Prerequisites

We recall some properties of conditioning over Gaussian vectors $(\mathbf{X},\mathbf{Y})\in\mathbb{R}^{N+K}$, where $\mathbf{X}\in\mathbb{R}^N$ and $\mathbf{Y}\in\mathbb{R}^K$ with respective means $\boldsymbol{\mu}_X$ and $\boldsymbol{\mu}_Y$ and covariance matrices $\boldsymbol{\Sigma}_{XX}$ and $\boldsymbol{\Sigma}_{YY}$. Denote by $\boldsymbol{\Sigma}_{XY}$ the cross-covariance between $\mathbf{X}$ and $\mathbf{Y}$. The conditional distribution of $\mathbf{X}$ given $\mathbf{Y}=\mathbf{y}$ remains Gaussian, with conditional mean given by

$$\mathbb{E}[\mathbf{X}\mid\mathbf{Y}=\mathbf{y}]=\boldsymbol{\mu}_X+\boldsymbol{\Sigma}_{XY}\boldsymbol{\Sigma}_{YY}^{-1}(\mathbf{y}-\boldsymbol{\mu}_Y) \tag{EC.1}$$

and conditional variance

$$\mathbb{V}[\mathbf{X}\mid\mathbf{Y}=\mathbf{y}]=\boldsymbol{\Sigma}_{XX}-\boldsymbol{\Sigma}_{XY}\boldsymbol{\Sigma}_{YY}^{-1}\boldsymbol{\Sigma}_{XY}^\top. \tag{EC.2}$$

The Woodbury matrix identity (Hager (1989)) will be used repeatedly: let $\boldsymbol{\Sigma}\in\mathbb{R}^{N\times N}$ and $\boldsymbol{\Omega}\in\mathbb{R}^{K\times K}$ be invertible, and $\mathbf{U}\in\mathbb{R}^{N\times K}$, $\mathbf{V}\in\mathbb{R}^{K\times N}$. Then

$$\left(\boldsymbol{\Sigma}+\mathbf{U}\boldsymbol{\Omega}\mathbf{V}\right)^{-1}=\boldsymbol{\Sigma}^{-1}-\boldsymbol{\Sigma}^{-1}\mathbf{U}\left(\boldsymbol{\Omega}^{-1}+\mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\boldsymbol{\Sigma}^{-1}.$$

EC.1. Section 3 – Proof of Proposition 1

The proof of the proposition can be split into two parts: We first derive the drift and volatility of the conditional log-returns process, then use these expressions to show that the process can be written as a solution of a Stochastic Differential Equation (SDE). Finally, we derive the SDE of the conditional price process using Itô's Lemma.

EC.1.1. Conditional Mean and Covariance of the Log-Returns

The price process $\mathbf{S}(t)$, log-returns $\mathbf{X}(t)$ and views $\mathbf{Y}(0,T)$ are given by equations (9)–(10), respectively. The investment horizon is T and Assumption 1 is assumed to hold.

The vector $(\mathbf{X}(t),\mathbf{Y}(0,T))$ is then Gaussian with

$$\begin{pmatrix}\mathbf{X}(t)\\ \mathbf{Y}(0,T)\end{pmatrix}\sim\mathcal{N}\left(\begin{pmatrix}t\,\boldsymbol{\mu}^x\\ T\,\mathbf{P}\,\boldsymbol{\mu}^x\end{pmatrix},\boldsymbol{M}\right),$$

where the covariance matrix $\boldsymbol{M}\in\mathbb{R}^{(N+K)\times(N+K)}$ is positive definite with block form

$$\boldsymbol{M}=\begin{pmatrix}t\,\boldsymbol{\Sigma} & t\,\boldsymbol{\Sigma}\,\mathbf{P}^\top\\ t\,\mathbf{P}\,\boldsymbol{\Sigma} & T\big(\mathbf{P}\,\boldsymbol{\Sigma}\,\mathbf{P}^\top+\boldsymbol{\Omega}\big)\end{pmatrix}.$$

Since the log-returns vector and views are jointly Gaussian, the distribution of any subset of the log-returns given any subset of the views remains Gaussian, fully specified by its conditional mean and covariance.

By (EC.1), the conditional mean is

$$\begin{aligned}\mathbb{E}\big[\mathbf{X}(t)\;\big|\;\mathbf{Y}(0,T)=\mathbf{y}\big]&=\mathbb{E}[\mathbf{X}(t)]+\mathrm{Cov}\big(\mathbf{X}(t),\mathbf{Y}(0,T)\big)\,\mathbb{V}\big[\mathbf{Y}(0,T)\big]^{-1}\big(\mathbf{y}-\mathbb{E}[\mathbf{Y}(0,T)]\big)\\&=t\,\boldsymbol{\mu}^x+t\,\boldsymbol{\beta}_1\big(\mathbf{y}-T\,\mathbf{P}\,\boldsymbol{\mu}^x\big),\end{aligned} \tag{EC.3}$$

where

$$\boldsymbol{\beta}_1 = \frac{1}{T}\boldsymbol{\Sigma}\mathbf{P}^\top \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}\right)^{-1}.$$

By (EC.2), the conditional covariance is

$$\begin{aligned}\mathbb{V}\big[\mathbf{X}(t) \;\big|\; \mathbf{Y}(0,T)=\mathbf{y}\big] &= \mathbb{V}[\mathbf{X}(t)] - \mathrm{Cov}\big(\mathbf{X}(t),\mathbf{Y}(0,T)\big)\,\mathbb{V}[\mathbf{Y}(0,T)]^{-1}\,\mathrm{Cov}\big(\mathbf{Y}(0,T),\mathbf{X}(t)\big)\\ &= t\,\boldsymbol{\Sigma} - \frac{t^2}{T}\boldsymbol{\Sigma}\mathbf{P}^\top \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}.\end{aligned}$$

A similar argument shows that for $t,\tau \le T$, the conditional covariance between $\mathbf{X}(t)$ and $\mathbf{X}(\tau)$ given $\mathbf{Y}(0,T)=\mathbf{y}$ is

$$\mathrm{Cov}\big(\mathbf{X}(t),\mathbf{X}(\tau) \;\big|\; \mathbf{Y}(0,T)=\mathbf{y}\big) = \min\{t,\tau\}\,\boldsymbol{\Sigma} - \frac{t\,\tau}{T}\boldsymbol{\Sigma}\mathbf{P}^\top \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}.$$

We now use these expressions to derive the distribution of the conditional dynamics $d\mathbf{X}(t) \;\big|\; \mathbf{Y}(0,T)=\mathbf{y}$.

EC.1.2. Conditional Dynamics

Consider an increment of the process $\mathbf{X}(t)$ over $[t, t+dt]$ given the history $\{\mathbf{X}(\tau), \tau \le t\}$ and views $\mathbf{Y}(0,T)=\mathbf{y}$:

$$d\mathbf{X}(t) \,|\, (\mathbf{Y}(0,T)=\mathbf{y}, \{\mathbf{X}(\tau)_{\tau\le t}\}) = \lim_{dt\to 0}\big(\mathbf{X}(t+dt)-\mathbf{X}(t)\big) \,|\, \big(\mathbf{Y}(0,T)=\mathbf{y}, \{\mathbf{X}(\tau)_{\tau\le t}\}\big).$$

Since the log-returns vector is driven by a Brownian motion $\mathbf{W}(t)$, it is Markovian and the information contained in the historical data $\{\mathbf{X}(\tau), \tau \le t\}$ is all stored in the last state $\mathbf{X}(t)$, thus

$$\big(\mathbf{X}(t+dt)-\mathbf{X}(t)\big) \,|\, \big(\mathbf{Y}(0,T)=\mathbf{y}, \{\mathbf{X}(\tau)_{\tau\le t}\}\big) = \big(\mathbf{X}(t+dt)-\mathbf{X}(t)\big) \,|\, \big(\mathbf{Y}(0,T)=\mathbf{y}, \mathbf{X}(t)\big).$$

Since $\mathbf{X}(t)$ and $\mathbf{Y}(0,T)$ are Gaussian, $\big(\mathbf{X}(t+dt) \,|\, \mathbf{Y}(0,T)=\mathbf{y}, \mathbf{X}(t)\big)$ is also Gaussian and fully identified by its mean and covariance matrix which we derive next.

Mean of the conditional dynamics. Consider the random variable $\mathbf{Z}(t) = (\mathbf{X}(t), \mathbf{Y}(0,T))^\top$, and its realization $\mathbf{z} = (\mathbf{x},\mathbf{y})^\top$. The vector $(\mathbf{X}(t+dt), \mathbf{Z}(t))^\top$ is jointly Gaussian. By (EC.1) the conditional expectation of the log-returns at time $t+dt$ given $\mathbf{Z}(t)=\mathbf{z}$ is

$$\begin{aligned}\mathbb{E}\big[\mathbf{X}(t+dt) \,|\, \mathbf{X}(t)=\mathbf{x}, \mathbf{Y}(0,T)=\mathbf{y}\big] &= \mathbb{E}\big[\mathbf{X}(t+dt) \,|\, \mathbf{Z}(t)=\mathbf{z}\big]\\ &= \mathbb{E}\big[\mathbf{X}(t+dt)\big] + \mathrm{Cov}(\mathbf{X}(t+dt),\mathbf{Z}(t))\mathbb{V}[\mathbf{Z}(t)]^{-1}(\mathbf{z}-\mathbb{E}[\mathbf{Z}(t)])\end{aligned}$$

where the covariance matrix of $\mathbf{Z}(t)$ is

$$\mathbb{V}[\mathbf{Z}(t)] = \begin{pmatrix} t\boldsymbol{\Sigma} & t\boldsymbol{\Sigma}\mathbf{P}^\top \\ t\mathbf{P}\boldsymbol{\Sigma} & T(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}), \end{pmatrix} \tag{EC.4}$$

the covariance between $\mathbf{X}(t+dt)$ and $\mathbf{Z}(t)$ is

$$\mathrm{Cov}(\mathbf{X}(t+dt),\mathbf{Z}(t)) = \begin{pmatrix} t\boldsymbol{\Sigma} \\ (t+dt)\boldsymbol{\Sigma}\mathbf{P}^\top \end{pmatrix}^\top$$

and

$$\mathbf{z} - \mathbb{E}[\mathbf{Z}(t)] = \begin{bmatrix} \mathbf{x} - t\boldsymbol{\mu}^x \\ \mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x \end{bmatrix}.$$

We now derive the explicit expression of the inverse of the covariance matrix (EC.4). By Lu and Shiou (2002), the inverse of a 2×2 block matrix $\mathbf{R}$ is

$$\mathbf{R}^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{pmatrix}$$

where $\mathbf{R}$ has size $N + K \times 2N$, $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$, $\mathbf{C}$ and $\mathbf{D}$ are $K \times N$, and $\mathbf{D}$ is a non-singular matrix. The Woodbury matrix identity gives

$$(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})\mathbf{C}\mathbf{A}^{-1}.$$

Letting $\mathbf{A} = \mathbf{B} = t\boldsymbol{\Sigma}$, $\mathbf{C} = t\mathbf{P}\boldsymbol{\Sigma}$, and $\mathbf{D} = T(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})$, we have

$$\mathbb{V}[\mathbf{Z}(t)] = \mathbb{V}[\mathbf{Z}(t)] = \begin{pmatrix} t\boldsymbol{\Sigma} & t\boldsymbol{\Sigma}\mathbf{P}^\top \\ t\mathbf{P}\boldsymbol{\Sigma} & T(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}) \end{pmatrix}$$

and hence

$$\begin{aligned} &\text{Cov}(\mathbf{X}(t+dt), \mathbf{Z}(t))\mathbb{V}[\mathbf{Z}(t)]^{-1} \\ &= \left[\mathbf{I}_N - dt\,\boldsymbol{\Sigma}\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P},\; dt\,\boldsymbol{\Sigma}\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\right]. \end{aligned}$$

It follows that the conditional mean is

$$\begin{aligned} &\mathbb{E}\big[\mathbf{X}(t+dt) - \mathbf{X}(t) \,|\, \mathbf{X}(t) = \mathbf{x}, \mathbf{Y}(0,T) = \mathbf{y}\big] \\ &= dt\Big(\boldsymbol{\mu}^x - \boldsymbol{\Sigma}\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P}(\mathbf{X}(t) - t\boldsymbol{\mu}^x) \\ &\quad + \boldsymbol{\Sigma}\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x)\Big). \end{aligned} \tag{EC.5}$$

Woodbury's matrix identity gives

$$\boldsymbol{\Sigma}\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1} = \big(\mathbf{I}_N + t\boldsymbol{\beta}_2(t)\big)\boldsymbol{\beta}_1$$

where

$$\boldsymbol{\beta}_1 = \frac{1}{T}\boldsymbol{\Sigma}\mathbf{P}^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}$$

and

$$\boldsymbol{\beta}_2(t) = \boldsymbol{\Sigma}\mathbf{P}^\top\left((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega}\right)^{-1}\mathbf{P}. \tag{EC.6}$$

Recalling (EC.3), it can be shown after a bit of algebra that (EC.5) and hence the drift of the conditional process $d\mathbf{X}(t) \,|\, \mathbf{Y}(0,T) = \mathbf{y}$ is

$$\mathbb{E}\big[\mathbf{X}(t+dt) - \mathbf{X}(t) \,|\, \mathbf{X}(t) = \mathbf{x}, \mathbf{Y}(0,T) = \mathbf{y}\big] = dt\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(t)(\mathbf{X}(t) - \mathbb{E}[\mathbf{X}(t) \,|\, \mathbf{y}])\big),$$

where $\mathbb{E}[\mathbf{X}(t) \,|\, \mathbf{Y}(0,T) = \mathbf{y}] = t\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\mu^x)$. Since $\mathbf{X}(t)$ is continuous and has finite increments, by taking the limit as $dt \to 0$, we obtain

$$\mathbb{E}\big[d\mathbf{X}(t) \,|\, \mathbf{X}(t) = \mathbf{x}, \mathbf{Y}(0,T) = \mathbf{y}\big] = dt\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(t)(\mathbf{X}(t) - \mathbb{E}[\mathbf{X}(t) \,|\, \mathbf{y}])\big). \tag{EC.7}$$

Covariance of the conditional dynamics. A similar argument shows that the covariance of the conditional dynamics is

$$\begin{aligned}\mathbb{V}\big[\mathbf{X}(t+dt)\,|\,\mathbf{X}(t)=\mathbf{x},\mathbf{Y}(0,T)=\mathbf{y}\big] &= \mathbb{V}\big[\mathbf{X}(t+dt)\,|\,\mathbf{Z}=\mathbf{z}\big]\\ &= \mathbb{V}\big[\mathbf{X}(t+dt)\big]-\mathrm{Cov}(\mathbf{X}(t+dt),\mathbf{Z})\mathbb{V}[\mathbf{Z}]^{-1}\,\mathrm{Cov}(\mathbf{X}(t+dt),\mathbf{Z})^{\top}\\ &= (dt)\boldsymbol{\Sigma}-(dt)^2\cdot\boldsymbol{\Sigma}\mathbf{P}^{\top}((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^{\top}+T\boldsymbol{\Sigma})^{-1}\mathbf{P}\boldsymbol{\Sigma}\\ &= (dt)\boldsymbol{\Sigma}+o(dt).\end{aligned}$$

By taking the limit as $dt\to 0$, we get

$$\mathbb{V}\big[d\mathbf{X}(t)\,|\,\mathbf{X}(t)=\mathbf{x},\mathbf{Y}(0,T)=\mathbf{y}\big]=(dt)\,\boldsymbol{\Sigma}+o(dt). \tag{EC.8}$$

Since $\mathbf{X}(t)$ has finite increments, equations (EC.7) and (EC.8) are well defined and they characterize the drift and v olatility of the conditional process $d\mathbf{X}(t)\,|\,(\mathbf{X}(t),\mathbf{Y}(0,T)=\mathbf{y})$. We now derive the SDE that is satisfied by the process $\mathbf{X}(t)\,|\,\mathbf{Y}(0,T)=\mathbf{y}$.

EC.1.3. SDE of the conditional log-returns process

We now show that the conditional process $\mathbf{X}(t)\,|\,\mathbf{Y}(0,T)=\mathbf{y}$ is the solution to an SDE with drift (EC.7) and volatility (EC.8). We first define

$$\boldsymbol{\beta}_3(t)=t\Big(\boldsymbol{\mu}^x+\boldsymbol{\beta}_1(\mathbf{Y}(0,T)-T\mathbf{P}\boldsymbol{\mu}^x)\Big)-\int_0^t\boldsymbol{\beta}_2(s)(\mathbf{X}(s)-\mathbb{E}[\mathbf{X}(s)\,|\,\mathbf{Y}(0,T)])\,ds,\ \text{ for } t\in[0,T].$$

By showing that

$$\mathbf{W}^y(t)=\mathbf{X}(t)-\boldsymbol{\beta}_3(t)$$

is a Brownian motion in the enlarged filtration $\mathcal{F}_t^{\mathbf{Y}}:=\sigma(\mathcal{F}_t\vee\sigma(\mathbf{Y}(0,T)))$, we obtain

$$d\mathbf{W}^y(t)=d\mathbf{X}(t)\,|\,\mathbf{y}-d\boldsymbol{\beta}_3(t),\ \text{ for } t\in[0,T]$$

which gives the SDE representation of $\mathbf{X}(t)\,|\,\mathbf{Y}(0,T)=\mathbf{y}$.

To prove that $\mathbf{W}^y(t)$ is a Brownian motion in the filtration $\mathcal{F}_t^{\mathbf{y}}$, we refer to Levy's Characterization of a Brownian motion (see for example Durrett (1996;2018;)).

THEOREM EC.1 **(Levy's characterization of a Brownian motion)**. *Let the stochastic process $\mathbf{W}^y=(W_1^y,\dots,W_N^y)$ be a N–dimensional local continuous martingale with $\mathbf{W}^y(0)=\mathbf{0}$. Then, the following is equivalent:*

1. $\mathbf{W}^y$ *is a Brownian motion on the underlying filtered probability space with* $\mathbf{W}^y(t)\sim\mathcal{N}\big(\mathbf{0},t\boldsymbol{\Sigma}\big)$.
2. $\mathbf{W}^y$ *has quadratic covariations* $[W_i^y(t),W_j^y(t)]=\Sigma_{ij}t$ *for* $1\le i,j\le N$.

Since $\mathbf{X}(t)$ and $\boldsymbol{\beta}_3(t)$ are both continuous, $\mathbf{W}^y$ is also continuous. Now we show that is it a local martingale in the filtration $\mathcal{F}_t^{\mathbf{Y}}$, that is for $s \le t$

$$\mathbb{E}[\mathbf{W}^y(t) - \mathbf{W}^y(s) \,|\, \mathcal{F}_s^{\mathbf{Y}}] = \mathbf{0}.$$

We first have

$$\begin{aligned}\mathbb{E}[\mathbf{X}(t) - \mathbf{X}(s) \,|\, \mathcal{F}_s^{\mathbf{Y}}] &\overset{(a)}{=} \mathbb{E}[\mathbf{X}(t) - \mathbf{X}(s) \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)] \\ &\overset{(b)}{=} (t-s)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(s)(\mathbf{X}(s) - \mathbb{E}[\mathbf{X}(s) \,|\, \mathbf{Y}(0,T)])\big)\end{aligned}$$

where (a) follows from the Markov property of $\mathbf{X}(t)$ and (b) is derived from (EC.5). It follows that

$$\begin{aligned}\mathbb{E}[\mathbf{W}^y(t) - \mathbf{W}^y(s) \,|\, \mathcal{F}_s^{\mathbf{Y}}] &= \mathbb{E}[\mathbf{W}^y(t) - \mathbf{W}^y(s) \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)] \\ &= \mathbb{E}[\mathbf{X}(t) - \mathbf{X}(s) \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)] - \mathbb{E}[\boldsymbol{\beta}_3(t) - \boldsymbol{\beta}_3(s) \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)] \\ &= \underbrace{-(t-s)\boldsymbol{\beta}_2(s)(\mathbf{X}(s) - \mathbb{E}[\mathbf{X}(s) \,|\, \mathbf{Y}(0,T)])}_{RHS_1} \\ &\quad \underbrace{+\,\mathbb{E}\Big[\int_s^t \boldsymbol{\beta}_2(u)(\mathbf{X}(u) - \mathbb{E}[\mathbf{X}(u) \,|\, \mathbf{Y}(0,T)]du \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)\Big]}_{RHS_2}.\end{aligned}$$

By Fubini's theorem, the second term of the right-hand side

$$\begin{aligned}RHS_2 &= \int_s^t \boldsymbol{\beta}_2(u)\mathbb{E}\Big[\Big(\mathbf{X}(u) - \mathbb{E}\big[\mathbf{X}(u) \,|\, \mathbf{Y}(0,T)\big]\Big) \,\Big|\, \mathbf{X}(s), \mathbf{Y}(0,T)\Big]du \\ &= \int_s^t \boldsymbol{\beta}_2(u)\Big(\mathbb{E}\big[\mathbf{X}(u) \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)\big] - \mathbb{E}\big[\mathbf{X}(u) \,|\, \mathbf{Y}(0,T)\big]\Big)du \\ &\overset{(c)}{=} \int_s^t \boldsymbol{\beta}_2(u)\Big(\mathbf{I}_N + (u-s)\boldsymbol{\beta}_2(s)\Big)\Big(\mathbf{X}(s) - \mathbb{E}\big[\mathbf{X}(s) \,|\, \mathbf{Y}(0,T)\big]\Big)du \\ &= \left\{\int_s^t \boldsymbol{\beta}_2(u)\Big(\mathbf{I}_N + (u-s)\boldsymbol{\beta}_2(s)\Big)du\right\}\Big(\mathbf{X}(s) - \mathbb{E}[\mathbf{X}(s) \,|\, \mathbf{Y}(0,T)]\Big)\end{aligned}$$

where (c) can be obtained (after a bit of algebra) from

$$\mathbb{E}\big[\mathbf{X}(u) \,|\, \mathbf{X}(s), \mathbf{Y}(0,T)\big] = \mathbf{X}(s) + (u-s)\cdot\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(s)(\mathbf{X}(s) - \mathbb{E}[\mathbf{X}(s) \,|\, \mathbf{Y}(0,T)])\big)$$

and

$$\mathbb{E}[\mathbf{X}(u) \,|\, \mathbf{Y}(0,T) = \mathbf{y}] = u\,\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x).$$

By showing

$$\boldsymbol{\beta}_2(u)(\mathbf{I}_N + (u-s)\boldsymbol{\beta}_2(s)) = \boldsymbol{\beta}_2(s), \;\; \forall u \ge s$$

we will have $RHS_1 + RHS_2 = 0$ and hence

$$\mathbb{E}[\mathbf{W}^y(t) - \mathbf{W}^y(s) \,|\, \mathcal{F}_s^{\mathbf{Y}}] = \mathbf{0}$$

and $\mathbf{W}^y(t)$ will be a local martingale in $\mathcal{F}_t^{\mathbf{Y}}$.

