in (0,a]$, recalling that c' is strictly increasing (by strict convexity of c). For underloaded equilibria, we first note that $p \le c'(0)$ implies $p < c'(a)$. Furthermore, $p^\dagger(v) > c'(0)$ by definition (see (EC.97)), so, $p \le c'(0)$ also implies $p < p^\dagger(v)$. Together, we have $p < \min\{c'(a), p^\dagger(v)\}$, so, the result follows from Lemma EC.28.

(b) $\mathbf{p > c'(0)}$**:** In this case, note that $(c')^{-1}(p) > 0$ is well-defined, since c' is strictly increasing.

(i): If $a > (c')^{-1}(p)$, then $p < c'(a)$. Thus, $c'(0) < p < c'(a)$. Recalling that c' is strictly increasing, it follows that $c'(\mu) = p$ must admit a unique solution in $(0,a)$, i.e., $n_o = 1$. On the other hand, if $a \le (c')^{-1}(p)$, then $c'(\mu) < c'(a) \le p$ for all $\mu \in (0,a)$. Therefore, $c'(\mu) = p$ has no solution in $(0,a)$, i.e., $n_o = 0$.

(ii): If $a = (c')^{-1}(p)$, then $c'(a) = p$, so $\mu = a$ is a solution to $c'(\mu) = p$, i.e., $n_c = 1$. Otherwise, $\mu = a$ is not a solution to $c'(\mu) = p$, i.e., $n_c = 0$.

(iii): If $p < p^\dagger(v)$, the result follows from combining the following two subcases:

(iii-1): If $a \le (c')^{-1}(p)$, Lemma EC.29 implies that $n_u = 1$.

(iii-2): If $a > (c')^{-1}(p)$, then $p < c'(a)$, so, from Lemma EC.28, there exists a unique $\overline{a}(p,v) > 0$ such that $n_u = 0$, 1, or 2 according to whether $a > \overline{a}(p,v)$, $a = \overline{a}(p,v)$, or $a < \overline{a}(p,v)$, respectively.

(iv): If $p \ge p^\dagger(v)$.

(iv-1): If $a < (c')^{-1}(p)$, then Lemma EC.29 implies that $n_u = 1$.

(iv-2): If $a \ge (c')^{-1}(p)$, then Lemma EC.30 implies that $n_u = 0$.

■

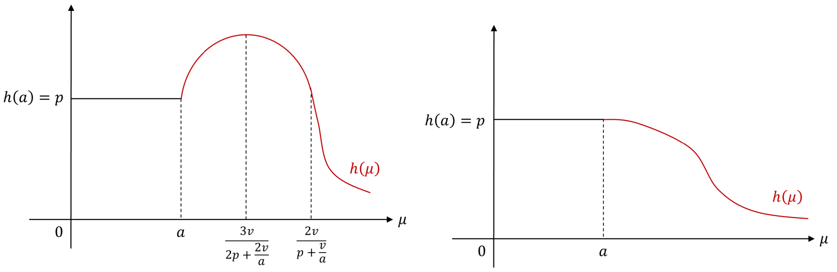
We now set up the necessary tools for the proofs of Lemmas EC.28, EC.29, and EC.30. Before proceeding, we note that, throughout these proofs, we heavily reference the definitions and results from the preliminary Sections EC.7.3 and EC.7.4. The reader might therefore find it useful to review these sections first.

To begin, recall, from (EC.94), that the limiting FOC (16) can be equivalently written as:

$$c'(\mu) = \begin{cases} p, & \mu < a, \\ h(\mu; a, p, v), & \mu \ge a, \end{cases} \tag{EC.124}$$

where $h(\mu;a,p,v)=\frac{a^2}{\mu^2}\left(p+\frac{v}{a}-\frac{v}{\mu}\right)$ is well-defined for all $\mu>0$. It follows from Lemma EC.21 (a)(b) and the fact that $\frac{3v}{2p+\frac{2v}{a}}<\frac{2v}{p+\frac{v}{a}}$ for any $a>0$, $p\geq 0$, and $v>0$; that $h(\mu)$ is strictly increasing and strictly concave in $\mu\in\left(0,\frac{3v}{2p+\frac{2v}{a}}\right)$; strictly decreasing and strictly concave in $\mu\in\left(\frac{3v}{2p+\frac{2v}{a}},\frac{2v}{p+\frac{v}{a}}\right)$; and then strictly decreasing and strictly convex in $\mu\in\left(\frac{2v}{p+\frac{v}{a}},\infty\right)$. It is useful to investigate how these properties of $h(\mu)$ affect the behavior of the right-hand side of (EC.124). This is because, in order to study the number of underloaded equilibria, it is important to understand how the right-hand side of (EC.124) interacts with $c'(\mu)$, a strictly convex and strictly increasing function, in the interval (a,∞). We consider the following two scenarios, illustrated in Figure EC.6:

- When $p<\frac{v}{2a}$, it is easy to see that $\frac{3v}{2p+\frac{2v}{a}}>a$, implying that $h(\mu)$ is strictly increasing and strictly concave in $\mu\in\left(a,\frac{3v}{2p+\frac{2v}{a}}\right)$, strictly decreasing and strictly concave in $\mu\in\left(\frac{3v}{2p+\frac{2v}{a}},\frac{2v}{p+\frac{v}{a}}\right)$, and then strictly decreasing and strictly convex in $\mu\in\left(\frac{2v}{p+\frac{v}{a}},\infty\right)$, as shown in Figure EC.6 (I).
- When $p\geq\frac{v}{2a}$, it is easy to see that $\frac{3v}{2p+\frac{2v}{a}}\leq a$, implying that $h(\mu)$ is strictly decreasing in $\mu\in(a,\infty)$, as shown in Figure EC.6 (II).



(I) $p<\frac{v}{2a}$ (II) $p\geq\frac{v}{2a}$

Figure EC.6 Illustration of the right-hand side (solid black and solid red curves) of the limiting FOC (EC.124).

To simplify the proof of Lemmas EC.28 and EC.29, we leverage the following generic result.

LEMMA EC.31. *Suppose $f(x)$ is a convex function of $x\in(0,\overline{x}]$. For any $0<x_1<x_2\leq\overline{x}$,*

(a) If $f(x_1)f(x_2)<0$, then $f(x)=0$ is obtained at exactly one $x\in(x_1,x_2)$.

(b) If $f(x_1)<0$ and $f(x_2)<0$, then $f(x)<0$ for all $x\in(x_1,x_2)$.

Proof of Lemma EC.31:

(a): Since $f(x_1)f(x_2)<0$, the intermediate value theorem implies that $f(x)=0$ is achieved at at least one $x\in(x_1,x_2)$. We prove uniqueness by contradiction. Suppose there exist two distinct x_0, x_0' such that $x_1<x_0<x_0'<x_2$ and $f(x_0)=x(x_0')=0$. Since $x_1<x_0<x_0'$, there exists $\alpha_0\in(0,1)$ such that $x_0=\alpha_0x_1+(1-\alpha_0)x_0'$. Similarly, since $x_0<x_0'<x_2$, there exists $\alpha_0'\in(0,1)$ such that $x_0'=\alpha_0'x_0+(1-\alpha_0')x_2$.

Case (I): Suppose $f(x_1) < 0$ and $f(x_2) > 0$. Since f is a convex function, we have $\alpha_0 f(x_1) + (1-\alpha_0) f(x_0') \geq f(\alpha_0 x_1 + (1-\alpha_0)x_0') = f(x_0)$, which implies that $f(x_1) \geq 0$ (recalling from the assumption that $f(x_0) = f(x_0') = 0$), a contradiction.

Case (II): Suppose $f(x_1) > 0$ and $f(x_2) < 0$. Since f is a convex function, we have $\alpha_0' f(x_0) + (1-\alpha_0') f(x_2) \geq f(\alpha_0' x_0 + (1-\alpha_0')x_2) = f(x_0')$, which implies that $f(x_2) \geq 0$ (recalling from the assumption that $f(x_0) = f(x_0') = 0$), a contradiction.

(b): Note that $x = \frac{x_2-x}{x_2-x_1}x_1 + \left(1 - \frac{x_2-x}{x_2-x_1}\right)x_2$, then since f is a convex function, it follows that $f(x) = f\left(\frac{x_2-x}{x_2-x_1}x_1 + \left(1-\frac{x_2-x}{x_2-x_1}\right)x_2\right) \leq \frac{x_2-x}{x_2-x_1}f(x_1) + \left(1-\frac{x_2-x}{x_2-x_1}\right)f(x_2) < 0$.

∎

Now, we are ready to prove Lemmas EC.28, EC.29, and EC.30.

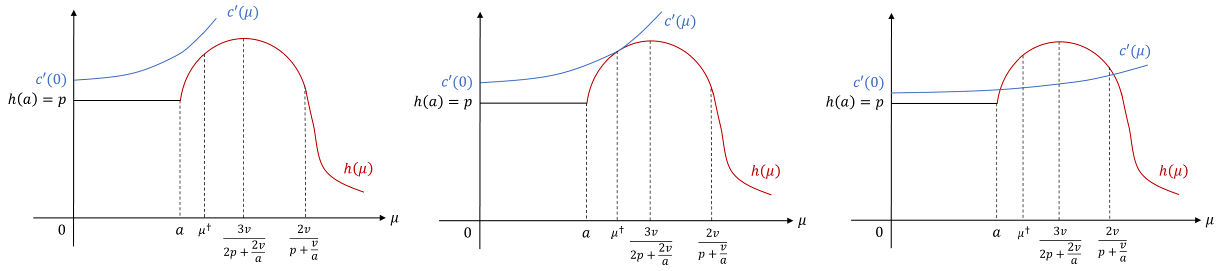
Proof of Lemma EC.28:

Suppose $0 < p < \min\{c'(a), p^\dagger(v)\}$. Since $p < p^\dagger(v)$, due to Remark EC.5, we can divide the interval $a \in (\overline{a}(p,v), \infty)$ into two subintervals $a \in \left(\overline{a}(p,v), \frac{v}{2p}\right)$ and $a \in \left[\frac{v}{2p}, \infty\right)$. In the latter subinter val, $p \geq \frac{v}{2a}$, and so $h(\mu)$ is strictly decreasing in $\mu \in (a,\infty)$ (recall Figure EC.6 (II)); then, since $p < c'(a)$, we have $c'(\mu) > c'(a) > p = h(a) > h(\mu)$ for all $\mu > a$, resulting in no underloaded equilibria, i.e., $n_u = 0$. Therefore, for the remainder of the proof, we need only focus on the case of $0 \leq p < \frac{v}{2a}$ (recall Figure EC.6 (I)) in studying underloaded equilibria.

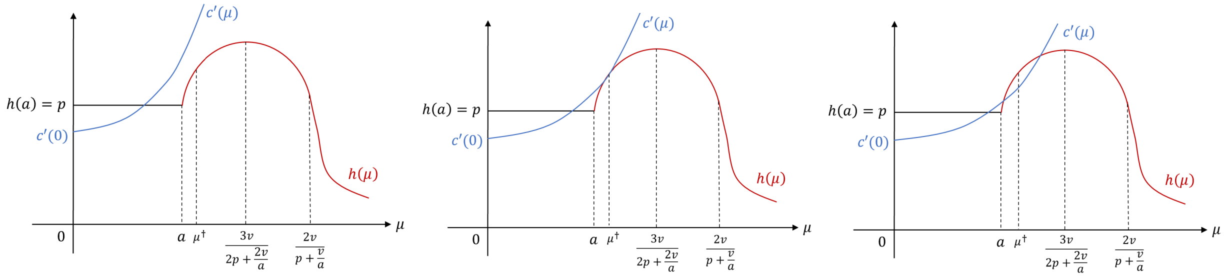
For any $p < c'(a)$, Lemma EC.27 states that there exists a unique $v > 0$ such that $p < p^\dagger(v)$, given by $v^\dagger(a,p)$. Recall the definitions of $\mu^\dagger(a,p)$ and $v^\dagger(a,p)$ from Definition EC.1, $p^\dagger(v)$ from Definition EC.2, and $\overline{a}(p,v)$ from Definition EC.3.

When $p < p^\dagger(v)$ for all $v > 0$, by Corollary EC.5, $a>\overline{a}(p,v)$, $a = \overline{a}(p,v)$ and $a<\overline{a}(p,v)$ are equivalent to $v<v^\dagger(a,p)$, $v = v^\dagger(a,p)$ and $v>v^\dagger(a,p)$, respectively. In the remainder of the proof, we use conditions in terms of $v^\dagger(a,p)$ instead of $\overline{a}(p,v)$, for ease of presentation.

- $v < v^\dagger(a,p)$ (Figures EC.7 (I)(IV)): First, recall from Lemma EC.21 (e) that $h(\mu;a,p,v)$ is strictly increasing in v for all $\mu > a$. Therefore, when $v < v^\dagger(a,p)$, $h(\mu;a,p,v) < h(\mu;a,p,v^\dagger(a,p))$ for all $\mu > a$. Moreover, from Lemma EC.24, when $p < c'(a)$, $h(\mu;a,p,v^\dagger(a,p)) \leq c'(\mu)$ for all $\mu > a$. Thus, it follows that when $v < v^\dagger(a,p)$, $h(\mu;a,p,v) < c'(\mu)$ for all $\mu > a$, i.e., $n_u = 0$.
- $v = v^\dagger(a,p)$ (Figure EC.7 (II)(V)): From Lemma EC.24, when $p < c'(a)$, $h(\mu;a,p,v^\dagger(a,p)) = c'(\mu)$ has exactly one solution in (a,∞), namely, $\mu^\dagger(a,p)$, i.e., $n_u = 1$.
- $v > v^\dagger(a,p)$ (Figure EC.7 (III)(VI)): First, recall from Lemma EC.21 (e) that $h(\mu;a,p,v)$ is strictly increasing in v for all $\mu > a$. Therefore, when $v > v^\dagger(a,p)$, $h(\mu;a,p,v) > h(\mu;a,p,v^\dagger(a,p))$ for all $\mu > a$. Thus, $h(\mu^\dagger(a,p);a,p,v) > h(\mu^\dagger(a,p);a,p,v^\dagger(a,p))$ (since $\mu^\dagger(a,p) > a$). Moreover, from Lemma EC.24, when $p < c'(a)$, $h(\mu^\dagger(a,p);a,p,v^\dagger(a,p)) =$



(I) $v < v^\dagger(a,p)$, $p \le c'(0)$ (II) $v = v^\dagger(a,p)$, $p \le c'(0)$ (III) $v > v^\dagger(a,p)$, $p \le c'(0)$



(IV) $v < v^\dagger(a,p)$, $p > c'(0)$ (V) $v = v^\dagger(a,p)$, $p > c'(0)$ (VI) $v > v^\dagger(a,p)$, $p > c'(0)$

Figure EC.7 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.124) when $p < \frac{v}{2a}$ and $0 \le p < \min\{c'(a), p^\dagger(v)\}$.

$c'(\mu^\dagger(a,p))$. Thus, it follows that, when $v > v^\dagger(a,p)$, $h(\mu^\dagger(a,p);a,p,v) > c'(\mu^\dagger(a,p))$. Since $h(a) = p < c'(a)$ and $\lim_{\mu\to\infty} h(\mu) = 0 < \infty = \lim_{\mu\to\infty} c'(\mu)$, by the intermediate value theorem, there must exist at least one underloaded equilibrium in $(a, \mu^\dagger(a,p))$ and at least one underloaded equilibrium in $(\mu^\dagger(a,p), \infty)$, i.e., $n_u \ge 2$.

Recalling that $h(\mu)$ is strictly concave in μ for $\mu \in \left(0, \frac{2v}{p+\frac{v}{a}}\right)$ and strictly decreasing in μ for $\mu \in \left[\frac{2v}{p+\frac{v}{a}}, \infty\right)$, we discuss the following two cases.

Case (I): If there exists an underloaded equilibrium in $\left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$ (e.g., Figure EC.7 (III)), then it is unique in $\left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$ because $c'(\mu)$ is a strictly increasing function of μ and $h(\mu)$ is a strictly decreasing function of μ in this range. Moreover, $c'\left(\frac{2v}{p+\frac{v}{a}}\right) < h\left(\frac{2v}{p+\frac{v}{a}}\right)$, by the intermediate value theorem and by noting that $\lim_{\mu\to\infty} c'(\mu) = \infty > 0 = \lim_{\mu\to\infty} h(\mu)$. Given that $c'\left(\frac{2v}{p+\frac{v}{a}}\right) < h\left(\frac{2v}{p+\frac{v}{a}}\right)$ and $h(a) = p < c'(a)$, which implies that $\left(c'\left(\frac{2v}{p+\frac{v}{a}}\right) - h\left(\frac{2v}{p+\frac{v}{a}}\right)\right)(c'(a) - h(a)) < 0$, together with the fact that $c' - h$ is convex in $\left(a, \frac{2v}{p+\frac{v}{a}}\right)$, Lemma EC.31 (a) implies that $c'(\mu) - h(\mu) = 0$ at exactly one $\mu \in \left(a, \frac{2v}{p+\frac{v}{a}}\right)$. Hence, $n_u = 2$, where the smaller underloaded equilibrium lies in $(a, \mu^\dagger(a,p))$.

Case (II): If all the underloaded equilibria lie in $\left(a, \frac{2v}{p+\frac{v}{a}}\right]$ (e.g., Figure EC.7 (VI)), then there are exactly two underloaded equilibria, because $c'(\mu)$, a strictly convex function and $h(\mu)$, a strictly concave function cannot intersect more than twice, i.e., $n_u = 2$, where the smaller underloaded equilibrium lies in $(a, \mu^\dagger(a,p))$ and the larger underloaded equilibrium lies in $\left(\mu^\dagger(a,p), \frac{2v}{p+\frac{v}{a}}\right]$.

Therefore, $n_u = 0$ if $a > \overline{a}(p,v)$; $n_u = 1$ if $a = \overline{a}(p,v)$; and $n_u = 2$ if $a < \overline{a}(p,v)$. ■

Proof of Lemma EC.29:

We address the cases $p > c'(a)$ and $p = c'(a) < p^\dagger(v)$ separately.

Case (I): When $p > c'(a)$, we consider two further subcases $p \geq \frac{v}{2a}$ and $p < \frac{v}{2a}$ separately.

