Electronic Companion

EC.1. Proofs of main results

EC.1.1. Proof of Theorem 1

Proof **Sufficiency:** Suppose the KKT conditions (5) hold and show that the UE conditions (3) are satisfied.

We first note that the problem becomes trivial if there is only one path. If there is more than one path, we can reformulate Eq. (5a) as

$$c_k^w - c_{k+1}^w + \theta_k^w(\pi_k^w - \pi_{k+1}^w) = \frac{\lambda_k^w - \lambda_{k+1}^w}{q_w g(\theta_k^w)}.$$

Suppose $f_k^w > 0$ for any given $k \in K_w, \forall w \in W$, and note that this implies $\theta_k^w > \theta_{k-1}^w$, i.e., $\mathcal{T}_k^w \neq \emptyset$. We shall show path k has the minimum generalized cost of all, i.e., $e_k^w(\theta) \leq e_l^w(\theta), \forall l \in K_w, l \neq k$ for any $\theta \in \mathcal{T}_k^w$. First note $\pi_k^w \geq \pi_{k+1}^w$ per the ranking of path tolls and $\theta \leq \theta_k^w, \forall \theta \in \mathcal{T}_k^w$ per definition. This implies

$$c_k^w - c_{k+1}^w + \theta(\pi_k^w - \pi_{k+1}^w) \leq c_k^w - c_{k+1}^w + \theta_k^w(\pi_k^w - \pi_{k+1}^w) = \frac{\lambda_k^w - \lambda_{k+1}^w}{q_w g(\theta_k^w)}. \tag{EC.1}$$

Given $f_k^w > 0$, Eq. (5b) leads to $\lambda_k^w = 0$. Since $\lambda_{k+1}^w \geq 0, \forall w \in W, k \in K_w$, we have

$$c_k^w + \theta\pi_k^w \leq c_{k+1}^w + \theta\pi_{k+1}^w \rightarrow e_k^w(\theta) \leq e_{k+1}^w(\theta), \forall \theta \in \mathcal{T}_k^w.$$

We proceed to show this relationship also holds for any other paths $l \neq k$. Applying the KKT condition (5a) to path $k+1$ yields

$$c_{k+1}^w - c_{k+2}^w + \theta(\pi_{k+1}^w - \pi_{k+2}^w) \leq c_{k+1}^w - c_{k+2}^w + \theta_{k+1}^w(\pi_{k+1}^w - \pi_{k+2}^w) = \frac{\lambda_{k+1}^w - \lambda_{k+2}^w}{q_w g(\theta_{k+1}^w)}, \ \forall \theta \in \mathcal{T}_k^w. \tag{EC.2}$$

Adding Eq. (19) to Eq. (20) yields

$$c_k^w - c_{k+2}^w + \theta(\pi_k^w - \pi_{k+2}^w) \leq \frac{\lambda_k^w - \lambda_{k+1}^w}{q_w g(\theta_k^w)} + \frac{\lambda_{k+1}^w - \lambda_{k+2}^w}{q_w g(\theta_{k+1}^w)}, \quad \forall \theta \in \mathcal{T}_k^w. \tag{EC.3}$$

There are two possibilities: path $k+1$ is not used, or path $k+1$ is used. In the first case, $\theta_k^w = \theta_{k+1}^w, g(\theta_k^w) = g(\theta_{k+1}^w)$, thus the above inequality becomes

$$c_k^w + \theta\pi_k^w - (c_{k+2}^w + \theta\pi_{k+2}^w) \leq \frac{\lambda_k^w - \lambda_{k+2}^w}{q_w g(\theta_k^w)}, \quad \forall \theta \in \mathcal{T}_k^w.$$

Given $\lambda_k^w = 0$ and $\lambda_{k+2}^w \geq 0$, it becomes

$$c_k^w + \theta\pi_k^w \leq c_{k+2}^w + \theta\pi_{k+2}^w \rightarrow e_k^w(\theta) \leq e_{k+2}^w(\theta), \quad \forall\theta \in \mathcal{T}_k^w.$$

If path $k+1$ is used, we have $\lambda_k^w = \lambda_{k+1}^w = 0$. Thus, Eq. (EC.3) still yields

$$e_k^w(\theta) \leq e_{k+2}^w(\theta), \quad \forall\theta \in \mathcal{T}_k^w.$$

The above derivation can be repeated for any path $k+1 < l \leq |K_w|$, leading to

$$c_k^w - c_l^w + \theta(\pi_k^w - \pi_l^w) \leq \frac{\lambda_k^w - \lambda_{k+1}^w}{q_w g(\theta_k^w)} + \frac{\lambda_{k+1}^w - \lambda_{k+2}^w}{q_w g(\theta_{k+1}^w)} + \cdots + \frac{\lambda_{l-1}^w - \lambda_l^w}{q_w g(\theta_{l-1}^w)} \leq 0, \quad \forall\theta \in \mathcal{T}_k^w,$$

whether these paths are used or not. Thus

$$e_k^w(\theta) \leq e_l^w(\theta), \forall\theta \in \mathcal{T}_k^w. \tag{EC.4}$$

Let's now consider path $l \leq k-1$. Applying the KKT condition (5a) to path $k-1$ yields

$$c_{k-1}^w - c_k^w + \theta_{k-1}^w(\pi_{k-1}^w - \pi_k^w) = \frac{\lambda_{k-1}^w - \lambda_k^w}{q_w g(\theta_{k-1}^w)}.$$

For any $\theta \in \mathcal{T}_k^w$, we have

$$c_{k-1}^w - c_k^w + \theta(\pi_{k-1}^w - \pi_k^w) \geq c_{k-1}^w - c_k^w + \theta_{k-1}^w(\pi_{k-1}^w - \pi_k^w) = \frac{\lambda_{k-1}^w - \lambda_k^w}{q_w g(\theta_{k-1}^w)}, \tag{EC.5}$$

by noting such θ is no less than θ_{k-1}^w and $\pi_{k-1}^w \geq \pi_k^w$ per definition. Since $\lambda_k^w = 0$ per the assumption, whether path $k-1$ is used or not, the right-hand side of Eq. (EC.5) is always non-negative. Thus, it can be rewritten as

$$c_k^w + \theta\pi_k^w \leq c_{k-1}^w + \theta\pi_{k-1}^w \rightarrow e_k^w(\theta) \leq e_{k-1}^w(\theta), \quad \forall\theta \in \mathcal{T}_k^w.$$

Similarly, repeating the above derivation for any path $1 \leq l < k-1$ yields

$$c_l^w - c_k^w + \theta(\pi_l^w - \pi_k^w) \geq \frac{\lambda_{k-1}^w - \lambda_k^w}{q_w g(\theta_{k-1}^w)} + \frac{\lambda_{k-2}^w - \lambda_{k-1}^w}{q_w g(\theta_{k-2}^w)} + \cdots + \frac{\lambda_l^w - \lambda_{l+1}^w}{q_w g(\theta_l^w)} \geq 0, \quad \forall\theta \in \mathcal{T}_k^w,$$

which implies

$$e_k^w(\theta) \leq e_l^w(\theta), \quad \forall\theta \in \mathcal{T}_k^w. \tag{EC.6}$$

Combining Eqs. (EC.4) and (EC.6) yields

$$e_k^w(\theta) \leq e_l^w(\theta), \quad \forall\theta \in \mathcal{T}_k^w \neq \emptyset,\ l \in K_w,\ l \neq k. \tag{EC.7}$$

This completes the proof of sufficiency.