Recall from EC.6 that

$$\boldsymbol{\beta}_2(u) = \boldsymbol{\Sigma}\mathbf{P}^\top((T-u)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P},\ \ \forall u \geq 0.$$

It follows that for $0 \leq s \leq u \leq T$

$$\boldsymbol{\beta}_2(u) = \boldsymbol{\Sigma}\mathbf{P}^\top((T-s)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega} - (u-s)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top)^{-1}\mathbf{P}.$$

Woodbury's identity implies

$$\boldsymbol{\beta}_2(u) = \boldsymbol{\beta}_2(s) + (u-s)\boldsymbol{\beta}_2(s)(\mathbf{I}_N - (u-s)\boldsymbol{\beta}_2(s))^{-1}\boldsymbol{\beta}_2(s) = \boldsymbol{\beta}_2(s)(\mathbf{I}_N - (u-s)\boldsymbol{\beta}_2(s))^{-1}$$

which gives

$$\boldsymbol{\beta}_2(u)(\mathbf{I}_N + (u-s)\boldsymbol{\beta}_2(s)) = \boldsymbol{\beta}_2(s), \quad \forall\, u \geq s$$

and hence

$$\mathbb{E}[\mathbf{W}^y(t) - \mathbf{W}^y(s)\,|\,\mathcal{F}_s^{\mathbf{Y}}] = \mathbf{0}, \quad \forall\, 0 \leq s \leq t \leq T,$$

so $\mathbf{W}^y$ is a local martingale in the filtration $\mathcal{F}_t^{\mathbf{Y}}$.

It follows from (EC.8) that the quadratic variation of $\mathbf{W}^y$ satisfies

$$[W_i^y(t), W_j^y(t)] = \Sigma_{ij}\, t, \quad \text{for } 1 \leq i,j \leq N.$$

Thus, by Theorem EC.1, $\mathbf{W}^y$ is a Brownian motion with respect to the filtration $\mathcal{F}_t^{\mathbf{Y}}$.

It follows from (EC.7) – (EC.8) that

$$\mathbf{X}(t)\,|\,\mathbf{y} = \mathbf{W}^y(t) + \boldsymbol{\beta}_3(t)\,|\,\mathbf{y},$$

and

$$\begin{aligned} d\mathbf{X}(t)\,|\,\mathbf{y} &= d\mathbf{W}^y(t) + d\boldsymbol{\beta}_3(t)\,|\,\mathbf{y} \\ &= dt\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(t)(\mathbf{X}(t)\,|\,\mathbf{y} - \mathbb{E}[\mathbf{X}(t)\,|\,\mathbf{y}])\big) + d\mathbf{W}^y(t). \end{aligned}$$

EC.1.4. Conditional Price Process

The stock price process can be obtained directly from the log-returns by noting that

$$\mathbf{S}(t)\,|\,(\mathbf{Y}(0,T) = \mathbf{y}) = \mathbf{S}(0)\exp\big(\mathbf{X}(t)\,|\,(\mathbf{Y}(0,T) = \mathbf{y})\big). \tag{EC.9}$$

We apply Itô's lemma to (EC.9). For $i \in [N]$, we have

$$\begin{aligned} dS_i(t)\,|\,\mathbf{y} &= S_i(0)e^{X_i(t)\,|\,\mathbf{y}}dX_i(t)\,|\,\mathbf{y} + \frac{1}{2}S_i(0)e^{X_i(t)\,|\,\mathbf{y}}(dX_i(t)\,|\,\mathbf{y})^2 \\ &= S_i(t)\,|\,\mathbf{y}\cdot(dX_i(t)\,|\,\mathbf{y} + (dX_i(t)\,|\,\mathbf{y})^2), \end{aligned}$$

with

$$(dX_i(t)\,|\,\mathbf{y})^2 = (dW_i^y(t))^2 = \sigma_i^2 dt.$$

Therefore, the conditional dynamics of asset prices are

$$\begin{aligned} d\mathbf{S}(t)\,|\,\mathbf{y} &= \mathbf{D}(\mathbf{S}(t)\,|\,\mathbf{y})\big(d\mathbf{X}(t)\,|\,\mathbf{y} + \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})dt\big) \\ &= \mathbf{D}(\mathbf{S}(t)\,|\,\mathbf{y})\big(\tilde{\boldsymbol{\mu}}(t,\mathbf{X}(t)\,|\,\mathbf{y})dt + d\mathbf{W}^y(t)\big), \end{aligned}$$

with drift

$$\begin{aligned} \tilde{\boldsymbol{\mu}}(t,\mathbf{x}) &= \boldsymbol{\mu}^x + \boldsymbol{\beta}_1\big(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x\big) - \boldsymbol{\beta}_2(t)\big(\mathbf{x} - \mathbb{E}[\mathbf{X}(t)\,|\,\mathbf{y}]\big) + \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma}) \\ &= \boldsymbol{\mu} + \boldsymbol{\beta}_1\big(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x\big) - \boldsymbol{\beta}_2(t)\big(\mathbf{x} - \mathbb{E}[\mathbf{X}(t)\,|\,\mathbf{y}]\big). \end{aligned}$$

This completes the proof. □

EC.1.5. Novikov's condition

For (14) to define a change of measure we require $\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$. A sufficient condition is Novikov's condition (see, e.g., Lemma 8.6.2 in Øksendal (2003))

$$\mathbb{E}_{\mathbb{P}}\left[\exp\Big(\frac{1}{2}\int_0^T |\,\mathbf{k}(t)\,|^2\,dt\Big)\right] < \infty.$$

Define

$$\tilde{\mathbf{X}}(t) := \mathbf{X}^y(t) - \mathbb{E}[\mathbf{X}^y(t)].$$

From (11), $\tilde{\mathbf{X}}(t)$ satisfies the SDE

$$d\tilde{\mathbf{X}}(t) = -\boldsymbol{\beta}_2(t)\tilde{\mathbf{X}}(t)\,dt + d\mathbf{W}(t), \quad \tilde{\mathbf{X}}(0) = \mathbf{0}. \tag{EC.10}$$

Moreover, since

$$\boldsymbol{\beta}_2(t) = \boldsymbol{\Sigma}\mathbf{P}^\top\Big((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega}\Big)^{-1}\mathbf{P}$$

is bounded on $[0,T]$, there exists a constant $M > 0$ such that

$$|\boldsymbol{\beta}_2(t)| \le M, \quad \forall\, t \in [0,T].$$

Thus, we get

$$|\mathbf{k}(t)|^2 \le 2\left|\boldsymbol{\beta}_1\big(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x\big)\right|^2 + 2\,|\boldsymbol{\beta}_2(t)|^2\,|\tilde{\mathbf{X}}(t)|^2 \le 2M^2\,|\tilde{\mathbf{X}}(t)|^2.$$

The Novikov condition reduces to showing that

$$\mathbb{E}_{\mathbb{P}}\left[\exp\Big(M^2\int_0^T |\tilde{\mathbf{X}}(t)|^2\,dt\Big)\right] < \infty.$$

The solution to (EC.10) is given by

$$\tilde{\mathbf{X}}(t) = \exp\Big(-\int_0^t \boldsymbol{\beta}_2(s)\,ds\Big)\int_0^t \exp\Big(\int_0^s \boldsymbol{\beta}_2(r)\,dr\Big)d\mathbf{W}(s),$$

which shows that $\tilde{\mathbf{X}}(t)$ is a zero–mean Gaussian process with uniformly bounded variance (see, e.g., Øksendal (2003), Revuz and Yor (1999)). Thus, the quadratic function $\int_0^T |\tilde{\mathbf{X}}(t)|^2\,dt$ admits finite exponential moments. Thus, we conclude that

$$\mathbb{E}_{\mathbb{P}}\left[\exp\Big(\frac{1}{2}\int_0^T |\mathbf{k}(t)|^2\,dt\Big)\right] < \infty,$$

which verifies the Novikov condition. □

EC.2. Section 4

EC.2.1. Proof of Proposition 3

Let $W(t)\in\mathbb{R}$ be a standard Brownian motion with variance $\mathbb{V}[W(t)] = t$ and initial value $W(0) = a$. At time $t = 0$, we observe a sample y of the random variable $Y(0,T) = W(T) + \epsilon$ where $\epsilon \sim \mathcal{N}\big(0, T\omega^2\big)$ is independent of W. The following result shows that the conditional process $\{B(t) = (W(t)\,|\,Y(0,T) = y), t\in[0,T]\}$ is a restriction of a Brownian bridge from a to y with hitting time $\tilde{T} = T(1+\omega^2)$ to the interval $[0,T]$.

From Proposition 2, a process $B(t)$ is defined as a Brownian bridge from a to y with hitting time $\tilde{T} = T(1+\omega^2)$ if it satisfies

1. $B(0) = a$, and $B(\tilde{T}) = y$ (with probability 1),
2. $\{B(t), t\in[0,\tilde{T}]\}$ is a Gaussian process,
3. $\mathbb{E}[B(t)] = a + \frac{t}{\tilde{T}}(y-a)$ for $t\in[0,\tilde{T}]$,
4. $\mathrm{Cov}\,\big(B(t), B(s)\big) = \min\{s,t\} - \dfrac{st}{\tilde{T}}$, for $s,t\in[0,\tilde{T}]$,
5. With probability 1, $t\to B(t)$ is continuous in $[0,\tilde{T}]$.

We now prove that $B(t) = (W(t)\,|\,Y(0,T) = y)$ satisfies the above properties for $t\in[0,T]$.

We first have $B(0) = W(0) = a$, and since the vector $(W(t), Y(0,T))$ is jointly Gaussian, the conditional process $B(t) = (W(t)\,|\,Y(0,T))$ is normally distributed and satisfies 2. It is therefore fully identified by its mean and variance, and we have

$$\begin{aligned}\mathbb{E}[B(t)] &= \mathbb{E}[W(t)\,|\,Y(0,T) = y]\\ &= \mathbb{E}[W(t)] + \mathrm{Cov}(W(t), Y(0,t))\mathbb{V}^{-1}[Y(0,T)](y - \mathbb{E}[Y(0,T)])\\ &= a + \frac{t}{\tilde{T}}(y-a), \ \text{ for } t\in[0,T],\end{aligned}$$

with $\tilde{T} = T(1+\omega^2)$. Thus, $B(t)$ satisfies 3. Now let $s,t\in\mathbb{R}$ with $s\le t$, we have

$$\begin{aligned}\mathrm{Cov}(B(t),B(s)) &= \mathrm{Cov}(W(t), W(s)\,|\,Y(0,T) = y)\\ &= \mathbb{E}\big[W(t)W(s)\,|\,Y(0,T) = y\big] - \mathbb{E}\big[W(s)\,|\,Y(0,T) = y\big]\mathbb{E}\big[W(t)\,|\,Y(0,T) = y\big],\end{aligned} \tag{EC.11}$$

by the law of total expectation we get

$$\mathbb{E}\big[W(t)W(s)\,|\,Y(0,T)=y\big]=\mathbb{E}\big[W(s)\mathbb{E}[W(t)\,|\,W(s),Y(0,T)=y]\,|\,Y(0,T)=y\big].$$

Since $W(t)$ and $Y(0,T)$ are jointly Gaussian, we write

$$\begin{aligned}\mathbb{E}[W(t)\,|\,W(s),Y(0,T)=y]&=\mathbb{E}[W(t)\,|\,W(s)]\\&\quad+\operatorname{Cov}\big(W(t),Y(0,T)\,\big|\,W(s)\big)\,\mathbb{V}\big[Y(0,T)\,\big|\,W(s)\big]^{-1}\\&\quad\cdot\Big[y-\mathbb{E}\big(Y(0,T)\,\big|\,W(s)\big)\Big]\\&=W(s)+\frac{t-s}{T(1+\omega^2)-s}\big(y-W(s)\big)\\&=\frac{\tilde{T}-t}{\tilde{T}-s}W(s)+\frac{t-s}{\tilde{T}-s}y.\end{aligned}$$

Therefore, the conditional expectation of the product $W(t)W(s)$ is

$$\mathbb{E}\big[W(t)W(s)\,|\,Y(0,T)=y\big]=\frac{\tilde{T}-t}{\tilde{T}-s}\mathbb{E}[W^2(s)\,|\,Y(0,T)=y]+\frac{t-s}{\tilde{T}-s}y\cdot\mathbb{E}[W(s)\,|\,Y(0,T)=y]. \tag{EC.12}$$

It follows (EC.11) and (EC.12) that

$$\operatorname{Cov}(B(t),B(s))=\min\{s,t\}-\frac{st}{\tilde{T}},\ \text{ for } t\in[0,T],$$

and $B(t)$ satisfies 4. This implies that $\mathbb{V}[B(\tilde{T})]=0$ so by 3, $B(\tilde{T})=y$. Since $W(t)$ has continuous sample paths, so too does $B(t)$ and hence 5 holds. Therefore, $\{B(t)=(W(t)\,|\,Y(0,T)=y),t\in[0,T]\}$ is a restriction of a Brownian bridge from a to y with hitting time $\tilde{T}=T(1+\omega^2)$ to the interval $[0,T]$. This concludes the proof. □

EC.2.2. Example 2: Detailed calculations

Let $W_1(t)$ and $W_2(t)$ be two standard Brownian motions with correlation $\rho\in(0,1]$. It can easily be shown that there exist a standard Brownian motion $W_3(t)$ such that:

1. $W_3(t)$ is independent of $W_1(t)$,
2. $W_2(t)=\rho W_1(t)+\sqrt{1-\rho^2}W_3(t)$.

Given a view y_2 sampled from $Y_2(0,T)=W_2(T)+\epsilon$ where ϵ is zero-mean Gaussian and with variance ω^2 and independent of $W_2(t)$, we show that the two processes $B_1(t)=\big(W_1(t)\,|\,Y_2(0,T)=y_2\big)$ and $B_2(t)=\big(W_2(t)\,|\,Y_2(0,T)=y_2\big)$ are Brownian bridges restricted to the interval $[0,T]$. The latter is directly deduced from Proposition 3, where $B_2(t)=(W_2(t)\,|\,Y_2(0,T)=y_2)$ is a restriction of a Brownian bridge from 0 to y_2 with hitting time $\tilde{T}_2=T+\omega^2$ to $[0,T]$. Here, we prove the same for $\{B_1(t),t\in[0,T]\}$.

We start by showing how the view y_2 of the Brownian motion $W_2(T)$ can be transformed to a view y_1 about the Brownian motion $W_1(T)$. Consider the random variable $Y_1(0,T)$ derived from $Y_2(0,T)$ such that

$$Y_1(0,T) = \frac{1}{\rho} Y_2(0,T).$$

We have

$$\begin{aligned} Y_1(0,T) &= \frac{1}{\rho}(W_2(T)+\epsilon) \\ &\overset{(a)}{=} W_1(T) + \frac{\sqrt{1-\rho^2}}{\rho} W_3(T) + \frac{1}{\rho}\epsilon \\ &= W_1(T) + \bar{\epsilon}, \end{aligned}$$

where (a) follows from the decomposition of the Brownian motion $W_2(t)$ into $W_1(t)$ and $W_3(t)$, and

$$\bar{\epsilon} \sim \mathcal{N}\big(0, \frac{\omega^2 + (1-\rho^2)T}{\rho^2}\big)$$

is the noise term in the view $Y_1(0,T)$. Therefore, we have

$$\begin{aligned} B_1(t) &= W_1(t) \,|\, (Y_2(0,T) = y_2) \\ &= W_1(t) \,|\, (Y_1(0,T) = y_1). \end{aligned}$$

From Proposition 3, the conditional process $B_1(t) = (W_1(t) \,|\, Y_1(0,T) = y_1)$ is a Brownian bridge from 0 to y_1 with hitting time $\tilde{T}_1$ restricted to the interval $[0,T]$, with

$$\begin{aligned} \tilde{T}_1 &= T + \mathbb{V}[\bar{\epsilon}] \\ &= \frac{\omega^2 + T}{\rho^2}. \end{aligned}$$

When $\rho = 0$, notice that $W_1(t)$ and $Y_2(0,T)$ are independent,and therefore

$$\begin{aligned} B_1(t) &= W_1(t) \,|\, (Y_2(0,T) = y_2) \\ &= W_1(t), \end{aligned}$$

this is also equivalent to having $\tilde{T}_1 = \infty$ (notice that a Brownian bridge with infinite hitting time is a Brownian motion). □

EC.2.3. Example 2: Plots

Figure EC.1a compares the hitting times of both Brownian bridges for a fixed level of view uncertainty ($\omega^2 = 4$). The hitting time of the Brownian bridge for asset 2 is $\tilde{T}_2 = T + \omega^2 = 14$, and that of asset 1 depends on the correlation between the two assets. When they are independent ($\rho = 0$), the hitting time $\tilde{T}_1$ is infinite because the view on asset 2 provides no information about asset 1. In this case, the Bb for asset 1 remains a Brownian motion. As the correlation between the two processes increases, the difference between the hitting times $\tilde{T}_1$ and $\tilde{T}_2$ diminishes because the view

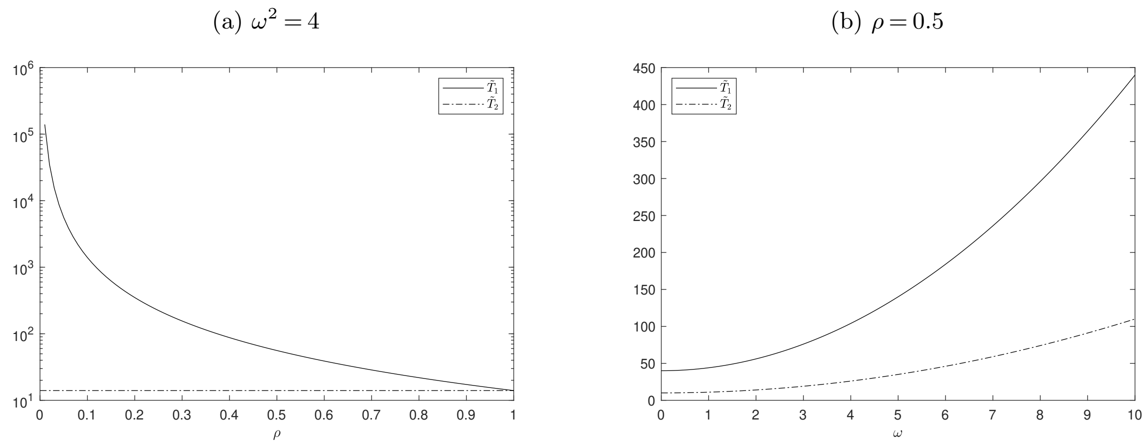

Figure EC.1 Impact of the noise in the view and correlation on the hitting time of the Bb. Comparison of the hitting times of the two Brownian bridges. (*a*) Varying correlation ρ with fixed noise $\omega^2 = 4$. (*b*) Varying noise ω^2 with fixed correlation $\rho = 0.5$.

of asset 2 provides information about asset 1. When the two assets are perfectly correlated ($\rho = 1$) the hitting times are equal.

In figure EC.1b, we fix the correlation coefficient ρ to be 0.5, and compare the hitting times of the Brownian bridges for varying degrees of uncertainty in the view ($\omega^2 \in [0, 10^2]$). When the view is certain ($\omega^2 = 0$), $\tilde{T}_2 = T$ as the terminal value of asset 2 is fixed. $\tilde{T}_1$ is always greater than T due to the imperfect correlation between the two assets. As we increase the view uncertainty, $\tilde{T}_1$ increases at a faster rate than $\tilde{T}_2$; The information about asset 1 disappears faster than the information about asset 2 due to the correlation between the two assets being less than 1.

EC.2.4. Proof of Proposition 4

Let $\mathbf{W}(t)$ be an $N-$dimensional Brownian motion starting at $\mathbf{a} \in \mathbb{R}^N$ with

$$\mathbf{W}(t) \sim \mathcal{N}\big(\mathbf{a}, t\boldsymbol{\Sigma}\big).$$

At $t = 0$, we have $\mathbf{Y}(0, T) = \mathbf{y}$ where

$$\mathbf{Y}(0, T) = \mathbf{P}\mathbf{W}(T) + \boldsymbol{\epsilon},$$

$\mathbf{P} \in \mathbb{R}^{K \times N}$ is a linear mapping such that $\mathbf{P}\mathbf{L}_j \neq \mathbf{0}$ for $j \in [N]$, and $\boldsymbol{\epsilon}$ is normally distributed with $\boldsymbol{\epsilon} \sim \mathcal{N}\big(\mathbf{0}, T\boldsymbol{\Omega}\big)$. We define the conditional process $\mathbf{B}(t) \in \mathbb{R}^N$ as the Brownian motion $\mathbf{W}(t)$ conditioned on the forward-looking views $\mathbf{Y}(0, T) = \mathbf{y}$

$$\mathbf{B}(t) = \mathbf{W}(t) \,|\, (\mathbf{Y}(0, T) = \mathbf{y}), \text{ for } t \in [0, T].$$

Since the vector $(\mathbf{W}(t), \mathbf{Y}(0,T))$ is Gaussian, the conditional process $\mathbf{B}(t)$ is also Gaussian, and with probability 1, we have

$$\mathbf{B}(0) = \mathbf{W}(0) \,|\, (\mathbf{Y}(0,T) = \mathbf{y}) = \mathbf{a}.$$

Additionally, the conditional expectation is

$$\begin{aligned}\mathbb{E}[\mathbf{B}(t)] &= \mathbb{E}[\mathbf{W}(t) \,|\, \mathbf{Y}(0,T) = \mathbf{y}] \\ &= \mathbb{E}[\mathbf{W}(t)] + \mathrm{Cov}(\mathbf{W}(t), \mathbf{Y}(0,T))\mathbb{V}^{-1}[\mathbf{Y}(0,T)]\big(\mathbf{y} - \mathbb{E}[\mathbf{Y}(0,T)]\big) \\ &= \mathbf{a} + \frac{t}{T}\boldsymbol{\Sigma}\mathbf{P}^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}(\mathbf{y} - \mathbf{P}\mathbf{a}).\end{aligned}$$

For $s, t \in \mathbb{R}$ with $s \leq t$, the covariance between $\mathbf{B}(t)$ and $\mathbf{B}(s)$ is

$$\begin{aligned}\mathrm{Cov}(\mathbf{B}(t), \mathbf{B}(s)) &= \mathrm{Cov}(\mathbf{W}(t), \mathbf{W}(s) \,|\, \mathbf{Y}(0,T) = \mathbf{y}) \\ &= \mathrm{Cov}(\mathbf{W}(t), \mathbf{W}(s)) - \mathrm{Cov}(\mathbf{W}(t), \mathbf{Y}(0,T))\mathbb{V}\big[\mathbf{Y}(0,T)\big]^{-1}\mathrm{Cov}(\mathbf{Y}(0,T), \mathbf{W}(s)) \\ &= s\boldsymbol{\Sigma} - \frac{st}{T}\boldsymbol{\Sigma}\mathbf{P}^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}\mathbf{P}\boldsymbol{\Sigma}.\end{aligned}$$

We define

$$\mathbf{H} := \frac{1}{T}(\mathbf{P}\mathbf{L})^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}\mathbf{P}\mathbf{L} \in \mathbb{R}^{N\times N},$$

and recall that $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^\top$, thus

$$\begin{aligned}\mathrm{Cov}(\mathbf{B}(t), \mathbf{B}(s)) &= s\boldsymbol{\Sigma} - \frac{st}{T}\boldsymbol{\Sigma}\mathbf{P}^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}\mathbf{P}\boldsymbol{\Sigma} \\ &= \mathbf{L}\big(s\mathbf{I}_N - st\mathbf{H}\big)\mathbf{L}^\top.\end{aligned}$$

As the covariance matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ are positive definite, and the linear mapping matrix $\mathbf{P}$ satisfies the condition $\mathbf{P}\mathbf{L}_i \neq \mathbf{0}$, for $i \in [N]$, it is easy to see that for a non-zero vector $\mathbf{z} \in \mathbb{R}^N$, we have

$$\mathbf{z}^\top\mathbf{H}\mathbf{z} = \frac{1}{T}(\mathbf{P}\mathbf{L}\mathbf{z})^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}\mathbf{P}\mathbf{L}\mathbf{z} \geq 0,$$

therefore, $\mathbf{H}$ is positive semi-definite[5]. Finally, as the Brownian motion $t \to W_i(t)$ is continuous for every $i \in [N]$, and the view $\mathbf{y}$ is given at time 0 and expires at time T (the view causes no jumps in the interval $(0,T)$), the process $t \to B_i(t) = (W_i(t) \,|\, \mathbf{Y}(0,T) = \mathbf{y})$ is also continuous for $t \in [0,T]$ and $i \in [N]$. This concludes the proof. □

EC.2.5. Proof of Theorem 1

We define the stochastic process $\{\bar{\mathbf{B}}(t), t \in [0,T]\}$ as

$$\bar{\mathbf{B}}(t) = \mathbf{L}^{-1}(\mathbf{B}(t) - \mathbb{E}[\mathbf{B}(t)]),$$

where $\{\mathbf{B}(t), t \in [0,T]\}$ satisfies the properties in Theorem 2. It is easy to see that

[5] It is not positive definite as there can exist a vector $\mathbf{z} \in \mathbb{R}^N$ such that $\mathbf{z} \neq \mathbf{0}$ and $\mathbf{P}\mathbf{L}\mathbf{z} = \mathbf{0}$.