Case (I-1): If $p \geq \frac{v}{2a}$ (Figure EC.8), then $h(\mu)$ is strictly decreasing in $\mu \in (a,\infty)$. Since $h(a) = p > c'(a)$, and $\lim_{\mu\to\infty} h(\mu) = 0 < \infty = \lim_{\mu\to\infty} c'(\mu)$, together with the fact that h is strictly decreasing and c' is strictly increasing, it follows that there exists exactly one intersection point in (a,∞), i.e., $n_u = 1$.

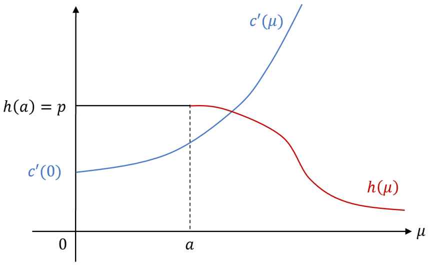


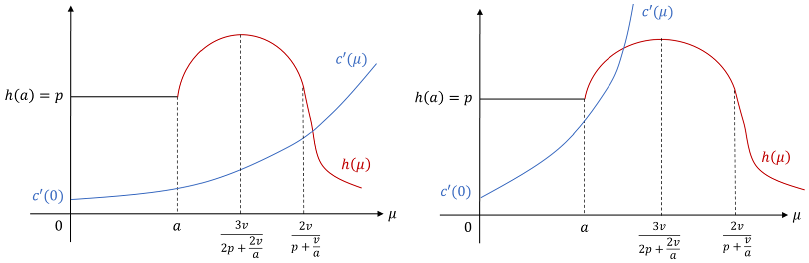
Figure EC.8 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.124) when $p \geq \frac{v}{2a}$ and $p > c'(a)$.

Case (I-2): If $p < \frac{v}{2a}$ (Figure EC.9), since $h(a) = p > c'(a)$ and $\lim_{\mu\to\infty} h(\mu) = 0 < \infty = \lim_{\mu\to\infty} c'(\mu)$, there must exist at least one underloaded equilibria in (a,∞), i.e., $n_u \geq 1$, by the intermediate value theorem. Recalling that $h(\mu)$ is strictly concave in μ for $\mu \in \left(0, \frac{2v}{p+\frac{v}{a}}\right)$ and strictly decreasing in μ for $\mu \in \left[\frac{2v}{p+\frac{v}{a}}, \infty\right)$, we discuss the following two cases.

Case (I-2-a): If there exists an underloaded equilibrium in $\left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$ (Figure EC.9 (I)), then it is unique in $\left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$ because $c'(\mu)$ is a strictly increasing function of μ and $h(\mu)$ is a strictly decreasing function of μ in this range. Moreover, $c'\left(\frac{2v}{p+\frac{v}{a}}\right) < h\left(\frac{2v}{p+\frac{v}{a}}\right)$, because, otherwise, $c'(\mu)$ and $h(\mu)$ would not intersect in $\left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$, which contradicts the assumption. Then, Lemma EC.31 (b) implies no equilibrium in $\left(a, \frac{2v}{p+\frac{v}{a}}\right]$, given that $c'(a) - h(a) < 0$, $c'\left(\frac{2v}{p+\frac{v}{a}}\right) - h\left(\frac{2v}{p+\frac{v}{a}}\right) < 0$ and the fact that $c' - h$ is convex in $\left(a, \frac{2v}{p+\frac{v}{a}}\right)$. Therefore, $n_u = 1$.

Case (I-2-b): If all the underloaded equilibria lie in $\left(a, \frac{2v}{p+\frac{v}{a}}\right]$ (Figure EC.9 (II)), then $c'\left(\frac{2v}{p+\frac{v}{a}}\right) \geq h\left(\frac{2v}{p+\frac{v}{a}}\right)$, because, otherwise, by the intermediate value theorem and the fact that $\lim_{\mu\to\infty} c'(\mu) = \infty > 0 = \lim_{\mu\to\infty} h(\mu)$, there would exist at least one underloaded equilibrium in $\left(\frac{2v}{p+\frac{v}{a}}, \infty\right)$, which contradicts the assumption. Since $c'\left(\frac{2v}{p+\frac{v}{a}}\right) \geq h\left(\frac{2v}{p+\frac{v}{a}}\right)$ and $h'\left(\frac{2v}{p+\frac{v}{a}}\right) < 0 < c''\left(\frac{2v}{p+\frac{v}{a}}\right)$, it follows that $c'\left(\frac{2v}{p+\frac{v}{a}} + \epsilon\right) > h\left(\frac{2v}{p+\frac{v}{a}} + \epsilon\right)$ for all small enough $\epsilon > 0$. Then, given

$h(a)=p>c'(a)$ and $c'\left(\frac{2v}{p+\frac{v}{a}}+\epsilon\right)>h\left(\frac{2v}{p+\frac{v}{a}}+\epsilon\right)$ for all small enough $\epsilon>0$, which implies that $\left(c'\left(\frac{2v}{p+\frac{v}{a}}+\epsilon\right)-h\left(\frac{2v}{p+\frac{v}{a}}+\epsilon\right)\right)(c'(a)-h(a))<0$, together with the fact that $c'-h$ is convex in $\left(a,\frac{2v}{p+\frac{v}{a}}\right)$, Lemma EC.31 (a) implies that $c'(\mu)-h(\mu)=0$ at exactly one $\mu\in\left(a,\frac{2v}{p+\frac{v}{a}}+\epsilon\right)$ for all small enough $\epsilon>0$. Hence, $n_u=1$.



(I) (II)

Figure EC.9 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.124) when $p<\frac{v}{2a}$ and $p>c'(a)$.

Case (II): If $p=c'(a)<p^{\dagger}(v)$ (Figure EC.10), then, recalling the definition of $p^{\dagger}(v)$ in (EC.97), $p<p^{\dagger}(v)$ becomes

$$
\begin{aligned}
& p(c')^{-1}(p)+\frac{1}{2}\left((c')^{-1}(p)\right)^2 c''((c')^{-1}(p))<\frac{v}{2}\\
\Rightarrow\quad & pa+\frac{1}{2}a^2c''(a)<\frac{v}{2}\\
\Rightarrow\quad & c''(a)<-\frac{2p}{a}+\frac{v}{a^2}\overset{(*)}{=}h'(a),
\end{aligned}
$$

where $(*)$ follows by substituting for $\mu=a$ into $h'(\mu)=-\frac{2a^2}{\mu^3}\left(p+\frac{v}{a}\right)+\frac{3a^2v}{\mu^4}$. Given $c'(a)=h(a)$ and $c''(a)<h'(a)$, $c'(a+\epsilon)<h(a+\epsilon)$ for all small enough $\epsilon>0$. Then, the same proof from Case (I-2) is applicable here, yielding a unique intersection of $c'(\mu)$ and $h(\mu)$ in $(a+\epsilon,\infty)$ for all small enough $\epsilon>0$, i.e., $n_u=1$.

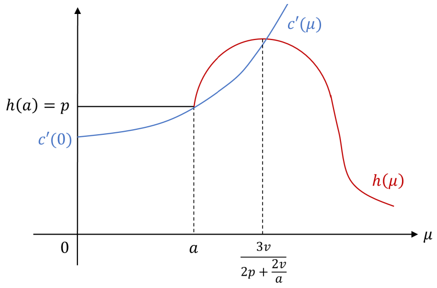


Figure EC.10 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.124) when $c'(0)<p<p^{\dagger}(v)$ and $a=(c')^{-1}(p)$.

Proof of Lemma EC.30:

If $p^\dagger(v) \le p \le c'(a)$, then, recalling the definition of $p^\dagger(v)$ in (EC.97), $p \ge p^\dagger(v)$ implies

$$p(c')^{-1}(p) + \frac{1}{2}\left((c')^{-1}(p)\right)^2 c''((c')^{-1}(p)) \ge \frac{v}{2},$$

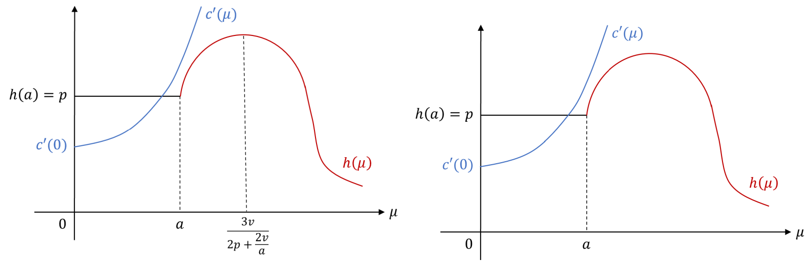
which implies

$$pa + \frac{1}{2}a^2 c''(a) \ge p(c')^{-1}(p) + \frac{1}{2}\left((c')^{-1}(p)\right)^2 c''((c')^{-1}(p)) \ge \frac{v}{2},$$

This above inequality implies that

$$\begin{aligned} & pa + \frac{1}{2}a^2 c''(a) \ge \frac{v}{2} \\ \Rightarrow \quad & c''(a) \ge -\frac{2p}{a} + \frac{v}{a^2} \overset{(*)}{=} h'(a), \end{aligned}$$

where $(*)$ follows by substituting for $\mu = a$ into $h'(\mu) = -\frac{2a^2}{\mu^3}\left(p + \frac{v}{a}\right) + \frac{3a^2 v}{\mu^4}$. Thus, $c''(\mu) \ge c''(a) \ge h'(a) \ge h'(\mu)$, for all $\mu \ge a$, recalling that c' is strictly increasing, and $h(\mu)$ is either (a) strictly decreasing in $\mu \in (a, \infty)$ (Figure EC.11 (I)), or (b) strictly increasing and strictly concave on $\mu \in \left(0, \frac{3v}{2p + \frac{2v}{a}}\right)$, and strictly decreasing on $\mu \in \left(\frac{3v}{2p + \frac{2v}{a}}, \infty\right)$ (Figure EC.11 (II)). Together $c''(\mu) \ge h'(\mu)$ for all $\mu \ge a$, and the fact that $c'(a) \ge p = h(a)$, it follows that $c'(\mu) \ge h(\mu)$ for all $\mu \ge a$, with equality held only possible at $\mu = a$. Thus, $n_u = 0$.



(I) (II)

Figure EC.11 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.124) when $p \ge p^\dagger(v)$ and $a \ge (c')^{-1}(p)$.

∎

EC.7.10. Proof of Proposition 6

Suppose $\mu_1^\star$, $\mu_2^\star$ are symmetric limiting equilibrium service rates such that $\mu_1^\star > \mu_2^\star > 0$. From (15), the limiting utility function is given by

$$U^\infty(\mu, \mu; a, p, v) = \begin{cases} p\mu - c(\mu), & \mu \le a, \\ pa + v\left(1 - \frac{a}{\mu}\right) - c(\mu), & \mu > a. \end{cases}$$

The limiting equilibrium service rates $\mu_1^\star$ and $\mu_2^\star$ satisfy the limiting FOC (16):

$$p\left(1-\left[1-\frac{a^2}{\mu^2}\right]^+\right)+v\frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^+=c'(\mu),\ \text{for } \mu=\mu_1^\star,\mu_2^\star. \tag{EC.125}$$

Moreover, strict convexity of c implies that

$$c(\mu_1^\star)-c(\mu_2^\star)\le(\mu_1^\star-\mu_2^\star)c'(\mu_1^\star)<\mu_1^\star c'(\mu_1^\star)-\mu_2^\star c'(\mu_2^\star),$$

where the second inequality follows from $c'(\mu_1^\star)>c'(\mu_2^\star)$ for $\mu_1^\star>\mu_2^\star$. Thus,

$$0<c(\mu_1^\star)-c(\mu_2^\star)<\mu_1^\star c'(\mu_1^\star)-\mu_2^\star c'(\mu_2^\star). \tag{EC.126}$$

Recalling from Theorem 4 that there can be at most one overloaded or critically loaded equilibrium, we consider the following two cases.

Case (I): If $\mu_1^\star,\mu_2^\star$ satisfy $\mu_1^\star\ge a\ge\mu_2^\star$, then

$$U^\infty(\mu_1^\star,\mu_1^\star)=pa+v\left(1-\frac{a}{\mu_1^\star}\right)-c(\mu_1^\star)\quad\text{and}\quad U^\infty(\mu_2^\star,\mu_2^\star)=p\mu_2^\star-c(\mu_2^\star), \tag{EC.127}$$

and, from (EC.125),

$$c'(\mu_1^\star)=\frac{a}{(\mu_1^\star)^2}\left(v\left(1-\frac{a}{\mu_1^\star}\right)+pa\right)\quad\text{and}\quad c'(\mu_2^\star)=p. \tag{EC.128}$$

Substituting for $c'(\mu_1^\star)$ and $c'(\mu_2^\star)$ into (EC.126) using (EC.128):

$$0<c(\mu_1^\star)-c(\mu_2^\star)<\frac{a}{\mu_1^\star}\left(v\left(1-\frac{a}{\mu_1^\star}\right)+pa\right)-p\mu_2^\star<v\left(1-\frac{a}{\mu_1^\star}\right)+pa-p\mu_2^\star,$$

where the second inequality follows because $\mu_1^\star>a$. Thus,

$$p\mu_2^\star-c(\mu_2^\star)<pa+v\left(1-\frac{a}{\mu_1^\star}\right)-c(\mu_1^\star),$$

which, from (EC.127), is equivalent to

$$U^\infty(\mu_2^\star,\mu_2^\star)<U^\infty(\mu_1^\star,\mu_1^\star).$$

Case (II): If $\mu_1^\star,\mu_2^\star$ satisfy $\mu_1^\star>\mu_2^\star\ge a$, then

$$U^\infty(\mu_1^\star,\mu_1^\star)=pa+v\left(1-\frac{a}{\mu_1^\star}\right)-c(\mu_1^\star)\quad\text{and}\quad U^\infty(\mu_2^\star,\mu_2^\star)=pa+v\left(1-\frac{a}{\mu_2^\star}\right)-c(\mu_2^\star), \tag{EC.129}$$

and, from (EC.125),

$$c'(\mu_1^\star)=\frac{a}{(\mu_1^\star)^2}\left(v\left(1-\frac{a}{\mu_1^\star}\right)+pa\right)\quad\text{and}\quad c'(\mu_2^\star)=\frac{a}{(\mu_2^\star)^2}\left(v\left(1-\frac{a}{\mu_2^\star}\right)+pa\right). \tag{EC.130}$$

Substituting for $c'(\mu_1^\star)$ and $c'(\mu_2^\star)$ into (EC.126) using (EC.130):

$$\begin{aligned}0 < c(\mu_1^\star) - c(\mu_2^\star) &< \frac{a}{\mu_1^\star}\left(v\left(1-\frac{a}{\mu_1^\star}\right)+pa\right) - \frac{a}{\mu_2^\star}\left(v\left(1-\frac{a}{\mu_2^\star}\right)+pa\right)\\ &< \frac{a}{\mu_1^\star}\left[\left(v\left(1-\frac{a}{\mu_1^\star}\right)+pa\right) - \left(v\left(1-\frac{a}{\mu_2^\star}\right)+pa\right)\right]\\ &< \left(v\left(1-\frac{a}{\mu_1^\star}\right)+pa\right) - \left(v\left(1-\frac{a}{\mu_2^\star}\right)+pa\right),\end{aligned}$$

where the second inequality follows because $\mu_1^\star > \mu_2^\star$, and the third inequality follows because $\mu_1^\star > a$. Thus,

$$pa + v\left(1-\frac{a}{\mu_2^\star}\right) - c(\mu_2^\star) < pa + v\left(1-\frac{a}{\mu_1^\star}\right) - c(\mu_1^\star),$$

which, from (EC.129), is equivalent to

$$U^\infty(\mu_2^\star,\mu_2^\star) < U^\infty(\mu_1^\star,\mu_1^\star).$$

■

EC.7.11. Proof of Proposition 7

Throughout the proof, we heavily reference the definitions in Section EC.7.4. The reader might therefore find it useful to review it first. Fix $v > 0$.

(a) Overloaded: From (16), the limiting overloaded equilibrium $\mu_o^\star(a,p;v) \in (0,a)$, when it exists, satisfies $c'(\mu) = p$, implying that $\mu_o^\star(a,p;v) = (c')^{-1}(p)$, which exists when $c'(0) < p < c'(a)$. Thus, $\mu_o(a,p;v)$ does not depend on a, and is strictly increasing in p, recalling that c' is strictly increasing (by strict convexity of c) and thus $(c')^{-1}$ is strictly increasing.

(b) Underloaded:

Recall from (EC.94) that limiting underloaded equilibria, when they exist, satisfy

$$c'(\mu) = h(\mu; a,p,v), \tag{EC.131}$$

where $h(\mu;a,p,v) = \frac{a^2}{\mu^2}\left(p + \frac{v}{a} - \frac{v}{\mu}\right)$.

- From Theorem 4 (a) and (b)(iii), when $p < \min\{c'(a), p^\dagger(v)\}$ and $a < \overline{a}(p,v)$ (corresponding to Areas 1,2, and 12 in Figure 6), $n_u = 2$. Denote the two underloaded equilibria by $\mu_1^\star(a,p;v) > \mu_2^\star(a,p;v) > a$. Notice that $p < \frac{v}{2a}$, because $a < \overline{a}(p,v) < \frac{v}{2p}$ (from Remark EC.5). We assert that $\mu_1^\star(a,p;v)$ and $\mu_2^\star(a,p;v)$ are continuously differentiable functions of a and p in this region.

Remark EC.6. Although $\mu_1^\star(a,p;v)$ exists when $a = \overline{a}(p,v)$ (from Theorem 4 (b)(iii)), we exclude it because $\mu_1^\star(a,p;v)$ is not defined for $a > \overline{a}(p,v)$ and therefore its monotonicity in terms of a at this boundary is meaningless. Also, although $\mu_1^\star(a,p;v)$ exists when $p = (\overline{a})^{-1}(a,v)$, we choose to exclude it when discussing its monotonicity in terms of p, noting that $\mu_1^\star(a,p;v)$ is not defined for $p < (\overline{a})^{-1}(a,v)$.

Claim EC.10. *For any* $p < \min\{c'(a), p^\dagger(v)\}$ *and* $a < \overline{a}(p,v)$, $a < \mu_2^\star(a,p;v) < \mu^\dagger(a,p) < \mu_1^\star(a,p;v)$.

- From Theorem 4 (b)(iii)-(iv), when $p \geq c'(a)$ and either (i) $p < p^\dagger(v)$ or (ii) $c'(a) \neq p \geq p^\dagger(v)$ (corresponding to Areas 5 and 51 in Figure 6), $n_u = 1$. Denote this unique underloaded equilibrium by $\mu_1^\star(a,p;v)$. We assert that $\mu_1^\star(a,p;v)$ is a continuously differentiable function of a and p in this region.