Necessity: Suppose the UE conditions (3) hold, prove the KKT conditions (5) are satisfied.

We shall show that one can always find a feasible vector of multipliers $\boldsymbol{\lambda}$ such that a C-BiTA equilibrium flow pattern satisfies the KKT conditions. Let $\boldsymbol{\theta}$ be the vector of boundary TEMs corresponding to a UE path flow pattern. For any UE path $k \in K_w$ ($\forall w \in W$), one of the following cases must arise. We shall show each and every case can be made consistent with the KKT conditions (5) with suitable multipliers.

(i) $f_k^w > 0$ and $f_{k+1}^w > 0$. Let us choose $\lambda_k^w = 0$ and $\lambda_{k+1}^w = 0$, then Condition (5b) is satisfied. Note that $f_k^w > 0$ leads to $c_k^w + \theta_k^w \cdot \pi_k^w \leq c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w$ and $f_{k+1}^w > 0$ leads to $c_k^w + \theta_k^w \cdot \pi_k^w \geq c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w$ according to (3). Thus, we must have $c_k^w + \theta_k^w \cdot \pi_k^w = c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w$, which satisfies Condition (5a).

(ii) $f_k^w > 0$ and $f_{k+1}^w = 0$. We choose $\lambda_k^w = 0$ and note that $f_{k+1}^w = 0$ implies $\theta_{k+1}^w = \theta_k^w$. Thus, Condition (5b) is satisfied. Since $f_k^w > 0$ leads to $c_k^w + \theta_k^w \cdot \pi_k^w \leq c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w$, we can satisfy Condition (5a) by setting $\lambda_{k+1}^w = q_w \cdot g\left(\theta_k^w\right) \cdot \left(c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w - c_k^w - \theta_k^w \cdot \pi_k^w\right)$.

(iii) $f_k^w = 0$ and $f_{k+1}^w > 0$. This case is similar to (ii) and can be proven similarly. We choose $\lambda_{k+1}^w = 0$ and $f_k^w = 0$ suggests that $\theta_k^w = \theta_{k-1}^w$, Condition (5b) will hold anyway. $f_{k+1}^w > 0$ leads to $c_k^w + \theta_k^w \cdot \pi_k^w \geq c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w$, then Condition (5a) will hold if we further set $\lambda_k^w = q_w \cdot g\left(\theta_k^w\right) \cdot \left(c_k^w + \theta_k^w \cdot \pi_k^w - c_{k+1}^w - \theta_k^w \cdot \pi_{k+1}^w\right)$.

(iv) $f_k^w = 0$ and $f_{k+1}^w = 0$. $f_k^w = 0$ suggests that $\theta_k^w = \theta_{k-1}^w$ and $f_{k+1}^w = 0$ implies $\theta_{k+1}^w = \theta_k^w$, which satisfies Condition (5b). Since there is no restriction on the value of $c_k^w + \theta_k^w \cdot \pi_k^w$ and $c_{k+1}^w + \theta_k^w \cdot \pi_{k+1}^w$ (as neither path is used), there are potentially an infinite number of values of λ_k^w and λ_{k+1}^w that can satisfy (5a).

This completes the proof of necessity.

EC.1.2. Proof of Theorem 2

Proof We shall first show that any stationary point of Problem (4) is a local minimum, and then prove the objective values corresponding to any two stationary points must be identical. Note that the objective function $Z(\boldsymbol{\theta})$ defined in (4a) is equivalent to the following (See Marcotte 1998, and refer to Remark 1 for details)

$$Z(\boldsymbol{\theta}) = \sum_{w \in W} \sum_{k \in K_w} \int_{\theta_{k-1}^w}^{\theta_k^w} \left(c_k^w(\boldsymbol{\theta}) + v\pi_k^w\right) q_w g(v) dv. \tag{EC.8}$$

Consider a feasible solution $\boldsymbol{\theta}^*$ that satisfies the KKT conditions. We construct another feasible solution $\boldsymbol{\theta}'$ by locally perturbing $\boldsymbol{\theta}^*$ as follows: choose any $w \in W$ and $k \in K_w, k < |K_w|$ such that

$\mathcal{T}_k^w = [\theta_{k-1}^{w*}, \theta_k^{w*}) \neq \emptyset$, and move θ_k^{w*} to the left by a small Δ. This effectively shifts a certain amount of flow from path k to path $k+1$. Denote the flows on path k and $k+1$ corresponding to $\boldsymbol{\theta}^*$ as f_k^{w*} and f_{k+1}^{w*} respectively. Similarly, associated with $\boldsymbol{\theta}'$ are the path flows $f_k^{w*} - \Delta_f$ on path k and $f_{k+1}^{w*} + \Delta_f$ on path $k+1$ where $\Delta_f = q_w \int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} g(v)dv > 0$.

Suppose $Z(\boldsymbol{\theta}^*) \geq Z(\boldsymbol{\theta}')$. We first claim the presumption implies the following must hold.

$$\int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} \left(c_k^w(\boldsymbol{\theta}^*) + v\pi_k^w\right) q_w g(v)dv \geq \int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} \left(c_{k+1}^w(\boldsymbol{\theta}') + v\pi_{k+1}^w\right) q_w g(v)dv. \tag{EC.9}$$

To see this, we reformulate (EC.8) as follows:

$$\begin{aligned}
Z(\boldsymbol{\theta}) &= Z_\pi(\boldsymbol{\theta}) + \sum_{w\in W}\sum_{k'\in K_w, k'\neq k,k+1} \int_0^{f_{k'}^w} c_{k'}^w(f)df + \left[\int_0^{f_k^w} c_k^w(f)df + \int_0^{f_{k+1}^w} c_{k+1}^w(f)df\right] \\
&= Z_\pi(\boldsymbol{\theta}) + \sum_{a\in A}\int_0^{\bar{x}_a} t_a(u)du \\
&+ \left[\sum_{a\in(A_k\setminus A_{k+1})}\int_{\bar{x}_a}^{\bar{x}_a+f_k^w} t_a(u)du + \sum_{a\in(A_{k+1}\setminus A_k)}\int_{\bar{x}_a}^{\bar{x}_a+f_{k+1}^w} t_a(u)du + \sum_{a\in(A_k\bigcap A_{k+1})}\int_{\bar{x}_a}^{\bar{x}_a+f_k^w+f_{k+1}^w} t_a(u)du\right],
\end{aligned}$$

where $Z_\pi(\boldsymbol{\theta}) = \sum_{w\in W}\sum_{k\in K_w} q_w\pi_k^w \int_{\theta_{k-1}^w}^{\theta_k^w} vg(v)dv$ and $\bar{x}_a$ is the link flow contributed by all paths $k' \neq k$ or $k+1$. Let us construct an infeasible solution, denoted as $\boldsymbol{\theta}^*\setminus\Delta$ by setting $f_k^w = f_k^{w*} - \Delta_f$ and $f_{k+1}^w = f_{k+1}^{w*}$ while keeping all other path flows intact. We then have