1. $\bar{\mathbf{B}}(0)=\mathbf{0}$ (with probability 1),
2. $\bar{\mathbf{B}}$ is a Gaussian process,
3. $\mathbb{E}[\bar{\mathbf{B}}(t)]=\mathbf{0}$, for $t\in[0,T]$,
4. $cov\big(\bar{B}_i(t),\bar{B}_j(s)\big)=\begin{cases}\min\{s,t\}-\frac{st}{\tilde{T}_i}, \text{ if } i=j,\\ -\frac{st}{H_{i,j}}, \text{ if } i\neq j,\end{cases}$
5. With probability 1, $t\to\bar{B}_i(t)$ is continuous in $[0,T]$ for $i\in[N]$.

It follows that each $\bar{B}_i(t)$, where $i\in[N]$, satisfies Definition 1 and is therefore a Brownian bridge from 0 to 0 with hitting time $\tilde{T}_i$ restricted to the interval $[0,T]$. Thus

$$d\bar{B}_i(t)=\frac{y-\bar{B}_i(t)}{\tilde{T}_i-t}dt+dW_i^y(t), \text{ for } i\in[N].$$

However, these Brownian bridges are correlated with

$$\mathrm{Cov}(\bar{B}_i(t),\bar{B}_j(t))=-\frac{t^2}{\tilde{T}_{ij}} \text{ for } i\neq j,\ t\in[0,T]$$

so the SDE of the multidimensional $\bar{\mathbf{B}}(t)$ is not obtained from "stacking" the marginals.

We now derive the SDE representation of $\bar{\mathbf{B}}(t)$. As in the proof of Proposition 1, we first derive the drift and volatility of $d\bar{\mathbf{B}}(t)$, then show that it admits an SDE representation. Let $\mathbf{V}(t)=[V_1(t),\dots,V_N(t)]^\top$ be a vector of N-independent Brownian motions with

$$\mathbf{V}(t)\sim\mathcal{N}\big(\mathbf{0},t\mathbf{I}_N\big),$$

and consider the forward-looking views

$$\bar{\mathbf{Y}}(0,T)=(\mathbf{PL})\mathbf{V}(T)+\boldsymbol{\epsilon},$$

where $\boldsymbol{\epsilon}\sim\mathcal{N}\big(\mathbf{0},T\boldsymbol{\Omega}\big)$. It can be easily proven that the process $\bar{\mathbf{B}}(t)$ has the same distribution as the Brownian motion $\mathbf{V}(t)$ conditioned on the views $\bar{\mathbf{Y}}(0,T)=\mathbf{0}$

$$\bar{\mathbf{B}}(t)\overset{d}{=}\mathbf{V}(t)\,|\,(\bar{\mathbf{Y}}(0,T)=\mathbf{0}).$$

Conditional on $\bar{\mathbf{B}}(t)$ at time t

$$\begin{aligned}\mathbb{E}[\bar{\mathbf{B}}(t+dt)\,|\,\bar{\mathbf{B}}(t)]&=\mathbb{E}[\mathbf{V}(t+dt)\,|\,\mathbf{V}(t),\mathbf{Y}(0,T)=\mathbf{0}]\\&=\big((t+dt)\mathbf{I}_N\ \ t(\mathbf{PL})^\top\big)\begin{pmatrix}t\mathbf{I}_N & t(\mathbf{PL})^\top\\(t+dt)\mathbf{PL} & T((\mathbf{PL})(\mathbf{PL})^\top+\boldsymbol{\Omega})\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{V}(t)\\\mathbf{0}\end{pmatrix}.\end{aligned}$$

Further calculations show

$$\mathbb{E}[\bar{\mathbf{B}}(t+dt)\,|\,\bar{\mathbf{B}}(t)]=\bar{\mathbf{B}}(t)-\frac{dt}{T}(\mathbf{PL})^\top\big((1-\frac{t}{T})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+\boldsymbol{\Omega})^{-1}\mathbf{PL}\bar{\mathbf{B}}(t).$$

Defining

$$\bar{\boldsymbol{\beta}}_2(t) = \frac{1}{T}(\mathbf{PL})^\top\big((1-\frac{t}{T})\mathbf{P\Sigma P}^\top + \boldsymbol{\Omega})^{-1}\mathbf{PL}$$

it follows that

$$\mathbb{E}[\bar{\mathbf{B}}(t+dt) - \bar{\mathbf{B}}(t) \,|\, \bar{\mathbf{B}}(t)] = -(dt)\bar{\boldsymbol{\beta}}_2(t)\bar{\mathbf{B}}(t),$$

by continuity of $\bar{\mathbf{B}}(t)$ we get by letting $dt \to 0$

$$\mathbb{E}[d\bar{\mathbf{B}}(t) \,|\, \bar{\mathbf{B}}(t)] = -(dt)\bar{\boldsymbol{\beta}}_2(t)\bar{\mathbf{B}}(t). \tag{EC.13}$$

A similar argument shows that the covariance of $\bar{\mathbf{B}}(t)$ is

$$\begin{aligned}
\mathbb{V}[\bar{\mathbf{B}}(t+dt) - \bar{\mathbf{B}}(t) \,|\, \bar{\mathbf{B}}(t)] &= \mathbb{V}[\mathbf{V}(t+dt) \,|\, \mathbf{V}(t), \mathbf{Y}(0,T) = \mathbf{0}] \\
&= \big(t\mathbf{I}_N \;\; t(\mathbf{PL})^\top\big) \begin{pmatrix} t\mathbf{I}_N & t(\mathbf{PL})^\top \\ (t+dt)\mathbf{PL} & T((\mathbf{PL})(\mathbf{PL})^\top + \boldsymbol{\Omega}) \end{pmatrix}^{-1} \begin{pmatrix} t\mathbf{I}_N \\ t\mathbf{PL} \end{pmatrix} \\
&= (dt) \cdot \mathbf{I}_N + o(dt),
\end{aligned}$$

thus, by letting $dt \to 0$, we get

$$\mathbb{V}[d\bar{\mathbf{B}}(t) \,|\, \bar{\mathbf{B}}(t)] = (dt)\mathbf{I}_N + o(dt). \tag{EC.14}$$

We show that $\bar{\mathbf{B}}(t)$ is a solution to a Stochastic Differential Equation where the drift (EC.13) and volatility (EC.14) using Levy's characterization of a Brownian motion as in Proposition 1. Recall that

$$\bar{\mathbf{B}}(t) = \mathbf{L}^{-1}(\mathbf{B}(t) - \mathbb{E}[\mathbf{B}(t)]).$$

Additionally, we can write

$$\begin{aligned}
\mathbf{X}^y(t) &= t\boldsymbol{\mu}^x + \mathbf{W}(t) \,|\, (\mathbf{Y}(0,T) = \mathbf{y}) \\
&= t\boldsymbol{\mu}^x + \mathbf{B}(t)
\end{aligned}$$

so

$$\bar{\mathbf{B}}(t) = \mathbf{L}^{-1}(\mathbf{X}^y(t) - \mathbb{E}[\mathbf{X}^y(t)]).$$

From (11), (EC.13), and (EC.14), $\bar{\mathbf{B}}(t)$ satisfies

$$d\bar{\mathbf{B}}(t) = -dt \cdot \bar{\boldsymbol{\beta}}_2(t)\bar{\mathbf{B}}(t)dt + d\mathbf{V}^y(t)$$

where $\mathbf{V}^y(t) = \mathbf{L}^{-1}\mathbf{W}^y(t)$ is a vector of N-independent Brownian motions, which concludes the proof. □

EC.2.6. Proof of Proposition 5

Consider the conditional process $\{\bar{\mathbf{B}}(t), t\in[0,T]\}$ satisfying (23), we showed in the proof of Theorem 1 that each element $\bar{B}_i(t)$, $i\in[N]$, is the restriction to $[0,T]$ of a Brownian bridge from 0 to 0 with hitting time

$$\tilde{T}_i = T\big((\mathbf{P}\mathbf{L}_i)^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+\boldsymbol{\Omega})^{-1}\mathbf{P}\mathbf{L}_i\big)^{-1} > 0,$$

Since $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ are positive definite, so too is $(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+\boldsymbol{\Omega})^{-1}$. Furthermore, as we assume that $\mathbf{P}\mathbf{L}_i\neq\mathbf{0}$, for $i\in[N]$, it follows that $\tilde{T}_i<\infty$. Now we show that the hitting times $\tilde{T}_i$ are strictly larger than the views horizon T.

For $i\in[N]$, we have

$$\tilde{T}_i = \frac{1}{H_{ii}}$$

where

$$\begin{aligned}\mathbf{H} &= \frac{1}{T}(\mathbf{P}\mathbf{L})^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+\boldsymbol{\Omega})^{-1}\mathbf{P}\mathbf{L}\\ &\overset{(a)}{=} \frac{1}{T}\big(\mathbf{I}_N-\mathbf{L}^{-1}(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P})^{-1}(\mathbf{L}^{-1})^\top\big)\end{aligned}$$

with (a) coming from the Woodbury matrix identity. Thus, the hitting time $\tilde{T}_i$ can be written as

$$\frac{1}{\tilde{T}_i} = \frac{1}{T}\big(1-\boldsymbol{\ell}_i^{-1}(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P})^{-1}(\boldsymbol{\ell}_i^{-1})^\top\big),$$

where $\boldsymbol{\ell}_i^{-1}$ is the i^{th} row of $\mathbf{L}^{-1}$, the inverse of the Cholesky decomposition matrix. Since $(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P})^{-1}$ is positive definite and $\boldsymbol{\ell}_i^{-1}\neq\mathbf{0}$ (because $\mathbf{L}^{-1}$ is invertible)

$$\boldsymbol{\ell}_i^{-1}(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P})^{-1}(\boldsymbol{\ell}_i^{-1})^\top > 0.$$

Therefore

$$\frac{1}{\tilde{T}_i} = \frac{1}{T}\Big(1-\boldsymbol{\ell}_i^{-1}(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P})^{-1}(\boldsymbol{\ell}_i^{-1})^\top\Big) < \frac{1}{T}. \tag{EC.15}$$

Since $\tilde{T}_i>0$ it follows that $\tilde{T}_i>T,\;$ for $i\in[N]$.

We now show that the hitting times are increasing in the covariance matrix $\boldsymbol{\Omega}$. Consider two positive definite matrices $\boldsymbol{\Omega}^1$ and $\boldsymbol{\Omega}^2$ such that $\boldsymbol{\Omega}^1\succeq\boldsymbol{\Omega}^2$ (the matrix $\boldsymbol{\Omega}^1-\boldsymbol{\Omega}^2$ is positive semi-definite). Let $\tilde{T}_i^1$ and $\tilde{T}_i^2$ be their respective hitting times. We first have $(\boldsymbol{\Omega}^2)^{-1}\succeq(\boldsymbol{\Omega}^1)^{-1}$. Since $\boldsymbol{\Sigma}$ is positive definite

$$(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top(\boldsymbol{\Omega}^1)^{-1}\mathbf{P})^{-1}\succeq(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top(\boldsymbol{\Omega}^2)^{-1}\mathbf{P})^{-1}$$

and hence

$$\boldsymbol{\ell}_i^{-1}(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top(\boldsymbol{\Omega}^1)^{-1}\mathbf{P})^{-1}(\boldsymbol{\ell}_i^{-1})^\top \geq \boldsymbol{\ell}_i^{-1}(\boldsymbol{\Sigma}^{-1}+\mathbf{P}^\top(\boldsymbol{\Omega}^2)^{-1}\mathbf{P})^{-1}(\boldsymbol{\ell}_i^{-1})^\top,\;\text{ for } i\in[N].$$

It follows from (EC.15) that $\tilde{T}_i^1\geq\tilde{T}_i^2,\;$ for $i\in[N]$. Furthermore, if $\boldsymbol{\Omega}^1\succ\boldsymbol{\Omega}^2$ (the matrix $\boldsymbol{\Omega}^1-\boldsymbol{\Omega}^2$ is strictly positive definite) $\tilde{T}_i^1>\tilde{T}_i^2,\;$ for $i\in[N]$. This concludes the proof. □

EC.2.7. Proof of the Results in Section 4.3

Consider the log-returns process satisfying (8)

$$\mathbf{X}(t) = t\boldsymbol{\mu}^x + \mathbf{W}(t),$$

where $\mathbf{W}(t) \sim \mathcal{N}\big(\mathbf{0}, t\boldsymbol{\Sigma}\big)$ a $N-$dimensional Brownian motion. Let $\mathbf{y}$ be the expert views vector sampled from (10). Conditional on $\mathbf{Y}(0,T) = \mathbf{y}$, we have

$$\begin{aligned}\mathbf{X}^y(t) &= \mathbf{X}(t) \,|\, (\mathbf{Y}(0,T) = \mathbf{y}) \\ &= t\boldsymbol{\mu}^x + \mathbf{W}(t) \,|\, (\mathbf{Y}(0,T) = \mathbf{y}) \\ &= t\boldsymbol{\mu}^x + \mathbf{B}(t),\end{aligned}$$

where $\mathbf{B}(t)$ satisfies (22). From Theorem 1, we can write

$$\mathbf{B}(t) = \mathbb{E}[\mathbf{B}(t)] + \mathbf{L}\bar{\mathbf{B}}(t),$$

where $\bar{\mathbf{B}}(t)$ is a zero mean stochastic process satisfying the SDE (23). Furthermore, we have

$$\begin{aligned}d\mathbf{X}^y(t) &= (dt)\boldsymbol{\mu}^x + d\mathbf{B}(t) \\ &\overset{(a)}{=} (dt)\boldsymbol{\mu}^x + (dt)\boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) + \mathbf{L}d\bar{\mathbf{B}}(t) \\ &= (dt)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \mathbf{L}\bar{\boldsymbol{\beta}}_2(t)\bar{\mathbf{B}}(t)\big) + \mathbf{L}d\mathbf{V}^y(t) \\ &\overset{(b)}{=} (dt)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \mathbf{L}\bar{\boldsymbol{\beta}}_2(t)\mathbf{L}^{-1}(\mathbf{B}(t) - \mathbb{E}[\mathbf{B}(t)])\big) + \mathbf{L}d\mathbf{V}^y(t),\end{aligned}$$

where (a) follows from (23) and (b) from (22). Since

$$\mathbf{X}^y(t) - \mathbb{E}[\mathbf{X}^y(t)] = \mathbf{B}(t) - \mathbb{E}[\mathbf{B}(t)],$$

and

$$\mathbf{L}\bar{\boldsymbol{\beta}}_2(t)\mathbf{L}^{-1} = \boldsymbol{\beta}_2(t),$$

the conditional log-returns is a solution to the following SDE

$$d\mathbf{X}^y(t) = (dt)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(t)(\mathbf{X}^y(t) - \mathbb{E}[\mathbf{X}^y(t)]\big) + \mathbf{L}d\mathbf{V}^y(t).$$

Since $\mathbf{V}^y(t)$ is defined by $\mathbf{L}\mathbf{V}^y(t) = \mathbf{W}^y(t)$ (see Equation (9)), it follows that

$$d\mathbf{X}^y(t) = (dt)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\beta}_2(t)(\mathbf{X}^y(t) - \mathbb{E}[\mathbf{X}^y(t)]\big) + d\mathbf{W}^y(t).$$

□

EC.3. Section 5

EC.3.1. Proof of Value function and Proposition 6

We show that the value function (29) with $\mathbf{A}(t)\in\mathbb{R}^{N\times N}$, $\mathbf{b}(t)\in\mathbb{R}^N$, and $c(t)$ being solutions of (30)–(32) is the solution of the HJB equation (27).

The optimal policy is the maximizer in the HJB (27):

$$\boldsymbol{\pi}^*(t) = -\frac{\nabla_z V}{z\nabla_z^2 V}\boldsymbol{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f\mathbf{1}_N) - \frac{1}{z\nabla_z^2 V}\nabla^2_{\mathbf{x},z}V$$

where

$$\begin{aligned}\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) &= \boldsymbol{\mu} + \boldsymbol{\beta}_1\big(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x\big) - \boldsymbol{\beta}_2(t)\big(\mathbf{x} - \mathbb{E}[\mathbf{X}(t)|\mathbf{y}]\big)\big) \\ &= \boldsymbol{\alpha}_t + \boldsymbol{\Sigma}\boldsymbol{\eta}_t\mathbf{x}\end{aligned}$$

with $\boldsymbol{\eta}_t : [0,T]\to\mathbb{R}^{N\times N}$ is

$$\begin{aligned}\boldsymbol{\eta}_t &= -\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}_2(t) \\ &= -\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P}\end{aligned} \tag{EC.16}$$

and $\boldsymbol{\alpha}_t : [0,T]\to\mathbb{R}^N$ is

$$\boldsymbol{\alpha}_t = \boldsymbol{\mu} + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) - \boldsymbol{\Sigma}\boldsymbol{\eta}_t\mathbb{E}[\mathbf{X}^y(t)]. \tag{EC.17}$$

Observe that $\boldsymbol{\eta}_t$ is a symmetric matrix. From (29), we have

$$\begin{cases}\dfrac{\partial V}{\partial t} &= (\dfrac{1}{2}\mathbf{x}^\top\mathbf{A}'(t)\mathbf{x} + \mathbf{x}^\top\mathbf{b}'(t) + c'(t))V, \\ \nabla_z V &= \dfrac{1-\gamma}{z}V, \\ \nabla_{\mathbf{x}}V &= (\mathbf{A}(t)\mathbf{x} + \mathbf{b}(t))V, \\ \nabla_z^2 V &= \dfrac{-\gamma(1-\gamma)}{z^2}V, \\ \nabla_{\mathbf{x}}^2 V &= (\mathbf{A}(t) + (\mathbf{A}(t)\mathbf{x} + \mathbf{b}(t))(\mathbf{A}(t)\mathbf{x} + \mathbf{b}(t))^\top)V, \\ \nabla^2_{\mathbf{x},z}V &= \dfrac{1-\gamma}{z}(\mathbf{A}(t)\mathbf{x} + \mathbf{b}(t))V.\end{cases}$$

The HJB equation (27) becomes

$$\begin{aligned}0 = \frac{1}{2}\mathbf{x}^\top&\left\{\mathbf{A}'(t) + \frac{1-\gamma}{\gamma}\boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t + \frac{1}{\gamma}(\mathbf{A}(t)\boldsymbol{\Sigma}\boldsymbol{\eta}_t + \boldsymbol{\eta}_t\boldsymbol{\Sigma}\mathbf{A}(t)) + \frac{1}{\gamma}\mathbf{A}(t)\boldsymbol{\Sigma}\mathbf{A}(t)\right\}\mathbf{x} \\ &+\mathbf{x}^\top\left\{\mathbf{b}'(t) + \frac{1}{\gamma}\big(\boldsymbol{\eta}_t + \mathbf{A}(t)\big)\boldsymbol{\Sigma}\mathbf{b}(t) + \frac{1-\gamma}{\gamma}\big(\boldsymbol{\eta}_t + \mathbf{A}(t)\big)(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N) + \mathbf{A}(t)\big(\boldsymbol{\alpha}_t - \frac{\mathrm{diag}(\boldsymbol{\Sigma})}{2}\big)\right\} \\ &+c'(t) + (1-\gamma)r_f + \frac{1}{2}\operatorname{Tr}\big(\mathbf{A}(t)\boldsymbol{\Sigma}\big) + \frac{1-\gamma}{2\gamma}\big(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N\big)^\top\boldsymbol{\Sigma}^{-1}\big(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N\big) + \big(\boldsymbol{\alpha}_t - \frac{\mathrm{diag}(\boldsymbol{\Sigma})}{2}\big)^\top\mathbf{b}(t) \\ &+\frac{1-\gamma}{\gamma}\big(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N\big)^\top\mathbf{b}(t) + \frac{1}{2\gamma}\mathbf{b}^\top(t)\boldsymbol{\Sigma}\mathbf{b}(t)\end{aligned}$$

with terminal conditions $\mathbf{A}(T) = \mathbf{0}$, $\mathbf{b}(T) = \mathbf{0}$ and $c(T) = 0$. It follows that if $\mathbf{A}(t)$, $\mathbf{b}(t)$ and $c(t)$ satisfy (30)–(32), that (29) is the solution of (27).