Later arguments will require that $\mu_1^\star(a,p;v)$ is continuous in a and p across the boundary $a = (c')^{-1}(p)$ for $p \in (c'(0), p^\dagger(v))$ when transitioning between Area 1 and Area 5. Specifically, we show the following claim, whose proof appears later.

Claim EC.11. *Fixing any* $a \in (0, \overline{a}(p^\dagger(v), v))$, $\lim_{p \uparrow c'(a)} \mu_1^\star(a,p;v) = \mu_1^\star(a, c'(a); v)$. *Fixing any* $p \in (c'(0), p^\dagger(v))$, $\lim_{a \downarrow (c')^{-1}(p)} \mu_1^\star(a,p;v) = \mu_1^\star((c')^{-1}(p), p; v)$.

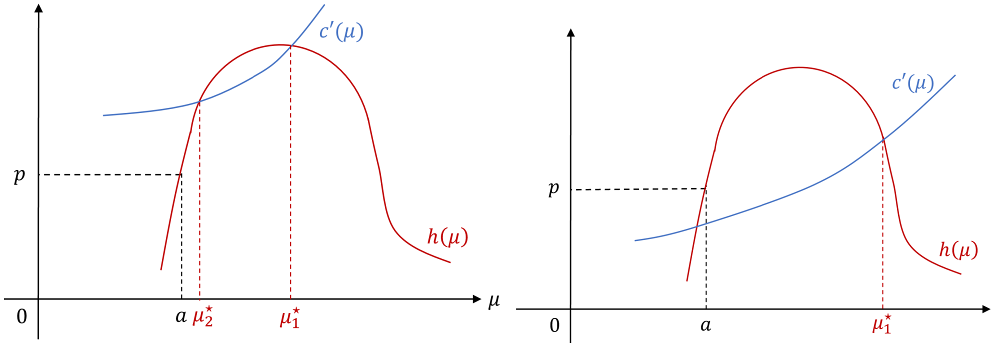
We might suppress the dependence of $\mu_1^\star(a,p;v)$ and $\mu_2^\star(a,p;v)$ on a, p and v henceforth, when the context is clear.

Since $\mu_1^\star(a,p;v)$ is the larger or the unique underloaded equilibrium, it must satisfy $h(\mu_1^\star) = c'(\mu_1^\star)$ and $h'(\mu_1^\star) < c''(\mu_1^\star)$ (see Figure EC.12 (I)(II)), which can be equivalently written as

$$\frac{a^2}{(\mu_1^\star)^2}\left(p + \frac{v}{a} - \frac{v}{\mu_1^\star}\right) = c'(\mu_1^\star) \quad \text{and} \quad \frac{2a^2}{(\mu_1^\star)^3}\left(\frac{3v}{2\mu_1^\star} - \frac{v}{a} - p\right) < c''(\mu_1^\star). \tag{EC.132}$$

Since $\mu_2^\star(a,p;v)$ is the smaller underloaded equilibrium, it satisfies $h(\mu_2^\star) = c'(\mu_2^\star)$ and $h'(\mu_2^\star) > c''(\mu_2^\star)$ (see Figure EC.12 (I)), which can be equivalently written as

$$\frac{a^2}{(\mu_2^\star)^2}\left(p + \frac{v}{a} - \frac{v}{\mu_2^\star}\right) = c'(\mu_2^\star) \quad \text{and} \quad \frac{2a^2}{(\mu_2^\star)^3}\left(\frac{3v}{2\mu_2^\star} - \frac{v}{a} - p\right) > c''(\mu_2^\star). \tag{EC.133}$$



(I) Two underloaded equilibria (II) One underloaded equilibrium

Figure EC.12 Illustration of the left-hand side (solid blue curve) and the right-hand side (solid black and solid red curves) of the limiting FOC (EC.131) when underloaded equilibria exist.

Before proceeding, the next result is useful for the rest of the proof.

Claim EC.12. $c''(\mu^\star) \neq h'(\mu^\star)$ *for any limiting underloaded equilibrium* $\mu^\star > a$.

Monotonicity in p:

(b)(i): Taking the partial derivative with respect to p on both sides of (EC.131) and due to Claim EC.12:

$$c''(\mu)\frac{\partial\mu}{\partial p} = h'(\mu)\frac{\partial\mu}{\partial p} + \frac{\partial h}{\partial p} \quad \Rightarrow \quad \frac{\partial\mu}{\partial p} = \frac{\frac{\partial h}{\partial p}}{c''(\mu) - h'(\mu)},$$

where

$$\frac{\partial h}{\partial p} = \frac{a^2}{\mu^2} \text{ and } h'(\mu) = \frac{a^2}{\mu^2}\cdot\frac{v}{\mu^2} - \frac{2a^2}{\mu^3}\left(p + \frac{v}{a} - \frac{v}{\mu}\right) = \frac{va^2}{\mu^4} - \frac{2}{\mu}h(\mu) = \frac{va^2}{\mu^4} - \frac{2}{\mu}c'(\mu).$$

Substituting for $\frac{\partial h}{\partial p}$ and $h'(\mu)$ using the above expressions yields

$$\frac{\partial\mu}{\partial p} = \frac{\frac{a^2}{\mu^2}}{c''(\mu) - \left(\frac{va^2}{\mu^4} - \frac{2}{\mu}c'(\mu)\right)} = \frac{1}{\frac{\mu}{a^2}\left(2c'(\mu) + \mu c''(\mu)\right) - \frac{v}{\mu^2}},$$

which implies that

$$\mu^2\left(\frac{\partial\mu}{\partial p}\right)^{-1} = \frac{\mu^3}{a^2}\left(2c'(\mu) + \mu c''(\mu)\right) - v. \tag{EC.134}$$

Substituting $c'(\mu_1^\star)$ and $c''(\mu_1^\star)$ using (EC.132) into (EC.134) yields

$$\begin{aligned}(\mu_1^\star)^2\left(\frac{\partial\mu_1^\star(a,p;v)}{\partial p}\right)^{-1} &= \frac{(\mu_1^\star)^3}{a^2}\left(2c'(\mu_1^\star) + \mu_1^\star c''(\mu_1^\star)\right) - v \\ &> \frac{(\mu_1^\star)^3}{a^2}\frac{2a^2}{(\mu_1^\star)^2}\left(p + \frac{v}{a} - \frac{v}{\mu_1^\star}\right) + \frac{(\mu_1^\star)^3}{a^2}\frac{2a^2}{(\mu_1^\star)^2}\left(\frac{3v}{2\mu_1^\star} - \frac{v}{a} - p\right) - v = 0.\end{aligned}$$

Hence, $\frac{\partial\mu_1^\star(a,p;v)}{\partial p} > 0$, i.e., $\mu_1^\star(a,p;v)$ is strictly increasing in p.

Similarly, when $\mu_2^\star(a,p;v)$ exists, substituting $c'(\mu_2^\star)$ and $c''(\mu_2^\star)$ using (EC.133) into (EC.134) yields

$$\begin{aligned}(\mu_2^\star)^2\left(\frac{\partial\mu_2^\star(a,p;v)}{\partial p}\right)^{-1} &= \frac{(\mu_2^\star)^3}{a^2}\left(2c'(\mu_2^\star) + \mu_2^\star c''(\mu_2^\star)\right) - v \\ &< \frac{(\mu_2^\star)^3}{a^2}\frac{2a^2}{(\mu_2^\star)^2}\left(p + \frac{v}{a} - \frac{v}{\mu_2^\star}\right) + \frac{(\mu_2^\star)^3}{a^2}\frac{2a^2}{(\mu_2^\star)^2}\left(\frac{3v}{2\mu_2^\star} - \frac{v}{a} - p\right) - v = 0.\end{aligned}$$

Hence, $\frac{\partial\mu_2^\star(a,p;v)}{\partial p} < 0$, i.e., $\mu_2^\star(a,p;v)$ is strictly decreasing in p.

(Non-)monotonicity in a:

Taking the partial derivative with respect to a on both sides of (EC.131) and due to Claim EC.12:

$$c''(\mu)\frac{\partial\mu}{\partial a} = h'(\mu)\frac{\partial\mu}{\partial a} + \frac{\partial h}{\partial a} \quad \Rightarrow \quad \frac{\partial\mu}{\partial a} = \frac{\frac{\partial h}{\partial a}}{c''(\mu) - h'(\mu)}, \tag{EC.135}$$

where

$$\begin{aligned}\frac{\partial h}{\partial a} &= \frac{a^2}{\mu^2}\left(-\frac{v}{a^2}\right) + \frac{2a}{\mu^2}\left(p + \frac{v}{a} - \frac{v}{\mu}\right) = \frac{2}{a}h(\mu) - \frac{v}{\mu^2} = \frac{2}{a}c'(\mu) - \frac{v}{\mu^2}, \text{ and} \\ h'(\mu) &= \frac{a^2}{\mu^2}\cdot\frac{v}{\mu^2} - \frac{2a^2}{\mu^3}\left(p + \frac{v}{a} - \frac{v}{\mu}\right) = \frac{va^2}{\mu^4} - \frac{2}{\mu}h(\mu) = \frac{va^2}{\mu^4} - \frac{2}{\mu}c'(\mu).\end{aligned} \tag{EC.136}$$

Substituting for $\frac{\partial h}{\partial a}$ and $h'(\mu)$ using the above expressions yields

$$\frac{\partial\mu}{\partial a} = \frac{\frac{2}{a}c'(\mu) - \frac{v}{\mu^2}}{c''(\mu) - \left(\frac{va^2}{\mu^4} - \frac{2}{\mu}c'(\mu)\right)} = \frac{\frac{2\mu^2 c'(\mu)}{a} - v}{\frac{a^2}{\mu^2}\left[\frac{\mu^3}{a^2}\left(2c'(\mu) + \mu c''(\mu)\right) - v\right]},$$

which, together with (EC.134), implies that

$$a^2 \frac{\partial \mu}{\partial a} = \frac{\frac{2\mu^2 c'(\mu)}{a} - v}{\frac{1}{\mu^2}\left[\frac{\mu^3}{a^2}\left(2c'(\mu) + \mu c''(\mu)\right) - v\right]} = \varphi(\mu; a, v) \cdot \frac{\partial \mu}{\partial p}, \tag{EC.137}$$

where $\varphi(\mu; a, v) := \frac{2\mu^2 c'(\mu)}{a} - v$. Note that, for all $\mu > a$,

$$\varphi(\mu; a, v) = \frac{2\mu^2 c'(\mu)}{a} - v < \frac{\mu}{a} \cdot \frac{2\mu^2 c'(\mu)}{a} - v < \mu^2 \left(\frac{\partial \mu}{\partial p}\right)^{-1}, \tag{EC.138}$$

where the last inequality follows from (EC.134). Thus, one can leverage the results about $\frac{\partial \mu}{\partial p}$ from (b)(i) to study $\varphi(\mu; a, v)$.

(b)(ii): When a smaller underloaded equilibrium $\mu_2^\star(a, p; v)$ exists, recall from (b)(i) that $\frac{\partial \mu_2^\star(a,p;v)}{\partial p} < 0$. Then, (EC.138) implies

$$\varphi(\mu_2^\star(a, p; v); a, v) < \left(\mu_2^\star(a, p; v)\right)^2 \left(\frac{\partial \mu_2^\star(a, p; v)}{\partial p}\right)^{-1} < 0, \tag{EC.139}$$

which, from (EC.137), implies that $\frac{\partial \mu_2^\star(a,p;v)}{\partial a} = \frac{1}{a^2}\varphi(\mu_2^\star(a, p; v); a, v)\frac{\partial \mu_2^\star(a,p;v)}{\partial p} > 0$, i.e., $\mu_2^\star(a, p; v)$, when it exists, is strictly increasing in a.

(b)(iii): Throughout the proof of (b)(iii), recall from (b)(i) that $\frac{\partial \mu_1^\star(a,p;v)}{\partial p} > 0$.

DEFINITION EC.4. For any $v > 0$, and $p \in [0, p^\ddagger(v)]$, $a^\dagger(p; v)$ is the unique solution for $a \in \left(0, \frac{v}{2p}\right]$ to $\varphi(\mu_1^\star(a, p; v); a, v) = 0$, which simplifies to

$$c'\left(\frac{v}{p + \frac{v}{2a}}\right) = \frac{av}{2}\left(\frac{p}{v} + \frac{1}{2a}\right)^2. \tag{EC.140}$$

REMARK EC.7. For any $v > 0$ and $p \in [0, p^\ddagger(v)]$, $a^\dagger(p; v) = \frac{v}{2p}$ if and only if $p = p^\ddagger(v)$.

REMARK EC.8. $\varphi(\mu_1^\star(a, p, v); a, v) > 0$ if $a \in (0, a^\dagger(p; v))$, and $\varphi(\mu_1^\star(a, p, v); a, v) < 0$ if $a \in \left(a^\dagger(p; v), \frac{v}{2p}\right]$.

Note that $a^\dagger(p; v)$ depends on p and v, but for simplicity, we may expose or suppress the dependence. The following lemma provides conditions for the existence of $a^\dagger$.

LEMMA EC.32 **(Validating Definition EC.4)**. *There exists a solution for* $a \in \left(0, \frac{v}{2p}\right]$ *that solves (EC.140) if and only if* $p \in [0, +p^\ddagger(v)]$, *where* $p^\ddagger(v)$ *is defined in Definition EC.2 (b). Furthermore, if such a solution exists, it is unique, and is denoted by* $a^\dagger(p; v)$.

LEMMA EC.33. *For any* $v > 0$, $a^\dagger(p; v)$ *is strictly increasing in* $p \in [0, p^\dagger(v)]$.

Let

$$g(a; p, v) := \begin{cases} 0, & a = 0, \\ \frac{2v^2}{a\left(p + \frac{v}{2a}\right)^2} c'\left(\frac{v}{p + \frac{v}{2a}}\right), & a > 0. \end{cases} \tag{EC.141}$$

Then, it is easy to see that (EC.140) is equivalent to $g(a;p,v)=v$; that is, $a^{\dagger}(p;v)\in\left(0,\frac{v}{2p}\right]$, if it exists, solves $g(a;p,v)=v$. Note that g is continuous at $a=0$, and it is straightforward to see that it is a strictly increasing function of a for $a\in\left(0,\frac{v}{2p}\right]$.

(b)(iii)(1): Using Remark EC.7 to exclude $p=p^{\ddagger}(v)$ from Lemma EC.32, when $0\le p<p^{\ddagger}(v)$, $a^{\dagger}(p;v)\in\left(0,\frac{v}{2p}\right)$. Recall from (EC.137) that $\frac{\partial\mu_1^\star(a,p;v)}{\partial a}=\frac{1}{a^2}\varphi(\mu_1^\star(a,p;v);a,v)\frac{\partial\mu_1^\star(a,p;v)}{\partial p}$, where $\frac{\partial\mu_1^\star(a,p;v)}{\partial p}>0$ from (b)(i).

Case (I): If $0\le p<p^{\dagger}(v)$, considering Remark EC.8, it suffices to show $a^{\dagger}(p;v)<\overline{a}(p,v)$, where $\overline{a}(p,v)<\frac{v}{2p}$ from Remark EC.5. (Recall from Theorem 4 (b)(iii) that $\mu_1^\star(a,p;v)$ exists only on $(0,\overline{a}(p,v))$.) We first establish the following claim:

Claim EC.13. *If $0\le p<p^{\dagger}(v)$, then $g(\overline{a}(p,v);p,v)>v$, where g is defined in (EC.141) and $\overline{a}(p,v)$ is defined in Theorem 4.*

Recalling that $g(a;p,v)$ is a strictly increasing function of a for $0<a\le\overline{a}(p,v)<\frac{v}{2p}$ with $g(0;p,v)=0$ and $g(\overline{a}(p,v);p,v)>v$ (from Claim EC.13), it follows that there exists a unique $a^{\dagger}(p;v)\in(0,\overline{a}(p,v))$ such that $g(a^{\dagger}(p;v);p,v)=v$. Therefore, $a^{\dagger}(p;v)<\overline{a}(p,v)$.

Case (II): If $p^{\dagger}(v)\le p<p^{\ddagger}(v)$, considering Remark EC.8, it suffices to show that $a^{\dagger}(p;v)<(c')^{-1}(p)$, where $(c')^{-1}(p)<\frac{v}{2p}$ when $p<p^{\ddagger}(v)$ from Definition EC.2 (b). (Recall from Theorem 4 (b)(iv) that $\mu_1^\star(a,p;v)$ exists only on $(0,(c')^{-1}(p))$.) It is straightforward to see that the right-hand side of (EC.140) is a strictly decreasing function of a for $0<a\le(c')^{-1}(p)<\frac{v}{2p}$, and the left-hand side of (EC.140) is strictly increasing in $a>0$. Thus, in order to show $a^{\dagger}(p;v)<(c')^{-1}(p)$, it suffices to show that the left-hand side of (EC.140) is strictly greater than its right-hand side at $a=(c')^{-1}(p)$, i.e.,

$$p^{\dagger}(v)\le p=c'(a)<p^{\ddagger}(v)\quad\text{implies}\quad c'\left(\frac{v}{c'(a)+\frac{v}{2a}}\right)>\frac{av}{2}\left(\frac{c'(a)}{v}+\frac{1}{2a}\right)^2. \tag{EC.142}$$

By definition of $p^{\dagger}(v)$ in Definition EC.2 (a) and $p^{\ddagger}(v)$ in Definition EC.2 (b), $p^{\dagger}(v)\le p<p^{\ddagger}(v)$ is equivalent to

$$p(c')^{-1}(p)<\frac{v}{2}\le p(c')^{-1}(p)+\frac{1}{2}\left((c')^{-1}(p)\right)^2c''\left((c')^{-1}(p)\right). \tag{EC.143}$$

Using (EC.143), (EC.142) is equivalent to

$$c'(a)<\frac{v}{2a}\le c'(a)+\frac{1}{2}ac''(a)\quad\text{implies}\quad c'\left(\frac{v}{c'(a)+\frac{v}{2a}}\right)>\frac{av}{2}\left(\frac{c'(a)}{v}+\frac{1}{2a}\right)^2. \tag{EC.144}$$

Note that $a<\frac{v}{p+\frac{v}{2a}}$ for any $a\in\left(0,\frac{v}{2p}\right)$. Then, by convexity of c' (Assumption 1), for any $a\in\left(0,\frac{v}{2p}\right)$,

$$c''(a)\le\frac{c'\left(\frac{v}{p+\frac{v}{2a}}\right)-c'(a)}{\frac{v}{p+\frac{v}{2a}}-a}\quad\Leftrightarrow\quad c'\left(\frac{v}{p+\frac{v}{2a}}\right)\ge c'(a)+c''(a)\left(\frac{v}{p+\frac{v}{2a}}-a\right). \tag{EC.145}$$

Moreover, $0<\frac{v}{2a}-c'(a)\le \frac{1}{2}ac''(a)$ implies

$$c''(a)\ge \frac{2}{a}\left(\frac{v}{2a}-c'(a)\right)\overset{(*)}{>}\frac{1}{2a}\left(\frac{v}{2a}-c'(a)\right)>0, \tag{EC.146}$$

where $(*)$ follows by noting that $\frac{v}{2a}-c'(a)>0$. Then, combining (EC.145) and (EC.146) implies that, when $a=(c')^{-1}(p)$,

$$\begin{aligned}
&c'\left(\frac{v}{p+\frac{v}{2a}}\right)>c'(a)+\frac{1}{2a}\left(\frac{v}{2a}-c'(a)\right)\left(\frac{v}{p+\frac{v}{2a}}-a\right)=c'(a)+\frac{1}{2}\frac{\left(\frac{v}{2a}-c'(a)\right)^2}{\frac{v}{2a}+c'(a)}\\
&\overset{(*)}{=}\frac{a}{2v}\left[\frac{\left(\frac{v}{2a}-c'(a)\right)^3}{\frac{v}{2a}+c'(a)}+\left(c'(a)+\frac{v}{2a}\right)^2\right]>\frac{a}{2v}\left(c'(a)+\frac{v}{2a}\right)^2,
\end{aligned}$$

where $(*)$ follows from algebra. Hence, (EC.144) is established.