$$\begin{aligned}
&Z(\boldsymbol{\theta}^*) - Z(\boldsymbol{\theta}^*\setminus\Delta) \\
&= \sum_{a\in A_k\setminus A_{k+1}}\int_{\bar{x}_a^*+\left(f_k^{w*}-\Delta_f\right)}^{\bar{x}_a^*+f_k^{w*}} t_a(u)du + \sum_{a\in(A_k\bigcap A_{k+1})}\int_{\bar{x}_a^*+\left(f_k^{w*}-\Delta_f\right)+f_{k+1}^{w*}}^{\bar{x}_a^*+f_k^{w*}+f_{k+1}^{w*}} t_a(u)du + q_w\pi_k^w\int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} vg(v)dv \\
&= \int_{f_k^{w*}-\Delta_f}^{f_k^{w*}} c_k^w(f)df + q_w\pi_k^w\int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} vg(v)dv = \int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} \left(c_k^w(\boldsymbol{\theta}^*) + v\pi_k^w\right) q_w g(v)dv,
\end{aligned}$$

and similarly

$$\begin{aligned}
&Z(\boldsymbol{\theta}') - Z(\boldsymbol{\theta}^*\setminus\Delta) \\
&= \sum_{a\in A_{k+1}\setminus A_k}\int_{\bar{x}_a^*+f_{k+1}^{w*}}^{\bar{x}_a^*+\left(f_{k+1}^{w*}+\Delta_f\right)} t_a(u)du + \sum_{a\in A_{k+1}\bigcap A_k}\int_{\bar{x}_a^*+f_k^{w*}+f_{k+1}^{w*}}^{\bar{x}_a^*+f_k^{w*}+\left(f_{k+1}^{w*}+\Delta_f\right)} t_a(u)du + q_w\pi_{k+1}^w\int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} vg(v)dv \\
&= \int_{f_{k+1}^{w*}}^{f_{k+1}^{w*}+\Delta_f} c_{k+1}^w(f)df + q_w\pi_{k+1}^w\int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} vg(v)dv = \int_{\theta_k^{w*}-\Delta}^{\theta_k^{w*}} \left(c_{k+1}^w(\boldsymbol{\theta}') + v\pi_{k+1}^w\right) q_w g(v)dv.
\end{aligned}$$

The last equality holds in the above two equations because it is exactly the same set of travelers (those whose TEM falls in $[\theta_k^{w*} - \Delta, \theta_k^{w*}]$) that cause the changes in the travel time integration term, regardless whether the impact on path flows manifests on path k or $k+1$. Thus, the presumption $Z(\boldsymbol{\theta}^*) \geq Z(\boldsymbol{\theta}')$ leads to (EC.9).

Since $g(v)$ is assumed to be positive everywhere, and the generalized cost on any path k, c_k^w, is determined by $\boldsymbol{\theta}^*$ (or $\boldsymbol{\theta}'$), Eq. (EC.9) implies

$$\exists \theta_0 \in [\theta_k^{w*} - \Delta, \theta_k^{w*}] \text{ such that } c_k^w(\boldsymbol{\theta}^*) + \theta_0 \pi_k^w \geq c_{k+1}^w(\boldsymbol{\theta}') + \theta_0 \pi_{k+1}^w. \tag{EC.10}$$

Since $\boldsymbol{\theta}^*$ satisfies the KKT conditions, we have

$$c_k^w(\boldsymbol{\theta}^*) + \theta_0 \pi_k^w \leq c_{k+1}^w(\boldsymbol{\theta}^*) + \theta_0 \pi_{k+1}^w < c_{k+1}^w(\boldsymbol{\theta}') + \theta_0 \pi_{k+1}^w, \forall \theta_0 \in [\theta_k^{w*} - \Delta, \theta_k^{w*}],$$

where the first inequality is per (EC.7) and the second is due to Assumption 1. This contradicts with (EC.10). Hence, we must reject the presumption and conclude $Z(\boldsymbol{\theta}^*) < Z(\boldsymbol{\theta}')$. Since any local perturbation at $\boldsymbol{\theta}^*$ is bound to increase the objective function value, it must be a local minimum.

Next consider two feasible solutions $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}^2$ that both satisfy the KKT conditions, and suppose $Z(\boldsymbol{\theta}^1) \neq Z(\boldsymbol{\theta}^2)$. Because $\boldsymbol{\theta}^1 \neq \boldsymbol{\theta}^2$ per assumption, we can always find an OD pair w and a path $k \in K_w$ such that there is a subset of $\mathcal{T}_k^w$ for $\boldsymbol{\theta}^1$, denoted as $[\theta, \theta + \Delta]$, that does not belong to $\mathcal{T}_k^w$ for $\boldsymbol{\theta}^2$. It follows $[\theta, \theta + \Delta]$ must belong to $\mathcal{T}_{\bar{k}}^w$ for $\boldsymbol{\theta}^2$, where $\bar{k} \neq k$. Then, $Z(\boldsymbol{\theta}^1) \neq Z(\boldsymbol{\theta}^2)$ implies

$$\int_\theta^{\theta+\Delta} \left(c_k^w(\boldsymbol{\theta}^1) + v\pi_k^w \right) q_w g(v) dv \neq \int_\theta^{\theta+\Delta} \left(c_{\bar{k}}^w(\boldsymbol{\theta}^2) + v\pi_{\bar{k}}^w \right) q_w g(v) dv.$$

Accordingly, there must exist $\theta_0 \in [\theta, \theta + \Delta]$ such that

$$c_k^w(\boldsymbol{\theta}^1) + \theta_0 \pi_k^w \neq c_{\bar{k}}^w(\boldsymbol{\theta}^2) + \theta_0 \pi_{\bar{k}}^w. \tag{EC.11}$$

Since $\boldsymbol{\theta}^1$ satisfies the KKT conditions, we have

$$c_k^w(\boldsymbol{\theta}^1) + \theta_0 \pi_k^w \leq c_{\bar{k}}^w(\boldsymbol{\theta}^1) + \theta_0 \pi_{\bar{k}}^w = c_{\bar{k}}^w(\boldsymbol{\theta}^2) + \theta_0 \pi_{\bar{k}}^w, \forall \theta_0 \in [\theta, \theta + \Delta], \tag{EC.12}$$

where the inequality is per (EC.7), and the equality is due to $c_{\bar{k}}^w(\boldsymbol{\theta}^1) = c_{\bar{k}}^w(\boldsymbol{\theta}^2)$, which holds because Problem (4) admits a unique total link flow solution under Assumption 1 (see Theorem 2.4 in (Marcotte and Zhu 1997)). Similarly, from the fact that $\boldsymbol{\theta}^2$ also satisfies the KKT conditions we have

$$c_{\bar{k}}^w(\boldsymbol{\theta}^2) + \theta_0 \pi_{\bar{k}}^w \leq c_k^w(\boldsymbol{\theta}^2) + \theta_0 \pi_k^w = c_k^w(\boldsymbol{\theta}^1) + \theta \pi_k^w, \forall \theta_0 \in [\theta, \theta + \Delta]. \tag{EC.13}$$

Combining (EC.12) and (EC.13) yields

$$c_k^w(\boldsymbol{\theta}^1) + \theta_0 \pi_k^w = c_{\bar{k}}^w(\boldsymbol{\theta}^2) + \theta_0 \pi_{\bar{k}}^w, \forall \theta_0 \in [\theta, \theta + \Delta],$$

a contradiction to (EC.11). As a result, we must have $Z(\boldsymbol{\theta}^1) = Z(\boldsymbol{\theta}^2)$. Therefore, every stationary point of Problem (4) must be one of its global solutions.