To see that $\mathbf{A}(t)$ is strictly negative definite when $t<T$, consider the linear quadratic problem

$$\tilde{V}(t,\mathbf{x}) = \min_{\mathbf{u}} \int_t^T \left\{ \mathbf{x}_t^\top \left(-\frac{1-\gamma}{\gamma} \boldsymbol{\eta}_t^\top \boldsymbol{\Sigma} \boldsymbol{\eta}_t \right) \mathbf{x}_t + \frac{1}{\gamma} \mathbf{u}_t^\top \boldsymbol{\Sigma} \mathbf{u}_t \right\} dt$$

where

$$\begin{aligned} d\mathbf{x}_t &= \frac{1}{\gamma} \boldsymbol{\Sigma} \boldsymbol{\eta}_t \mathbf{x}_t + \frac{1}{\gamma} \boldsymbol{\Sigma} \mathbf{u}_t \\ \mathbf{x}_t &= \mathbf{x}. \end{aligned}$$

Since $-\frac{1-\gamma}{\gamma} \boldsymbol{\eta}_t^\top \boldsymbol{\Sigma} \boldsymbol{\eta}_t \succeq \mathbf{0}$ and $\boldsymbol{\Sigma} \succ \mathbf{0}$ when $t<T$, $\tilde{V}(t,\mathbf{x})>0$ unless $\mathbf{x}=\mathbf{0}$. It can be shown that $\tilde{V}(t,\mathbf{x}) = -\mathbf{x}^\top \mathbf{A}(t)\mathbf{x}$ which implies that $\mathbf{A}(t)$ is strictly negative definite when $t<T$.

Given (29), the optimal investment policy is

$$\boldsymbol{\pi}^*(t) = \frac{1}{\gamma} \big(\boldsymbol{\Sigma}^{-1} (\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f \mathbf{1}_N) + \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t) \big).$$

which completes the proof. □

EC.3.2. Proof of Theorem 2

We derive explicit expressions for the ODEs (30), (31) and the hedging demand (35).

Explicit solution of Riccati equation (30). The following results will be useful.

LEMMA EC.1. *Suppose Assumption 1 holds and $\boldsymbol{\eta}_t$ is given by (EC.16). Then $\boldsymbol{\eta}_t$ is symmetric and negative semi-definite for $t\in[0,T]$ and*

$$\begin{aligned} \frac{d\boldsymbol{\eta}_t}{dt} &= -\boldsymbol{\eta}_t \boldsymbol{\Sigma} \boldsymbol{\eta}_t \\ \boldsymbol{\eta}_T &= -\mathbf{P}^\top (T\boldsymbol{\Omega})^{-1} \mathbf{P}. \end{aligned}$$

If

$$\boldsymbol{\zeta}_t = -\mathbf{P}^\top \left(\frac{1}{\gamma}(T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega} \right)^{-1} \mathbf{P} \tag{EC.18}$$

then

$$\begin{cases} \boldsymbol{\zeta}_t' + \frac{1}{\gamma} \boldsymbol{\zeta}_t \boldsymbol{\Sigma} \boldsymbol{\zeta}_t = \mathbf{0} \\ \boldsymbol{\zeta}_T = \boldsymbol{\eta}_T. \end{cases} \tag{EC.19}$$

Proof It is clear that $\boldsymbol{\eta}_t$ is symmetric and negative semi-definite. Additionally, for an invertible matrix $\mathbf{R}(t) \in \mathbb{R}^{K\times K}$

$$\frac{d}{dt}(\mathbf{R}^{-1}(t)) = -\mathbf{R}^{-1}(t) \frac{d}{dt}(\mathbf{R}(t)) \mathbf{R}^{-1}(t). \tag{EC.20}$$

If

$$\mathbf{R}(t) = (T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega} \in \mathbb{R}^{K\times K},$$

$$\begin{aligned}\frac{d}{dt}\boldsymbol{\eta}_t &= -\mathbf{P}^\top \frac{d}{dt}(\mathbf{R}^{-1}(t))\mathbf{P}\\ &= \mathbf{P}^\top \mathbf{R}^{-1}(t)\frac{d}{dt}(\mathbf{R}(t))\mathbf{R}^{-1}(t)\mathbf{P}\\ &= -\boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t.\end{aligned}$$

The ODE for $\boldsymbol{\zeta}_t$ can be derived similarly. □

Let

$$\mathbf{A}(t) = \boldsymbol{\zeta}_t - \boldsymbol{\eta}_t. \tag{EC.21}$$

Since $\mathbf{A}(T) = \mathbf{0}$ and

$$\begin{aligned}\frac{d}{dt}\mathbf{A}(t) &= -\frac{1}{\gamma}\boldsymbol{\zeta}_t\boldsymbol{\Sigma}\boldsymbol{\zeta}_t + \boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t\\ &= -\frac{1-\gamma}{\gamma}\boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t - \frac{1}{\gamma}\Big(\mathbf{A}(t)\boldsymbol{\Sigma}\boldsymbol{\eta}_t + \boldsymbol{\eta}_t\boldsymbol{\Sigma}\mathbf{A}(t)\Big) - \frac{1}{\gamma}\mathbf{A}(t)\boldsymbol{\Sigma}\mathbf{A}(t)\end{aligned}$$

it follows that the solution of (30) is (EC.21).

Now we show that

$$\mathbf{A}(t) = \boldsymbol{\zeta}_t - \boldsymbol{\eta}_t = \mathbf{M}(t)\boldsymbol{\eta}_t$$

where

$$\mathbf{M}(t) = (\gamma-1)(1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\Big(\gamma\boldsymbol{\Sigma}^{-1} + (1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\Big)^{-1} \in \mathbb{R}^{N\times N}. \tag{EC.22}$$

Observe from (EC.21) that

$$\mathbf{A}(t) = -\mathbf{P}^\top(\mathbf{C}(t)^{-1} - \mathbf{F}(t)^{-1})\mathbf{P},$$

where

$$\begin{aligned}\mathbf{C}(t) &= \frac{T-t}{\gamma}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega},\\ \mathbf{F}(t) &= (T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega}.\end{aligned}$$

Since

$$\begin{aligned}\mathbf{C}(t)^{-1} &= \Big(\frac{1-\gamma}{\gamma}(T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \mathbf{F}(t)\Big)^{-1}\\ &\overset{(a)}{=} \mathbf{F}(t)^{-1} - \mathbf{F}(t)^{-1}\mathbf{P}\Big(\frac{\gamma}{(1-\gamma)(T-t)}\boldsymbol{\Sigma}^{-1} + \mathbf{P}^\top\mathbf{F}(t)^{-1}\mathbf{P}\Big)^{-1}\mathbf{P}^\top\mathbf{F}(t)^{-1},\end{aligned}$$

where (a) follows from the Woodbury identity, it follows that

$$\begin{aligned}\mathbf{A}(t) &= -\mathbf{P}^\top(\mathbf{C}(t)^{-1} - \mathbf{F}(t)^{-1})\mathbf{P}\\ &= \mathbf{P}^\top\mathbf{F}(t)^{-1}\mathbf{P}\Big(\frac{\gamma}{(1-\gamma)(T-t)}\boldsymbol{\Sigma}^{-1} + \mathbf{P}^\top\mathbf{F}(t)^{-1}\mathbf{P}\Big)^{-1}\mathbf{P}^\top\mathbf{F}(t)^{-1}\mathbf{P}\\ &= -\boldsymbol{\eta}_t\Big(\frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\eta}_t\Big)^{-1}\boldsymbol{\eta}_t.\end{aligned}$$

The Woodbury identity also implies that

$$\begin{aligned}-\boldsymbol{\eta}_t\Big(\frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\eta}_t\Big)^{-1} &= (\gamma-1)(1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\Big(\gamma\boldsymbol{\Sigma}^{-1} + (1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\Big)^{-1}\\ &= \mathbf{M}(t)\end{aligned} \tag{EC.23}$$

so $\mathbf{A}(t) = \mathbf{M}(t)\boldsymbol{\eta}_t$, where $\mathbf{M}(t)$ satisfies (EC.22).

Explicit solution of ODE (31). The following result will be useful.

LEMMA EC.2. *Let*

$$\mathbf{s}(t)=\Big(\frac{\gamma}{(\gamma-1)(T-t)}\mathbf{\Sigma}^{-1}+\boldsymbol{\eta}_t\Big)^{-1}\in\mathbb{R}^{N\times N}.$$

Then the matrices $\boldsymbol{\eta}_t\mathbf{\Sigma}\in\mathbb{R}^{N\times N}$ *and* $\boldsymbol{\eta}_t\mathbf{s}(t)\in\mathbb{R}^{N\times N}$ *commute for all* $t\in[0,T)$*, i.e.,*

$$(\boldsymbol{\eta}_t\mathbf{\Sigma})(\boldsymbol{\eta}_t\mathbf{s}(t))=(\boldsymbol{\eta}_t\mathbf{s}(t))(\boldsymbol{\eta}_t\mathbf{\Sigma}),\ \textit{for}\ t\in[0,T).$$

Proof We have

$$\mathbf{s}(t)=\big(\lambda_t\mathbf{\Sigma}^{-1}+\boldsymbol{\eta}_t\big)^{-1}\in\mathbb{R}^{N\times N},$$

with $\lambda_t=\frac{\gamma}{(\gamma-1)(T-t)}$. We can then write

$$\begin{aligned}\boldsymbol{\eta}_t\mathbf{s}(t)\boldsymbol{\eta}_t\mathbf{\Sigma}&=\boldsymbol{\eta}_t\mathbf{s}(t)\big(\boldsymbol{\eta}_t+\lambda_t\mathbf{\Sigma}^{-1}-\lambda_t\mathbf{\Sigma}^{-1}\big)\mathbf{\Sigma}\\&=\boldsymbol{\eta}_t\mathbf{\Sigma}(\mathbf{I}_N-\lambda_t\mathbf{\Sigma}^{-1}\mathbf{s}(t))\\&=\boldsymbol{\eta}_t\mathbf{\Sigma}\boldsymbol{\eta}_t\mathbf{s}(t)\end{aligned}$$

which completes the proof. □

We now show that

$$\mathbf{b}(t)=\mathbf{M}(t)\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha}_t-r_f\mathbf{1}_N)$$

is the unique solution of (31). (Recall that $\mathbf{A}(t)=\mathbf{M}(t)\boldsymbol{\eta}_t$).

Since $\mathbf{M}(T)=\mathbf{0}$, $\mathbf{b}(T)=\mathbf{0}$, the terminal condition is satisfied. Differentiating with respect to t

$$\mathbf{b}'(t)=\mathbf{M}'(t)\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha}_t-r_f\mathbf{1}_N)+\mathbf{M}(t)\mathbf{\Sigma}^{-1}\frac{\partial\boldsymbol{\alpha}_t}{\partial t}.$$

From (EC.17) and Lemma EC.1

$$\frac{d\boldsymbol{\alpha}_t}{dt}=-\boldsymbol{\eta}_t(\boldsymbol{\alpha}_t-\frac{1}{2}\operatorname{diag}(\mathbf{\Sigma})),$$

so

$$\mathbf{b}'(t)=\mathbf{M}'(t)\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha}_t-r_f\mathbf{1}_N)-\mathbf{A}(t)(\boldsymbol{\alpha}_t-\frac{1}{2}\operatorname{diag}(\mathbf{\Sigma})).$$

Therefore, we show $\mathbf{b}(t)$ satisfies (37) by showing

$$\mathbf{M}'(t)\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha}_t-r_f\mathbf{1}_N)=-\frac{1}{\gamma}\big(\boldsymbol{\eta}_t+\mathbf{A}(t)\big)\mathbf{\Sigma}\mathbf{b}(t)-\frac{1-\gamma}{\gamma}\big(\boldsymbol{\eta}_t+\mathbf{A}(t)\big)(\boldsymbol{\alpha}_t-r_f\mathbf{1}_N).\qquad\text{(EC.24)}$$

Define

$$\mathbf{s}(t)=\big(\frac{\gamma}{(\gamma-1)(T-t)}\mathbf{\Sigma}^{-1}+\boldsymbol{\eta}_t\big)^{-1}\in\mathbb{R}^{N\times N}.$$

Observe $\mathbf{s}(t)$ is invertible and $\mathbf{M}(t) = -\boldsymbol{\eta}_t\mathbf{s}(t)$ (EC.23). It follows that

$$\begin{aligned}\mathbf{M}'(t) &= -(\boldsymbol{\eta}_t\mathbf{s}(t))' \\ &\overset{(a)}{=} \boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t - \boldsymbol{\eta}_t\mathbf{s}'(t) \\ &\overset{(b)}{=} \boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t + \boldsymbol{\eta}_t\mathbf{s}(t)(\mathbf{s}^{-1}(t))'\mathbf{s}(t) \\ &= \boldsymbol{\eta}_t\boldsymbol{\Sigma}\boldsymbol{\eta}_t - \frac{1-\gamma}{\gamma}\boldsymbol{\eta}_t\big(\mathbf{I}_N - \mathbf{s}(t)\boldsymbol{\eta}_t\big)\boldsymbol{\Sigma}\big(\mathbf{I}_N - \boldsymbol{\eta}_t\mathbf{s}(t)\big),\end{aligned} \tag{EC.25}$$

where (a) follow from Lemma EC.1 and (b) from the expression of the derivative of the inverse. Because of Lemma EC.2

$$\begin{aligned}\mathbf{M}'(t) &\overset{(b)}{=} -\frac{1}{\gamma}(\boldsymbol{\eta}_t + \mathbf{M}(t)\boldsymbol{\eta}_t)\boldsymbol{\Sigma}\mathbf{M}(t) - \frac{1-\gamma}{\gamma}(\boldsymbol{\eta}_t + \mathbf{M}(t)\boldsymbol{\eta}_t)\boldsymbol{\Sigma} \\ &\overset{(c)}{=} -\frac{1}{\gamma}(\boldsymbol{\eta}_t + \mathbf{A}(t))\boldsymbol{\Sigma}\mathbf{M}(t) - \frac{1-\gamma}{\gamma}(\boldsymbol{\eta}_t + \mathbf{A}(t))\boldsymbol{\Sigma},\end{aligned}$$

where (b) from the definition of $\mathbf{s}(t)$ and (c) from the expression of $\mathbf{A}(t)$. It follows that

$$\begin{aligned}\mathbf{M}'(t)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N) &= -\frac{1}{\gamma}(\boldsymbol{\eta}_t + \mathbf{A}(t))\boldsymbol{\Sigma}\mathbf{M}(t)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N) - \frac{1-\gamma}{\gamma}(\boldsymbol{\eta}_t + \mathbf{A}(t))(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N) \\ &= -\frac{1}{\gamma}(\boldsymbol{\eta}_t + \mathbf{A}(t))\boldsymbol{\Sigma}\mathbf{b}(t) - \frac{1-\gamma}{\gamma}(\boldsymbol{\eta}_t + \mathbf{A}(t))(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N).\end{aligned}$$

which shows that (EC.24) holds and hence that $\mathbf{b}(t)$ is a solution of (31). Since (31) is a linear ODE with coefficients that are bounded, this solution is unique.

Hedging demand (35). From (37) the hedging demand can be written

$$\begin{aligned}\frac{1}{\gamma}\frac{\partial g}{\partial \mathbf{x}}(t,\mathbf{x}) &= \frac{1}{\gamma}(\mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)) \\ &= \frac{1}{\gamma}(\mathbf{M}(t)\boldsymbol{\eta}_t\mathbf{x} + \mathbf{M}(t)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}_t - r_f\mathbf{1}_N)) \\ &\overset{(b)}{=} \frac{1}{\gamma}(\mathbf{M}(t)\boldsymbol{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f\mathbf{1}_N)) \\ &\overset{(c)}{=} \mathbf{M}(t)\boldsymbol{\pi}^*_{MV}(t),\end{aligned}$$

where (b) follows from the definition of $\boldsymbol{\alpha}_t$ and $\boldsymbol{\eta}_t$, and (c) from the definition of the mean-variance term

$$\boldsymbol{\pi}^*_{MV}(t) = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f\mathbf{1}_N).$$

□

EC.3.3. Proof of Corollary 1

From (39) in Theorem 2

$$\frac{\partial g}{\partial \mathbf{x}} = \mathbf{M}(t)\boldsymbol{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f\mathbf{1}_N).$$

where $\mathbf{M}(t)$ is given by (EC.22). It follows from (38) that

$$\boldsymbol{\pi}^*(t) = \frac{1}{\gamma}\big(\mathbf{I}_N + \mathbf{M}(t)\big)\boldsymbol{\Sigma}^{-1}(\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f\mathbf{1}_N).$$

Now we show that

$$\Big(\big(\mathbf{I}_N + \mathbf{M}(t)\big)\boldsymbol{\Sigma}^{-1}\Big)^{-1} \equiv \big(\boldsymbol{\Sigma}^{-1} + (1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\big)^{-1} + \frac{1}{\gamma}\big(\boldsymbol{\Sigma} - \big(\boldsymbol{\Sigma}^{-1} + (1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\big)^{-1}\big)$$
$$= \boldsymbol{\Sigma}_{\text{DBL}}.$$

From (EC.23)

$$\mathbf{M}(t) = -\boldsymbol{\eta}_t\big(\frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\eta}_t\big)^{-1}$$
$$= -\mathbf{I}_N + \frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1}\big(\frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\eta}_t\big)^{-1}.$$

Since $\mathbf{M}(t)$ is positive semi-definite when $\gamma > 1$ (see (EC.22)), follows that $\mathbf{I}_N + \mathbf{M}(t)$ is positive definite invertible. We can then write

$$\Big(\big(\mathbf{I}_N + \mathbf{M}(t)\big)\boldsymbol{\Sigma}^{-1}\Big)^{-1} = \boldsymbol{\Sigma}\big(\mathbf{I}_N + \mathbf{M}(t)\big)^{-1}$$
$$= \frac{(\gamma-1)(T-t)}{\gamma}\boldsymbol{\Sigma}\left(\frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1} - \mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P}\right)\boldsymbol{\Sigma}.$$

Woodbury's identity implies

$$\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P} = \frac{1}{T-t}\boldsymbol{\Sigma}^{-1} - \frac{1}{T-t}\boldsymbol{\Sigma}^{-1}\big(\boldsymbol{\Sigma}^{-1} + (1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\big)^{-1}\boldsymbol{\Sigma}^{-1}$$

from which it follows that

$$\Big(\big(\mathbf{I}_N + \mathbf{M}(t)\big)\boldsymbol{\Sigma}^{-1}\Big)^{-1} = \frac{(\gamma-1)(T-t)}{\gamma}\boldsymbol{\Sigma}\left(\frac{\gamma}{(\gamma-1)(T-t)}\boldsymbol{\Sigma}^{-1} - \mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega})^{-1}\mathbf{P}\right)\boldsymbol{\Sigma}$$
$$= \frac{1}{\gamma}\boldsymbol{\Sigma} + (1-\frac{1}{\gamma})\left(\boldsymbol{\Sigma}^{-1} + (1-\frac{t}{T})\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}$$
$$= \boldsymbol{\Sigma}_{\text{DBL}}.$$

Therefore, we have

$$\boldsymbol{\pi}^*(t) = \frac{1}{\gamma}\boldsymbol{\Sigma}_{\text{DBL}}^{-1}(\tilde{\boldsymbol{\mu}}(t,\mathbf{x}) - r_f\mathbf{1}_N),$$

which completes the proof. □

EC.3.4. Single-Period Black-Litterman

From (8), the log-returns follow

$$\mathbf{X}(t) = t\boldsymbol{\mu}^x + \mathbf{W}(t),$$

where $\mathbf{W}(t)$ is a Brownian motion with covariance matrix $\boldsymbol{\Sigma}$. Then for $T \geq t$ we have

$$\mathbf{X}(T) = \mathbf{X}(t) + (T-t)\boldsymbol{\mu}^x + \big(\mathbf{W}(T) - \mathbf{W}(t)\big). \tag{EC.26}$$

The investor's views are modeled by

$$\mathbf{Y}(0,T) = \mathbf{P}\,\mathbf{X}(T) + \boldsymbol{\epsilon},$$

with $\boldsymbol{\epsilon}$ being independent Gaussian noise with $\mathbf{0}$ mean and covariance $T\boldsymbol{\Omega}$. Inserting (EC.26) into the view, we write

$$\mathbf{Y}(0,T) = \mathbf{P}\Big(\mathbf{X}(t) + (T-t)\boldsymbol{\mu}^x + \big(\mathbf{W}(T) - \mathbf{W}(t)\big)\Big) + \boldsymbol{\epsilon}. \tag{EC.27}$$

Since $\mathbf{X}(T)$ and $\mathbf{Y}(0,T)$ are jointly Gaussian, the conditional expectation is given by

$$\begin{aligned}\mathbb{E}\Big[\mathbf{X}(T)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big] &= \mathbb{E}\Big[\mathbf{X}(T)\,\Big|\,\mathbf{X}(t)\Big] \\ &+ \mathrm{Cov}\Big(\mathbf{X}(T),\mathbf{Y}(0,T)\,|\,\mathbf{X}(t)\Big)\mathbb{V}\Big[\mathbf{Y}(0,T)\,|\,\mathbf{X}(t)\Big]^{-1}\Big(\mathbf{y} - \mathbb{E}\big[\mathbf{Y}(0,T)\,|\,\mathbf{X}(t)\big]\Big).\end{aligned} \tag{EC.28}$$

Derivation of the Conditional Mean From (EC.26) we have

$$\mathbb{E}\Big[\mathbf{X}(T)\,\Big|\,\mathbf{X}(t)\Big] = \mathbf{X}(t) + (T-t)\boldsymbol{\mu}^x.$$

It follows from (EC.27) that

$$\begin{aligned}\mathbb{E}\Big[\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big] &= \mathbf{P}\Big(\mathbf{X}(t) + (T-t)\boldsymbol{\mu}^x\Big), \\ \mathbb{V}\Big[\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big] &= (T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega},\end{aligned}$$

and

$$\mathrm{Cov}\Big(\mathbf{X}(T),\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big) = \mathrm{Cov}\Big(\mathbf{W}(T)-\mathbf{W}(t),\,\mathbf{P}\big(\mathbf{W}(T)-\mathbf{W}(t)\big)\Big) = (T-t)\boldsymbol{\Sigma}\mathbf{P}^\top.$$

Substituting in (EC.28) it can be shown that

$$\mathbb{E}\Big[\mathbf{X}(T) - \mathbf{X}(t)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big] = (T-t)\,\tilde{\boldsymbol{\mu}}^x(t,\mathbf{X}(t)) \tag{EC.29}$$

where

$$\begin{aligned}\tilde{\boldsymbol{\mu}}^x(t,\mathbf{x}) &= \boldsymbol{\mu}^x + \boldsymbol{\beta}_1(\mathbf{y} - T\mathbf{P}\boldsymbol{\mu}^x) + \boldsymbol{\beta}_2(t)(\mathbb{E}[\mathbf{X}^y(t)] - \mathbf{x}) \\ \boldsymbol{\beta}_1 &= \frac{1}{T}\boldsymbol{\Sigma}\mathbf{P}^\top(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega})^{-1}, \\ \boldsymbol{\beta}_2(t) &= \boldsymbol{\Sigma}\mathbf{P}^\top\left((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\end{aligned}$$

Derivation of the Conditional Covariance For jointly Gaussian variables the conditional covariance is given by

$$\begin{aligned}\mathbb{V}\Big[\mathbf{X}(T)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big] &= \mathbb{V}\Big[\mathbf{X}(T)\,\Big|\,\mathbf{X}(t)\Big] \\ &- \mathrm{Cov}\Big(\mathbf{X}(T),\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big)\mathbb{V}\Big[\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big]^{-1}\mathrm{Cov}\Big(\mathbf{Y}(0,T),\mathbf{X}(T)\,\Big|\,\mathbf{X}(t)\Big).\end{aligned}$$

Since $\mathbb{V}\Big[\mathbf{X}(T)\,\Big|\,\mathbf{X}(t)\Big] = (T-t)\boldsymbol{\Sigma}$, and using the expressions

$$\mathrm{Cov}\Big(\mathbf{X}(T),\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big) = (T-t)\boldsymbol{\Sigma}\mathbf{P}^\top, \quad \mathbb{V}\Big[\mathbf{Y}(0,T)\,\Big|\,\mathbf{X}(t)\Big] = (T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + T\boldsymbol{\Omega},$$

we deduce that

$$\mathbb{V}\Big[\mathbf{X}(T)-\mathbf{X}(t)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big]=(T-t)\boldsymbol{\Sigma}-(T-t)^2\boldsymbol{\Sigma}\mathbf{P}^\top\Big((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+T\boldsymbol{\Omega}\Big)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{EC.30}$$

By applying Woodbury's matrix identity to the right-hand side of (EC.30), we write

$$\mathbb{V}\Big[\mathbf{X}(T)-\mathbf{X}(t)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big]=(T-t)\left(\boldsymbol{\Sigma}^{-1}+\left(1-\frac{t}{T}\right)\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}.$$

Thus, we define $\boldsymbol{\Sigma}_{BL|t}=\left(\boldsymbol{\Sigma}^{-1}+\left(1-\frac{t}{T}\right)\mathbf{P}^\top\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}$.