(b)(iii)(2): When $p\ge p^{\ddagger}(v)$, it suffices to show that $\varphi(\mu_1^{\star};a,v)>0$ for all $a\in(0,(c')^{-1}(p))$, recalling from (EC.137) that $\frac{\partial\mu_1^{\star}(a,p;v)}{\partial a}=\frac{1}{a^2}\varphi(\mu_1^{\star}(a,p;v);a,v)\frac{\partial\mu_1^{\star}(a,p;v)}{\partial p}$, where $\frac{\partial\mu_1^{\star}(a,p;v)}{\partial p}>0$ from (b)(i). (Recall from Theorem 4 (b)(iv) that $\mu_1^{\star}(a,p;v)$ exists only for $a\in(0,(c')^{-1}(p))$.) We discuss two cases depending on whether there exists $a>0$ that solves (EC.140).

- **Case (I):** If there does not exist $a>0$ that solves (EC.140), then $\varphi(\mu_1^{\star};a,v)>0$ for all $a\in(0,(c')^{-1}(p))$, because $\lim_{a\downarrow 0}\varphi(\mu_1^{\star}(a,p;v);a,v)=\infty>0$.
- **Case (II):** If there exists $a>0$ that solves (EC.140), then denote the smallest solution by $a^{\dagger}(p;v)$. Then, it suffices to show that $a^{\dagger}(p;v)\ge(c')^{-1}(p)$. Then, $\varphi(\mu_1^{\star};a,v)>0$ for all $a\in(0,(c')^{-1}(p))$, because $\lim_{a\downarrow 0}\varphi(\mu_1^{\star}(a,p;v);a,v)=\infty>0$.
  - **Case (II-1):** If $p=p^{\ddagger}(v)$, Remark EC.7 implies that $a^{\dagger}(p;v)=\frac{v}{2p}=(c')^{-1}(p)$ by definition of $p^{\ddagger}(v)$ in Definition EC.2 (b).
  - **Case (II-2):** If $p>p^{\ddagger}(v)$, then it is straightforward that $(c')^{-1}(p)>\frac{v}{2p}$ (by definition of $p^{\ddagger}(v)$ in Definition EC.2 (b)). By Lemma EC.32 and Remark EC.7, when $p>p^{\ddagger}(v)$, $a^{\dagger}(p;v)$, if exists, must satisfy $a^{\dagger}(p;v)>\frac{v}{2p}$. This can be equivalently written as

$$a^{\dagger}(p;v)>\frac{v}{p+\frac{v}{2a^{\dagger}(p;v)}},$$

which implies that

$$c'\left(a^{\dagger}(p;v)\right)>c'\left(\frac{v}{p+\frac{v}{2a^{\dagger}(p;v)}}\right),$$

recalling that c' is a strictly increasing function (by strict convexity of c). Note that

$$c'\left(\frac{v}{p+\frac{v}{2a^{\dagger}(p;v)}}\right)\overset{(i)}{=}\frac{a^{\dagger}(p;v)v}{2}\left(\frac{p}{v}+\frac{1}{2a^{\dagger}(p;v)}\right)^2\overset{(ii)}{>}\frac{\frac{v}{2p}v}{2}\left(\frac{p}{v}+\frac{1}{2\frac{v}{2p}}\right)^2=p,$$

where (i) follows because $a^{\dagger}(p;v)$ solves (EC.140), and (ii) follows because $a^{\dagger}(p;v)>\frac{v}{2p}$ and $\frac{a(p+\frac{v}{2a})^2}{2v}$ is increasing in $a\in\left(\frac{v}{2p},\infty\right)$. Hence, from the above two displays,

$$c'(a^{\dagger}(p;v))>p\quad\Leftrightarrow\quad a^{\dagger}(p;v)>(c')^{-1}(p),$$

which is desired.

∎

Below we prove Lemmas EC.32-EC.33, Remarks EC.7-EC.8, and Claims EC.10-EC.13.

Proof of Lemma EC.32: By definition of $a^\dagger$ in Definition EC.4, $\varphi(\mu_1^\star; a^\dagger, v) = 0$ implies

$$\frac{2(\mu_1^\star)^2 c'(\mu_1^\star)}{a^\dagger} = v. \tag{EC.147}$$

Since $\mu_1^\star$ satisfies the limiting FOC (EC.131), substituting for $c'(\mu_1^\star)$ using (EC.131) in the above display yields

$$\mu_1^\star(a^\dagger, p; v) = \frac{v}{p + \frac{v}{2a^\dagger}}. \tag{EC.148}$$

Recall from (EC.139) that when $\mu_2^\star(a,p;v)$ exists, $\varphi(\mu_2^\star; a, v) < 0$, including when $\mu_2^\star(a^\dagger, p; v)$ exists.

Substitution into (EC.147) using (EC.148) shows that $a^\dagger$ satisfies

$$\frac{2v^2}{a\left(p + \frac{v}{2a}\right)^2} c'\left(\frac{v}{p + \frac{v}{2a}}\right) = v \quad \Leftrightarrow \quad c'\left(\frac{v}{p + \frac{v}{2a}}\right) = \frac{av}{2}\left(\frac{p}{v} + \frac{1}{2a}\right)^2,$$

which establishes (EC.140).

It is straightforward to see that $g(a;p,v)$ is a strictly increasing function of a for $a \in \left(0, \frac{v}{2p}\right]$ with $g(0;p,v) = 0$. Thus, $a^\dagger \in \left(0, \frac{v}{2p}\right]$ that solves $g(a;p,v) = v$ exists if and only if $g\left(\frac{v}{2p}; p, v\right) \geq v$, i.e.,

$$g\left(\frac{v}{2p}; p, v\right) = \frac{v}{p} c'\left(\frac{v}{2p}\right) \geq v,$$

which holds if and only if $0 \leq p \leq p^\ddagger(v)$, recalling the definition of $p^\ddagger(v)$ in Definition EC.2 (b). From Remark EC.7, $a^\dagger(p;v) = \frac{v}{2p}$ when $p = p^\ddagger(v)$. Moreover, from Lemma EC.33, $a^\dagger(p;v)$ is strictly increasing in $p \in [0, p^\ddagger(v)]$. Hence, $a^\dagger(p;v) \leq \frac{v}{2p}$ for $p \in [0, p^\ddagger(v)]$.

Finally, since $g(a;p,v)$ is a strictly increasing function of a, $a^\dagger$, which is a solution to $g(a;p,v) = v$, must be unique, if it exists. ∎

Proof of Lemma EC.33: By definition, $a^\dagger(p;v)$ solves (EC.140). Denote the left-hand side of (EC.140) by $LHS(a,p;v)$, and the right-hand side of (EC.140) by $RHS(a,p;v)$. Note that

(1) $LHS(a,p;v)$ is strictly increasing in $a \in (0,\infty)$, and $RHS(a,p;v)$ is strictly decreasing in $a \in \left(0, \frac{v}{2p}\right)$ and strictly increasing in $a \in \left(\frac{v}{2p}, \infty\right)$;

(2) $LHS(a,p;v)$ is strictly decreasing in p, and $RHS(a,p;v)$ is strictly increasing in p.

Then, for any $0 \leq p_1 < p_2 \leq p^\ddagger(v)$, we have $LHS(a^\dagger(p_1;v), p_1; v) = RHS(a^\dagger(p_1;v), p_1; v)$. Then, (2) implies that $LHS(a^\dagger(p_1;v), p_2; v) < LHS(a^\dagger(p_1;v), p_1; v)$ and $RHS(a^\dagger(p_1;v), p_2; v) > RHS(a^\dagger(p_1;v), p_2; v)$, implying that $LHS(a^\dagger(p_1;v), p_2; v) < RHS(a^\dagger(p_1;v), p_2; v)$. Then, (1) implies that $a^\dagger(p_1;v) < a^\dagger(p_2;v)$. Therefore, $a^\dagger(p;v)$ is strictly increasing in $p \in [0, p^\ddagger(v)]$. ∎

Proof of Remark EC.7: Plugging in $a = \frac{v}{2p}$ into (EC.140) yields $c'\left(\frac{v}{2p}\right) = p$, which implies that $p = p^{\ddagger}(v)$ by uniqueness and by the definition of $p^{\ddagger}(v)$ (in Definition EC.2 (b)). ■

Proof of Remark EC.8: We evaluate the partial derivative of $\varphi(\mu_1^\star(a,p;v);a,v)$ with respect to a at $a = a^\dagger(p;v)$:

$$\begin{aligned}
&\frac{\partial}{\partial a}\varphi(\mu_1^\star(a,p;v);a,v)\bigg|_{a=a^\dagger} = \frac{\partial}{\partial a}\left(\frac{2\mu_1^\star(a,p;v)^2 c'(\mu_1^\star(a,p;v))}{a} - v\right)\bigg|_{a=a^\dagger} \\
=&\frac{1}{(a^\dagger)^2}\left[a^\dagger \cdot \left(4\mu_1^\star c'(\mu_1^\star)\, \frac{\partial \mu_1^\star(a,p;v)}{\partial a}\bigg|_{a=a^\dagger} + 2(\mu_1^\star)^2 c''(\mu_1^\star)\, \frac{\partial \mu_1^\star(a,p;v)}{\partial a}\bigg|_{a=a^\dagger}\right) - 2(\mu_1^\star)^2 c'(\mu_1^\star)\right] \\
=&-\frac{2}{(a^\dagger)^2}(\mu_1^\star)^2 c'(\mu_1^\star) < 0,
\end{aligned}$$

which implies that $\varphi(\mu_1^\star(a,p;v);a,v)$ is strictly decreasing in a at $a = a^\dagger$. Since $a^\dagger$ is the unique $a \in \left(0, \frac{v}{2p}\right]$ that satisfies $\varphi(\mu_1^\star(a,p;v);a,v) = 0$, it follows that $\varphi(\mu_1^\star(a,p;v);a,v) > 0$ for all $a \in (0, a^\dagger(p;v))$, and $\varphi(\mu_1^\star(a,p;v);a,v) < 0$ for all $a \in \left(a^\dagger(p;v), \frac{v}{2p}\right]$. ■

Proof of Claim EC.10: We first note that $0 < a < \overline{a}(p,v)$ is equivalent to $v > v^\dagger(a,p)$ (from Corollary EC.5). When $v > v^\dagger(a,p)$, we have $h(\mu^\dagger(a,p);a,p,v) > c'(\mu^\dagger(a,p))$, because $h(\mu^\dagger(a,p);a,p,v) > h(\mu^\dagger(a,p);a,p,v^\dagger(a,p)) = c'(\mu^\dagger(a,p))$ (from Lemma EC.24), recalling that $h(\mu;a,p,v)$ is strictly increasing in v when $\mu > a$ (from Lemma EC.21 (e)). Then, since $h(a) = p < c'(a)$, $h(\mu^\dagger(a,p);a,p,v) > c'(\mu^\dagger(a,p))$ and $\lim_{\mu\to\infty} h(\mu) = 0 < \infty = \lim_{\mu\to\infty} c'(\mu)$, the intermediate value theorem implies that there exists $\mu_2^\star(a,p;v)$ in $(a, \mu^\dagger(a,p))$ and $\mu_1^\star(a,p;v)$ in $(\mu^\dagger(a,p), \infty)$; that is, $a < \mu_2^\star(a,p;v) < \mu^\dagger(a,p) < \mu_1^\star(a,p;v)$. ■

Proof of Claim EC.11: We formally prove the first part of the statement, and the second part follows by similar arguments. The proof follows three steps:

(i) $\mu_1^\star(a,p;v)$ has a limit as $p \uparrow c'(a)$;

(ii) $\lim_{p\uparrow c'(a)} \mu_1^\star(a,p;v)$ solves the limiting FOC (EC.131);

(iii) $\lim_{p\uparrow c'(a)} \mu_1^\star(a,p;v) = \mu_1^\star(a, c'(a); v)$.

Proof of (i): Recall that $\mu_1^\star(a,p;v)$ and $\mu_2^\star(a,p;v)$ are continuous in p for $p < \min\{c'(a), p^\dagger(v)\}$ when $a \in (0, \overline{a}(p^\dagger(v), v))$ (i.e., within Area 1). This implies that $\mu_1^\star(a,p;v)$ and $\mu_2^\star(a,p;v)$ have limits as $p \uparrow c'(a)$, denoted by $L_1 := \lim_{p\uparrow c'(a)} \mu_1^\star(a,p;v)$ and $L_2 := \lim_{p\uparrow c'(a)} \mu_2^\star(a,p;v)$.

Proof of (ii): Let $\varphi(\mu,a,p;v) := c'(\mu) - \frac{a^2}{\mu^2}\left(p + \frac{v}{a} - \frac{v}{\mu}\right)$. Any limiting equilibrium must satisfy the limiting FOC (EC.131), and so

$$\varphi\left(\mu_j^\star(a,p;v), a, p; v\right) = 0,\ j \in \{1,2\},$$

which implies

$$\lim_{p\uparrow c'(a)} \varphi\left(\mu_j^\star(a,p;v),a,p;v\right)=0,\ j\in\{1,2\}. \tag{EC.149}$$

Since $\varphi(\mu,a,p;v)$ is a continuous function of p, it follows that

$$\lim_{p\uparrow c'(a)} \varphi(\mu_j^\star(a,p;v),a,p;v)=\varphi(L_j,a,c'(a);v),\ j\in\{1,2\}. \tag{EC.150}$$

From (EC.149) and (EC.150), we conclude that $\varphi(L_j,a,c'(a);v)=0$, $j\in\{1,2\}$, which implies that both L_1 and L_2 solve the limiting FOC (EC.131).

Proof of (iii): From Theorem 4 (b), when $p=c'(a)$ and $p<p^\dagger(v)$ (i.e., Area 51), there exist an underloaded equilibrium, namely, $\mu_1^\star(a,c'(a);v)>a$, and a critically loaded equilibrium a. Therefore, L_1 and L_2 must be either $\mu_1^\star(a,c'(a);v)$ or a.

- Suppose $L_1=a$ and $L_2=\mu_1^\star(a,c'(a);v)$. Then, since $\mu_1^\star(a,c'(a);v)>a$, there must exist $p<c'(a)$ such that $\mu_2^\star(a,p;v)>\mu_1^\star(a,p;v)$, which contradicts the definition of $\mu_1^\star$ as the larger limiting equilibrium. Hence, $L_1=a$ and $L_2=\mu_1^\star(a,c'(a);v)$ cannot happen.
- Suppose $L_1=L_2=a$. Then, $\lim_{p\uparrow c'(a)}\mu_o^\star(a,p;v)=\mu_1^\star(a,c'(a);v)$, recalling from Theorem 4 (b) that there exists an overloaded equilibrium $\mu_o^\star(a,p;v)<a$ when $p<\min\{c'(a),p^\dagger(v)\}$ and $a\in(0,\overline{a}(p^\dagger(v),v))$ (i.e., within Area 1). Since $\mu_1^\star(a,c'(a);v)>a$, there must exist $p<c'(a)$ such that $\mu_o^\star(a,p;v)>\mu_1^\star(a,p;v)$ and $\mu_o^\star(a,p;v)>\mu_2^\star(a,p;v)$, which contradicts the definition of $\mu_o^\star(a,p;v)$ as an overloaded equilibrium, and $\mu_1^\star(a,p;v)$ and $\mu_2^\star(a,p;v)$ as underloaded equilibria. Hence, $L_1=L_2=a$ cannot happen.