EC.1.3. Proof of Proposition 1

Proof With Condition (i), the link flows at UE, as well as the corresponding link travel times, are unique, see Theorem 2.4 in (Marcotte and Zhu 1997) or Theorem 3 in (Leurent 1993). Thus, at UE, each path $k \in K_w$ must have a unique value for both π_k^w and c_k^w. In other words, if we plot π_k^w against c_k^w as in Figure 1(a), a path must always be located at the same position. This implies that the shape of the efficient frontier at UE must be unique for any O-D pair. Per Condition (ii), no two paths on the efficient frontier could share the same position. It follows that the boundaries on the efficient frontier must be uniquely determined.

EC.1.4. Proof of Proposition 2

Proof **Sufficiency:** If the KKT conditions (5) hold, then Conditions (7) must be satisfied. If there is more than one path in K_w, consider any given GAP (i, j) where $i, j \in K_w, i < j$ such that $\theta_i^w = \theta_{i+1}^w = \cdots = \theta_{j-1}^w$. The proof is trivial if there is no path between i and j. If there are unused paths between i and j, then applying the KKT condition Eq. (5a) to every pair of adjacent paths between i and j (i.e., $(i, i+1), \cdots, (j-1, j)$) yields

$$\begin{aligned} c_i^w - c_{i+1}^w + \theta_i^w(\pi_i^w - \pi_{i+1}^w) &= \frac{\lambda_i^w - \lambda_{i+1}^w}{q_w g(\theta_i^w)}, \\ &\cdots \\ c_{j-1}^w - c_j^w + \theta_{j-1}^w(\pi_{j-1}^w - \pi_j^w) &= \frac{\lambda_{j-1}^w - \lambda_j^w}{q_w g(\theta_{j-1}^w)}. \end{aligned}$$

Given $\theta_i^w = \theta_{i+1}^w = \cdots = \theta_{j-1}^w$ and $g(\theta_i^w) = g(\theta_{i+1}^w) = \cdots = g(\theta_{j-1}^w)$, adding the above equations yields

$$c_i^w - c_j^w + \theta_i^w(\pi_i^w - \pi_j^w) = \frac{\lambda_i^w - \lambda_j^w}{q_w g(\theta_i^w)}.$$

The sufficiency proof is completed by noting Condition (7b) is identical to (5b).

Necessity: If Conditions (7) hold, then the KKT conditions (5) must be satisfied. The proof is trivial, since every pair of adjacent paths $k, k+1 \in K_w, \forall w \in W$ is also a GAP according to Definition 2.

EC.1.5. Proof of Proposition 3

Proof **Necessity:** If a feasible solution $\boldsymbol{\theta}$ is optimal, then no GAP satisfies either (8) or (9). If $\boldsymbol{\theta}$ is an optimal solution to Problem (4), then it must satisfy the general KKT conditions (7) as per Theorem 1 and Proposition 2. Consequently, the proof of necessity can be established by enumerating all possible feasible scenarios and identifying those violating Conditions (7). For any GAP (i, j) where $i, j \in K_w, \forall w \in W$, one of the following cases must arise:

(1) : $\theta_i^w - \theta_{i-1}^w > 0, \theta_j^w - \theta_{j-1}^w > 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) > 0$ or $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) < 0$.

(2) : $\theta_i^w - \theta_{i-1}^w > 0$, $\theta_j^w - \theta_{j-1}^w > 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) = 0$.

(3) : $\theta_i^w - \theta_{i-1}^w = 0$, $\theta_j^w - \theta_{j-1}^w > 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) < 0$.

(4) : $\theta_i^w - \theta_{i-1}^w = 0$, $\theta_j^w - \theta_{j-1}^w > 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) \geq 0$.

(5) : $\theta_i^w - \theta_{i-1}^w > 0$, $\theta_j^w - \theta_{j-1}^w = 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) > 0$.

(6) : $\theta_i^w - \theta_{i-1}^w > 0$, $\theta_j^w - \theta_{j-1}^w = 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) \leq 0$.

(7) : $\theta_i^w - \theta_{i-1}^w = 0$, $\theta_j^w - \theta_{j-1}^w = 0$.

Cases (2), (4), (6), and (7) evidently satisfy the general KKT conditions (7), whereas the other three cover all possible violations. It is easy to verify that Cases (1), (3) and (5) are equivalent to Conditions (8) and (9). This establishes the necessity.

Sufficiency: If no GAP in a feasible solution $\boldsymbol{\theta}$ satisfies Condition (8) or (9), $\boldsymbol{\theta}$ is optimal. To see this, note that if no GAP satisfies Condition (8) or (9), then every GAP must conform to one of the cases (2), (4), (6), and (7). Consequently, sufficiency is established by showing that any of the four cases leads to Conditions (7) when suitable multipliers are chosen.

Case (2): $\theta_i^w - \theta_{i-1}^w > 0$, $\theta_j^w - \theta_{j-1}^w > 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) = 0$. Setting $\lambda_i^w = 0$ and $\lambda_j^w = 0$ leads to Conditions (7).

Case (4): $\theta_i^w - \theta_{i-1}^w = 0$, $\theta_j^w - \theta_{j-1}^w > 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) \geq 0$. If we choose $\lambda_i = q_w g(\theta_i^w)(c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w))$ and $\lambda_j^w = 0$, then Conditions (7) are satisfied.

Case (6): $\theta_i^w - \theta_{i-1}^w > 0$, $\theta_j^w - \theta_{j-1}^w = 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) \leq 0$. Choosing $\lambda_i = 0$ and $\lambda_j^w = q_w g(\theta_i^w)(c_j^w + \theta_i^w \pi_j^w - (c_i^w + \theta_i^w \pi_i^w))$ leads to Conditions (7).

Case (7): $\theta_i^w - \theta_{i-1}^w = 0$, $\theta_j^w - \theta_{j-1}^w = 0$, and $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) \leq 0$ or $c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w) > 0$. Conditions (7) are satisfied if we set $\lambda_i^w - \lambda_j^w = q_w g(\theta_i^w)(c_i^w + \theta_i^w \pi_i^w - (c_j^w + \theta_i^w \pi_j^w))$.

This completes the proof.