Optimal Portfolio Policy In the single-period setting, an investor with risk-aversion parameter $\gamma>0$ maximizes the objective

$$\begin{aligned}\max_{\boldsymbol{\pi}}\quad &\boldsymbol{\pi}^\top\,\mathbb{E}\Big[\mathbf{X}(T)-\mathbf{X}(t)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big]+\Big(1-\boldsymbol{\pi}^\top\mathbf{1}_N\Big)r_f\\ &-\frac{\gamma}{2}\boldsymbol{\pi}^\top\,\mathbb{V}\Big[\mathbf{X}(T)-\mathbf{X}(t)\,\Big|\,\mathbf{X}(t),\,\mathbf{Y}(0,T)=\mathbf{y}\Big]\boldsymbol{\pi},\end{aligned}$$

where r_f is the risk-free rate. Using (EC.29) and (EC.30), the optimal portfolio is

$$\boldsymbol{\pi}^*_{BL|t}=\frac{1}{\gamma}\,(\boldsymbol{\Sigma}_{BL|t})^{-1}\Big(\tilde{\boldsymbol{\mu}}^x(t,\mathbf{x})-r_f\mathbf{1}_N\Big).$$

□

EC.4. Section 6

EC.4.1. Section 6.1

EC.4.1.1. Proof of Proposition 7 Recall that the state variable $\mathbf{I}(t)=\mathbf{Y}^j(t_j,T)$ is the most recent expert view and

$$\bar{\mathbf{X}}(t)=\mathbf{X}(t)-\mathbf{X}(t_j)$$

is the log-returns over the horizon $[t_j,t_{j+1})$. The following result shows that $\bar{\mathbf{X}}(t)$ given the most recent view $\mathbf{I}(t)$ is independent of earlier views.

Lemma EC.3. *For $t\in[t_j,t_{j+1})$, given the most recent view $\mathbf{Y}^j(t_j,T)$, the process $\bar{\mathbf{X}}(t)$ is independent of all previous views $\mathbf{Y}^k(t_k,T)$, $k\in\{0,\dots,j-1\}$.*

The proof is given in Appendix EC.5.

From Lemma EC.3, it follows that

$$\bar{\mathbf{X}}^y(t)\triangleq\bar{\mathbf{X}}(t)\,|\,\mathcal{F}^{\mathbf{Y}}_t=\bar{\mathbf{X}}(t)\,|\,\mathbf{I}(t).$$

At time $t=t_j$, the investor receives the view $\mathbf{Y}^j(t_j,T)=\mathbf{y}^j$ with covariance matrix $\boldsymbol{\Omega}^j$. It follows from Proposition 1 that the dynamics of $\bar{\mathbf{X}}^y(t)$ on $[t_j,t_{j+1})$ are given by

$$d\bar{\mathbf{X}}^y(t)=\Big(\boldsymbol{\mu}^x+\boldsymbol{\beta}_1^j\big(\mathbf{y}^j-(T-t_j)\mathbf{P}\boldsymbol{\mu}^x\big)+\boldsymbol{\beta}_2^j(t)\big(\mathbb{E}[\bar{\mathbf{X}}^y(t)]-\bar{\mathbf{X}}^y(t)\big)\Big)dt+d\mathbf{W}^y(t) \tag{EC.31}$$

where

$$\boldsymbol{\beta}_1^j = \boldsymbol{\Sigma}\mathbf{P}^\top\big((T-t_j)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\big)^{-1} \in \mathbb{R}^{N\times K},$$

$$\boldsymbol{\beta}_2^j(t) = \boldsymbol{\Sigma}\mathbf{P}^\top\big((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\big)^{-1}\mathbf{P} \in \mathbb{R}^{N\times N},$$

$$\mathbb{E}[\bar{\mathbf{X}}^y(t)] = (t-t_j)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1^j\big(\mathbf{y}^j - (T-t_j)\mathbf{P}\boldsymbol{\mu}^x\big)\big).$$

This completes the proof. □

EC.4.1.2. Proof of Theorem 3 We show the value function is of the form

$$V(t,z,\bar{\mathbf{x}},\mathbf{y}) = U(z)\exp\left(g^j(t,\bar{\mathbf{x}},\mathbf{y})\right), \quad t\in[t_j,t_{j+1})$$

where $g^j : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^K \to \mathbb{R}$ is quadratic in the log-returns vector $\mathbf{x}$ and views $\mathbf{y}$

$$g^j(t,\bar{\mathbf{x}},\mathbf{y}) = \frac{1}{2}\bar{\mathbf{x}}^\top\mathbf{A}^j(t)\bar{\mathbf{x}} + \bar{\mathbf{x}}^\top\left(\mathbf{B}^j(t)\mathbf{y} + \bar{\mathbf{b}}^j(t)\right) + \left(-\frac{1}{2}\mathbf{y}^\top\bar{\mathbf{C}}^j(t)\mathbf{y} + \mathbf{y}^\top\hat{\mathbf{c}}^j(t) + \bar{c}^j(t)\right). \tag{EC.32}$$

We derive ODEs for the coefficients of this function using dynamic programming; specifically, the value function solves the HJB equation in each interval (t_j,t_{j+1}) with the terminal condition $V(t_{j+1}^-,\bar{\mathbf{x}},\mathbf{y})$ determined from the value function for the next interval $[t_{j+1},t_{j+2}]$ at time t_{j+1} (48). The terminal condition for the last interval $[t_M,T]$ is $V(T,\bar{\mathbf{x}},\mathbf{y})=0$.

We begin with the last interval $[t_M,T]$ $(j=M)$. Since $V(T,\bar{\mathbf{x}},\mathbf{y})=0$, then $g(T,\bar{\mathbf{x}},\mathbf{y})=0$, and

$$\mathbf{A}^M(T)=\mathbf{0}_{N\times N},\ \mathbf{B}^M(T)=\mathbf{0}_{N\times K},\ \mathbf{C}^M(T)=\mathbf{0}_{K\times K},\ \bar{\mathbf{b}}^M(T)=\mathbf{0}_N,\ \hat{\mathbf{c}}^M(T)=\mathbf{0}_K,\ \bar{c}^M(T)=0.$$

For the last interval $[t_M,T]$, we can use Proposition 6 directly with the views $\mathbf{Y}(t_M,T)=\mathbf{y}$ and views covariance $\boldsymbol{\Omega}^j$. It follows that $\mathbf{A}^j(t)$ $(j=M)$ is the solution of the Riccati equation (30), $\mathbf{B}^j(t)\mathbf{y}+\bar{\mathbf{b}}^j(t)$ satisfies the ODE (31), and

$$-\frac{1}{2}\mathbf{y}^\top\bar{\mathbf{C}}^j(t)\mathbf{y} + \mathbf{y}^\top\hat{\mathbf{c}}^j(t) + \bar{c}^j(t)$$

solves (32). That is, $\mathbf{A}^j(t)\in\mathbb{R}^{N\times N}$ is negative semi-definite with

$$\mathbf{A}^{j\prime}(t) + \frac{1-\gamma}{\gamma}\boldsymbol{\eta}_t^j\boldsymbol{\Sigma}\boldsymbol{\eta}_t^j + \frac{1}{\gamma}\left(\mathbf{A}^j(t)\boldsymbol{\Sigma}\boldsymbol{\eta}_t^j + \boldsymbol{\eta}_t^j\boldsymbol{\Sigma}\mathbf{A}^j(t)\right) + \frac{1}{\gamma}\mathbf{A}^j(t)\boldsymbol{\Sigma}\mathbf{A}^j(t) = \mathbf{0} \tag{EC.33}$$

where

$$\boldsymbol{\eta}_t^j = -\mathbf{P}^\top((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j)^{-1}\mathbf{P};$$

$\mathbf{B}^j(t)\in\mathbb{R}^{N\times K}$ and $\bar{\mathbf{b}}^j(t)\in\mathbb{R}^N$ are solutions of

$$\mathbf{B}^{j\prime}(t) + \frac{1}{\gamma}(\boldsymbol{\eta}_t^j + \mathbf{A}^j(t))\boldsymbol{\Sigma}\mathbf{B}^j(t) + \frac{1-\gamma}{\gamma}\boldsymbol{\eta}_t^j\boldsymbol{\alpha}_1^j(t) + \frac{1}{\gamma}\mathbf{A}^j(t)\boldsymbol{\alpha}_1^j(t) = \mathbf{0}, \tag{EC.34}$$

and

$$\bar{\mathbf{b}}^{j'}(t) + \frac{1}{\gamma}(\boldsymbol{\eta}_t^j + \mathbf{A}^j(t))\left(\boldsymbol{\Sigma}\bar{\mathbf{b}}^j(t) + \boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right) - \boldsymbol{\eta}_t^j\left(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right) + \mathbf{A}^j(t)\left(r_f\mathbf{1}_N - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\right) = \mathbf{0}, \tag{EC.35}$$

where

$$\begin{cases}\boldsymbol{\alpha}_1^j(t) = -\boldsymbol{\Sigma}\mathbf{P}^\top\left((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1} \in \mathbb{R}^{N\times K},\\ \boldsymbol{\alpha}_0^j(t) = \boldsymbol{\mu} - (T-t_j)\boldsymbol{\beta}_1^j\mathbf{P}\boldsymbol{\mu}^x - (t-t_j)\boldsymbol{\Sigma}\boldsymbol{\eta}_t^j\left(\boldsymbol{\mu}^x - (T-t_j)\boldsymbol{\beta}_1^j\mathbf{P}\boldsymbol{\mu}^x\right) \in \mathbb{R}^N;\end{cases}$$

and $\mathbf{C}^j(t) \in \mathbb{R}^{K\times K}$, $\hat{\mathbf{c}}^j(t) \in \mathbb{R}^K$ and $\bar{c}^j(t) \in \mathbb{R}$ are solutions of

$$\mathbf{C}^{j'}(t) - \frac{1}{\gamma}\left(\mathbf{B}^j(t) + \boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}_1^j(t)\right)^\top\boldsymbol{\Sigma}\left(\mathbf{B}^j(t) + \boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}_1^j(t)\right) + \boldsymbol{\alpha}_1^j(t)^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}_1^j(t) = \mathbf{0}, \tag{EC.36}$$

$$\begin{aligned}&\hat{\mathbf{c}}^{j'}(t) + \frac{1-\gamma}{\gamma}\boldsymbol{\alpha}_1^j(t)\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right) + \frac{1}{\gamma}\boldsymbol{\alpha}_1^j(t)^\top\bar{\mathbf{b}}^j(t) + \mathbf{B}^j(t)^\top(r_f\mathbf{1}_N - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma}))\\ &\quad + \frac{1}{\gamma}\mathbf{B}^j(t)^\top\left(\boldsymbol{\Sigma}\bar{\mathbf{b}}^j(t) + \boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right) = \mathbf{0},\end{aligned} \tag{EC.37}$$

and

$$\begin{aligned}&\bar{c}^{j'}(t) + (1-\gamma)r_f + \frac{1}{2}\operatorname{Tr}(\mathbf{A}^j(t)\boldsymbol{\Sigma}) + \frac{1-\gamma}{2\gamma}\left(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right)^\top\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right)\\ &\quad + \left(\boldsymbol{\alpha}_0^j(t) - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\right)^\top\bar{\mathbf{b}}^j(t) + \frac{1-\gamma}{\gamma}\left(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N\right)^\top\bar{\mathbf{b}}^j(t) + \frac{1}{2\gamma}\bar{\mathbf{b}}^j(t)^\top\boldsymbol{\Sigma}\bar{\mathbf{b}}^j(t) = 0.\end{aligned} \tag{EC.38}$$

For earlier time intervals $[t_j, t_{j+1})$ $(j = 0, \cdots, M-1)$, $g^j(t, \bar{\mathbf{x}}, \mathbf{y})$ can be derived in a similar manner by solving the dynamic programming equation over the interval $[t_j, t_{j+1}]$. This gives us the same ODEs for $\mathbf{A}^j(t)$, $\mathbf{B}^j(t)$, $\bar{\mathbf{b}}^j(t)$, $\mathbf{C}^j(t)$, $\hat{\mathbf{c}}^j(t)$ and $\bar{c}^j(t)$ when $j = M$ though with different terminal conditions because they depend on the view $\mathbf{Y}^{j+1}(t_{j+1}, T)$ and the non-zero value function $V(t, z, \bar{\mathbf{x}}, \mathbf{y})$ at t_{j+1} through the Principle of Optimality (48). The following result computes the expectation in (48). The proof can be found in Appendix EC.5.

Lemma EC.4.

$$\begin{aligned}V(t_{j+1}^-, z, \bar{\mathbf{x}}, \mathbf{y}) &= \mathbb{E}[V(t_{j+1}, z, \mathbf{0}, \mathbf{Y}^{j+1}) \mid \bar{\mathbf{X}}^y(t_{j+1}^-) = \bar{\mathbf{x}}, \mathbf{I}(t) = \mathbf{y}], \ \textit{for } j \in \{0, \ldots, M-1\}\\ &= U(z)\exp\left(g^j(t_{j+1}, \bar{\mathbf{x}}, \mathbf{y})\right), \quad t \in [t_j, t_{j+1})\end{aligned} \tag{EC.39}$$

where

$$g^j(t_{j+1}, \bar{\mathbf{x}}, \mathbf{y}) = \frac{1}{2}\bar{\mathbf{x}}^\top\mathbf{A}^j(t_{j+1})\bar{\mathbf{x}} + \bar{\mathbf{x}}^\top\left(\mathbf{B}^j(t_{j+1})\mathbf{y} + \bar{\mathbf{b}}^j(t_{j+1})\right) + \left(-\frac{1}{2}\mathbf{y}^\top\bar{\mathbf{C}}^j(t_{j+1})\mathbf{y} + \mathbf{y}^\top\hat{\mathbf{c}}^j(t_{j+1}) + \bar{c}^j(t_{j+1})\right)$$

with

$$\begin{aligned}\mathbf{C}^j(t_{j+1}) &= \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\mid j}\ \right)^{-1}\bar{\boldsymbol{\beta}}_0^j,\\ \hat{\mathbf{c}}^j(t_{j+1}) &= \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\mid j}\right)^{-1}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1}\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \bar{\boldsymbol{\alpha}}_0^j\right),\\ \bar{c}^j(t_{j+1}) &= \bar{c}^{j+1}(t_{j+1}) + (\bar{\boldsymbol{\beta}}_1^j)^\top\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \frac{1}{2}\bar{\boldsymbol{\beta}}_1^{j\top}\mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\beta}}_1^j + \ln\left(\det(\mathbf{I}_K + \bar{\boldsymbol{\Omega}}^{j+1\mid j}\mathbf{C}^{j+1}(t_{j+1}))^{-\frac{1}{2}}\right)\\ &\quad + \frac{1}{2}(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\beta}}_1^j)^\top\left((\bar{\boldsymbol{\Omega}}^{j+1\mid j})^{-1} + \mathbf{C}^{j+1}(t_{j+1})\right)^{-1}(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\beta}}_1^j),\end{aligned} \tag{EC.40}$$

$$\mathbf{A}^j(t_{j+1}) = -\mathbf{P}^\top \mathbf{C}^j(t_{j+1})\mathbf{P} \tag{EC.41}$$

and

$$\begin{aligned} \bar{\mathbf{b}}^j(t_{j+1}) &= -\mathbf{P}^\top \hat{\mathbf{c}}^j(t_{j+1}), \\ \mathbf{B}^j(t_{j+1}) &= \mathbf{P}^\top \mathbf{C}^j(t_{j+1}), \end{aligned} \tag{EC.42}$$

where

$$\begin{aligned} \bar{\boldsymbol{\alpha}}_0^j &= (T-t_{j+1})(\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}\mathbf{P}\boldsymbol{\mu}^x, \\ \bar{\boldsymbol{\beta}}_0^j &= \mathbf{I}_K - (\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}, \\ \bar{\boldsymbol{\beta}}_1^j &= (T-t_{j+1})(\mathbf{I}_K - \bar{\boldsymbol{\beta}}_0^j)\mathbf{P}\boldsymbol{\mu}^x, \\ \bar{\boldsymbol{\Omega}}^{j+1|j} &= (\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^{j+1}\right), \end{aligned}$$

are constants.

For the interval $[t_j, t_{j+1}]$ $(j = 0, \cdots, M-1)$ (EC.40) gives the terminal conditions for the ODEs (EC.36)–(EC.38), (EC.41) for the Riccati equation (EC.33), and (EC.42) for the ODEs (EC.34)–(EC.35). This provides us with enough information to solve the 6 ODEs (EC.33)–(EC.38).

The terminal condition for $\mathbf{A}^j(t)$, $\mathbf{B}^j(t)$ and $\bar{\mathbf{b}}^j(t)$ suggest that it might be possible to express the solutions of their respective ODEs in terms of $\mathbf{C}^j(t)$ and $\hat{\mathbf{c}}^j(t)$. The following result shows that this is indeed the case, so only 3 of these ODEs instead of 6 need to be solved numerically to compute $g^j(t, \bar{\mathbf{x}}, \mathbf{y})$ on $[t_j, t_{j+1}]$. The proof can be found in Appendix EC.5.

Lemma EC.5. *For $t \in [t_j, t_{j+1})$ and $j \in \{0, \ldots, M\}$, let $\mathbf{A}^j(t) \in \mathbb{R}^{N\times N}$, $\mathbf{B}^j(t) \in \mathbb{R}^{N\times K}$, and $\mathbf{C}^j(t) \in \mathbb{R}^{K\times K}$ be the solutions of (EC.33), (EC.34), and (EC.36), respectively, with boundary conditions specified in Lemma EC.4. Then,*

$$\begin{aligned} \mathbf{A}^j(t) &= -\mathbf{P}^\top \mathbf{C}^j(t)\mathbf{P}, \quad \textit{for} \quad t \in [t_j, t_{j+1}),\ j \in \{0, \ldots, M\}, \\ \mathbf{B}^j(t) &= \mathbf{P}^\top \mathbf{C}^j(t), \quad \textit{for} \quad t \in [t_j, t_{j+1}),\ j \in \{0, \ldots, M\}. \end{aligned}$$

Furthermore, if $\bar{\mathbf{b}}^j(t)$ and $\hat{\mathbf{c}}^j(t)$ are the solutions of (EC.35) and (EC.37), respectively,

$$\bar{\mathbf{b}}^j(t) = -\mathbf{P}^\top \hat{\mathbf{c}}^j(t), \quad \textit{for} \quad t \in [t_j, t_{j+1}),\ j \in \{0, \ldots, M\}.$$

It follows that

$$\begin{aligned} g^j(t, \bar{\mathbf{x}}, \mathbf{y}) &= \frac{1}{2}\bar{\mathbf{x}}^\top \mathbf{A}^j(t)\bar{\mathbf{x}} + \bar{\mathbf{x}}^\top\left(\mathbf{B}^j(t)\mathbf{y} + \bar{\mathbf{b}}^j(t)\right) + \left(-\frac{1}{2}\mathbf{y}^\top \bar{\mathbf{C}}^j(t)\mathbf{y} + \mathbf{y}^\top \hat{\mathbf{c}}^j(t) + \bar{c}^j(t)\right) \\ &= -\frac{1}{2}(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y})^\top \mathbf{C}^j(t)(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y}) - (\mathbf{P}\bar{\mathbf{x}} - \mathbf{y})^\top \hat{\mathbf{c}}^j(t) + \bar{c}^j(t) \end{aligned}$$

and the hedging demand is

$$\begin{aligned} \frac{1}{\gamma}\frac{\partial g^j}{\partial \bar{\mathbf{x}}}(t, \bar{\mathbf{x}}, \mathbf{y}) &= \frac{1}{\gamma}\left(\mathbf{A}^j(t)\bar{\mathbf{x}} + \mathbf{B}^j(t)\mathbf{y} + \bar{\mathbf{b}}^j(t)\right) \\ &= \frac{1}{\gamma}\mathbf{P}^\top\left(\mathbf{C}^j(t)(\mathbf{y} - \mathbf{P}\bar{\mathbf{x}}) - \hat{\mathbf{c}}^j(t)\right). \end{aligned}$$

Hence, the hedging demand depends only on two coefficients, $\mathbf{C}^j(t)$ and $\hat{\mathbf{c}}^j(t)$ obtained by solving (EC.36)–(EC.37) with boundary conditions (EC.40).