Therefore, we conclude that $L_1=\mu_1^\star(a,c'(a);v)$ and $L_2=a$, or $L_1=L_2=\mu_1^\star(a,c'(a);v)$. In either case, $L_1=\mu_1^\star(a,c'(a);v)$, which completes Step (iii). ∎

Proof of Claim EC.12: We prove by contradiction. Suppose that $c''(\mu)-h'(\mu)=0$ for some $\mu_o>a$. Then, from the left equation in (EC.135), $\frac{\partial h}{\partial a}=0$ at μ_o. Using the expressions for $h'(\mu)$ and $\frac{\partial h}{\partial a}$ in (EC.136), it follows that

$$c''(\mu_o)-\frac{va^2}{\mu_o^4}+\frac{2}{\mu_o}c'(\mu_o)=0 \quad\Leftrightarrow\quad \mu_o^3c''(\mu_o)=\frac{va^2}{\mu_o}-2\mu_o^2c'(\mu_o),\ \text{and} \tag{EC.151}$$

$$\frac{2}{a}c'(\mu_o)-\frac{v}{\mu_o^2}=0 \quad\Leftrightarrow\quad \mu_o^2c'(\mu_o)=\frac{1}{2}av. \tag{EC.152}$$

Substituting for $\mu_o^2c'(\mu_o)$ using (EC.152) into (EC.151) yields

$$\mu_o^3c''(\mu_o)=\frac{va^2}{\mu_o}-av=va\left(\frac{a}{\mu_o}-1\right)<0,$$

which contradicts $c''>0$ (by strict convexity of c). Hence, $c''(\mu)\neq h'(\mu)$ for all $\mu>a$. ∎

Proof of Claim EC.13: Substituting for $\mu c''(\mu)$ using (EC.95) in Definition EC.1 (a) into (EC.96) yields

$$\begin{aligned} v^\dagger(a,p) =& \frac{(\mu^\dagger)^3}{a^2}\left(2c'(\mu^\dagger) + \frac{\left(3a-2\mu^\dagger\right)c'(\mu^\dagger) - \frac{a^3 p}{(\mu^\dagger)^2}}{\mu^\dagger - a}\right) \\ =& \frac{\mu^\dagger}{a(\mu^\dagger - a)}\left((\mu^\dagger)^2 c'(\mu^\dagger) - a^2 p\right), \end{aligned}$$

which implies

$$\frac{(\mu^\dagger)^2 c'(\mu^\dagger)}{a} + \frac{a v^\dagger(a,p)}{\mu^\dagger} = v^\dagger(a,p) + ap, \quad \forall a>0, p\geq 0. \tag{EC.153}$$

Let $\phi(\mu;a,v) := \frac{\mu^2 c'(\mu)}{a} + \frac{av}{\mu}$. Then, note that

$$\frac{\mu^2}{a}\phi'(\mu;a,v) = \frac{\mu^3}{a^2}\left(2c'(\mu) + \mu c''(\mu)\right) - v.$$

Plugging in $a=\overline{a}$ into the above equation:

$$\begin{aligned} \frac{\mu^2}{\overline{a}}\phi'(\mu;\overline{a},v) =& \frac{\mu^3}{\overline{a}^2}\left(2c'(\mu) + \mu c''(\mu)\right) - v \\ =& \frac{\mu^3}{\overline{a}^2}\left(2c'(\mu) + \mu c''(\mu)\right) - \frac{(\mu^\dagger(\overline{a},p))^3}{\overline{a}^2}\left(2c'(\mu^\dagger(\overline{a},p)) + \mu^\dagger(\overline{a},p)c''(\mu^\dagger(\overline{a},p))\right), \end{aligned}$$

where the second equality follows from $v = v^\dagger(\overline{a},p)$ by the definition of $\overline{a}(p,v)$ when $0\leq p < p^\dagger(v)$ (see Definition EC.3) and from the definition of $v^\dagger(a,p)$ in Definition EC.1 (b). Since $\frac{\mu^3}{a^2}(2c'(\mu) + \mu c''(\mu))$ is a strictly increasing function of μ, it is clear from the above display that $\phi'(\mu;\overline{a},v)$ is strictly negative when $\mu < \mu^\dagger(\overline{a},p)$, and is strictly positive when $\mu > \mu^\dagger(\overline{a},p)$. That is, $\phi(\mu;\overline{a},v)$ is strictly decreasing in $\mu \in (0,\mu^\dagger(\overline{a},p))$, strictly increasing in $\mu\in(\mu^\dagger(\overline{a},p),\infty)$, and thus, is minimized at $\mu = \mu^\dagger(\overline{a},p)$. Hence,

$$\phi(\mu;\overline{a},v) > \phi(\mu^\dagger(\overline{a},p);\overline{a},v), \quad \forall \mu \neq \mu^\dagger(\overline{a},p). \tag{EC.154}$$

Recall from Lemma EC.22 and Definition EC.3 that $\mu^\dagger(\overline{a},p)$ satisfies $h'\big(\mu^\dagger(\overline{a},p);\overline{a},p,v\big) = c''\big(\mu^\dagger(\overline{a},p)\big) > 0$, hence $\mu^\dagger(a,p)$ lies on the increasing portion of $h(\mu;\overline{a},p,v)$; that is, $\mu^\dagger(\overline{a},p) < \frac{3v}{2p+\frac{2v}{\overline{a}}}$, recalling from Lemma EC.21 (a). Moreover, it is easy to see that $\frac{3v}{2p+\frac{2v}{\overline{a}}} < \frac{v}{p+\frac{v}{2\overline{a}}}$ given that $\overline{a} < \frac{v}{2p}$ from Remark EC.5. Thus, $\mu^\dagger(\overline{a},p) > \frac{v}{p+\frac{v}{2\overline{a}}}$, and hence, from (EC.154),

$$\begin{aligned} \phi\left(\frac{v}{p+\frac{v}{2\overline{a}}};\overline{a},v\right) >& \phi(\mu^\dagger(\overline{a},v);\overline{a},v) \\ \Leftrightarrow \frac{\left(\frac{v}{p+\frac{v}{2\overline{a}}}\right)^2 c'\left(\frac{v}{p+\frac{v}{2\overline{a}}}\right)}{\overline{a}} + \frac{\overline{a}v}{\frac{v}{p+\frac{v}{2\overline{a}}}} >& \frac{(\mu^\dagger(\overline{a},p))^2 c'(\mu^\dagger(\overline{a},p))}{\overline{a}} + \frac{\overline{a}v}{\mu^\dagger(\overline{a},p)}. \end{aligned}$$

Then, from (EC.153) by plugging in $a=\overline{a}$, the above display is equivalent to

$$\frac{1}{2}\cdot g(\overline{a};p,v) + \overline{a}p + \frac{v}{2} > v^\dagger(\overline{a},p) + \overline{a}p = v + \overline{a}p \quad \Leftrightarrow \quad g(\overline{a};p,v) > v.$$

■

EC.7.12. Proof of Proposition 8

By definition in (17), $\mu^\star_{\max}(p,v)$ is the supremum (limiting) equilibrium service rate over all possible values of a, given p and v. To determine this, we derive the supremum underloaded equilibrium and the supremum overloaded equilibrium separately; then, $\mu^\star_{\max}(p,v)$ is the maximum of these two. (We do not consider the critically loaded equilibrium separately, because every critically loaded equilibrium at $a=(c')^{-1}(p)$ is also an overloaded equilibrium for all $a>(c')^{-1}(p)$.)

(I) $\mathbf{0 \le p < p^\ddagger(v)}$**:** From Proposition 7 (b)(iii)(1), the maximum underloaded equilibrium is obtained at $a=a^\dagger(p;v)\in(0,\frac{v}{2p})$, which satisfies

$$c'\left(\frac{v}{p+\frac{v}{2a^\dagger}}\right)=\frac{a^\dagger v}{2}\left(\frac{p}{v}+\frac{1}{2a^\dagger}\right)^2. \tag{EC.155}$$

Note that plugging $\mu=\frac{v}{p+\frac{v}{2a^\dagger}}$ into the limiting FOC (EC.131) with $a=a^\dagger$ yields (EC.155), which means that $\mu=\frac{v}{p+\frac{v}{2a^\dagger}}$ is an underloaded equilibrium. This can only be $\mu_1^\star$ because even if $\mu_2^\star$ exists, $\frac{\partial\mu_2^\star(a,p;v)}{\partial a}|_{a=a^\dagger}>0$ (from Theorem 7 (b)(ii)), contradicting the definition of $a^\dagger$ for which this partial derivative is zero. Thus, the maximum underloaded equilibrium is given by

$$\mu_1^\star(a^\dagger,p;v)=\frac{v}{p+\frac{v}{2a^\dagger}}. \tag{EC.156}$$

From Proposition 7 (a)(ii), the overloaded equilibrium $\mu_o^\star(a,p;v)$, when it exists, does not depend on a, and is given by $\mu_o^\star(a,p;v)=(c')^{-1}(p)$ (from the limiting FOC (16)).

It remains to compare the maximum underloaded equilibrium and the overloaded equilibrium. We argue that the former is larger. To see this, it suffices to show that

$$(c')^{-1}(p)<\frac{v}{p+\frac{v}{2a^\dagger}} \quad\overset{(i)}{\Leftrightarrow}\quad p<c'\left(\frac{v}{p+\frac{v}{2a^\dagger}}\right) \quad\overset{(ii)}{\Leftrightarrow}\quad p<\frac{a^\dagger v}{2}\left(\frac{p}{v}+\frac{1}{2a^\dagger}\right)^2, \tag{EC.157}$$

where (i) follows because c' is strictly increasing (by strict convexity of c), and (ii) follows from (EC.155). Using Remark EC.7 to exclude $p=p^\ddagger(v)$ from Lemma EC.32, $a^\dagger\in\left(0,\frac{v}{2p}\right)$ when $0\le p<p^\ddagger(v)$. Additionally, it is straightforward to see that $\frac{av}{2}\left(\frac{p}{v}+\frac{1}{2a}\right)^2$ is strictly decreasing in a for $a\in\left(0,\frac{v}{2p}\right)$. Thus, it follows that

$$\frac{a^\dagger v}{2}\left(\frac{p}{v}+\frac{1}{2a^\dagger}\right)^2>\frac{\frac{v}{2p}v}{2}\left(\frac{p}{v}+\frac{1}{2\frac{v}{2p}}\right)^2=p,$$

which establishes (EC.157).

Hence, the maximum equilibrium is the maximum underloaded equilibrium, namely,

$$\mu^\star_{\max}(p,v)=\mu_1^\star(a^\dagger,p;v)=\frac{v}{p+\frac{v}{2a^\dagger}}=\left(\frac{p}{v}+\frac{1}{2a^\dagger}\right)^{-1}.$$

(II) $\mathbf{p \ge p^\ddagger(v)}$**:** From Proposition 7 (b)(iii)(2), the underloaded equilibrium $\mu_1^\star(a,p;v)$, when it exists, is strictly increasing in $a\in\left(0,(c')^{-1}(p)\right)$. Using similar arguments that prove the first part of Claim EC.11, it can be shown that $\lim_{a\to(c')^{-1}(p)}\mu_1^\star(a,p;v)$ solves the limiting FOC, which has the unique solution $\mu=a$ by Theorem 4 (b)(ii). Thus, $\mu_1^\star(a,p;v)\to(c')^{-1}(p)$ when $a\to(c')^{-1}(p)$.

On the other hand, identical to the analysis in (I) above, the overloaded equilibrium $\mu_o^\star(a,p;v)$, when it exists, is given by $\mu_o^\star(a,p;v) = (c')^{-1}(p)$. Hence, the maximum equilibrium is given by $\mu^\star_{\max}(p,v) = (c')^{-1}(p)$.

Finally, we verify the equivalence of this statement to Proposition 3.

- When $0 \le p < p^\ddagger(v)$,

$$\mu^\star_{\max} c'(\mu^\star_{\max}) = \frac{v}{p+\frac{v}{2a^\dagger}} c'\left(\frac{v}{p+\frac{v}{2a^\dagger}}\right) \overset{(*)}{=} \frac{v}{p+\frac{v}{2a^\dagger}} \frac{a^\dagger\left(p+\frac{v}{2a^\dagger}\right)^2}{2v} = \frac{a^\dagger}{2}\left(p+\frac{v}{2a^\dagger}\right).$$

where $(*)$ follows from (EC.155). Moreover,

$$\frac{v^2}{4(v-p\mu^\star_{\max})} = \frac{v^2}{4\left(v-p\frac{v}{p+\frac{v}{2a^\dagger}}\right)} = \frac{a^\dagger}{2}\left(p+\frac{v}{2a^\dagger}\right).$$

The above two displays imply that $\mu^\star_{\max} c'(\mu^\star_{\max}) = \frac{v^2}{4(v-p\mu^\star_{\max})}$ when $0 \le p < p^\ddagger(v)$.

- When $p \ge p^\ddagger(v)$, $c'(\mu^\star_{\max}) = p$.

Therefore, (18) satisfies the equations that characterize $\mu^\star_{\max}$ for the loss system in Proposition 3.

$\blacksquare$

EC.7.13. Proofs of Theorems 5 and 6

From (6), the prelimit FOC is given by

$$c'(\mu) = p\left(1 - I^\lambda(\mu,\mu)\right) + (v-p\mu)\left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu}. \tag{EC.158}$$

Set $LHS(\mu)$ and $RHS^\lambda(\mu)$ equal to the left-hand side and right-hand side of the above display, respectively. From (16), the limiting FOC is given by

$$c'(\mu) = p\left(1 - \left[1-\frac{a}{\mu}\right]^+\right) + (v-p\mu)\frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^+. \tag{EC.159}$$

Recall the definition of $h(\mu)$ from (EC.93) in Section EC.7.3:

$$h(\mu) := \frac{a^2}{\mu^2}\left(p+\frac{v}{a}-\frac{v}{\mu}\right) = p\frac{a}{\mu} + (v-p\mu)\frac{a}{\mu^2}\left(1-\frac{a}{\mu}\right), \quad \mu > 0.$$

Then, (EC.159) can be written as

$$c'(\mu) = \begin{cases} p, & \text{when } \mu < a, \\ h(\mu), & \text{when } \mu \ge a. \end{cases} \tag{EC.160}$$

Set $RHS(\mu)$ equal to the right-hand side of the above display. Observe that LHS, RHS, and RHS^λ are all continuous functions of μ for $\mu \in (0,\infty)$.

In order to analyze the convergence of prelimit equilibria (i.e., solutions to (EC.158)) to limiting equilibria (i.e., solutions to (EC.159)), we rely on the uniform convergence of RHS^λ to RHS. Formally, we need the next lemma, whose proof appears at the end.

Lemma EC.34. *$RHS^\lambda(\mu)$ converges uniformly on $[0,\infty)$ to $RHS(\mu)$.*

Proof of Theorem 5 (a): $n_c = n_o = 0$ implies that $LHS(\mu) \neq RHS(\mu)$ for all $\mu \in (0, a]$. Moreover, $LHS(0) = c'(0) \neq p = RHS(0)$. Therefore, $LHS(\mu) \neq RHS(\mu)$ for all $\mu \in [0, a]$. Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(\mu) \neq RHS^\lambda(\mu)$ for all $\mu \in [0, a]$, implying that there does not exist a solution to the prelimit FOC (EC.158) in $(0, a]$, i.e., $n_o^\lambda = 0$.

Proof of Theorem 5 (b): Recall, from Lemma 3, that $\mu^\star_{\max}(p, v)$ is a finite, constant upper bound on pre-limit equilibria for all λ. $n_u = n_c = 0$ implies that $LHS(\mu) \neq RHS(\mu)$ for all $\mu \in [a, \mu^\star_{\max}(p, v)]$. Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(\mu) \neq RHS^\lambda(\mu)$ for all $\mu \in [a, \mu^\star_{\max}(p, v)]$, implying that there does not exist a solution to the prelimit FOC (EC.158) in $(a, \mu^\star_{\max}(p, v)]$, i.e., $n_u^\lambda = 0$.

Proof of Theorem 5 (c): $n_c = 0$ implies that $LHS(a) \neq RHS(a)$. Then, by continuity of LHS and RHS, there exists $\delta \in (0, a]$ such that $LHS(\mu) \neq RHS(\mu)$ for all $\mu \in [a - \delta, a + \delta]$. Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(\mu) \neq RHS^\lambda(\mu)$ for all $\mu \in [a - \delta, a + \delta]$. Moreover, $\lim_{\lambda\to\infty} \frac{\lambda}{N^\lambda} = a$ implies that for all large enough λ, $\frac{\lambda}{N^\lambda} \in [a - \delta, a + \delta]$. Therefore, for all large enough λ, the critically loaded service rate $\frac{\lambda}{N^\lambda}$ is not a solution to the prelimit FOC (EC.158), i.e., $n_c^\lambda = 0$.

Proof of Theorem 5 (d): First, it follows from Theorem 4 (a) and (b)(iii)(iv) that when $p < c'(a)$, $n_u = 1$ implies that (i) $p < p^\dagger(v)$, and (ii) $a = \overline{a}$, or, equivalently, from Definition EC.3, $v = v^\dagger$. Next, recall, from Definition EC.1 and Remark EC.1, that when $p < c'(a)$, $\mu^\dagger \in \left(a, \frac{3}{2}a\right)$ is the unique limiting underloaded equilibrium for which $c'(\mu^\dagger)$ (i.e., $LHS(\mu^\dagger)$) and $h(\mu^\dagger)$ (i.e., $RHS(\mu^\dagger)$) are tangent and $v^\dagger$ is the unique value of v for which this phenomenon occurs; see Figure EC.5 (I) for an illustration.

In order to prove Theorem 5 (d), it suffices to show that if $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} < 0$, then there exists some $\delta \in (0, \mu^\dagger - a)$ such that $RHS^\lambda(\mu) - RHS(\mu) < 0$ for all $\mu \in [\mu^\dagger - \delta, \mu^\dagger + \delta]$ for all large enough λ. This is because, from Lemma EC.24 (noting that, when $\mu \geq a$, $c'(\mu)$ and $h(\mu)$ are the same as $LHS(\mu)$ and $RHS(\mu)$ respectively), it would then follow that

- $LHS(\mu) \geq RHS(\mu) > RHS^\lambda(\mu)$ for all $\mu \in [\mu^\dagger - \delta, \mu^\dagger + \delta]$ for all large enough λ; and
- $LHS(\mu) > RHS(\mu)$ for all $\mu \in [a, \mu^\dagger - \delta] \cup [\mu^\dagger + \delta, \infty)$, implying that, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), $LHS(\mu) > RHS^\lambda(\mu)$ for all $\mu \in [a, \mu^\dagger - \delta] \cup [\mu^\dagger + \delta, \infty)$ for all large enough λ.

To proceed, we need the following result, whose proof appears at the end.

CLAIM EC.14. *If* $\lim_{\lambda\to\infty} N^\lambda - \frac{\lambda}{a} < 0$, *then there exists* $\delta \in (0, \mu^\dagger - a)$ *such that the following hold for all* $\mu \in [\mu^\dagger - \delta, \mu^\dagger + \delta]$, *where* $\mu^\dagger$ *is defined in Definition EC.1 (a):*

(i) $I^\lambda(\mu, \mu) - \left(1 - \frac{a}{\mu}\right) < 0$ *for all large enough* λ, *and*

(ii) $\lim_{\lambda\to\infty} \frac{I^{\lambda}(\mu,\mu)^2 ErlC\left(N^{\lambda},\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^{\lambda}-N^{\lambda}} i\left(\frac{\lambda}{N^{\lambda}\mu}\right)^i}{N^{\lambda}\left(\left(1-\frac{a}{\mu}\right)-I^{\lambda}(\mu,\mu)\right)} = 0.$

Claim EC.14 guarantees the existence of some $\delta_1 \in (0, \mu^{\dagger} - a)$ such that for all $\mu \in [\mu^{\dagger} - \delta_1, \mu^{\dagger} + \delta_1]$, the statements Claim EC.14 (i)(ii) are true. In addition, the following statements are also true:

- $\mu^{\dagger} - \delta_1 > \frac{\lambda}{N^{\lambda}}$ for all large enough λ, and
- $\mu^{\dagger} + \delta_1 < 2\mu^{\dagger} - a < 2\left(\frac{3}{2}a\right) - a = 2a < \frac{v^{\dagger}}{p}$, where the last two inequalities follow from Lemma EC.23 and Remark EC.2, respectively.