EC.1.6. Proof of Lemma 1

Proof This lemma is proved by showing the objective function (10a) is either unimodal or monotonic in the feasible set. We first show that the objective is an unimodal function when there exists an interior solution $\theta_i^{w*} \in (\bar{\theta}_{i-1}^w, \bar{\theta}_j^w)$. Note that the first-order condition implies

$$c_i^* + \theta_i^{w*} \cdot \pi_i^w = c_j^* + \theta_i^{w*} \cdot \pi_j^w,$$

where c_i^* is the travel time on path i corresponding to $\theta_i^w = \theta_i^{w*}$. To see such a stationary point must be unique, suppose $\tilde{\theta}_i^w : \tilde{\theta}_i^w \neq \theta_i^{w*}$ is another stationary point, which leads to $\tilde{c}_i + \tilde{\theta}_i^w \cdot \pi_i^w = \tilde{c}_j + \tilde{\theta}_i^w \cdot \pi_j^w$, where $\tilde{c}_i$ is the travel time on path i when $\theta_i^w = \tilde{\theta}_i^w$. Some algebra leads to $\pi_i^w - \pi_j^w = (c_j^* - c_i^*)/\theta_i^{w*}$ and $\pi_i^w - \pi_j^w = (\tilde{c}_j - \tilde{c}_i)/\tilde{\theta}_i^w$, and accordingly

$$(c_j^* - c_i^*)/\theta_i^{w*} = (\tilde{c}_j - \tilde{c}_i)/\tilde{\theta}_i^w. \tag{EC.14}$$

Supposing $\tilde{\theta}_i^w > \theta_i^{w*}$ we have $f_i(\tilde{\theta}_i^w) > f_i(\theta_i^{w*})$ per Eq. (2) and $f_j(\tilde{\theta}_i^w) < f_j(\theta_i^{w*})$ (since the total flow of the two paths is a constant, as dictated by the boundary condition). Because path travel time strictly increases with path flow by Assumption 1, we must have $\tilde{c}_i > c_i^*$ and $\tilde{c}_j < c_j^*$. Given $\tilde{\theta}_i^w > \theta_i^{w*} > 0$ per assumption, we have

$$(c_j^* - c_i^*)/\theta_i^{w*} > (\tilde{c}_j - \tilde{c}_i)/\tilde{\theta}_i^w, \tag{EC.15}$$

which contradicts with Eq. (EC.14). If, instead, $\tilde{\theta}_i^w < \theta_i^{w*}$, we can arrive at $(c_j^* - c_i^*)/\theta_i^{w*} < (\tilde{c}_j - \tilde{c}_i)/\tilde{\theta}_i^w$, another contradiction. Thus, there is only one unique stationary point.

To prove the unique stationary point must be a minimum, we proceed to show (10a) is strictly convex in its neighborhood. Consider the second-order derivative of (10a) shown in Eq. (12). The first term is always positive (per Assumption 1); the second term is non-negative per definition; and the third term vanishes at the stationary point (per the first-order condition). Given the objective function is continuous, there must exist a neighborhood of the stationary point within which the second-order derivative is positive. We next examine the case where no stationary points exist within the feasible set. Thus, either $\theta_i^{w*} = \bar{\theta}_{i-1}^w$ (all flows corresponding to the lower and upper bounds assigned to path j) or $\theta_i^{w*} = \bar{\theta}_j^w$ (all flows assigned to path i). Consider the former case first. For any $\theta_i^w \in (\bar{\theta}_{i-1}^w, \bar{\theta}_j^w)$, we have $\pi_i^w - \pi_j^w \geq (c_j^* - c_i^*)/\theta_i^{w*} > (c_j - c_i)/\theta_i^w$, because $c_i > c_i^*$ and $c_j < c_j^*$ as per Assumption 1. This implies the derivative $v_i = q \cdot g\left(\theta_i^w\right) \cdot \left(c_i - c_j + \theta_i^w \left(\pi_i^w - \pi_j^w\right)\right) > 0$ for any $\theta_i^w \in (\bar{\theta}_{i-1}^w, \bar{\theta}_j^w)$. In other words, the objective function strictly increases with θ_i^w as it moves from $\bar{\theta}_{i-1}^w = \theta_i^{w*}$ to the right, which ensures $\theta_i^w = \bar{\theta}_{i-1}^w$ is a global minimum. The latter case can be proven by showing that, when $\theta_i^w = \bar{\theta}_j^w$, the objective function strictly decreases as θ_i^w approaches $\bar{\theta}_j^w$ from the left. Detail is omitted here for brevity. This completes the proof.

EC.1.7. Proof of Lemma 2

Proof Case(i): Applying the single-boundary adjustment to θ_i^w results in an update of θ_i^w to $\theta_i^w + \beta\bar{\Delta}$ as per (15), where $\bar{\Delta} = -v_i^w/s_i^w$ or $-(\theta_i^w - \theta_{i-1}^w)$, and β is the step size determined using the Armijo rule. Accordingly, the objective function reduction can be evaluated by

$$\Delta_{Z(\theta_i^w)} = \tilde{Z}(\theta_i^w) - \tilde{Z}(\theta_i^w + \beta\bar{\Delta}) \geq -\sigma\beta v_i^w \bar{\Delta},$$

where the inequality holds due to the application of the Armijo rule. Here, σ is a constant parameter associated with the rule, with $\sigma \in (0,1)$. When $\bar{\Delta} = -v_i^w / s_i^w$, the reduction in the objective function is at least $\sigma\beta q_w^2 g(\theta_i^w)^2 \delta^2 / s_i^w$; on the other hand, when $\bar{\Delta} = -(\theta_i^w - \theta_{i-1}^w)$, the reduction is at least $\sigma\beta q_w g(\theta_i^w)\delta\Delta$.

Case (ii): Applying the single-boundary adjustment leads to $\theta_i^w = \theta_i^w + \beta\bar{\Delta}$ as per (15), where $\bar{\Delta} = -v_i^w / s_i^w$ or $\theta_j^w - \theta_{j-1}^w$. Accordingly, the objective function reduction can be evaluated by

$$\Delta_{Z(\theta_i^w)} = \tilde{Z}(\theta_i^w) - \tilde{Z}(\theta_i^w + \beta\bar{\Delta}) \geq -\sigma\beta v_i^w \bar{\Delta}.$$

Similarly, when $\bar{\Delta} = -v_i^w / s_i^w$, the reduction in the objective function is at least $\sigma\beta q_w^2 g(\theta_i^w)^2 \delta^2 / s_i^w$; on the other hand, when $\bar{\Delta} = \theta_j^w - \theta_{j-1}^w$, the reduction is at least $\sigma\beta q_w g(\theta_i^w)\delta\Delta$.

In summary, the reduction in the objective function resulting from the single boundary adjustment in the above two cases is guaranteed to be at least

$$\varepsilon(\Delta, \delta) = \min\{\sigma\beta q_w^2 g(\theta_i^w)^2 \delta^2 / s_i^w, \sigma\beta q_w g(\theta_i^w)\delta\Delta\} > 0.$$