The following result shows that $\mathbf{C}^j(t)$ and $\hat{\mathbf{c}}^j(t)$ admit explicit solutions, which we derive. The proof can be found in Appendix EC.5.

LEMMA EC.6. *Let $\mathbf{C}^j(t) \in \mathbb{R}^{K\times K}$ and $\hat{\mathbf{c}}^j(t) \in \mathbb{R}^K$ be the solutions to (EC.36) and (EC.37), respectively, with boundary conditions (EC.40). Then*

$$\mathbf{C}^j(t) = \bar{\mathbf{M}}^j(t)\mathbf{P}^\top \bar{\boldsymbol{\eta}}_t^j,$$
$$\hat{\mathbf{c}}^j(t) = \bar{\mathbf{M}}^j(t)\left(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_N) + (T-t)\mathbf{P}^\top \bar{\boldsymbol{\eta}}_t^j \mathbf{P}\boldsymbol{\mu}^x\right),$$

where

$$\bar{\mathbf{M}}^j(t) = -(\gamma-1)(T-t)(\boldsymbol{\Omega}^j)^{-1}\mathbf{P}\Big(\gamma\boldsymbol{\Sigma}^{-1} + (T-t)\mathbf{P}^\top(\boldsymbol{\Omega}^j)^{-1}\mathbf{P}\Big)^{-1} \in \mathbb{R}^{K\times N}.$$

It follows from Lemma EC.6 and (EC.32) that

$$\begin{aligned}\frac{1}{\gamma}\frac{\partial g^j}{\partial \bar{\mathbf{x}}}(t,\bar{\mathbf{x}},\mathbf{y}) &= \frac{1}{\gamma}\mathbf{P}^\top\left(\mathbf{C}^j(t)(\mathbf{y} - \mathbf{P}\bar{\mathbf{x}}) - \hat{\mathbf{c}}^j(t)\right)\\ &= \frac{1}{\gamma}\mathbf{P}^\top\mathbf{M}^j(t)\boldsymbol{\Sigma}^{-1}\left(\tilde{\boldsymbol{\mu}}^j(t,\bar{\mathbf{x}},\mathbf{y}) - r_f\mathbf{1}_N\right),\end{aligned}$$

where $\mathbf{M}^j(t) = -\mathbf{P}\bar{\mathbf{M}}^j(t)$. The optimal policy is

$$\boldsymbol{\pi}^{j*}(t,\bar{\mathbf{x}},\mathbf{y}) = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\tilde{\boldsymbol{\mu}}^j(t,\bar{\mathbf{x}},\mathbf{y}) - r_f\mathbf{1}_N\right) + \frac{1}{\gamma}\frac{\partial g^j}{\partial \bar{\mathbf{x}}}(t,\bar{\mathbf{x}},\mathbf{y}), \text{ for } t\in[t_j,t_{j+1}),$$

where

$$\tilde{\boldsymbol{\mu}}^j(t,\bar{\mathbf{x}},\mathbf{y}) = \boldsymbol{\mu} + \boldsymbol{\beta}_1^j(\mathbf{y} - (T-t_j)\mathbf{P}\boldsymbol{\mu}^x) + \boldsymbol{\beta}_2^j(t)(\mathbb{E}[\bar{\mathbf{X}}^y(t)] - \bar{\mathbf{x}}), \text{ for } t\in[t_j,t_{j+1})$$

is the drift of the conditional asset price. This concludes the proof. □

EC.4.2. Section 6.2

EC.4.2.1. Proofs of Conditional Market Dynamics

Proof of Proposition 8. Assume views $\{\mathbf{Y}^j(T_j,T_{j+1}),\, j\in\{0,\dots,M\}\}$ satisfy (51)–(52). Let

$$\bar{\mathbf{X}}^y(t) \triangleq \mathbf{X}(t)\,|\,\left(\mathbf{Y}^0(0,T_1),\dots,\mathbf{Y}^j(T_j,T_{j+1})\right),\ t\in[T_j,T_{j+1})$$

be the conditional log-returns and refined views $\{\bar{\mathbf{Y}}^j(T_j,T_{j+1}),\, j\in\{0,\dots,M\}\}$ be defined by (54). It follows that

$$\bar{\mathbf{Y}}^j(T_j,T_{j+1})\,|\,(\mathbf{X}(T_j),\mathbf{X}(T_{j+1})) = \mathbf{P}\left(\mathbf{X}(T_{j+1}) - \mathbf{X}(T_j)\right) + \boldsymbol{\epsilon}^{j,0} \sim \mathcal{N}\left(\mathbf{P}\left(\mathbf{X}(T_{j+1}) - \mathbf{X}(T_j)\right), \boldsymbol{\Omega}^{j,0}\right).$$

Since $\{\boldsymbol{\epsilon}^{j,0},\, j\in\{1,\dots,M\}\}$ are independent and $\mathbf{X}(t)$ has independent increments,

$$\{\bar{\mathbf{Y}}^j(T_j,T_{j+1})\,|\,j\in\{0,\dots,M\}\}$$

are mutually independent. Let

$$\bar{\mathbf{X}}(t) = \mathbf{X}(t) - \mathbf{X}(T_j), \quad t \in [T_j, T_{j+1})$$

denote the log-return between the time of the last view T_j and the current time t. Since the increments of $\mathbf{X}(t)$ are independent, $\bar{\mathbf{X}}(t)$ is independent of

$$\{\bar{\mathbf{Y}}^k(T_k, T_{k+1}) \mid k \in \{0, \dots, j-1\}\}.$$

It follows that

$$\begin{aligned}\bar{\mathbf{X}}(t) \mid \big(\mathbf{Y}^0(0,T_1), \dots, \mathbf{Y}^j(T_j,T_{j+1})\big) &= \bar{\mathbf{X}}(t) \mid \big(\bar{\mathbf{Y}}^0(0,T_1), \dots, \bar{\mathbf{Y}}^j(T_j,T_{j+1})\big) \\ &= \bar{\mathbf{X}}(t) \mid \bar{\mathbf{Y}}^j(T_j,T_{j+1}).\end{aligned}$$

This concludes the proof. □

Proof of Proposition 9. From Proposition 8, we have $\bar{\mathbf{X}}^y(t) \triangleq \bar{\mathbf{X}}(t) \mid \bar{\mathbf{Y}}^j(T_j, T_{j+1})$ is the log-returns at time t conditional on noisy views (56).

This setting is analogous to that in Section 3 except: (1) The horizon is now $[T_j, T_{j+1}]$ instead of $[0,T]$, (2) the log-returns are given by $\bar{\mathbf{X}}(t)$ instead of $\mathbf{X}(t)$, and (3) the views vector is $\bar{\mathbf{Y}}^j(T_j, T_{j+1}) = \bar{\mathbf{y}}^j$ instead $\mathbf{Y}(0,T) = \mathbf{y}$. It follows from Proposition 1 that

$$d\bar{\mathbf{X}}^y(t) = \left(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1^j\big(\bar{\mathbf{y}}^j - (T_{j+1} - T_j)\mathbf{P}\boldsymbol{\mu}^x\big) + \boldsymbol{\beta}_2^j(t)\big(\mathbb{E}[\bar{\mathbf{X}}^y(t)] - \bar{\mathbf{X}}^y(t)\big)\right)dt + d\mathbf{W}^y(t), \text{ for } t \in [T_j, T_{j+1})$$

where $\bar{\mathbf{y}}^j = \mathbf{y}^j - \sum_{i=1}^{\bar{p}} \boldsymbol{\Phi}^i(\mathbf{y}^{j-i} - \mathbf{P}\bar{\mathbf{x}}^{j-i})$, and

$$\begin{aligned}\boldsymbol{\beta}_1^j &= \boldsymbol{\Sigma}\mathbf{P}^\top\big((T_{j+1} - T_j)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^{j,0}\big)^{-1} \in \mathbb{R}^{N \times K}, \\ \boldsymbol{\beta}_2^j(t) &= \boldsymbol{\Sigma}\mathbf{P}^\top\big((T_{j+1} - t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^{j,0}\big)^{-1}\mathbf{P} \in \mathbb{R}^{N \times N}, \\ \mathbb{E}[\bar{\mathbf{X}}^y(t)] &= (t - T_j)\big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1^j\big(\bar{\mathbf{y}}^j - (T_{j+1} - T_j)\mathbf{P}\boldsymbol{\mu}^x\big)\big).\end{aligned}$$

This completes the proof. □

EC.4.2.2. Proof of Theorem 4 The overall structure follows closely the proof of Theorem 3. For each $j \in \{0, \dots, M\}$ we consider the control problem on the interval $[T_j, T_{j+1})$ while the transformed view $\bar{\mathbf{Y}}^j(T_j, T_{j+1})$ is fixed.

Assume

$$V(t, z, \bar{\mathbf{x}}, \bar{\mathbf{y}}) = U(z)\exp\big(g^j(t, \bar{\mathbf{x}}, \bar{\mathbf{y}})\big), \quad t \in [T_j, T_{j+1}),$$

where

$$g^j(t, \bar{\mathbf{x}}, \bar{\mathbf{y}}) = \frac{1}{2}\bar{\mathbf{x}}^\top \mathbf{A}^j(t)\bar{\mathbf{x}} + \bar{\mathbf{x}}^\top\big(\mathbf{B}^j(t)\bar{\mathbf{y}} + \bar{\mathbf{b}}^j(t)\big) - \frac{1}{2}\bar{\mathbf{y}}^\top \mathbf{C}^j(t)\bar{\mathbf{y}} + \bar{\mathbf{y}}^\top \hat{\mathbf{c}}^j(t) + \bar{c}^j(t). \tag{EC.43}$$

On the last interval $j=M$, the terminal condition $g^M(T,\cdot)=0$ forces $\mathbf{A}^M(T)=\mathbf{0}$, $\mathbf{B}^M(T)=\mathbf{0}$, $\mathbf{C}^M(T)=\mathbf{0}$, $\bar{\mathbf{b}}^M(T)=\mathbf{0}$, $\hat{\mathbf{c}}^M(T)=\mathbf{0}$, and $\bar{c}^M(T)=0$.

Substituting (EC.43) into the HJB equation gives exactly the six coupled ODEs (EC.33)–(EC.38) after replacing T by T_{M+1} and $\boldsymbol{\Omega}^M$ by $\boldsymbol{\Omega}^{M,0}$. Because these equations coincide with those solved in Appendix EC.4.1.2, their solution is not repeated here.

For $j<M$, continuity of the value function at $t=T_{j+1}$ and independence of $\big(\bar{\mathbf{X}}(t),\bar{\mathbf{Y}}^j(T_j,T_{j+1})\big)$ across quarters imply

$$V\big(T_{j+1}^-,z,\bar{\mathbf{x}},\bar{\mathbf{y}}\big)=\mathbb{E}\Big[V\big(T_{j+1},z,\mathbf{0},\bar{\mathbf{Y}}^{j+1}\big)\Big]=U(z)\exp g^j\big(T_{j+1},\bar{\mathbf{x}},\bar{\mathbf{y}}\big). \tag{EC.44}$$

The Gaussian conditional expectation is similar to Lemma EC.4 and yields the boundary values

$$\mathbf{A}^j(T_{j+1})=\mathbf{B}^j(T_{j+1})=\mathbf{C}^j(T_{j+1})=\bar{\mathbf{b}}^j(T_{j+1})=\hat{\mathbf{c}}^j(T_{j+1})=\mathbf{0},$$

while the scalar term satisfies

$$\begin{aligned}
&\bar{c}^{j'}(t)+(1-\gamma)r_f+\frac{1}{2}\operatorname{Tr}(\mathbf{A}^j(t)\boldsymbol{\Sigma})+\frac{1-\gamma}{2\gamma}\left(\boldsymbol{\alpha}_0^j(t)-r_f\mathbf{1}_N\right)^\top\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\alpha}_0^j(t)-r_f\mathbf{1}_N\right)\\
&\quad+\left(\boldsymbol{\alpha}_0^j(t)-\frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\right)^\top\bar{\mathbf{b}}^j(t)+\frac{1-\gamma}{\gamma}\left(\boldsymbol{\alpha}_0^j(t)-r_f\mathbf{1}_N\right)^\top\bar{\mathbf{b}}^j(t)+\frac{1}{2\gamma}\bar{\mathbf{b}}^j(t)^\top\boldsymbol{\Sigma}\bar{\mathbf{b}}^j(t)=0.
\end{aligned} \tag{EC.45}$$

with terminal condition

$$\begin{aligned}
\bar{c}^j(T_{j+1})=&\ln\det\big(\mathbf{I}_K+\bar{\boldsymbol{\Omega}}^{j+1}\mathbf{C}^{j+1}(T_{j+1})\big)^{-1/2}\\
&+\tfrac{1}{2}\big(\hat{\mathbf{c}}^{j+1}(T_{j+1})-\mathbf{C}^{j+1}(T_{j+1})\bar{\boldsymbol{\alpha}}_0^{j+1}\big)^\top\left(\mathbf{C}^{j+1}(T_{j+1})^{-1}+\bar{\boldsymbol{\Omega}}^{j+1}\right)^{-1}\big(\hat{\mathbf{c}}^{j+1}(T_{j+1})-\mathbf{C}^{j+1}(T_{j+1})\bar{\boldsymbol{\alpha}}_0^{j+1}\big)\\
&+\tfrac{1}{2}\bar{\boldsymbol{\alpha}}_0^{j+1\top}\mathbf{C}^{j+1}(T_{j+1})\bar{\boldsymbol{\alpha}}_0^{j+1}+\bar{\boldsymbol{\alpha}}_0^{j+1\top}\left(\hat{\mathbf{c}}^{j+1}(T_{j+1})-\mathbf{C}^{j+1}(T_{j+1})\bar{\boldsymbol{\alpha}}_0^{j+1}\right)+\bar{c}^{j+1}(T_{j+1}),
\end{aligned} \tag{EC.46}$$

where $\bar{\boldsymbol{\alpha}}_0^{j+1}=(T_{j+2}-T_{j+1})\mathbf{P}\boldsymbol{\mu}^x$ and $\bar{\boldsymbol{\Omega}}^{j+1}=(T_{j+2}-T_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+\boldsymbol{\Omega}^{j+1,0}$.

As in Theorem 3, we can prove that $\mathbf{A}^j=-\mathbf{P}^\top\mathbf{C}^j\mathbf{P}$, $\mathbf{B}^j=\mathbf{P}^\top\mathbf{C}^j$, $\bar{\mathbf{b}}^j=-\mathbf{P}^\top\hat{\mathbf{c}}^j$. Therefore only $\mathbf{C}^j,\hat{\mathbf{c}}^j,\bar{c}^j$ remain to be integrated.

Following the same steps as in the proof of Lemma EC.5, we can show that $\mathbf{C}^j(t)$ and $\hat{\mathbf{c}}^j(t)$ are given explicitly by

$$\begin{aligned}
\mathbf{C}^j(t)&=\bar{\mathbf{M}}^j(t)\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j,\\
\hat{\mathbf{c}}^j(t)&=\bar{\mathbf{M}}^j(t)\Big[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-r_f\mathbf{1}_N)+(T_{j+1}-t)\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j\mathbf{P}\boldsymbol{\mu}^x\Big],
\end{aligned}$$

where

$$\bar{\mathbf{M}}^j(t)=-(\gamma-1)(T_{j+1}-t)(\boldsymbol{\Omega}^{j,0})^{-1}\mathbf{P}\Big[\gamma\boldsymbol{\Sigma}^{-1}+(T_{j+1}-t)\mathbf{P}^\top(\boldsymbol{\Omega}^{j,0})^{-1}\mathbf{P}\Big]^{-1},\quad \bar{\boldsymbol{\eta}}_t^j=-\mathbf{P}^\top\left((T_{j+1}-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top+\boldsymbol{\Omega}^{j,0}\right)^{-1}\mathbf{P}.$$

Using these expressions in (EC.43) one obtains

$$g^j(t,\bar{\mathbf{x}},\bar{\mathbf{y}})=-\tfrac{1}{2}(\mathbf{P}\bar{\mathbf{x}}-\bar{\mathbf{y}})^\top\mathbf{C}^j(t)(\mathbf{P}\bar{\mathbf{x}}-\bar{\mathbf{y}})-(\mathbf{P}\bar{\mathbf{x}}-\bar{\mathbf{y}})^\top\hat{\mathbf{c}}^j(t)+\bar{c}^j(t).$$

Hence

$$\boldsymbol{\pi}^{j*}(t,\bar{\mathbf{x}},\bar{\mathbf{y}}) = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\big(\tilde{\boldsymbol{\mu}}^j(t,\bar{\mathbf{x}},\bar{\mathbf{y}}) - r_f\mathbf{1}_N\big) + \frac{1}{\gamma}\mathbf{P}^\top\mathbf{M}^j(t)\boldsymbol{\Sigma}^{-1}\big(\tilde{\boldsymbol{\mu}}^j(t,\bar{\mathbf{x}},\bar{\mathbf{y}}) - r_f\mathbf{1}_N\big),$$

with $\mathbf{M}^j(t) = -(\gamma-1)\bar{\mathbf{M}}^j(t)$ and

$$\tilde{\boldsymbol{\mu}}^j(t,\bar{\mathbf{x}},\bar{\mathbf{y}}) = \boldsymbol{\mu} + \boldsymbol{\beta}_1^j\big(\bar{\mathbf{y}} - (T_{j+1}-T_j)\mathbf{P}\boldsymbol{\mu}^x\big) + \boldsymbol{\beta}_2^j(t)\big(\mathbb{E}[\bar{\mathbf{X}}(t)] - \bar{\mathbf{x}}\big).$$

This completes the proof. □

EC.5. Proofs: Results in EC.4.1

Proof of Lemma EC.3. Revised views (41) satisfy

$$\mathbf{Y}^j(t_j,T)\,|\,(\mathbf{X}(t_j),\mathbf{X}(T)) = \mathbf{P}\left(\mathbf{X}(T)-\mathbf{X}(t_j)\right) + \boldsymbol{\epsilon}^j \sim \mathcal{N}\big(\mathbf{P}\left(\mathbf{X}(T)-\mathbf{X}(t_j)\right), \boldsymbol{\Omega}^j\big)$$

where log-returns $\mathbf{X}(t) = t\boldsymbol{\mu}^x + \mathbf{W}(t)$ from (9) has independent increments. For each $j\in\{1,\ldots,M\}$ and $t\in[t_j,t_{j+1})$ (with $t_{M+1}=T$) define $\bar{\mathbf{X}}(t) = \mathbf{X}(t)-\mathbf{X}(t_j)$. Since $\mathbf{X}(t)$ has independent increments, $\bar{\mathbf{X}}(t)$ is independent of $\{\mathbf{Y}^0(0,T),\ldots,\mathbf{Y}^{j-1}(t_{j-1},T)\}$, which concludes the proof. □

Proof of Lemma EC.4 From the Principle of Optimality

$$V\big(t_{j+1}^-,z,\bar{\mathbf{x}},\mathbf{y}\big) = \mathbb{E}_{\mathbf{Y}^{j+1}}\Big[V\big(t_{j+1},z,\mathbf{0},\mathbf{Y}^{j+1}\big)\,\Big|\,Z(t_{j+1}^-)=z,\bar{\mathbf{X}}^y\big(t_{j+1}^-\big)=\bar{\mathbf{x}},\,\mathbf{I}(t_{j+1}^-)=\mathbf{y}\Big].$$

We now evaluate this expectation. Recall that

$$\begin{aligned} V(t_{j+1},z,\mathbf{0},\mathbf{y}) &= U(z)\exp\left(g^j(t_{j+1},\bar{\mathbf{x}},\mathbf{y})\right) \\ &= U(z)\exp\left(-\frac{1}{2}\mathbf{y}^\top\mathbf{C}^j(t_{j+1})\mathbf{y} + \mathbf{y}^\top\hat{\mathbf{c}}^j(t_{j+1}) + \bar{c}^j(t_{j+1})\right). \end{aligned} \tag{EC.47}$$