In other words, for all large enough λ, in addition to the two statements guaranteed by Claim EC.14 (i)(ii), it is also true that

$$\frac{\lambda}{N^{\lambda}} < \mu < \frac{v^{\dagger}}{p} \quad \forall \mu \in [\mu^{\dagger} - \delta_1, \mu^{\dagger} + \delta_1]. \tag{EC.161}$$

Next, by definition of RHS^{λ} and RHS,

$$RHS^{\lambda}(\mu) - RHS(\mu) = -p\left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right] + \left(\frac{v^{\dagger}}{\mu} - p\right)\left[\mu\,\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right]. \tag{EC.162}$$

Using (EC.19) from Corollary EC.1, we can evaluate the term multiplying $\left(\frac{v^{\dagger}}{\mu} - p\right)$ in the above display as follows:

$$\begin{aligned}
\mu\,\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\bigg|_{\mu_1=\mu} - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right) &= I^{\lambda}(\mu,\mu)\left(1 - I^{\lambda}(\mu,\mu)\right) + I^{\lambda}(\mu,\mu)^2 \frac{ErlC\left(N^{\lambda},\frac{\lambda}{\mu}\right)}{N^{\lambda}} \sum_{i=1}^{k^{\lambda}-N^{\lambda}} i\left(\frac{\lambda}{N^{\lambda}\mu}\right)^i - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\\
&= \left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]\left(\frac{a}{\mu} - I^{\lambda}(\mu,\mu)\right) + I^{\lambda}(\mu,\mu)^2 \frac{ErlC\left(N^{\lambda},\frac{\lambda}{\mu}\right)}{N^{\lambda}} \sum_{i=1}^{k^{\lambda}-N^{\lambda}} i\left(\frac{\lambda}{N^{\lambda}\mu}\right)^i\\
&= \left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]\left[\left(\frac{2a}{\mu} - 1\right) - \left(I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right)\right]\\
&\qquad + I^{\lambda}(\mu,\mu)^2 \frac{ErlC\left(N^{\lambda},\frac{\lambda}{\mu}\right)}{N^{\lambda}} \sum_{i=1}^{k^{\lambda}-N^{\lambda}} i\left(\frac{\lambda}{N^{\lambda}\mu}\right)^i.
\end{aligned} \tag{EC.163}$$

Substituting (EC.163) into (EC.162) yields

$$\begin{aligned}
RHS^{\lambda}(\mu) - RHS(\mu) &\le -p\left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]\\
&\quad + \left(\frac{v^{\dagger}}{\mu} - p\right)\left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]\left[\left(\frac{2a}{\mu} - 1\right) - \left(I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right)\right]\\
&\quad + \left(\frac{v^{\dagger}}{\mu} - p\right) I^{\lambda}(\mu,\mu)^2 \frac{ErlC\left(N^{\lambda},\frac{\lambda}{\mu}\right)}{N^{\lambda}} \sum_{i=1}^{k^{\lambda}-N^{\lambda}} i\left(\frac{\lambda}{N^{\lambda}\mu}\right)^i\\
&= \left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]\left[\frac{2a}{\mu}\left(\frac{v^{\dagger}}{\mu} - \frac{v^{\dagger}}{2a} - p\right) - \left(\frac{v^{\dagger}}{\mu} - p\right)\left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]\right]\\
&\quad + \left(\frac{v^{\dagger}}{\mu} - p\right) I^{\lambda}(\mu,\mu)^2 \frac{ErlC\left(N^{\lambda},\frac{\lambda}{\mu}\right)}{N^{\lambda}} \sum_{i=1}^{k^{\lambda}-N^{\lambda}} i\left(\frac{\lambda}{N^{\lambda}\mu}\right)^i.
\end{aligned}$$

Claim EC.14 (i) guarantees that $\left(1-\frac{a}{\mu}\right) - I^{\lambda}(\mu,\mu) > 0$; dividing throughout by this term yields

$$\frac{RHS^{\lambda}(\mu) - RHS(\mu)}{\left(1-\frac{a}{\mu}\right) - I^{\lambda}(\mu,\mu)} \le -\frac{2a}{\mu}\left(\frac{v^{\dagger}}{\mu} - \frac{v^{\dagger}}{2a} - p\right) + \left(\frac{v^{\dagger}}{\mu} - p\right)\left[I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right]$$

$$+\left(\frac{v^\dagger}{\mu}-p\right)\frac{I^\lambda(\mu,\mu)^2 ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda} i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{N^\lambda\left(\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\right)}$$

$$\overset{(*)}{<}-\frac{2a}{\mu}\left(\frac{v^\dagger}{\mu}-\frac{v^\dagger}{2a}-p\right)+\left(\frac{v^\dagger}{\mu}-p\right)\frac{I^\lambda(\mu,\mu)^2 ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda} i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{N^\lambda\left(\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\right)},$$

where $(*)$ follows from Claim EC.14 (i). Taking the limit as $\lambda\to\infty$ and applying Claim EC.14 (ii), we obtain, for all $\mu\in[\mu^\dagger-\delta_1,\mu^\dagger+\delta_1]$,

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{RHS^\lambda(\mu)-RHS(\mu)}{\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)} &\le -\frac{2a}{\mu}\left(\frac{v^\dagger}{\mu}-\frac{v^\dagger}{2a}-p\right)\\ &=-\frac{2av^\dagger}{\mu}\left(\frac{1}{\mu}-\frac{\frac{2v^\dagger}{a}+2p}{3v^\dagger}\right)-\frac{a}{3\mu}\left(\frac{v^\dagger}{a}-2p\right)\\ &=-\frac{2av^\dagger}{\mu}f(\mu)-\frac{a}{3}g(\mu),\end{aligned}\tag{EC.164}$$

where $f(\mu)=\frac{1}{\mu}-\frac{\frac{2v^\dagger}{a}+2p}{3v^\dagger}$ and $g(\mu)=\frac{1}{\mu}\left(\frac{v^\dagger}{a}-2p\right)$. We resolve the signs of $f(\mu)$ and $g(\mu)$ as follows:

- From Lemma EC.23, we know that $f(\mu^\dagger)>0$, which, due to the continuity of f, implies that there exists a small enough $\delta\in(0,\delta_1)$ such that $f(\mu)>0$ for all $\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]$.
- From Remark EC.2, we know that $v^\dagger>2ap$, which implies that $g(\mu)>0$ for all $\mu>0$.

Therefore, (EC.164) implies that

$$\lim_{\lambda\to\infty}\frac{RHS^\lambda(\mu)-RHS(\mu)}{\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)}<0,\quad \forall\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta].$$

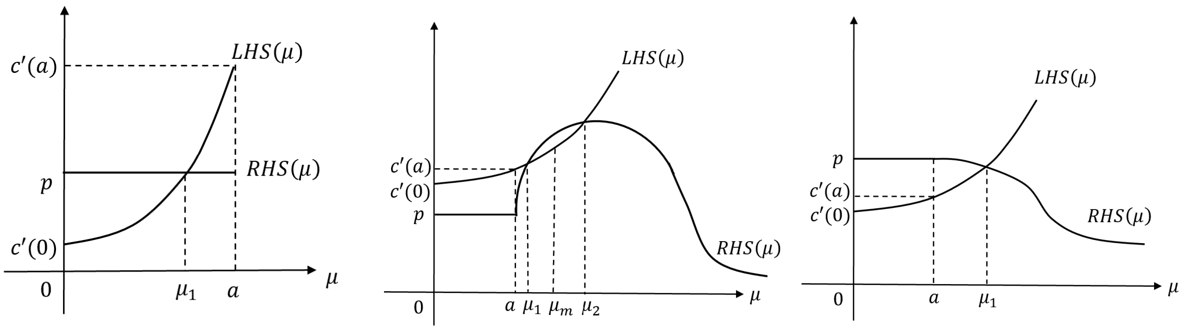
Then, using the guarantee from Claim EC.14 (i) that $\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)>0$ for all $\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]$ for all large enough λ, it follows that

$$RHS^\lambda(\mu)-RHS(\mu)<0,\quad\forall\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]\quad\text{for all large enough }\lambda,$$

as desired.

Proof of Theorem 6 (a): From Theorem 4 (a) and (b)(i), $n_o=1$ if and only if $c'(0)<p<c'(a)$ (Figure EC.13 (a)); that is, $LHS(0)=c'(0)<p=RHS(0)$ and $LHS(a)=c'(a)>p=RHS(a)$. Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(0)<RHS^\lambda(0)$ and $LHS(a)>RHS^\lambda(a)$; the intermediate value theorem then guarantees the existence of at least one solution to the prelimit FOC (EC.158) in $(0,a)$, i.e., $n_o^\lambda\ge 1$.

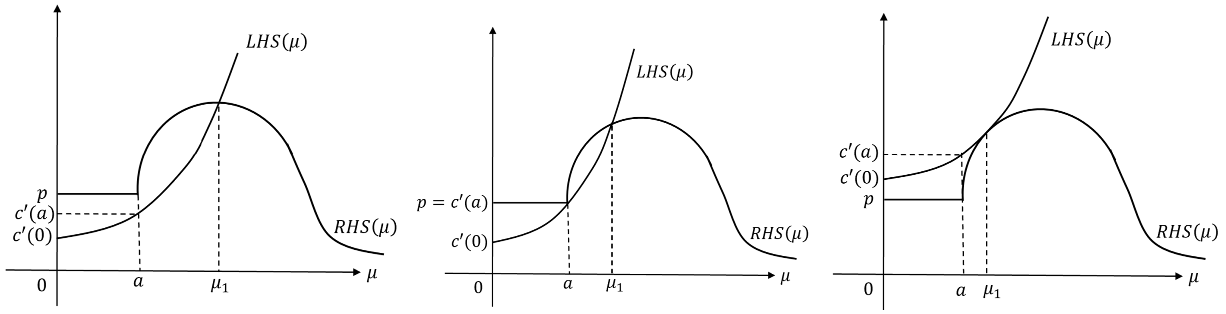
Proof of Theorem 6 (b)(i): From Theorem 4 (a) and (b)(iii)(iv), $n_u=2$ only if $LHS(a)=c'(a)>p=RHS(a)$ (Figure EC.13 (b)). Then, because there is more than one underloaded equilibrium, Lemmas EC.22 and EC.24 together imply that LHS and RHS cannot be tangent at any $\mu\in(a,\infty)$. Let $\mu_2>\mu_1>a$ be the two distinct limiting underloaded equilibria that solve $LHS(\mu)=RHS(\mu)$. Since $\lim_{\mu\to\infty}LHS(\mu)=\infty>0=\lim_{\mu\to\infty}RHS(\mu)$, it must be that $LHS(\mu)>RHS(\mu)$ for all $\mu>\mu_2$, and in particular, for some $\overline{\mu}>\mu_2$. Together with $LHS(a)>RHS(a)$, this means that



(a) Theorem 6 (a)

(b) Theorem 6 (b)(i)

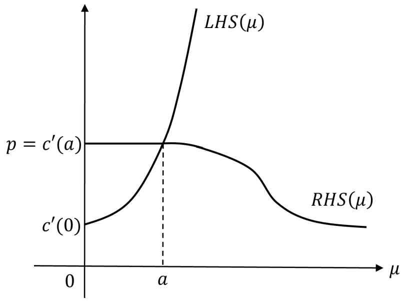
(c) Theorem 6 (b)(ii)



(d) Theorem 6 (b)(ii)

(e) Theorem 6 (b)(ii)

(f) Theorem 6 (b)(iii)



(g) Theorem 6 (c)

Figure EC.13 Existence of the solutions to the prelimit FOC (EC.158) in Theorem 6.

there must exist some $\mu_m \in (\mu_1, \mu_2)$ for which $LHS(\mu_m) < RHS(\mu_m)$. Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(a) > RHS^\lambda(a)$, $LHS(\mu_m) < RHS^\lambda(\mu_m)$, and $LHS(\overline{\mu}) > RHS^\lambda(\overline{\mu})$; the intermediate value theorem then guarantees the existence of at least two solutions to the prelimit FOC (EC.158) in $(a, \overline{\mu})$, i.e., $n_u^\lambda \geq 2$.

Proof of Theorem 6 (b)(ii): Here, we are given that $p \geq c'(a)$ and $n_u = 1$ (Figures EC.13 (c)-(e)). Let $\mu_1 > a$ denote the unique limiting underloaded equilibrium. Recall, from Lemmas EC.22 and EC.24 (noting that, when $\mu \geq a$, $c'(\mu)$ and $h(\mu)$ are the same as $LHS(\mu)$ and $RHS(\mu)$ respectively), that $LHS(\mu)$ and $RHS(\mu)$ are tangent for some $\mu \geq a$ if and only if $p \leq c'(a)$ and $\mu = \mu^\dagger(a,p)$; moreover, when this happens, there are no solutions to $LHS(\mu) = RHS(\mu)$ in $[a, \infty)$ other than $\mu^\dagger$. Also recall, from Remark EC.1, that $p = c'(a)$ if and only if $\mu^\dagger = a$. Therefore, we can conclude that if $p \geq c'(a)$, then $LHS(\mu)$ and $RHS(\mu)$ are never tangent at any $\mu \in (a, \infty)$, and in particular, at μ_1. Since $\lim_{\mu\to\infty} LHS(\mu) = \infty > 0 = \lim_{\mu\to\infty} RHS(\mu)$, it must be that $LHS(\mu) > RHS(\mu)$ for all $\mu > \mu_1$, and in particular, for some $\overline{\mu} > \mu_1$. Therefore, there exists a small enough $\delta \in (0, \mu_1 - a)$ such that $LHS(\mu_1 - \delta) < RHS(\mu_1 - \delta)$. Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(\mu_1 - \delta) < RHS^\lambda(\mu_1 - \delta)$ and $LHS(\overline{\mu}) > RHS^\lambda(\overline{\mu})$; the intermediate value theorem then guarantees the existence of at least one solution to the prelimit FOC (EC.158) in $(\mu_1 - \delta, \overline{\mu})$, i.e., $n_u^\lambda \geq 1$.

Proof of Theorem 6 (b)(iii): The proof technique is similar to that of Theorem 5 (d).

First, it follows from Theorem 4 (a) and (b)(iii)(iv) that when $p < c'(a)$, $n_u = 1$ implies that (i) $p < p^\dagger(v)$, and (ii) $a = \overline{a}$, or, equivalently, from Definition EC.3, $v = v^\dagger$. Next, recall, from Definition EC.1 and Remark EC.1, that when $p < c'(a)$, $\mu^\dagger \in \left(a, \frac{3}{2}a\right)$ is the unique limiting underloaded equilibrium for which $c'(\mu^\dagger)$ (i.e., $LHS(\mu^\dagger)$) and $h(\mu^\dagger)$ (i.e., $RHS(\mu^\dagger)$) are tangent and $v^\dagger$ is the unique value of v for which this phenomenon occurs; see Figure EC.5 (I) for an illustration. Also recall, from Remark EC.2, that $v^\dagger > 2ap$.

In order to prove Theorem 6 (b)(iii), it suffices to show that if $N^\lambda - \frac{\lambda}{a} \geq 0$ for all large enough λ, then $RHS^\lambda(\mu^\dagger) - RHS(\mu^\dagger) > 0$ for all large enough λ. This is because, from Lemmas EC.24 (noting that, when $\mu \geq a$, $c'(\mu)$ and $h(\mu)$ are the same as $LHS(\mu)$ and $RHS(\mu)$ respectively), it would then follow that

- $LHS(\mu) > RHS(\mu)$ for all $\mu > \mu^\dagger$, and in particular, for some $\overline{\mu} > \mu^\dagger$;
- $LHS(\mu^\dagger) = RHS(\mu^\dagger) < RHS^\lambda(\mu^\dagger)$ for all large enough λ; and
- $LHS(a) = c'(a) > p = RHS(a)$.

Then, due to the uniform convergence of RHS^λ to RHS (from Lemma EC.34), for all large enough λ, $LHS(a) > RHS^\lambda(a)$ and $LHS(\overline{\mu}) > RHS^\lambda(\overline{\mu})$; together with $LHS(\mu^\dagger) < RHS^\lambda(\mu^\dagger)$, the intermediate value theorem then guarantees the existence of at least two solutions to the prelimit FOC (EC.158) in $(a, \overline{\mu})$, i.e., $n_u^\lambda \geq 2$.