EC.1.8. Proof of Theorem 3

Proof We first note that Z^* must be finite because the existence of a solution to Problem (4) is guaranteed (Theorem 2.1 Marcotte 1998). Per Lemma 2, Algorithm 1 always produces a strict reduction in the objective function value of Problem (4) when it detects a GAP that violates the equilibrium conditions. Hence, the sequence $\{Z(\boldsymbol{\theta}^l)\}_{l=0}^{\infty}$ is non-increasing and bounded below by Z^*. Because the feasible set of (4) is compact, $\{\boldsymbol{\theta}^l\}_{l=0}^{\infty}$ must have a converging sub-sequence $\{\boldsymbol{\theta}^{l_n}\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \boldsymbol{\theta}^{l_n} = \bar{\boldsymbol{\theta}}$. Suppose that $\bar{\boldsymbol{\theta}}$ is not a global solution to Problem (4), i.e., it does not satisfy the equilibrium conditions. Per Theorem 1, Proposition 3, and Lemma 2, there exists, without loss of generality, an O-D pair w with a GAP (i, j) where $i, j \in K_w, i < j$ such that $\bar{\theta}_i - \bar{\theta}_{i-1} = \tilde{\Delta} > 0$ and $c_i + \bar{\theta}_i \pi_i - (c_j + \bar{\theta}_i \pi_j) = \tilde{\delta} > 0$. By assumption, each path cost c_i is continuous in $\boldsymbol{\theta}$. Hence Δ and δ are both continuous in $\boldsymbol{\theta}$ according to Eq. (8) and (9). In particular, if $\boldsymbol{\theta}^{l_n} \to \bar{\boldsymbol{\theta}}$ then $\Delta(\boldsymbol{\theta}^{l_n}) \to \Delta(\bar{\boldsymbol{\theta}})$ and $\delta(\boldsymbol{\theta}^{l_n}) \to \delta(\bar{\boldsymbol{\theta}})$, which justifies choosing n_0 so that for all $n > n_0$ we have $\Delta(\boldsymbol{\theta}^{l_n}) > \tilde{\Delta}/2$ and $\delta(\boldsymbol{\theta}^{l_n}) > \tilde{\delta}/2$. Therefore, in each of these iterations applying Algorithm 1 leads to a reduction in the objective function no less than $\varepsilon(\tilde{\Delta}/2, \tilde{\delta}/2)$. Because the number of these iterations is infinite, the cumulative reduction is bound to reduce Z^{l_n} below Z^*, a contradiction.

Remark 1 (Equivalence of the objective functions) *Our objective function $Z(\boldsymbol{\theta})$ is equivalent to Marcotte's, since*

$$\sum_{w\in W}\sum_{k\in K_w} q_w \int_{\theta_{k-1}^w}^{\theta_k^w} c_k^w(\boldsymbol{v})g(v)dv = \sum_{w\in W}\sum_{k\in K_w}\int_0^{f_k^w} c_k^w(\boldsymbol{f})df$$
$$= \sum_{w\in W}\sum_{k\in K_w}\int_0^{f_k^w}\sum_{a\in A} t_a(\boldsymbol{f})\delta_{ak}^w df = \sum_{a\in A}\int_0^{x_a=\sum_{w\in W}\sum_{k\in K_w}\delta_{ak}^w f_k^w} t_a(u)du = \sum_{a\in A}\int_0^{x_a} t_a(u)\,du.$$

Note that the first transition is on the basis of the integration by substitution rule (which is given by $\int_a^b f(h(x))h'(x)dx = \int_{h(a)}^{h(b)} f(u)du$, where $u = h(x)$), where we have $f_k^w = q_w\int_{\theta_{k-1}^w}^{\theta_k^w} g(v)dv$ and $0 = q_w\int_{\theta_{k-1}^w}^{\theta_{k-1}^w} g(v)dv$.

EC.2. MATLAB source code of the illustrative example

```
function r2 = int_BPR(flow, t0, Ca)
r2 = (t0*flow + 0.03*t0*(flow^5)/(Ca^4));
end
```

```
function r = Gtheta(theta)
theta0 = 1/30; theta3 = 1/6;
r = 1/(1/theta0 - 1/theta3)*(1/theta0-1/theta);
end
```

```
function  r2 = int_theta_gtheta(theta)
theta0 = 1/30; theta3 = 1/6;
r2 = log(theta)/(1/theta0-1/theta3);
end
```

```
q_w = 40000;
mk1 = 10; mk2 = 8; mk3 = 6;
ta1 = 8/60; ta2 = 12/60; ta3 = 8/60; ta4 = 4/60; ta5 = 3/60;
Ca1 = 10000; Ca2 = 15000; Ca3 = 10000; Ca4 = 10000; Ca5 = 10000;
theta0 = 1/30; theta3 = 1/6;
x0 = [1/30, 1/30];
fun = @(x) int_BPR(q_w * (Gtheta(x(2)) - Gtheta(x(1)))...
    + q_w * (Gtheta(theta3) - Gtheta(x(2))), ta1, Ca1) ...
    + int_BPR(q_w * (Gtheta(theta3) - Gtheta(x(2))), ta2, Ca2) ...
    + int_BPR(q_w * (Gtheta(x(1)) - Gtheta(theta0)), ta3, Ca3) ...
    + int_BPR(q_w * (Gtheta(x(1)) - Gtheta(theta0))...
    + q_w * (Gtheta(x(2)) - Gtheta(x(1))), ta4, Ca4) ...
    + int_BPR(q_w * (Gtheta(x(2)) - Gtheta(x(1))), ta5, Ca5) ...
    +q_w*(mk1*(int_theta_gtheta(x(1))-int_theta_gtheta(theta0)) ...
    + mk2*(int_theta_gtheta(x(2))-int_theta_gtheta(x(1))) ...
    + mk3*(int_theta_gtheta(theta3)-int_theta_gtheta(x(2))));
A = [-1, 0; 1, -1; 0, 1];
b = [-theta0; 0; theta3];
x = fmincon(fun, x0, A, b);
```

```
x = [0.054060173586594,0.054060173659230]
```

EC.3. Derivation of $\partial \tilde{Z}/\partial \theta_i^w$ and $\partial^2 \tilde{Z}/\partial (\theta_i^w)^2$ for Problem (10)

Let us consider the first and second terms in the objective function (10a) separately for convenience. Set

$$\tilde{Z}^{[1]}(\theta_i^w) = \sum_{a \in A} \int_0^{x_a} t_a(u) du, \tilde{Z}^{[2]}(\theta_i^w) = q_w \left(\pi_i^w \int_{\bar{\theta}_{i-1}^w}^{\theta_i^w} v g(v) dv + \pi_j^w \int_{\theta_{j-1}^w}^{\bar{\theta}_j^w} v g(v) dv \right).$$

For clarity, we drop the O-D subscript w in the following derivation. We first derive

$$\frac{\partial \tilde{Z}^{[1]}}{\partial \theta_i} = \sum_{a \in A} \frac{\partial \tilde{Z}^{[1]}(\theta_i)}{\partial x_a} \cdot \frac{\partial x_a}{\partial \theta_i} = \sum_{a \in A} t_a(x_a) \cdot q \cdot \left(\delta_{ai} \cdot g(\theta_i) - \delta_{aj} \cdot g(\theta_i) \right) = q \cdot g(\theta_i) \cdot \left(c_i - c_j \right), \tag{EC.16}$$