From (41) – (42), the random variable $\mathbf{Y}^{j+1}(t_{j+1},T)\,|\,\big(\bar{\mathbf{X}}(t_{j+1}^-)=\bar{\mathbf{x}},\mathbf{Y}^j(t_j,T)=\mathbf{y}\big)$ is Gaussian with conditional mean and covariance

$$\begin{aligned} \bar{\boldsymbol{\mu}}^{j+1|j} &= \bar{\boldsymbol{\alpha}}_0^j - \bar{\boldsymbol{\beta}}_0^j(\mathbf{P}\bar{\mathbf{x}}-\mathbf{y}), \\ \bar{\boldsymbol{\Omega}}^{j+1|j} &= (\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^{j+1}\right) \end{aligned}$$

where

$$\begin{aligned} \bar{\boldsymbol{\alpha}}_0^j &= (T-t_{j+1})(\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}\mathbf{P}\boldsymbol{\mu}^x, \\ \bar{\boldsymbol{\beta}}_0^j &= \mathbf{I}_K - (\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\left((T-t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} &V(t_{j+1}^-,z,\bar{\mathbf{x}},\mathbf{y}) = \mathbb{E}_{\mathbf{Y}^{j+1}}\Big[V\big(t_{j+1},z,\mathbf{0},\mathbf{Y}^{j+1}\big)\,\Big|\,Z(t_{j+1}^-)=z,\bar{\mathbf{X}}^y\big(t_{j+1}^-\big)=\bar{\mathbf{x}},\,\mathbf{Y}^j\big(t_j,T\big)=\mathbf{y}\Big] \\ &= \int(2\pi)^{-K/2}\det(\bar{\boldsymbol{\Omega}}^{j+1|j})^{-1/2}\exp\left(-\frac{1}{2}(\mathbf{y}^{j+1}-\bar{\boldsymbol{\mu}}^{j+1|j})^\top(\bar{\boldsymbol{\Omega}}^{j+1|j})^{-1}(\mathbf{y}^{j+1}-\bar{\boldsymbol{\mu}}^{j+1|j})\right)V(t_{j+1},z,\mathbf{0})d\mathbf{y}^{j+1} \\ &= U(z)\int(2\pi)^{-K/2}\det(\bar{\boldsymbol{\Omega}}^{j+1|j})^{-1/2}\exp\big(-\frac{1}{2}(\mathbf{y}^{j+1}-\bar{\boldsymbol{\mu}}^{j+1|j})^\top(\bar{\boldsymbol{\Omega}}^{j+1|j})^{-1}(\mathbf{y}^{j+1}-\bar{\boldsymbol{\mu}}^{j+1|j}) \\ &\quad -\frac{1}{2}\mathbf{y}^{j+1\top}\mathbf{C}^j(t_{j+1})\mathbf{y}^{j+1} + \mathbf{y}^{j+1\top}\hat{\mathbf{c}}^j(t_{j+1}) + \bar{c}^j(t_{j+1}))\big)d\mathbf{y}^{j+1} \end{aligned}$$

where the last equality follows from (EC.47). By rearranging the terms in the exponential

$$V(t_{j+1}^-, z, \bar{\mathbf{x}}, \mathbf{y}) = U(z)\det\left(\mathbf{I}_N + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\mathbf{C}^{j+1}(t_{j+1})\right)^{-1/2}\exp\left(\boldsymbol{\epsilon}(\mathbf{y},\bar{\mathbf{x}})\right) \tag{EC.48}$$
$$\times \int (2\pi)^{-K/2}\det(\hat{\boldsymbol{\Omega}}^{j+1\,|\,j})^{-1/2}\exp\left(-\frac{1}{2}(\mathbf{y}^{j+1} - \hat{\boldsymbol{\mu}}^{j+1\,|\,j})^\top(\hat{\boldsymbol{\Omega}}^{j+1\,|\,j})^{-1}(\mathbf{y}^{j+1} - \hat{\boldsymbol{\mu}}^{j+1\,|\,j})\right)d\mathbf{y}^{j+1}$$

where

$$\begin{aligned}\boldsymbol{\epsilon}(\mathbf{y},\bar{\mathbf{x}}) = &-\frac{1}{2}\left(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y}\right)^\top\bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\bar{\boldsymbol{\beta}}_0^{j}\left(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y}\right)\\ &- \left(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y}\right)^\top\bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1}\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \bar{\boldsymbol{\alpha}}_0^j\right) + \epsilon_0,\end{aligned} \tag{EC.49}$$

$$\begin{aligned}\epsilon_0 = &\frac{1}{2}\left(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\alpha}}_0^j\right)^\top\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\left(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\alpha}}_0^j\right)^\top\\ &+ \frac{1}{2}\bar{\boldsymbol{\alpha}}_0^{j\top}\mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\alpha}}_0^j + \bar{\boldsymbol{\alpha}}_0^{j\top}\left(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\alpha}}_0^j\right) + \bar{c}^{j+1}(t_{j+1}),\end{aligned}$$

and

$$\begin{aligned}\hat{\boldsymbol{\Omega}}^{j+1\,|\,j} &= \left((\bar{\boldsymbol{\Omega}}^{j+1\,|\,j})^{-1} + \mathbf{C}^{j+1}(t_{j+1})\right)^{-1} \in \mathbb{R}^{K\times K},\\ \hat{\boldsymbol{\mu}}^{j+1\,|\,j} &= \hat{\boldsymbol{\Omega}}^{j+1\,|\,j}\left(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\left(\boldsymbol{\alpha}_0^j - \boldsymbol{\beta}_0^j(\mathbf{P}\bar{\mathbf{x}} - \mathbf{y})\right)\right) + \bar{\boldsymbol{\mu}}^{j+1\,|\,j} \in \mathbb{R}^K.\end{aligned}$$

Notice that the term inside the integral in (EC.48) is the density function of a multivariate Gaussian random variable with mean $\hat{\boldsymbol{\mu}}^{j+1\,|\,j}$ and covariance $\hat{\boldsymbol{\Omega}}^{j+1\,|\,j}$ so it integrates to 1. It follows that

$$V(t_{j+1}^-, z, \bar{\mathbf{x}}, \mathbf{y}) = U(z)\det\left(\mathbf{I}_N + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\mathbf{C}^{j+1}(t_{j+1})\right)^{-1/2}\exp\left(\boldsymbol{\epsilon}(\mathbf{y},\bar{\mathbf{x}})\right) \tag{EC.50}$$

where $\boldsymbol{\epsilon}(\mathbf{y},\bar{\mathbf{x}})$ given by (EC.49). We now derive the boundary conditions of the ODEs (EC.33)–(EC.38). From (EC.50)

$$\begin{aligned}&V(t_{j+1}^-, z, \bar{\mathbf{x}}, \mathbf{y})\\ &= U(z)\exp\left(\frac{1}{2}\bar{\mathbf{x}}^\top\mathbf{A}^j(t_{j+1})\mathbf{x} + \bar{\mathbf{x}}^\top\left(\mathbf{B}^j(t_{j+1})\mathbf{y} + \bar{\mathbf{b}}^j(t_{j+1})\right) - \frac{1}{2}\mathbf{y}^\top\mathbf{C}^j(t_{j+1})\mathbf{y} + \mathbf{y}^\top\hat{\mathbf{c}}^j(t_{j+1}) + \bar{c}^j(t_{j+1})\right)\\ &= U(z)\det\left(\mathbf{I}_N + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\mathbf{C}^{j+1}(t_{j+1})\right)^{-1/2}\exp\left(\boldsymbol{\epsilon}(\mathbf{y},\bar{\mathbf{x}})\right).\end{aligned}$$

It follows that

$$\frac{1}{2}\bar{\mathbf{x}}^\top\mathbf{A}^j(t_{j+1})\bar{\mathbf{x}} + \bar{\mathbf{x}}^\top\left(\mathbf{B}^j(t_{j+1})\mathbf{y} + \bar{\mathbf{b}}^j(t_{j+1})\right) - \frac{1}{2}\mathbf{y}^\top\mathbf{C}^j(t_{j+1})\mathbf{y} + \mathbf{y}^\top\hat{\mathbf{c}}^j(t_{j+1}) = \boldsymbol{\epsilon}(\mathbf{y},\bar{\mathbf{x}}) - \epsilon_0, \tag{EC.51}$$

and

$$\exp\left(\bar{c}^j(t_{j+1})\right) = \det\left(\mathbf{I}_N + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\mathbf{C}^{j+1}(t_{j+1})\right)^{-1/2}\exp\left(\epsilon_0\right). \tag{EC.52}$$

It follows from (EC.51) that

$$\begin{aligned}\mathbf{A}^j(t_{j+1}) &= -\mathbf{P}^\top\bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\bar{\boldsymbol{\beta}}_0^j\mathbf{P},\\ \mathbf{B}^j(t_{j+1}) &= \mathbf{P}^\top\bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\bar{\boldsymbol{\beta}}_0^j,\\ \mathbf{C}^j(t_{j+1}) &= \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\bar{\boldsymbol{\beta}}_0^j,\\ \bar{\mathbf{b}}^j(t_{j+1}) &= -\mathbf{P}^\top\bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1}\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \bar{\boldsymbol{\alpha}}_0^j\right),\\ \hat{\mathbf{c}}^j(t_{j+1}) &= \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1}\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \bar{\boldsymbol{\alpha}}_0^j\right).\end{aligned}$$

Thus, we have $\mathbf{A}^j(t_{j+1}) = -\mathbf{P}^\top \mathbf{C}^j(t_{j+1})\mathbf{P}$, $\mathbf{B}^j(t_{j+1}) = \mathbf{C}^j(t_{j+1})\mathbf{P}$ and $\bar{\mathbf{b}}^j(t_{j+1}) = -\mathbf{P}^\top \hat{\mathbf{c}}^j(t_{j+1})$. Additionally, from (EC.52) we directly get

$$\begin{aligned}
&\bar{c}^j(t_{j+1}) = \bar{c}^{j+1}(t_{j+1}) + (\bar{\boldsymbol{\beta}}_1^j)^\top \hat{\mathbf{c}}^{j+1}(t_{j+1}) \\
&\quad -\frac{1}{2}\bar{\boldsymbol{\beta}}_1^{j\top} \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\beta}}_1^j + \ln\Big(\det(\mathbf{I}_K + \bar{\boldsymbol{\Omega}}^{j+1|j}\mathbf{C}^{j+1}(t_{j+1}))^{-\frac{1}{2}}\Big) \\
&\qquad +\frac{1}{2}\big(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\beta}}_1^j\big)^\top \Big((\bar{\boldsymbol{\Omega}}^{j+1|j})^{-1} + \mathbf{C}^{j+1}(t_{j+1})\Big)^{-1}\big(\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \mathbf{C}^{j+1}(t_{j+1})\bar{\boldsymbol{\beta}}_1^j\big).
\end{aligned}$$

This concludes the proof. □

Proof of Lemma EC.5

From Lemma EC.4 the matrix $\mathbf{A}^j(t) \in \mathbb{R}^{N\times N}$ ($t \in [t_j, t_{j+1})$, $j \in \{1,\dots,M\}$) and $\mathbf{B}^j(t) \in \mathbb{R}^{N\times K}$ are solutions of

$$\begin{cases}
\mathbf{A}^{j\prime}(t) + \dfrac{1-\gamma}{\gamma}\boldsymbol{\eta}_t^j \boldsymbol{\Sigma}\,\boldsymbol{\eta}_t^j + \dfrac{1}{\gamma}\Big(\mathbf{A}^j(t)\boldsymbol{\Sigma}\,\boldsymbol{\eta}_t^j + \boldsymbol{\eta}_t^j\boldsymbol{\Sigma}\,\mathbf{A}^j(t)\Big) + \dfrac{1}{\gamma}\mathbf{A}^j(t)\boldsymbol{\Sigma}\,\mathbf{A}^j(t) = \mathbf{0}, \\
\mathbf{A}^j(t_{j+1}) = -\mathbf{P}^\top \mathbf{C}^j(t_{j+1})\mathbf{P}, \\
\mathbf{A}^M(T) = \mathbf{0}_{N\times N},
\end{cases} \tag{EC.53}$$

and

$$\begin{cases}
\mathbf{B}^{j\prime}(t) + \dfrac{1}{\gamma}\Big(\boldsymbol{\eta}_t^j + \mathbf{A}^j(t)\Big)\boldsymbol{\Sigma}\,\mathbf{B}^j(t) + \dfrac{1-\gamma}{\gamma}\boldsymbol{\eta}_t^j\boldsymbol{\alpha}_1^j(t) + \dfrac{1}{\gamma}\mathbf{A}^j(t)\boldsymbol{\alpha}_1^j(t) = \mathbf{0}, \\
\mathbf{B}^j(t_{j+1}) = \mathbf{P}^\top \mathbf{C}^j(t_{j+1}), \\
\mathbf{B}^M(T) = \mathbf{0}_{N\times K},
\end{cases} \tag{EC.54}$$

respectively, where

$$\begin{aligned}
\boldsymbol{\alpha}_1^j(t) &= -\boldsymbol{\Sigma}\mathbf{P}^\top\Big((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\Big), \\
\boldsymbol{\eta}_t^j &= -\mathbf{P}^\top\Big((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\Big)\mathbf{P} = \boldsymbol{\Sigma}^{-1}\,\boldsymbol{\alpha}_1^j(t)\mathbf{P}.
\end{aligned}$$

By direct substitution, it is easy to show that

$$\mathbf{A}^j(t) = -\mathbf{B}^j(t)\mathbf{P}, \quad \text{for } t \in [t_j, t_{j+1}],\ j \in \{0,\dots,M\}. \tag{EC.55}$$

is a solution of (EC.53). It follows that (EC.54) can be rewritten as

$$\begin{cases}
\mathbf{B}^{j\prime}(t) + \dfrac{1}{\gamma}\Big(\boldsymbol{\eta}_t^j - \mathbf{B}^j(t)\mathbf{P}\Big)\boldsymbol{\Sigma}\,\mathbf{B}^j(t) + \dfrac{1-\gamma}{\gamma}\boldsymbol{\eta}_t^j\boldsymbol{\alpha}_1^j(t) - \dfrac{1}{\gamma}\mathbf{B}^j(t)\mathbf{P}\boldsymbol{\alpha}_1^j(t) = \mathbf{0}, \\
\mathbf{B}^j(t_{j+1}) = \mathbf{P}^\top \mathbf{C}^j(t_{j+1}), \\
\mathbf{B}^M(T) = \mathbf{0}_{N\times K}.
\end{cases} \tag{EC.56}$$

Next, from Lemma EC.4 we have that $\mathbf{C}^j(t) \in \mathbb{R}^{K\times K}$ is the solution of

$$\mathbf{C}^{j\prime}(t) - \frac{1}{\gamma}\Big(\mathbf{B}^j(t) + \boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}_1^j(t)\Big)^\top \boldsymbol{\Sigma}\Big(\mathbf{B}^j(t) + \boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}_1^j(t)\Big) + \boldsymbol{\alpha}_1^j(t)^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}_1^j(t) = \mathbf{0} \tag{EC.57}$$

with boundary conditions given by (EC.40) and $\mathbf{C}^M(T) = \mathbf{0}_{K\times K}$. It is easy to show that

$$\mathbf{B}^j(t) = \mathbf{P}^\top \mathbf{C}^j(t), \quad t \in [t_j, t_{j+1}],\ j \in \{0,\dots,M\}.$$

is a solution of (EC.56) and hence by (EC.55), that $\mathbf{A}^j(t) = -\mathbf{P}^\top, \mathbf{C}^j(t)\mathbf{P}$.

Similarly, recall from Lemma EC.4 that

$$\begin{cases} \bar{\mathbf{b}}^{j\prime}(t) + \frac{1}{\gamma}\Big(\boldsymbol{\eta}_t^j + \mathbf{A}^j(t)\Big)\Big(\boldsymbol{\Sigma}\,\bar{\mathbf{b}}^j(t) + \boldsymbol{\alpha}_0^j(t) - r_f\,\mathbf{1}_N\Big) - \boldsymbol{\eta}_t^j\Big(\boldsymbol{\alpha}_0^j(t) - r_f\,\mathbf{1}_N\Big) \\ \qquad\qquad + \mathbf{A}^j(t)\Big(r_f\,\mathbf{1}_N - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\Big) = \mathbf{0}, \\ \bar{\mathbf{b}}^j(t_{j+1}) = -\mathbf{P}^\top\hat{\mathbf{c}}^j(t_{j+1}), \\ \bar{\mathbf{b}}^M(T) = \mathbf{0}, \end{cases} \tag{EC.58}$$

with

$$\boldsymbol{\alpha}_0^j(t) = \boldsymbol{\mu} - (T - t_j)\boldsymbol{\beta}_1^j\mathbf{P}\boldsymbol{\mu}^x - (t - t_j)\boldsymbol{\Sigma}\,\boldsymbol{\eta}_t^j\Big(\boldsymbol{\mu}^x - (T - t_j)\boldsymbol{\beta}_1^j\mathbf{P}\boldsymbol{\mu}^x\Big) \in \mathbb{R}^N.$$

If $\hat{\mathbf{c}}^j(t)$ is the solution to

$$\begin{aligned} &\hat{\mathbf{c}}^{j\prime}(t) + \frac{1-\gamma}{\gamma}\,\boldsymbol{\alpha}_1^j(t)\boldsymbol{\Sigma}^{-1}\Big(\boldsymbol{\alpha}_0^j(t) - r_f\,\mathbf{1}_N\Big) + \frac{1}{\gamma}\,\boldsymbol{\alpha}_1^j(t)^\top\bar{\mathbf{b}}^j(t) + \mathbf{B}^j(t)^\top\Big(r_f\,\mathbf{1}_N - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\Big) \\ &\qquad + \frac{1}{\gamma}\,\mathbf{B}^j(t)^\top\Big(\boldsymbol{\Sigma}\,\bar{\mathbf{b}}^j(t) + \boldsymbol{\alpha}_0^j(t) - r_f\,\mathbf{1}_N\Big) = \mathbf{0}, \end{aligned} \tag{EC.59}$$

with terminal condition $\hat{\mathbf{c}}^M(T) = \mathbf{0}$, then

$$\bar{\mathbf{b}}^j(t) = -\mathbf{P}^\top\hat{\mathbf{c}}^j(t), \quad t \in [t_j, t_{j+1}],\ j \in \{0, \dots, M\}.$$

is a solution to (EC.58). This completes the proof. ☐

Proof of Lemma EC.6. Recall from Lemma EC.5 that $\mathbf{C}^j(t) \in \mathbb{R}^{K\times K}$ $j \in \{0, \dots, M\}$ satisfies the Riccati equation

$$\begin{cases} \mathbf{C}^{j\prime}(t) - \dfrac{1-\gamma}{\gamma}\,\bar{\boldsymbol{\eta}}_t^j\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j + \dfrac{1}{\gamma}\Big(\mathbf{C}^j(t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j + \bar{\boldsymbol{\eta}}_t^j\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\mathbf{C}^j(t)\Big) - \dfrac{1}{\gamma}\mathbf{C}^j(t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\mathbf{C}^j(t) = \mathbf{0}, \\ \mathbf{C}^j(t_{j+1}) = \bar{\boldsymbol{\beta}}_0^{j\top}\Big(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\Big)^{-1}\bar{\boldsymbol{\beta}}_0^j, \end{cases} \tag{EC.60}$$

where

$$\begin{aligned} \bar{\boldsymbol{\eta}}_t^j &= -\Big((T-t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\Big)^{-1}, \\ \bar{\boldsymbol{\beta}}_0^j &= \mathbf{I}_K - (\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\Big((T - t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\Big)^{-1}, \\ \bar{\boldsymbol{\Omega}}^{j+1\,|\,j} &= (\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\Big((T - t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\Big)^{-1}\Big((T - t_{j+1})\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^{j+1}\Big). \end{aligned}$$

The terminal condition is given by $\mathbf{C}^M(T) = \mathbf{0}_{K\times K}$. We first show that

$$\mathbf{C}^j(t) = \bar{\boldsymbol{\eta}}_t^j + \left(\frac{T-t}{\gamma}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1} \tag{EC.61}$$

is a solution of (EC.60). To begin, observe that $\bar{\boldsymbol{\eta}}_t^j$ satisfies the ODE

$$\frac{d\bar{\boldsymbol{\eta}}_t^j}{dt} = -\bar{\boldsymbol{\eta}}_t^j\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j.$$

It follows that $\mathbf{C}^j(t) = \bar{\boldsymbol{\eta}}_t^j + \mathbf{Q}^j(t)$ is the solution of (EC.60) if a symmetric matrix $\mathbf{Q}^j(t) \in \mathbb{R}^{K\times K}$ for $j = \{0, \cdots, M\}$ such that

$$\begin{cases} \mathbf{Q}^{j'}(t) - \dfrac{1}{\gamma}\mathbf{Q}^j(t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\mathbf{Q}^j(t) = \mathbf{0}, \\ \mathbf{Q}^j(t_{j+1}) = -\boldsymbol{\eta}_{t_{j+1}}^j + \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\bar{\boldsymbol{\beta}}_0^j \end{cases} \tag{EC.62}$$

with terminal condition $\mathbf{Q}^M(T) = (\boldsymbol{\Omega}^M)^{-1}$ can be found. To establish (EC.61) we show that

$$\mathbf{Q}^j(t) = \left(\frac{T-t}{\gamma}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1}. \tag{EC.63}$$

First, it can be easily verified that

$$\frac{d}{dt}\mathbf{Q}^j(t) = \frac{d}{dt}\left(\frac{T-t}{\gamma}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1} = \frac{1}{\gamma}\mathbf{Q}^j(t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\mathbf{Q}^j(t),$$

therefore, (EC.63) satisfies the differential equation in (EC.62). We now show that (EC.63) satisfies the required terminal and boundary conditions. We can see from (EC.63) that $\mathbf{Q}^M(T) = (\boldsymbol{\Omega}^j)^{-1}$ so the terminal condition is satisfied so (EC.63) is the solution of (EC.62) and $\mathbf{C}^M(t) = \bar{\boldsymbol{\eta}}_t^M + \mathbf{Q}^M(t)$ on the interval $[t_M, T]$.

Suppose there is a $j \in \{0, \cdots, M-1\}$ such that $\mathbf{Q}^{j+1}(t)$ is the solution of (EC.62) on $[t_{j+1}, t_{j+2}]$ and hence

$$\mathbf{C}^{j+1}(t) = \bar{\boldsymbol{\eta}}_t^{j+1} + \mathbf{Q}^{j+1}(t),\ t \in [t_{j+1}, t_{j+2}]. \tag{EC.64}$$

Using (EC.64), the definitions of $\bar{\boldsymbol{\eta}}_t^j$, $\bar{\boldsymbol{\Omega}}^{j+1|j}$ and $\bar{\boldsymbol{\beta}}_0^j$, and the Woodbury identity

$$-\bar{\boldsymbol{\eta}}_{t_{j+1}}^j + \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\bar{\boldsymbol{\beta}}_0^j = \left(\frac{T-t_{j+1}}{\gamma}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top + \boldsymbol{\Omega}^j\right)^{-1} = \mathbf{Q}^j(t_{j+1})$$

so (EC.63) is the solution of (EC.62) and $\mathbf{C}^j(t) + \bar{\boldsymbol{\eta}}_t^j + \mathbf{Q}^j(t)$ on $[t_j, t_{j+1}]$. It follows from induction that (EC.63) is the solution of (EC.62). Finally, it follows from the Woodbury identity that

$$\mathbf{C}^j(t) = \bar{\mathbf{M}}^j(t)\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j,$$

where $\bar{\mathbf{M}}^j(t) = -(\gamma-1)(T-t)(\boldsymbol{\Omega}^j)^{-1}\mathbf{P}\Big(\gamma\boldsymbol{\Sigma}^{-1} + (T-t)\mathbf{P}^\top(\boldsymbol{\Omega}^j)^{-1}\mathbf{P}\Big)^{-1}$, which completes the first half of the proof of Lemma EC.6.