First, we observe that $\mu^\dagger < \left(\frac{3}{2}\right) a < \frac{3}{4}\frac{v^\dagger}{p} < \frac{v^\dagger}{p}$, where the first two inequalities follow from Lemma EC.23 and Remark EC.2, respectively. In other words, we have

$$\frac{v^\dagger}{\mu^\dagger} - p > 0. \tag{EC.165}$$

Next, by definition of RHS^λ and RHS,

$$RHS^\lambda(\mu^\dagger) - RHS(\mu^\dagger) = -p\left[I^\lambda(\mu^\dagger, \mu^\dagger) - \left(1 - \frac{a}{\mu^\dagger}\right)\right] + \left(\frac{v^\dagger}{\mu^\dagger} - p\right)\left[\mu^\dagger \left.\frac{\partial I^\lambda(\mu_1, \mu^\dagger)}{\partial \mu_1}\right|_{\mu_1 = \mu^\dagger} - \frac{a}{\mu^\dagger}\left(1 - \frac{a}{\mu^\dagger}\right)\right]. \tag{EC.166}$$

Using (EC.19) from Corollary EC.1, we can infer that $\mu \left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} \geq I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right)$ for all $\mu \in (0,\infty)$ and for all λ. This observation can be used to bound the term multiplying $\left(\frac{v^\dagger}{\mu^\dagger} - p\right)$ in the above display as follows:

$$\mu^\dagger \left.\frac{\partial I^\lambda(\mu_1, \mu^\dagger)}{\partial \mu_1}\right|_{\mu_1 = \mu^\dagger} - \frac{a}{\mu^\dagger}\left(1 - \frac{a}{\mu^\dagger}\right) \geq I^\lambda(\mu^\dagger, \mu^\dagger)\left(1 - I^\lambda(\mu^\dagger, \mu^\dagger)\right) - \frac{a}{\mu^\dagger}\left(1 - \frac{a}{\mu^\dagger}\right)$$

$$
\begin{aligned}
&= \left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]\left(\frac{a}{\mu^{\dagger}} - I^{\lambda}(\mu^{\dagger},\mu^{\dagger})\right)\\
&= \left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]\left[\left(\frac{2a}{\mu^{\dagger}}-1\right) - \left(I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right)\right]. \qquad \text{(EC.167)}
\end{aligned}
$$

Substituting (EC.167) into (EC.166) and given (EC.165), we obtain:

$$
\begin{aligned}
RHS^{\lambda}(\mu^{\dagger}) - RHS(\mu^{\dagger}) &\geq -p\left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]\\
&\quad + \left(\frac{v^{\dagger}}{\mu^{\dagger}} - p\right)\left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]\left[\left(\frac{2a}{\mu^{\dagger}}-1\right) - \left(I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right)\right]\\
&= \left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]\left[\frac{2a}{\mu^{\dagger}}\left(\frac{v^{\dagger}}{\mu^{\dagger}} - \frac{v^{\dagger}}{2a} - p\right) - \left(\frac{v^{\dagger}}{\mu^{\dagger}} - p\right)\left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]\right]. \qquad \text{(EC.168)}
\end{aligned}
$$

In what follows, we want to divide both sides by $I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)$. To single out the case when $I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right) = 0$, we consider a subsequence on which $k^{\lambda} = \infty$ for all large enough λ, and a subsequence on which $k^{\lambda} < \infty$ for all large enough λ, separately.

Case(I): If $k^{\lambda} = \infty$ for all large enough λ, then Lemma EC.8 implies that $I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right) = 0$ for all large enough λ (when $k^{\lambda} = \infty$). Moreover, from (EC.19) in Corollary EC.1, $\mu \left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} > I^{\lambda}(\mu,\mu)\,(1 - I^{\lambda}(\mu,\mu))$ for all $\mu \in (0,\infty)$ and for all large enough λ (when $k^{\lambda} = \infty$). This implies that (EC.168) holds with strict inequality for all large enough λ. Therefore, $RHS^{\lambda}(\mu^{\dagger}) - RHS(\mu^{\dagger}) > 0$ for all large enough λ.

Case(II): If $k^{\lambda} < \infty$ for all large enough λ, the proof proceeds as follows. Note that, for all large enough λ,

$$
\begin{aligned}
I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right) &= I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{\lambda}{N^{\lambda}\mu^{\dagger}}\right) + \left(\frac{a}{\mu^{\dagger}} - \frac{\lambda}{N^{\lambda}\mu^{\dagger}}\right)\\
&= I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{\lambda}{N^{\lambda}\mu^{\dagger}}\right) + \frac{a}{N^{\lambda}\mu^{\dagger}}\left(N^{\lambda} - \frac{\lambda}{a}\right) > 0, \qquad \text{(EC.169)}
\end{aligned}
$$

where $I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{\lambda}{N^{\lambda}\mu^{\dagger}}\right) \geq 0$ for all λ (from Lemma EC.8) and $N^{\lambda} - \frac{\lambda}{a} \geq 0$ for all large enough λ (by assumption). Then, (EC.168) implies that, for all large enough λ,

$$
\frac{RHS^{\lambda}(\mu^{\dagger}) - RHS(\mu^{\dagger})}{I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)} \geq \frac{2a}{\mu^{\dagger}}\left(\frac{v^{\dagger}}{\mu^{\dagger}} - \frac{v^{\dagger}}{2a} - p\right) - \left(\frac{v^{\dagger}}{\mu^{\dagger}} - p\right)\left[I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)\right]
$$

Taking the limit as $\lambda \to \infty$ and applying Lemma 5, we obtain

$$
\begin{aligned}
\lim_{\lambda\to\infty} \frac{RHS^{\lambda}(\mu^{\dagger}) - RHS(\mu^{\dagger})}{I^{\lambda}(\mu^{\dagger},\mu^{\dagger}) - \left(1-\frac{a}{\mu^{\dagger}}\right)} &\geq \frac{2a}{\mu^{\dagger}}\left(\frac{v^{\dagger}}{\mu^{\dagger}} - \frac{v^{\dagger}}{2a} - p\right)\\
&\overset{(i)}{>} \frac{2a}{\mu^{\dagger}}\left(v^{\dagger}\cdot\frac{2p+\frac{2v^{\dagger}}{a}}{3v^{\dagger}} - \frac{v^{\dagger}}{2a} - p\right) = \frac{2a}{3\mu^{\dagger}}\left(\frac{v^{\dagger}}{2a} - p\right)\\
&\overset{(ii)}{>} 0,
\end{aligned}
$$

where (i) follows from Lemma EC.23 and (ii) follows from Remark EC.2. Hence, recalling (EC.169), $RHS^{\lambda}(\mu^{\dagger}) - RHS(\mu^{\dagger}) > 0$ for all large enough λ.

Proof of Theorem 6 (c): From Theorem 4 (a) and (b)(ii), $n_c = 1$ if and only if $p = c'(a)$. Then, $LHS(0) = c'(0) < c'(a) = p = RHS(0)$. Moreover, since $\lim_{\mu\to\infty} LHS(\mu) = \infty > 0 = \lim_{\mu\to\infty} RHS(\mu)$, it must be that $LHS(\overline{\mu}) > RHS(\overline{\mu})$ for some large enough $\overline{\mu} > a$. Then, due to the uniform convergence of RHS^{λ} to RHS (from Lemma EC.34), for all large enough λ, $LHS(0) < RHS^{\lambda}(0)$ and $LHS(\overline{\mu}) > RHS^{\lambda}(\overline{\mu})$; the intermediate value theorem then guarantees the existence of at least one solution to the prelimit FOC (EC.158) in $(0,\overline{\mu})$, i.e., $n_u^{\lambda} + n_c^{\lambda} + n_o^{\lambda} \geq 1$.

■

Proof of Lemma EC.34: It suffices to show that $\lim_{\lambda\to\infty} \sup_{\mu\in[0,\infty)} |RHS^{\lambda}(\mu) - RHS(\mu)| = 0$. Recall from (EC.158) that

$$
\begin{aligned}
&\sup_{\mu\in[0,\infty)} |RHS^{\lambda}(\mu) - RHS(\mu)| \\
&= \sup_{\mu\in[0,\infty)} \left| -p\cdot I^{\lambda}(\mu,\mu) + (v-p\mu)\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} + p\left[1-\frac{a}{\mu}\right]^{+} - (v-p\mu)\frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+} \right| \\
&= \sup_{\mu\in[0,\infty)} \left| -p\left(I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right) + (v-p\mu)\left(\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+}\right) \right| \\
&\leq \sup_{\mu\in[0,\infty)} \left| -p\left(I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right)\right| + \left|(v-p\mu)\left(\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+}\right) \right| \\
&\leq \sup_{\mu\in[0,\infty)} \left| p\left(I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right)\right| + |v+p\mu|\left|\left(\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+}\right) \right| \\
&= \max\left\{ \sup_{\mu\in[0,a]} \left| p\left(I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right)\right|, \sup_{\mu\in[a,\infty)} \left| p\left(I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right)\right| \right\} \\
&\quad + \max\left\{ \sup_{\mu\in[0,a]} |v+p\mu|\left|\left(\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+}\right)\right|, \sup_{\mu\in[a,\infty)} \left|\frac{v}{\mu}+p\right|\left|\left(\mu\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left[1-\frac{a}{\mu}\right]^{+}\right)\right| \right\} \\
&\leq p\cdot\max\left\{ \sup_{\mu\in[0,a]} \left| I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right|, \sup_{\mu\in[a,\infty)} \left| I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right| \right\} \\
&\quad + \max\left\{ (v+pa)\sup_{\mu\in[0,a]} \left|\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+}\right|, \left(\frac{v}{a}+p\right)\sup_{\mu\in[a,\infty)} \left|\mu\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left[1-\frac{a}{\mu}\right]^{+}\right| \right\}.
\end{aligned}
$$

Thus, it suffices to show that all four suprema in the above display converge to zero, as $\lambda\to\infty$:

$$\text{(A)} \lim_{\lambda\to\infty} \sup_{\mu\in[0,a]} \left| I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right| = 0 \ \Leftrightarrow\ \lim_{\lambda\to\infty} \sup_{\mu\in[0,a]} \left| I^{\lambda}(\mu,\mu)\right| = 0$$

$$\text{(B)} \lim_{\lambda\to\infty} \sup_{\mu\in[0,a]} \left| \left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu^2}\left[1-\frac{a}{\mu}\right]^{+}\right| = 0 \ \Leftrightarrow\ \lim_{\lambda\to\infty} \sup_{\mu\in[0,a]} \left| \left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu}\right| = 0$$

$$\text{(C)} \lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| I^{\lambda}(\mu,\mu) - \left[1-\frac{a}{\mu}\right]^{+}\right| = 0 \ \Leftrightarrow\ \lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| I^{\lambda}(\mu,\mu) - \left(1-\frac{a}{\mu}\right)\right| = 0$$

$$\text{(D)} \lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| \mu\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left[1-\frac{a}{\mu}\right]^{+}\right| = 0 \ \Leftrightarrow\ \lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| \mu\left.\frac{\partial I^{\lambda}(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| = 0.$$

Proof of (A):

$$\lim_{\lambda\to\infty}\sup_{\mu\in[0,a]}\left|I^\lambda(\mu,\mu)\right| \overset{(i)}{=} \lim_{\lambda\to\infty}\sup_{\mu\in[0,a]} I^\lambda(\mu,\mu) \overset{(ii)}{=} \lim_{\lambda\to\infty} I^\lambda(a,a) \overset{(iii)}{=} 0,$$

where (i) follows from the fact that $I^\lambda(\mu,\mu)\ge 0$ for all $\lambda,\mu>0$; (ii) follows because $I^\lambda(\mu,\mu)\ge 0$ is strictly increasing in $\mu\in(0,\infty)$ for all λ by applying Lemma EC.4 (a) (N times iterating on each μ_i, $i\in[N]$); and (iii) follows from Lemma 5.

Proof of (B): The notation $I^\lambda(\mu_1,\mu)$ is a shorthand for $I(\mu_1,\mu;\lambda,k^\lambda,N^\lambda)$.

$$\begin{aligned}
&\lim_{\lambda\to\infty}\sup_{\mu\in[0,a]}\left|\left.\frac{\partial I(\mu_1,\mu;\lambda,k^\lambda,N^\lambda)}{\partial\mu_1}\right|_{\mu_1=\mu}\right|\\
\overset{(i)}{=}&\lim_{\lambda\to\infty}\sup_{\mu\in[0,a]}\left.\frac{\partial I(\mu_1,\mu;\lambda,k^\lambda,N^\lambda)}{\partial\mu_1}\right|_{\mu_1=\mu}\\
\overset{(ii)}{\le}&\lim_{\lambda\to\infty}\sup_{\mu\in[0,a]}\frac{I(\mu,\mu;\lambda,k^\lambda,N^\lambda)}{\mu}+\frac{2\sqrt{N^\lambda}}{\lambda}\\
\overset{(iii)}{\le}&\lim_{\lambda\to\infty}\sup_{\mu\in[0,a]}\frac{I(\mu,\mu;\lambda,N^\lambda,N^\lambda)}{\mu}+\frac{2\sqrt{N^\lambda}}{\lambda}\\
\overset{(iv)}{\le}&\lim_{\lambda\to\infty}\frac{I(a,a;\lambda,N^\lambda,N^\lambda)}{a}+\frac{2\sqrt{N^\lambda}}{\lambda}\\
\overset{(v)}{=}&0,
\end{aligned}$$

where (i) follows from the fact that $\left.\frac{\partial I(\mu_1,\mu;\lambda,k^\lambda,N^\lambda)}{\partial\mu_1}\right|_{\mu_1=\mu}\ge 0$ for all $\lambda,\mu>0$ (Lemma EC.4 (a)), (ii) follows from Corollary EC.2 (b), (iii) follows from Lemma EC.4 (b), (iv) follows from Lemma EC.10, noting that, for all large enough λ, $a<\frac{5\lambda}{4N^\lambda}$ and $N^\lambda\ge 6$, and (v) follows from Lemma 5 and because $\frac{\sqrt{N^\lambda}}{\lambda}\to 0$ as $\lambda\to\infty$.

Hence,

$$\lim_{\lambda\to\infty}\sup_{\mu\in[0,a]}\left|\left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}\right|=0.$$

Proof of (C): Consider a subsequence λ' on which $\lim_{\lambda'\to\infty}\operatorname{sgn}\left(N^{\lambda'}-\frac{\lambda'}{a}\right)$ is either ≥ 0 or <0. We simply use λ rather than λ' to denote the subsequence. We discuss the following two cases based on $\lim_{\lambda\to\infty}\operatorname{sgn}\left(N^\lambda-\frac{\lambda}{a}\right)$. For ease of presentation, let $f^\lambda(\mu):=I^\lambda(\mu,\mu)-\left(1-\frac{a}{\mu}\right)$ and $g^\lambda(\mu):=I^\lambda(\mu,\mu)-\left(1-\frac{\lambda}{N^\lambda\mu}\right)$ for $\mu\in(0,\infty)$.

Case (I): Suppose $\lim_{\lambda\to\infty}\operatorname{sgn}\left(N^\lambda-\frac{\lambda}{a}\right)\ge 0$, i.e., $N^\lambda-\frac{\lambda}{a}\ge 0$ for all $\lambda>\Lambda$ for some $\Lambda\in(0,\infty)$, then it follows that, for all $\mu\in(0,\infty)$,

$$f^\lambda(\mu)-g^\lambda(\mu)=\frac{a}{\mu}-\frac{\lambda}{N^\lambda\mu}=\frac{N^\lambda-\frac{\lambda}{a}}{N^\lambda}\frac{a}{\mu}\ge 0,\ \forall\lambda>\Lambda,$$

which implies that

$$f^\lambda(\mu)\ge g^\lambda(\mu)\ge 0,\ \forall\lambda>\Lambda,$$

where the last inequality follows from Lemma EC.8 for all λ. In addition, note that

$$\left(f^\lambda(\mu)\right)' - \left(g^\lambda(\mu)\right)' = -\frac{a}{\mu^2} + \frac{\lambda}{N^\lambda \mu^2} = -\frac{N^\lambda - \frac{\lambda}{a}}{N^\lambda}\frac{a}{\mu^2} \le 0,\ \forall \lambda > \Lambda,$$

which implies that

$$\left(f^\lambda(\mu)\right)' \le \left(g^\lambda(\mu)\right)' \le 0,\ \forall \lambda > \Lambda,$$

where the last inequality follows from Lemma EC.8 for all λ.

In summary, for all $\lambda > \Lambda$, $f^\lambda(\mu) \ge 0$ and is (not necessarily strictly) decreasing in μ for $\mu \in (0,\infty)$. Hence,

$$\sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu) - \left(1 - \frac{a}{\mu}\right)\right| = \sup_{\mu\in[a,\infty)} \left|f^\lambda(\mu)\right| = f^\lambda(a) = I^\lambda(a,a) - \left(1 - \frac{a}{a}\right),$$

which implies that

$$\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu) - \left(1 - \frac{a}{\mu}\right)\right| = \lim_{\lambda\to\infty} I^\lambda(a,a) = 0,$$

where the last step follows from Lemma 5.

Case (II): Suppose $\lim_{\lambda\to\infty} \operatorname{sgn}\left(N^\lambda - \frac{\lambda}{a}\right) < 0$, i.e., $N^\lambda - \frac{\lambda}{a} < 0$ for all $\lambda > \Lambda$ for some $\Lambda \in (0,\infty)$. For all $\lambda > \Lambda$,

$$\begin{aligned}
\sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu) - \left(1 - \frac{a}{\mu}\right)\right| &= \sup_{\mu\in[a,\infty)} \left| g^\lambda(\mu) + \left(1 - \frac{\lambda}{N^\lambda \mu}\right) - \left(1 - \frac{a}{\mu}\right)\right| \\
&= \sup_{\mu\in[a,\infty)} \left| g^\lambda(\mu) + \frac{1}{\mu}\left(a - \frac{\lambda}{N^\lambda}\right)\right| \\
&\le \sup_{\mu\in[a,\infty)} \left|g^\lambda(\mu)\right| + \sup_{\mu\in[a,\infty)} \frac{1}{\mu}\left|a - \frac{\lambda}{N^\lambda}\right| \\
&= g^\lambda(a) + \frac{1}{a}\left(\frac{\lambda}{N^\lambda} - a\right),
\end{aligned}$$

where the last inequality follows because $g^\lambda(\mu)$ is (not necessarily strictly) decreasing in $\mu \in (0,\infty)$ for all λ from Lemma EC.8, $\frac{1}{\mu}$ is strictly decreasing in μ, and also $a < \frac{\lambda}{N^\lambda}$ for all $\lambda > \Lambda$. Then,

$$\lim_{\lambda\to\infty} g^\lambda(a) + \frac{1}{a}\left(\frac{\lambda}{N^\lambda} - a\right) = \lim_{\lambda\to\infty} I^\lambda(a,a) - \left(1 - \frac{\lambda}{N^\lambda a}\right) + \left(\frac{\lambda}{N^\lambda a} - 1\right) = 0,$$

from Lemma 5 and because $\lim_{\lambda\to\infty} \frac{\lambda}{N^\lambda} = a$.