$$\begin{aligned} \frac{\partial^2 \tilde{Z}^{[1]}}{\partial (\theta_i)^2} =& q \cdot g(\theta_i) \cdot \left(\frac{\partial \sum_{a \in A} t_a(x_a) \delta_{ai}}{\partial \theta_i} - \frac{\partial \sum_{a \in A} t_a(x_a) \delta_{aj}}{\partial \theta_i} \right) + q \cdot g'(\theta_i) \cdot (c_i - c_j) \\ =& q^2 \left(g(\theta_i) \right)^2 \cdot \left(\sum_{a \in A_i \backslash A_j} \frac{dt_a(x_a)}{dx_a} \cdot \delta_{ai} + \sum_{a \in A_j \backslash A_i} \frac{dt_a(x_a)}{dx_a} \cdot \delta_{a,j} \right) + q \cdot g'(\theta_i) \cdot (c_i - c_j) \\ =& q^2 \left(g(\theta_i) \right)^2 \sum_{a \in \left[\left(A_i \backslash A_j \right) \cup \left(A_j \backslash A_i \right) \right]} \frac{dt_a(x_a)}{dx_a} + q \cdot g'(\theta_i) \cdot (c_i - c_j) \end{aligned} \tag{EC.17}$$

where A_i represents the set of links used by path i, $A_i \backslash A_j$ represents the set of links used by path i but not by path j. Thus, $\left[\left(A_i \backslash A_j \right) \cup \left(A_j \backslash A_i \right) \right]$ represent the set of links used by either i or j but not both. To see how the second equality is obtained in Eq. (EC.17), note that

$$\begin{aligned} \frac{\partial \sum_{a \in A} t_a(x_a) \delta_{ai}}{\partial \theta_i} &= \sum_{b \in A} \frac{\partial \sum_{a \in A} t_a(x_a) \delta_{ai}}{\partial x_b} \cdot \frac{\partial x_b}{\partial \theta_i} = \sum_{b \in A} \frac{dt_b(x_b)}{dx_b} \cdot \delta_{bi} \cdot q \cdot \left(\delta_{bi} \cdot g(\theta_i) - \delta_{bj} \cdot g(\theta_i) \right) \\ &= q \cdot g(\theta_i) \sum_{a \in A} \left(\frac{dt_a(x_a)}{dx_a} \cdot \delta_{ai} - \frac{dt_a(x_a)}{dx_a} \cdot \delta_{ai} \cdot \delta_{aj} \right) = q \cdot g(\theta_i) \sum_{a \in A_i \backslash A_j} \frac{dt_a(x_a)}{dx_a} \cdot \delta_{ai}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \sum_{a \in A} t_a(x_a) \delta_{aj}}{\partial \theta_i} &= \sum_{b \in A} \frac{\partial \sum_{a \in A} t_a(x_a) \delta_{aj}}{\partial x_b} \cdot \frac{\partial x_b}{\partial \theta_i} = \sum_{b \in A} \frac{dt_b(x_b)}{dx_b} \cdot \delta_{bj} \cdot q \cdot \left(\delta_{bi} \cdot g(\theta_i) - \delta_{bj} \cdot g(\theta_i) \right) \\ &= q \cdot g(\theta_i) \sum_{a \in A} \left(\frac{dt_a(x_a)}{dx_a} \cdot \delta_{ai} \cdot \delta_{aj} - \frac{dt_a(x_a)}{dx_a} \cdot \delta_{aj} \right) = -q \cdot g(\theta_i) \sum_{a \in A_j \backslash A_i} \frac{dt_a(x_a)}{dx_a} \cdot \delta_{aj}. \end{aligned}$$

Similarly, we have

$$\frac{\partial \tilde{Z}^{[2]}}{\partial \theta_i} = q \cdot \pi_i \cdot \theta_i \cdot g\left(\theta_i\right) - q \cdot \pi_j \cdot \theta_i \cdot g\left(\theta_i\right) = q \cdot \theta_i \cdot g\left(\theta_i\right)\left(\pi_i - \pi_j\right), \tag{EC.18}$$

and

$$\frac{\partial^2 \tilde{Z}^{[2]}}{\partial (\theta_i)^2} = q \cdot g\left(\theta_i\right) \cdot \left(\pi_i - \pi_j\right) + q \cdot \theta_i \cdot g'(\theta_i) \cdot (\pi_i - \pi_j). \tag{EC.19}$$

Adding Eqs. (EC.16) and (EC.18) results in Eq. (11), while adding Eqs. (EC.17) and (EC.19) leads to Eq. (12).

EC.4. Armijo rule

Definition 1 (Armijo rule) *Consider an iterative algorithm for minimizing a function* $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *over a set* X *that generates a sequence* $\{x_l\}$, *the Armijo rule can determine a step size* $\beta_l = \rho^{m_l}\beta^0$, *where* m_l *is the first positive integer* $m \in [1, 2, 3, ...]$ *to ensure the following condition is satisfied:*

$$f(x_l) - f\left(x_l - \rho^m \beta^0 D_l \nabla f(x_l)\right) \geq \sigma \rho^m \beta^0 \nabla f(x_l)' D_l \nabla f(x_l). \tag{EC.20}$$

In our implementation, the parameters ρ, σ, and β^0 take the value of 0.5, 10^{-4} and 1.0, respectively (the reader can validate these values satisfy the general conditions required for convergence), and D_l is defined as the reciprocal of the diagonal matrix of $\nabla^2 f(x_l)$, which is approximated by (14). The reader is referred to p. 69 (Bertsekas 2015) for more discussions of the Armijo rule.

EC.5. Bi-criteria parametric shortest path algorithm

Algorithm EC.1 describes a parametric shortest path algorithm that identifies the efficient path set $\{\dot{K}_w\}_{w \in W_r}$ and the corresponding boundary TEMs $\{\dot{\mathcal{T}}_k^w\}_{k \in \dot{K}_w, w \in W_r}$ for all O-D pairs originating from the same origin r. It is based on Marcotte et al. (1996) and Dial (1997).

Algorithm EC.1 A one-to-all bi-criteria parametric shortest path algorithm.

1: **Input:** Network $G(N, A)$, demand $\boldsymbol{q} = \{q_w\}$, an origin r, and the TEM distribution $g(\theta)$ with a support $\mathcal{T} = \left[\underline{\theta}, \overline{\theta}\right]$.
2: Set a destination queue $\mathcal{D} = \emptyset$ and a check node queue $\mathcal{H} = \emptyset$.
3: **Initialization**: *Lines 4–11*.
4: Set the tree lower bound $\theta^{lb} = \underline{\theta}$ and the tree upper bound $\theta^{ub} = \underline{\theta}$.
5: **for** each $w \in W_r$ **do**
6: Initialize the efficient path set $\dot{K}_w = \emptyset$ and initialize the boundary TEM set $\dot{V}_w = \emptyset$.
7: Push $\underline{\theta}$ into $\dot{V}_w$.
8: **end for**

9: Set the generalized cost on each link a as $t_a + \underline{\theta} \cdot m_a$.

10: Compute a shortest path tree $\Gamma(A_\Gamma, N_\Gamma)$ rooted at origin r based on the link generalized cost.

11: Let D^n, E^n be the time and toll from r to n on Γ for all $n \in N_\Gamma$.

12: **Main Loop**: *Lines 13–37*.

13: **while** $\theta^{ub} < \overline{\theta}$ **do**

14:     **for** each $a = (n_1, n_2) \in (A \setminus A_\Gamma)$ **do**

15:

$$\theta_a = \begin{cases} \text{ceil}\left(\frac{D^{n_2} - D^{n_1} - t_a}{E^{n_1} + m_a - E^{n_2}}, M\right), & \text{if } E^{n_1} + m_a \neq E^{n_2}, \\ \overline{\theta}, & \text{otherwise,} \end{cases}$$

ceil (a, M) rounds a up to the Mth decimal place. For example, ceil $(2.134, 2) = 2.14$.