For the second half, recall from Lemma EC.5 that for $t \in [t_j, t_{j+1})$, $\hat{\mathbf{c}}^j(t)$ is the solution of

$$\begin{cases} \hat{\mathbf{c}}^{j'}(t) - \dfrac{1-\gamma}{\gamma}\bar{\boldsymbol{\eta}}_t^j\mathbf{P}(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N) + \dfrac{1}{\gamma}\bar{\boldsymbol{\eta}}_t^j\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\hat{\mathbf{c}}^j(t) - \dfrac{1}{\gamma}\mathbf{C}^j(t)\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\top\hat{\mathbf{c}}^j(t) + \dfrac{1-\gamma}{\gamma}\mathbf{C}^j(t)\mathbf{P}(\boldsymbol{\alpha}_0^j(t) - r_f\mathbf{1}_N) \\ \qquad + \mathbf{C}^j(t)\mathbf{P}(r_f\mathbf{1}_N - \dfrac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})) = \mathbf{0}, \\ \hat{\mathbf{c}}^j(t_{j+1}) = \bar{\boldsymbol{\beta}}_0^{j\top}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1} + \bar{\boldsymbol{\Omega}}^{j+1\,|\,j}\right)^{-1}\left(\mathbf{C}^{j+1}(t_{j+1})^{-1}\hat{\mathbf{c}}^{j+1}(t_{j+1}) - \bar{\boldsymbol{\alpha}}_0^j\right), \end{cases} \tag{EC.65}$$

where $\mathbf{C}^j(t) = \bar{\mathbf{M}}^j(t)\,\mathbf{P}^\top \bar{\boldsymbol{\eta}}_t^j$, and

$$\boldsymbol{\alpha}_0^j(t) = \boldsymbol{\mu} + (T-t)\,\boldsymbol{\Sigma}\,\mathbf{P}^\top \bar{\boldsymbol{\eta}}_t^j\,\mathbf{P}\boldsymbol{\mu}^x,$$

$$\bar{\boldsymbol{\alpha}}_0^j = -(T-t_{j+1})\,(\boldsymbol{\Omega}^j - \boldsymbol{\Omega}^{j+1})\,\bar{\boldsymbol{\eta}}_{t_{j+1}}^j\,\mathbf{P}\boldsymbol{\mu}^x,$$

$$\bar{\mathbf{M}}^j(t) = -(\gamma-1)(T-t)(\boldsymbol{\Omega}^j)^{-1}\mathbf{P}\Big(\gamma\boldsymbol{\Sigma}^{-1} + (T-t)\mathbf{P}^\top(\boldsymbol{\Omega}^j)^{-1}\mathbf{P}\Big)^{-1}.$$

with the terminal condition $\hat{\mathbf{c}}^M(T) = \mathbf{0}$. It can be shown that

$$\hat{\mathbf{c}}^j(t) = \bar{\mathbf{M}}^j(t)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_N) + (T-t)\,\mathbf{C}^j(t)\,\mathbf{P}\boldsymbol{\mu}^x = \bar{\mathbf{M}}^j(t)\Big(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_N) + (T-t)\mathbf{P}^\top\bar{\boldsymbol{\eta}}_t^j\,\mathbf{P}\boldsymbol{\mu}^x\Big),$$

is the solution of (EC.65) by a direct substitution of this expression into the differential equation and boundary conditions. □

EC.6. Extension – Views over Different Time Horizons

We now explore a scenario where the expert gives forward-looking views with varying horizons.

EC.6.1. Views Model

At time $t = 0$, the investor receives a vector of K views on future log-returns, where each view $j \in [K]$ is a noisy observation of the log-returns vector $\mathbf{X}(t)$ at time $t = T_j$. Specifically, conditional on $\mathbf{X}(T_j)$, we assume

$$Y_j(0,T_j)\,|\,\mathbf{X}(T_j) = \mathbf{p}_j^\top\mathbf{X}(T_j) + \sqrt{T_j}\epsilon_j \sim \mathcal{N}\big(\mathbf{p}_j^\top\mathbf{X}(T_j), T_j\omega_{jj}\big),\ j \in [K], \tag{EC.66}$$

where $\mathbf{p}_j \in \mathbb{R}^N$ is a linear mapping from log-returns to the j-th view, and $\omega_{jj} > 0$ is the view's noise variance.

Non-expired views. Without loss of generality we order the views by their respective time horizons ($T_i \le T_j$ for $i \le j$), and denote $T_0 = 0$. Thus, on the j-th interval $[T_{j-1}, T_j]$, we consider the non-expired views $j, j+1, \dots, K$. Let

$$\mathbf{Y}^j(0, T_{j:K}) = \big(Y_j(0,T_j), \dots, Y_K(0,T_K)\big)^\top$$

be the vector of these non-expired views during the j-th interval $[T_{j-1}, T_j]$, with $\mathbf{T}_{j:K} := (T_j, \dots, T_K)^\top \in \mathbb{R}^{K-j+1}$ and $\boldsymbol{\epsilon}^j := \big(\epsilon_j, \cdots, \epsilon_K\big)^\top \sim \mathcal{N}\big(\mathbf{0}, \boldsymbol{\Omega}^j\big)$, the associated noise vector. $\boldsymbol{\Omega}^j \in \mathbb{R}^{(K-j+1)\times(K-j+1)}$ is positive-definite, its entries ω_{ik}, for $i,k \in \{j, \dots, K\}$, represent the covariances among the noise terms ϵ_i and ϵ_k.

Figure EC.2 illustrates the discrete-time Bayesian network in which the views $\{Y_j(0,T_j)\}_{j=1}^K$ are observed at $t = 0$. Each $Y_j(0,T_j)$ is a noisy observation of $\mathbf{X}(T_j)$ and, via the Markov property of $\mathbf{X}$, it is also an observation about all earlier states (e.g. $Y_2(0,T_2)$ is an observation of $\mathbf{X}(T_1)$). Therefore the views vector $\mathbf{Y}^j(0,T_{j:K}) = \big(Y_j(0,T_j), \dots, Y_K(0,T_K)\big)^\top$, is a noisy observation of past returns and we can directly use Proposition 1 to obtain the posterior dynamics of the log-returns.

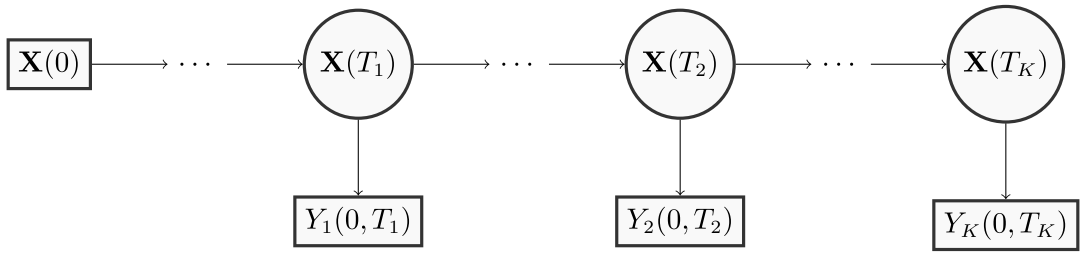

Figure EC.2 Bayesian network of the Dynamic Black-Litterman model with Different Views Horizons. The figure shows a discrete time version of the problem where $t=0,\dots,T_K$, and the noisy views $Y_j(0,T_j)$ of the log-return $\mathbf{X}(T_j)$ for $j\in[K]$ are revealed at $t=0$

EC.6.2. Market Dynamics Under Expert Views

The following Proposition shows how views from different time horizons can be transformed into views regarding the same time horizon.

PROPOSITION EC.1. *Consider a stock price following equation (8), and expert views following (EC.66). For $j\in[K]$, define the transformation*

$$\bar{\mathbf{Y}}^j(0,T_j)=\mathbf{Y}^j(0,T_{j:K})-\bar{\boldsymbol{\mu}}^j(T_j,T_{j:K}) \tag{EC.67}$$

where $\bar{\boldsymbol{\mu}}^j(T_j,T_{j:K})\in\mathbb{R}^{K-j+1}$ with elements

$$\bar{\mu}^j_i(T_j,T_{j:K})=(T_i-T_j)\mathbf{p}_i^\top\boldsymbol{\mu}^x,\ \textit{for } i\in\{j,\dots,K\}.$$

Then, conditional on the true realization of $\mathbf{X}(T_j)$, the views vector $\bar{\mathbf{Y}}^j(0,T_j)$ is Gaussian with

$$\bar{\mathbf{Y}}^j(0,T_j)|\mathbf{X}(T_j)=\mathbf{P}_{j:K}\mathbf{X}(T_j)+\bar{\boldsymbol{\epsilon}}^j\sim\mathcal{N}\big(\mathbf{P}_{j:K}\mathbf{X}(T_j),T_j\bar{\boldsymbol{\Omega}}^j\big),$$

where $\mathbf{P}_{j:K}=(\mathbf{p}_j,\dots,\mathbf{p}_K)^\top\in\mathbb{R}^{(K-j+1)\times N}$, and $\bar{\boldsymbol{\Omega}}^j$ is positive definite with

$$\bar{\boldsymbol{\Omega}}^j=\mathbf{P}_{j:K}\bar{\boldsymbol{\Omega}}^{jW}(\mathbf{P}_{j:K})^\top+\bar{\boldsymbol{\Omega}}^{jV}$$

where for $i,k\in\{1,\dots,K-j+1\}$, $\bar{\Omega}^{jV}_{ik}=\dfrac{\sqrt{T_{j+i-1}T_{j+k-1}}}{T_j}\Omega_{j+i-1,j+k-1}$, and $\bar{\Omega}^{jW}_{ik}=\dfrac{1}{T_j}\min\{T_{j+i-1}-T_j,T_{j+k-1}-T_j\}\Sigma_{j+i-1,j+k-1}$.

Furthermore

$$\mathbf{X}(t)\,|\,\mathbf{Y}^j(0,T_{j:K})=\mathbf{X}(t)\,|\,\bar{\mathbf{Y}}^j(0,T_j),\quad \textit{for } t\in[T_{j-1},T_j],\ j\in[K].$$

The conditional dynamics of the log-returns follow directly from Proposition 1.

Corollary EC.1. *Suppose that the price process satisfies* (8) *and expert views* $\mathbf{Y}(0,T)$ *satisfy* (EC.66). *Assume that* $\mathbf{PL}_j \neq \mathbf{0}$ *for* $j \in [N]$. *Conditional on* $\mathbf{Y}(0,T)=\mathbf{y}$, *the log-returns* $\mathbf{X}(t)$ *satisfy*

$$d\mathbf{X}^y(t) = \Big(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1^j\big(\bar{\mathbf{y}}^j - T_j\mathbf{P}_{j:K}\boldsymbol{\mu}^x\big) + \boldsymbol{\beta}_2^j(t)\big(\mathbb{E}[\mathbf{X}^y(t)] - \mathbf{X}^y(t)\big)\Big)dt + d\mathbf{W}^y(t) \tag{EC.68}$$

where $\bar{\mathbf{y}}^j$ *is sampled from the transformed views vector* (EC.67), *with*

$$\boldsymbol{\beta}_1^j = \frac{1}{T_j}\boldsymbol{\Sigma}(\mathbf{P}_{j:K})^\top(\mathbf{P}_{j:K}\boldsymbol{\Sigma}(\mathbf{P}_{j:K})^\top + \bar{\boldsymbol{\Omega}}^j)^{-1} \in \mathbb{R}^{N\times K},$$

$$\boldsymbol{\beta}_2^j(t) = \boldsymbol{\Sigma}(\mathbf{P}_{j:K})^\top\left((T_j - t)\mathbf{P}_{j:K}\boldsymbol{\Sigma}(\mathbf{P}_{j:K})^\top + \bar{\boldsymbol{\Omega}}^j\right)^{-1}\mathbf{P}_{j:K} \in \mathbb{R}^{N\times N},$$

$$\mathbb{E}[\mathbf{X}^y(t)] = t\left(\boldsymbol{\mu}^x + \boldsymbol{\beta}_1^j\big(\bar{\mathbf{y}}^j - T_j\mathbf{P}_{j:K}\boldsymbol{\mu}^x\big)\right)$$

is the expected log-return over the horizon $[0,t]$ *given* $\mathbf{Y}(0,T)=\mathbf{y}$. $\mathbf{W}^y(t) \sim \mathcal{N}\big(\mathbf{0}, t\boldsymbol{\Sigma}\big)$ *is a* $N-$*dimensional Brownian motion adapted to the filtration* $\mathcal{F}_t^{\mathbf{Y}}$.

Corollary EC.1 shows that having access to views with varying time horizons affect the conditional dynamics solely through the covariance matrix $\boldsymbol{\Omega}$.

EC.6.3. Optimal Policy

Let $T = T_K$ be the investor's investment horizon, and

$$V(t,z,\mathbf{x}) = \max_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}\big[U(Z(T))|\mathbf{X}(t)=\mathbf{x},\, Z(t)=z,\, Y_1(0,T_1)=y_1,\,\ldots,\, Y_K(0,T_K)=y_k\big]$$

her value function at time $t \in [T_{j-1}, T_j)$, where the wealth process $Z(t)$ satisfies

$$dZ(t) = Z(t)\Big(r_f dt + \boldsymbol{\pi}(t)^\top\big(\tilde{\boldsymbol{\mu}}(t,\mathbf{X}^y(t),\bar{\mathbf{y}}^j) - r_f\mathbf{1}_N\big)dt + \boldsymbol{\pi}(t)^\top d\mathbf{W}^y(t)\Big),$$

with

$$\tilde{\boldsymbol{\mu}}^j(t,\mathbf{x},\bar{\mathbf{y}}^j) = \boldsymbol{\mu} + \boldsymbol{\beta}_1^j(\bar{\mathbf{y}}^j - T_j\mathbf{P}_{j:K}\boldsymbol{\mu}^x) + \boldsymbol{\beta}_2^j(t)(\mathbb{E}[\mathbf{X}^y(t)] - \mathbf{x}),\; t\in[T_{j-1},T_j).$$

The log-returns $\mathbf{X}^y(t)$ satisfy (EC.68).

Proposition EC.2. *Suppose* $\gamma > 1$. *For* $j \in [K]$ *and* $t \in [T_{j-1}, T_j)$, *the solution to the HJB equation is*

$$V(t,z,\mathbf{x}) = \frac{z^{1-\gamma}}{1-\gamma}\exp(g^j(t,\mathbf{x})),\; t\in[T_{j-1},T_j)$$

where

$$g^j(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top\mathbf{A}^j(t)\mathbf{x} + \mathbf{x}^\top\mathbf{b}^j(t) + c^j(t),\; t\in[T_{j-1},T_j).$$

The matrix $\mathbf{A}^j(t)$ *is symmetric negative semi-definite for* $t \in [T_{j-1}, T_j)$ *and satisfies a Riccati equation*

$$\begin{cases} \mathbf{A}^{j\prime}(t) + \dfrac{1-\gamma}{\gamma}\boldsymbol{\eta}_t^j\boldsymbol{\Sigma}\boldsymbol{\eta}_t^j + \dfrac{1}{\gamma}(\mathbf{A}(t)\boldsymbol{\Sigma}\boldsymbol{\eta}_t^j + \boldsymbol{\eta}_t^j\boldsymbol{\Sigma}\mathbf{A}(t)) + \dfrac{1}{\gamma}\mathbf{A}^j(t)\boldsymbol{\Sigma}\mathbf{A}^j(t) = \mathbf{0}, \\ \mathbf{A}^j(T_j) = \mathbf{A}^{j+1}(T_j), \end{cases} \tag{EC.69}$$

with terminal condition $\mathbf{A}^K(T) = \mathbf{0}$*, and*

$$\boldsymbol{\eta}_t^j = -(\mathbf{P}_{j:K})^\top((T_j - t)\mathbf{P}_{j:K}\boldsymbol{\Sigma}(\mathbf{P}_{j:K})^\top + T_j\bar{\boldsymbol{\Omega}}^j)^{-1}\mathbf{P}_{j:K},\ t \in [T_{j-1}, T_j). \tag{EC.70}$$

$\mathbf{b}^j(t)$ *solves a system of linear ODEs*

$$\begin{cases}\mathbf{b}^{j\prime}(t) + \frac{1}{\gamma}\big(\boldsymbol{\eta}_t^j + \mathbf{A}^j(t)\big)\boldsymbol{\Sigma}\mathbf{b}^j(t) + \frac{1-\gamma}{\gamma}\big(\boldsymbol{\eta}_t^j + \mathbf{A}^j(t)\big)(\boldsymbol{\alpha}_t^j - r_f\mathbf{1}_N) + \mathbf{A}^j(t)\big(\boldsymbol{\alpha}_t^j - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\big) = \mathbf{0}, \\ \mathbf{b}^j(T_j) = \mathbf{b}^{j+1}(T_j), \end{cases} \tag{EC.71}$$

with terminal condition $\mathbf{b}^K(T) = \mathbf{0}$*, and*

$$\boldsymbol{\alpha}_t^j = \boldsymbol{\mu} + \frac{1}{T_j}\boldsymbol{\beta}_1^j(\bar{\mathbf{y}}^j - T_j\mathbf{P}_{j:K}\boldsymbol{\mu}^x) - \boldsymbol{\Sigma}\boldsymbol{\eta}_t^j\mathbb{E}[\mathbf{X}^y(t)].$$

$c^j(t)$ *is the solution of*

$$\begin{cases} c^{j\prime}(t) + (1-\gamma)r_f + \frac{1}{2}\operatorname{Tr}\left(\mathbf{A}^j(t)\boldsymbol{\Sigma}\right) + \frac{1-\gamma}{2\gamma}\big(\boldsymbol{\alpha}_t^j - r_f\mathbf{1}_N\big)^\top\boldsymbol{\Sigma}^{-1}\big(\boldsymbol{\alpha}_t^j - r_f\mathbf{1}_N\big) + \big(\boldsymbol{\alpha}_t^j - \frac{1}{2}\operatorname{diag}(\boldsymbol{\Sigma})\big)^\top\mathbf{b}^j(t) \\ \qquad + \frac{1-\gamma}{\gamma}\big(\boldsymbol{\alpha}_t^j - r_f\mathbf{1}_N\big)^\top\mathbf{b}^j(t) + \frac{1}{2\gamma}\mathbf{b}^{j\top}(t)\boldsymbol{\Sigma}\mathbf{b}^j(t) = 0, \\ c^j(T_j) = c^{j+1}(T_j). \end{cases} \tag{EC.72}$$

with $c^K(T) = 0$*.*

There exists a unique optimal allocation policy

$$\boldsymbol{\pi}^{j*}(t, \mathbf{x}, \bar{\mathbf{y}}^j) = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\big(\tilde{\boldsymbol{\mu}}^j(t, \mathbf{x}, \bar{\mathbf{y}}^j) - r_f\mathbf{1}_N\big) + \frac{1}{\gamma}\frac{\partial g^j}{\partial \mathbf{x}}(t, \mathbf{x}, \bar{\mathbf{y}}^j),\ t \in [T_{j-1}, T_j), \tag{EC.73}$$

where

$$\frac{\partial g^j}{\partial \mathbf{x}}(t, \mathbf{x}, \bar{\mathbf{y}}^j) = \mathbf{A}^j(t)\mathbf{x} + \mathbf{b}^j(t),\ t \in [T_{j-1}, T_j).$$

EC.6.4. Proofs of the results

Proof of Proposition EC.1. As views are ordered according to their horizon, let $k, j \in [K]$ such that $k \geq j$, we can write

$$\begin{aligned}\mathbf{X}(T_k) &= T_j\boldsymbol{\mu}^x + \mathbf{W}(T_j) + (T_k - T_j)\boldsymbol{\mu}^x + \mathbf{W}(T_k) - \mathbf{W}(T_j) \\ &= \mathbf{X}(T_j) + \underbrace{(T_k - T_j)\boldsymbol{\mu}^x + \mathbf{W}(T_k) - \mathbf{W}(T_j)}_{\text{Adjustments}}.\end{aligned}$$

It follows that for $k \in [K]$, the view $Y_k(0, T_k)$ can be transformed to a view about the log-return realization at time $T_j \leq T_k$

$$Y_k(0, T_k) = \mathbf{p}_k^\top\mathbf{X}(T_j) + \bar{\mu}_k^j(T_j, T_{j:K}) + \bar{\epsilon}_k^j,$$

with

$$\begin{aligned}\bar{\mu}_k^j(T_j, T_{j:K}) &= (T_k - T_j)\mathbf{p}_k^\top\boldsymbol{\mu}^x, \\ \bar{\epsilon}_k^j &= \mathbf{p}_k^\top(\mathbf{W}(T_k) - \mathbf{W}(T_j)) + \sqrt{T_k}\epsilon_k.\end{aligned}$$

Now given the views horizons vector $\mathbf{T}_{j:K} = (T_j, \ldots, T_K)^\top \in \mathbb{R}^{K-j+1}$, and the linear mapping $\mathbf{P}_{j:K} = (\mathbf{p}_j, \ldots, \mathbf{p}_K)^\top \in \mathbb{R}^{(K-j+1)\times N}$, we can write

$$\mathbf{Y}^j(0, T_{j:K}) = \mathbf{P}_{j:K}\mathbf{X}(T_j) + \bar{\boldsymbol{\mu}}^j(T_j, T_{j:K}) + \bar{\boldsymbol{\epsilon}}^j,$$

where $\bar{\boldsymbol{\epsilon}}^j \in \mathbb{R}^{K-j+1\times K-j+1}$ is Gaussian with $\bar{\boldsymbol{\epsilon}}^j \sim \mathcal{N}\big(\mathbf{0}, T_j\bar{\boldsymbol{\Omega}}^j\big)$, and the covariance matrix $\bar{\boldsymbol{\Omega}}^j$ takes the form

$$\bar{\boldsymbol{\Omega}}^j = \bar{\boldsymbol{\Omega}}^{Vj} + \mathbf{P}_{j:K}\bar{\boldsymbol{\Omega}}^{Wj}(\mathbf{P}_{j:K})^\top,$$

where for $i, k \in \{1, \ldots, K-j+1\}$, we have $\bar{\Omega}^{jV}_{ik} = \dfrac{\sqrt{T_{j+i-1}T_{j+k-1}}}{T_j}\Omega_{j+i-1,j+k-1}$, and $\bar{\Omega}^{jW}_{ik} = \dfrac{1}{T_j}\min\{T_{j+i-1} - T_j, T_{j+k-1} - T_j\}\Sigma_{j+i-1,j+k-1}$. It follows that

$$\bar{\mathbf{Y}}^j(0, T_j) = \mathbf{Y}^j(0, T_{j:K}) - \bar{\boldsymbol{\mu}}^j(T_j, T_{j:K}) = \mathbf{P}_{j:K}\mathbf{X}(T_j) + \bar{\boldsymbol{\epsilon}}^j \sim \mathcal{N}\big(\mathbf{P}_{j:K}\mathbf{X}(T_j), T_j\bar{\boldsymbol{\Omega}}^j\big).$$

Furthermore, since $\bar{\mathbf{Y}}^j(0, T_j)$ contain the same information as $\mathbf{Y}^j(0, T_{j:K})$, it follows that

$$\mathbf{X}(t) \,|\, \mathbf{Y}^j(0, T_{j:K}) = \mathbf{X}(t) \,|\, \bar{\mathbf{Y}}^j(0, T_j), \quad t \in [T_{j-1}, T_j].$$

□

Proof of Proposition EC.2. The proof of this proposition is straightforward. We apply the results from Section 5, substituting the horizon T with T_j and the view $\mathbf{y}$ with $\bar{\mathbf{y}}^j$ for each $j \in [K]$. This directly leads to the derivation of ODEs (EC.69)–(EC.72). Furthermore, the boundary conditions are established by the continuation of the value function and the Principle of Optimality. Therefore, $\mathbf{A}^j(T_j) = \mathbf{A}^{j+1}(T_j)$, $\mathbf{b}^j(T_j) = \mathbf{b}^{j+1}(T_j)$, and $c^j(T_j) = c^{j+1}(T_j)$. This concludes the proof. □

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