Hence, we conclude

$$\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu) - \left(1 - \frac{a}{\mu}\right)\right| = 0. \tag{EC.170}$$

Proof of (D):

$$\begin{aligned}
&\sup_{\mu\in[a,\infty)} \left| \mu \left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left(1 - \frac{a}{\mu}\right)\right| \\
\overset{(i)}{=} &\sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right) + I^\lambda(\mu,\mu)^2 \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{N^\lambda} \sum_{i=1}^{k^\lambda - N^\lambda} i \left(\frac{\lambda}{N^\lambda \mu}\right)^i - \frac{a}{\mu}\left(1 - \frac{a}{\mu}\right)\right|
\end{aligned}$$

$$\begin{aligned}
&\overset{(ii)}{\leq} \sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu)^2 \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{N^\lambda} \sum_{i=1}^{k^\lambda - N^\lambda} i \left(\frac{\lambda}{N^\lambda \mu}\right)^i \right| + \left| I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right) - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| \\
&= \sup_{\mu\in[a,\infty)} I^\lambda(\mu,\mu)^2 \frac{ErlC\left(N^\lambda, \frac{\lambda}{\mu}\right)}{N^\lambda} \sum_{i=1}^{k^\lambda - N^\lambda} i \left(\frac{\lambda}{N^\lambda \mu}\right)^i + \left| I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right) - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| \\
&\overset{(iii)}{\leq} \sup_{\mu\in[a,\infty)} \frac{2\sqrt{N^\lambda}}{N^\lambda} + \left| I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right) - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| \\
&= \frac{2}{\sqrt{N^\lambda}} + \sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right) - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right|,
\end{aligned}$$

where (i) follows from (EC.19) in Corollary EC.1, (ii) follows from the triangle inequality, and (iii) follows from Lemma EC.6 (a). Since $\frac{2}{\sqrt{N^\lambda}} \to 0$ as $\lambda \to \infty$, it suffices to show that the second term in the above display converges to 0 as $\lambda \to \infty$. From (EC.170),

$$\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu) - \left(1 - \frac{a}{\mu}\right)\right| = 0.$$

Let $\zeta(x) := x(1-x)$ for $x \in (0,1)$. Since $\zeta(x)$ is a continuous function,

$$\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| \zeta\left(I^\lambda(\mu,\mu)\right) - \zeta\left(1 - \frac{a}{\mu}\right)\right| = 0,$$

which can be equivalently written as

$$\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| I^\lambda\left(1 - I^\lambda(\mu,\mu)\right) - \frac{a}{\mu}\left(1 - \frac{a}{\mu}\right)\right| = \infty,$$

as required. Therefore,

$$\begin{aligned}
&\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| \mu \left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| \\
&\leq \lim_{\lambda\to\infty} \left\{ \frac{2}{\sqrt{N^\lambda}} + \sup_{\mu\in[a,\infty)} \left| I^\lambda(\mu,\mu)\left(1 - I^\lambda(\mu,\mu)\right) - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| \right\} = 0,
\end{aligned}$$

noting that $N^\lambda \to \infty$ as $\lambda \to \infty$.

Hence,

$$\lim_{\lambda\to\infty} \sup_{\mu\in[a,\infty)} \left| \mu \left.\frac{\partial I^\lambda(\mu_1,\mu)}{\partial \mu_1}\right|_{\mu_1=\mu} - \frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\right| = 0.$$

∎

Proof of Claim EC.14:

(i): By definition of $\mu^\dagger$ (in Definition EC.1 (a)), $\mu^\dagger > a$. Then, there exists $\delta \in (0, \mu^\dagger - a)$ such that $\mu > a$ for all $\mu \in [\mu^\dagger - \delta, \mu^\dagger + \delta]$. Then, for all large enough λ, $\mu > \frac{\lambda}{N^\lambda}$ for all $\mu \in [\mu^\dagger - \delta, \mu^\dagger + \delta]$.

Note that

$$\begin{aligned}
I^\lambda(\mu,\mu) - \left(1 - \frac{a}{\mu}\right) &= I^\lambda(\mu,\mu) - \left(1 - \frac{\lambda}{N^\lambda \mu}\right) + \left(\frac{a}{\mu} - \frac{\lambda}{N^\lambda \mu}\right) \\
&= I^\lambda(\mu,\mu) - \left(1 - \frac{\lambda}{N^\lambda \mu}\right) + \frac{a}{N^\lambda \mu}\left(N^\lambda - \frac{\lambda}{a}\right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(i)}{=} \frac{\left(1-\frac{\lambda}{N^\lambda\mu}\right)}{1-ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}} - \left(1-\frac{\lambda}{N^\lambda\mu}\right)+\frac{a}{N^\lambda\mu}\left(N^\lambda-\frac{\lambda}{a}\right)\\
&= \frac{\left(1-\frac{\lambda}{N^\lambda\mu}\right)}{1-ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}} ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}+\frac{a}{N^\lambda\mu}\left(N^\lambda-\frac{\lambda}{a}\right)\\
&\overset{(ii)}{=} I^\lambda(\mu,\mu)ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}+\frac{a}{N^\lambda\mu}\left(N^\lambda-\frac{\lambda}{a}\right),
\end{aligned}
$$

where (i) and (ii) follow from (EC.18) in Corollary EC.1. Thus, for all large enough λ, for all $\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]$,

$$
\begin{aligned}
I^\lambda(\mu,\mu)-\left(1-\frac{a}{\mu}\right) &= I^\lambda(\mu,\mu)ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}+\frac{a}{N^\lambda\mu}\left(N^\lambda-\frac{\lambda}{a}\right)\\
&\overset{(*)}{\le} I^\lambda(\mu,\mu)ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{N^\lambda\mu}\right)+\frac{a}{N^\lambda\mu}\left(N^\lambda-\frac{\lambda}{a}\right) \qquad \text{(EC.171)}\\
&= \frac{1}{N^\lambda}\left(\frac{\lambda}{a}-N^\lambda\right)\left[I^\lambda(\mu,\mu)\frac{\lambda}{N^\lambda\mu}\frac{ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}{\frac{1}{N^\lambda}\left(\frac{\lambda}{a}-N^\lambda\right)}-\frac{a}{\mu}\right],
\end{aligned}
$$

where $(*)$ follows from the fact that $k^\lambda\ge N^\lambda$ and $\frac{\lambda}{N^\lambda\mu}<1$ for all $\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]$, for all large enough λ. Since $\frac{\lambda}{a}-N^\lambda>0$ for all large enough λ (by assumption), it suffices to show that $I^\lambda(\mu,\mu)\frac{\lambda}{N^\lambda\mu}\frac{ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}{\frac{1}{N^\lambda}\left(\frac{\lambda}{a}-N^\lambda\right)}<\frac{a}{\mu}$ for all large enough λ, for which, in turn, it suffices to show that $\lim_{\lambda\to\infty}I^\lambda(\mu,\mu)\frac{\lambda}{N^\lambda\mu}\frac{ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}{\frac{1}{N^\lambda}\left(\frac{\lambda}{a}-N^\lambda\right)}<\frac{a}{\mu}$. Note that

$$
\begin{aligned}
&\lim_{\lambda\to\infty}I^\lambda(\mu,\mu)\frac{\lambda}{N^\lambda\mu}\frac{ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}{\frac{1}{N^\lambda}\left(\frac{\lambda}{a}-N^\lambda\right)}=\frac{a}{\mu}\left(1-\frac{a}{\mu}\right)\left(\lim_{\lambda\to\infty}\frac{ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}{\frac{1}{N^\lambda}\left(\frac{\lambda}{a}-N^\lambda\right)}\right)<\frac{a}{\mu}\\
&\Leftrightarrow \lim_{\lambda\to\infty}\frac{\frac{\lambda}{a}-N^\lambda}{N^\lambda ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}>1-\frac{a}{\mu}, \qquad \text{(EC.172)}
\end{aligned}
$$

which is true because the left-hand side is $+\infty$, noting that $N^\lambda ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\to 0$ for all $\mu\ge\mu^\dagger-\delta>a$ (from Lemma EC.13 (b)) and $\lim_{\lambda\to\infty}\frac{\lambda}{a}-N^\lambda>0$ (by assumption).

(ii): From (EC.171), for all $\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]$ and for all large enough λ,

$$
\begin{aligned}
&\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\ge\frac{a}{N^\lambda\mu}\left(\frac{\lambda}{a}-N^\lambda\right)-I^\lambda(\mu,\mu)ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\frac{\lambda}{N^\lambda\mu}\\
&\Leftrightarrow\frac{I^\lambda(\mu,\mu)^2ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda}i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{N^\lambda\left(\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\right)}\le\frac{I^\lambda(\mu,\mu)^2ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda}i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{\frac{a}{\mu}\left(\frac{\lambda}{a}-N^\lambda\right)-I^\lambda(\mu,\mu)N^\lambda ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\frac{\lambda}{N^\lambda\mu}}. \qquad \text{(EC.173)}
\end{aligned}
$$

Taking the limit as $\lambda\to\infty$ and observing that $\lim_{\lambda\to\infty}\frac{\lambda}{N^\lambda\mu}=\frac{a}{\mu}$, we obtain, for all $\mu\in[\mu^\dagger-\delta,\mu^\dagger+\delta]$,

$$
\lim_{\lambda\to\infty}\frac{I^\lambda(\mu,\mu)^2ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda}i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{N^\lambda\left(\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\right)}\le\lim_{\lambda\to\infty}\frac{\frac{\mu}{a}I^\lambda(\mu,\mu)^2ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda}i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{\left(\frac{\lambda}{a}-N^\lambda\right)-I^\lambda(\mu,\mu)N^\lambda ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}
$$

$$\overset{(*)}{=} \frac{\left(\lim_{\lambda\to\infty}\frac{1}{N^\lambda}\right)\left(\frac{\mu}{a}\left(1-\frac{a}{\mu}\right)^2\lim_{\lambda\to\infty}\sum_{i=1}^{k^\lambda-N^\lambda} i\left(\frac{\lambda}{N^\lambda\mu}\right)^i\right)}{\left(\lim_{\lambda\to\infty}\frac{\frac{\lambda}{a}-N^\lambda}{N^\lambda ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}-\left(1-\frac{a}{\mu}\right)\right)}$$
$$= \frac{T_1T_2}{T_3}, \tag{EC.174}$$

where $(*)$ follows because $I^\lambda(\mu,\mu)\to 1-\frac{a}{\mu}$ as $\lambda\to\infty$ (from Lemma 5), $T_1=\lim_{\lambda\to\infty}\frac{1}{N^\lambda}$, $T_2=\frac{\mu}{a}\left(1-\frac{a}{\mu}\right)^2\lim_{\lambda\to\infty}\sum_{i=1}^{k^\lambda-N^\lambda} i\left(\frac{\lambda}{N^\lambda\mu}\right)^i$, and $T_3=\lim_{\lambda\to\infty}\frac{\frac{\lambda}{a}-N^\lambda}{N^\lambda ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)}-\left(1-\frac{a}{\mu}\right)$. Clearly, $T_3>0$ from (EC.172) and $T_1=0$. Using the finite summation formula,

$$\begin{aligned}0\le T_2 &= \frac{\mu}{a}\left(1-\frac{a}{\mu}\right)^2\lim_{\lambda\to\infty}\frac{\frac{\lambda}{N^\lambda\mu}}{\left(1-\frac{\lambda}{N^\lambda\mu}\right)^2}\left[1-\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda}\left(1+(k^\lambda-N^\lambda)\left(1-\frac{\lambda}{N^\lambda\mu}\right)\right)\right]\\ &= \lim_{\lambda\to\infty}\left[1-\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda}\left(1+(k^\lambda-N^\lambda)\left(1-\frac{\lambda}{N^\lambda\mu}\right)\right)\right]\\ &<\infty,\end{aligned}$$

since $\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda}<1$ for all large enough λ and $\lim_{\lambda\to\infty}(k^\lambda-N^\lambda)\left(\frac{\lambda}{N^\lambda\mu}\right)^{k^\lambda-N^\lambda+1}<\infty$ because, even if $\lim_{\lambda\to\infty}k^\lambda-N^\lambda=\infty$, exponential decay would dominate linear growth in terms of $k^\lambda-N^\lambda$. As a result, (EC.174) implies that

$$\lim_{\lambda\to\infty}\frac{I^\lambda(\mu,\mu)^2 ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda} i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{N^\lambda\left(\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\right)}\le 0.$$

However, the left-hand side is non-negative, since $I^\lambda(\mu,\mu)-\left(1-\frac{a}{\mu}\right)<0$ for all large enough λ from Claim EC.14 (i). Therefore, it must be that

$$\lim_{\lambda\to\infty}\frac{I^\lambda(\mu,\mu)^2 ErlC\left(N^\lambda,\frac{\lambda}{\mu}\right)\sum_{i=1}^{k^\lambda-N^\lambda} i\left(\frac{\lambda}{N^\lambda\mu}\right)^i}{N^\lambda\left(\left(1-\frac{a}{\mu}\right)-I^\lambda(\mu,\mu)\right)}=0.$$

∎

EC.8. Proofs from Section 6

EC.8.1. Proof of Proposition 9

Under utility function (20), the FOC (6), when $k=N$, $p=0$, $v=1$ and $\mu_1=\mu$, is given by

$$\mu c'(\mu)=\alpha I(\mu,\mu)^{\alpha-1}\left.\frac{\partial I(\mu_1,\mu)}{\partial\mu_1}\right|_{\mu_1=\mu}=\alpha I(\mu,\mu)^\alpha(1-I(\mu,\mu)). \tag{EC.175}$$

recalling Corollary EC.3 (b). Note that the derivative of the right-hand side of (EC.175) is

$$\alpha I(\mu,\mu)^{\alpha-1}\left(I(\mu,\mu)\right)'\left(\alpha(1-I(\mu,\mu))-I(\mu,\mu)\right).$$

Since $(I(\mu,\mu))'>0$ (either by directly calculating $\frac{dI(\mu,\mu)}{d\mu}$ using Corollary EC.3 (a) or by applying Lemma EC.4 (a) twice), the above display is strictly positive when $I(\mu,\mu)\in\left(0,\frac{\alpha}{\alpha+1}\right)$ and strictly negative when $I(\mu,\mu) \in \left(\frac{\alpha}{\alpha+1}, 1\right)$. This implies that the right-hand side of (EC.175) is maximized when $I(\mu,\mu) = \frac{\alpha}{\alpha+1}$, and the associated maximum value is $\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$.

Moreover, since the left-hand side of (EC.175), $\mu c'(\mu)$, is a strictly increasing function of μ (recalling that c is strictly convex), the maximum candidate server equilibrium $\mu^{\star ?}_{\max}$, which satisfies (EC.175), is attained when $\mu^{\star ?}_{\max} c'(\mu^{\star ?}_{\max}) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$, that is, $\mu^{\star ?}_{\max}$ is the unique solution to

$$\mu c'(\mu) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

and satisfies

$$I(\mu^{\star ?}_{\max}, \mu^{\star ?}_{\max}) = \frac{\alpha}{\alpha+1}. \tag{EC.176}$$

To complete the proof, it remains to be shown that $\mu_1 = \mu^{\star ?}_{\max}$ is a local maximum of $U\left(\mu_1, \mu^{\star ?}_{\max}\right)$ for all $\alpha > 0$ and a global maximum when $\alpha \in (0,1]$. For this, we investigate the second partial derivative of the generalized utility function $U\left(\mu_1, \mu\right)$ with respect to μ_1. For ease of presentation, we denote $I(\mu_1,\mu)$, $\frac{\partial I(\mu_1,\mu)}{\partial \mu_1}$ and $\frac{\partial^2 I(\mu_1,\mu)}{\partial \mu_1^2}$ simply by I, I' and I''. Note that

$$(I^\alpha)'' = \left(\alpha I^{\alpha-1} I'\right)' = \alpha(\alpha-1) I^{\alpha-2} \left(I'\right)^2 + \alpha I^{\alpha-1} I'' = \alpha I^{\alpha-2} \left\{ (\alpha-1) \left(I'\right)^2 + I I'' \right\}. \tag{EC.177}$$

From Lemma EC.9 (b)(c),

$$I' = \frac{I(1-I)}{\mu_1} \qquad \text{and} \qquad I'' = -\frac{2I^2(1-I)}{\mu_1^2} = -\frac{2II'}{\mu_1}.$$

Substituting these expressions for I' and I'' into (EC.177), we get

$$(I^\alpha)'' = \alpha I^{\alpha-2} I' \left((\alpha-1) \frac{I(1-I)}{\mu_1} - \frac{2I^2}{\mu_1} \right) = -\frac{\alpha(\alpha+1) I^{\alpha-1} I'}{\mu_1} \left(I - \frac{\alpha-1}{\alpha+1} \right). \tag{EC.178}$$

From (EC.176) and (EC.178), it follows that

$$\begin{aligned}
&\frac{\partial^2}{\partial \mu_1^2} \left(I(\mu_1, \mu^{\star ?}_{\max})^\alpha \right) \Big|_{\mu_1 = \mu^{\star ?}_{\max}} \\
&= -\frac{\alpha(\alpha+1) I(\mu^{\star ?}_{\max}, \mu^{\star ?}_{\max})^{\alpha-1} I'(\mu^{\star ?}_{\max}, \mu^{\star ?}_{\max})}{\mu_1} \left(I(\mu^{\star ?}_{\max}, \mu^{\star ?}_{\max}) - \frac{\alpha-1}{\alpha+1} \right) \\
&= -\frac{\alpha(\alpha+1) I(\mu^{\star ?}_{\max}, \mu^{\star ?}_{\max})^{\alpha-1} I'(\mu^{\star ?}_{\max}, \mu^{\star ?}_{\max})}{\mu_1} \left(\frac{\alpha}{\alpha+1} - \frac{\alpha-1}{\alpha+1} \right) < 0, \quad \forall \alpha > 0.
\end{aligned}$$

Recalling that $c''(\mu) > 0$ for all $\mu > 0$ (since c is strictly convex),

$$\frac{\partial^2 U(\mu_1, \mu^{\star ?}_{\max})}{\partial \mu_1^2} \Big|_{\mu_1 = \mu^{\star ?}_{\max}} = \frac{\partial^2}{\partial \mu_1^2} \left(I(\mu_1, \mu^{\star ?}_{\max})^\alpha \right) \Big|_{\mu_1 = \mu^{\star ?}_{\max}} - c''(\mu^{\star ?}_{\max}) < 0,$$

which implies that $\mu_1 = \mu^{\star ?}_{\max}$ is a local maximum of $U\left(\mu_1, \mu^{\star ?}_{\max}\right)$. In particular, when $\alpha \in (0,1]$, $(EC.178) \le 0$ because $\frac{\alpha-1}{\alpha+1} \le 0$, implying that $\frac{\partial^2 U(\mu_1,\mu)}{\partial \mu_1^2} < 0$ for all $\mu_1, \mu > 0$, i.e., the utility function is concave and $\mu^{\star ?}_{\max}$ is a global maximum, for $\alpha \in (0,1]$. Therefore, when $\alpha \in (0,1]$, $\mu^\star > 0$ is an equilibrium if and only if it satisfies the FOC (EC.175).

∎