16:     **end for**

17:     Update $\theta^{ub} = \min_{a \notin \Gamma} \left\{\theta_a : \theta^{lb} < \theta_a \leq \overline{\theta}\right\}$ and add n_1 to $\mathcal{H}$ if $\theta_a = \theta^{ub}$ for all $a = (n_1, n_2) \notin \Gamma$.

18:     Set $\Gamma' = \Gamma$.

19:     **while** $\mathcal{H} \neq \emptyset$ **do**

20:         Take a node n' out of $\mathcal{H}$.

21:         **for** each outgoing link $a = (n', n) \in (A \setminus A_\Gamma)$ **do**

22:             Set $D' = D^{n'} + t_a$ and $E' = E^{n'} + m_a$.

23:             **if** $D' + \theta^{ub} E' < D^n + \theta^{ub} E^n$ or $\left(\left(D' + \theta^{ub} E' = D^n + \theta^{ub} E^n\right) \,\&\, (E' < E^n)\right)$ **then**

24:                 Set $D^n = D'$ and $E^n = E'$.

25:                 Replace the incumbent link on Γ heading to n with a and add n to $\mathcal{H}$.

26:             **end if**

27:         **end for**

28:         If $n' \in S_r$, then push n' into $\mathcal{D}$.

29:     **end while**

30:     If $\theta^{ub} = \overline{\theta}$, then set $\mathcal{D} = \{S_r \setminus \{r\}\}$.

31:     **while** $\mathcal{D} \neq \emptyset$ **do**

32:         Take a node s out of $\mathcal{D}$ and let $w = (r, s)$.

33:         Retrieve the shortest path $\dot{k}$ from r to s in tree Γ'.

34:         If $\dot{k} \notin \dot{K}_w$, then add $\dot{k}$ to $\dot{K}_w$ and add θ^{ub} to $\dot{V}_w$.

35:     **end while**

36:     Set the tree lower bound $\theta^{lb} = \theta^{ub}$.

37: **end while**

38: **for** each $w \in W_r$ **do**

39:     Rank all elements in $\dot{V}_w$ in ascending order, let θ^w_k be the kth ($k = 0, 1, \ldots, |\dot{K}_w|$) smallest value in $\dot{V}_w$, and then set

$$\dot{\mathcal{T}}^w_k = \begin{cases} [\theta^w_{k-1}, \theta^w_k), & k = 1, \ldots, |\dot{K}_w| - 1, \\ [\theta^w_{k-1}, \theta^w_k], & k = |\dot{K}_w|. \end{cases}$$

40: **end for**

41: **Output:** Efficient path set $\{\dot{K}_w\}_{w \in W_r}$ and boundary TEMs $\{\dot{\mathcal{T}}^w_k\}_{k \in \dot{K}_w, w \in W_r}$.

EC.6. Transformation between the distribution of α and θ

Recall that $h(\cdot)$ and $H(\cdot)$ denote the PDF and CDF of VOT $\tilde{\alpha}$, and $g(\cdot)$ and $G(\cdot)$ denote the PDF and CDF of TEM $\tilde{\theta} = 1/\tilde{\alpha}$. Thus, we have

$$G(\theta) = Prob\left(\tilde{\theta} \le \theta\right) = Prob\left(\frac{1}{\tilde{\alpha}} \le \theta\right) = Prob(\tilde{\alpha} \ge \frac{1}{\theta}) = 1 - H\left(\frac{1}{\theta}\right) = 1 - H(\alpha) \qquad \text{(EC.21a)}$$

$$\Longrightarrow g(\theta) = \frac{d\left(1 - H\left(\frac{1}{\theta}\right)\right)}{d\theta} = \left(-h\left(\frac{1}{\theta}\right)\right) \cdot \left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2} \cdot h\left(\frac{1}{\theta}\right) = \alpha^2 h(\alpha). \qquad \text{(EC.21b)}$$

If, for example, $\tilde{\alpha}$ follows a uniform distribution, i.e., $h(\alpha) = 1/\left(\overline{\alpha} - \underline{\alpha}\right)$, then we have $H(\alpha) = 1/\left(\overline{\alpha} - \underline{\alpha}\right) \cdot \left(\alpha - \underline{\alpha}\right)$, $g(\theta) = 1/\left(\theta^2\left(\overline{\alpha} - \underline{\alpha}\right)\right)$, $G(\theta) = \left(\overline{\alpha} - 1/\theta\right)/\left(\overline{\alpha} - \underline{\alpha}\right)$, where $\underline{\alpha}$ and $\overline{\alpha}$ are the parameters for the uniform distribution. Figure 1 shows the PDF and CDF for $\tilde{\alpha}$ and $\tilde{\theta}$ when $\underline{\alpha} = 6$ and $\overline{\alpha} = 30$.

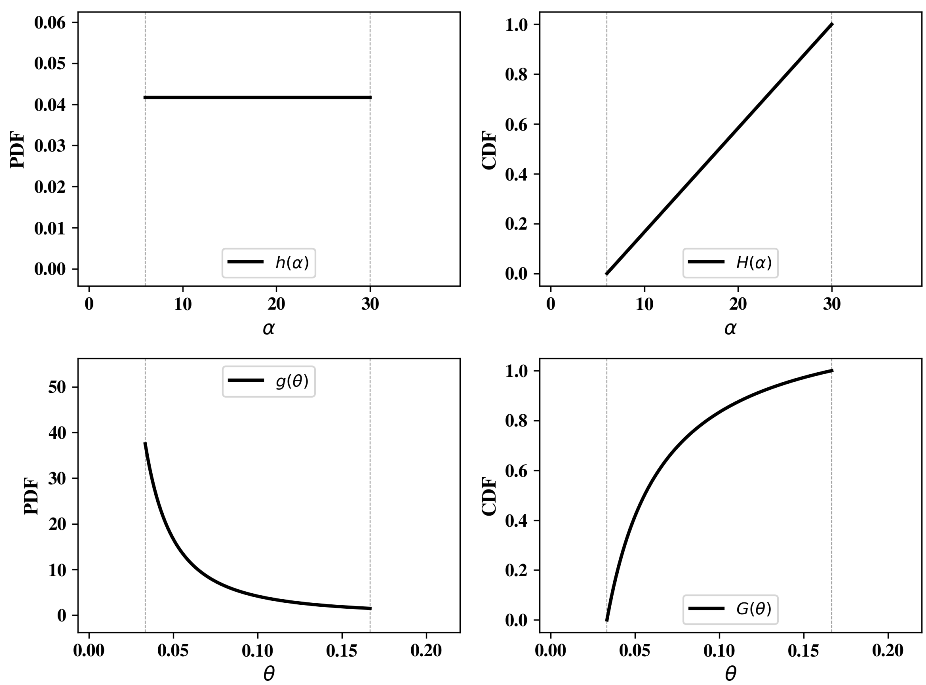

Figure 1 PDF and CDF of VOT ($\tilde{\alpha}$) and TEM ($\tilde{\theta}$) when VOT follows a uniform distribution between 6 and 30.

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