

Electronic Companion for Pre-Hedging

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Appendix A: Proofs

A.1. Proof of Theorem 1

Throughout the proof, we fix an arbitrary value δ of the dealer's spread and solve for the optimal trading strategy in the three stages of the model by backward induction. We directly consider the case of two-way RFQs from Section 3.4; Theorem 1 then is obtained as the special case where the client buys with probability $\pi = 1$ if there is a trade.

A.1.1. Post-Hedging Stage First, we consider the post-hedging stage $[T + \tau, \infty)$. To ease notation, we set $\mathcal{T} := T + \tau$. Irrespective of whether the dealer does or does not win the trade with the client, the dealer solves an infinite-horizon optimal execution problem as in Schied and Schöneborn (2009); the various cases only differ in the dealer's initial holdings. Before the trade with the client, these are given by the same pre-hedge $X_{\mathcal{T}-}$ in each case. If no trade happens then this is the initial

condition for the dealer's execution problem. If the client purchases (resp. sells) X_* shares then the initial holdings are $X_{\mathcal{T}} = X_{\mathcal{T}-} - X_*$ (resp. $X_{\mathcal{T}} = X_{\mathcal{T}-} + X_*$). In each case, the permanent and transient price impacts remain at the same levels $\kappa X_{\mathcal{T}-}$ and $I_{\mathcal{T}} = I_{\mathcal{T}-}$ as before the trade with the client. The first step to solve the dealer's post-hedging problem is the following standard reformulation of the corresponding P&L:

LEMMA 1. *For each $\mathcal{T}' > \mathcal{T}$, the expected cash balance of the dealer from trading on $[\mathcal{T}, \mathcal{T}']$ is*

$$\mathbb{E}_{\mathcal{T}} \left[P_{\mathcal{T}}^* X_{\mathcal{T}} - P_{\mathcal{T}'}^* X_{\mathcal{T}'} + \frac{\kappa}{2} (X_{\mathcal{T}}^2 - X_{\mathcal{T}'}^2) + \kappa (X_{\mathcal{T}} - X_{\mathcal{T}-} + X_0) (X_{\mathcal{T}'} - X_{\mathcal{T}}) - \int_{\mathcal{T}}^{\mathcal{T}'} \left(\frac{\nu}{2} \dot{X}_t^2 + I_t \dot{X}_t \right) dt \right]. \quad (1)$$

Proof Using two integrations by parts, the dealer's cash balance can be rewritten as

$$\begin{aligned} & - \int_{\mathcal{T}}^{\mathcal{T}'} \left(P_t^* + I_t + \frac{\nu}{2} \dot{X}_t \right) dX_t - \int_{\mathcal{T}}^{\mathcal{T}'} (\kappa (X_t - X_{\mathcal{T}} + X_{\mathcal{T}-} - X_0)) dX_t \\ & = P_{\mathcal{T}}^* X_{\mathcal{T}} - P_{\mathcal{T}'}^* X_{\mathcal{T}'} + \int_{\mathcal{T}}^{\mathcal{T}'} X_t dP_t^* + \frac{\kappa}{2} (X_{\mathcal{T}}^2 - X_{\mathcal{T}'}^2) \\ & \quad + \kappa (X_{\mathcal{T}} - X_{\mathcal{T}-} + X_0) (X_{\mathcal{T}'} - X_{\mathcal{T}}) - \int_{\mathcal{T}}^{\mathcal{T}'} \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 \right) dt. \end{aligned}$$

By the martingale property of the unaffected price P_t^* , it follows that the corresponding expected cash balance is indeed given by (1).

Under the transversality condition that the dealer eventually liquidates their position ($X_{\mathcal{T}'} \rightarrow 0$ as $\mathcal{T}' \rightarrow \infty$) and this happens sufficiently fast to ensure $P_{\mathcal{T}'}^* X_{\mathcal{T}'} \rightarrow 0$ in \mathbb{L}_1 ,¹ the dealer's expected cash balance converges to

$$\mathbb{E}_{\mathcal{T}} \left[P_{\mathcal{T}}^* X_{\mathcal{T}} + \frac{\kappa}{2} X_{\mathcal{T}}^2 + \kappa (X_{\mathcal{T}-} - X_{\mathcal{T}} - X_0) X_{\mathcal{T}} - \int_{\mathcal{T}}^{\infty} \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 \right) dt \right], \quad \text{as } \mathcal{T}' \rightarrow \infty.$$

Since there is no external randomness left in the argument, we can drop the expectation and focus on deterministic trading strategies without loss of generality. After taking into account the dealer's running inventory cost $\frac{\gamma}{2} X_t^2 dt$, the optimal execution trajectory then needs to maximize

$$\begin{aligned} & P_{\mathcal{T}}^* X_{\mathcal{T}} + \frac{\kappa}{2} X_{\mathcal{T}}^2 + \kappa (X_{\mathcal{T}-} - X_{\mathcal{T}} - X_0) X_{\mathcal{T}} \\ & - \int_{\mathcal{T}}^{\infty} \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 + \frac{\gamma}{2} X_t^2 \right) dt. \end{aligned} \quad (2)$$

¹ Without transient price impact and for an unaffected price process following Brownian motion, Schied and Schöneborn (2009) provide a convenient sufficient condition for this via the law of the iterated logarithm.

LEMMA 2. *The optimal value for the execution problem (2) is*

$$P_{\mathcal{T}}^* X_{\mathcal{T}} + \frac{\kappa}{2} X_{\mathcal{T}}^2 + \kappa(X_{\mathcal{T}-} - X_{\mathcal{T}} - X_0)X_{\mathcal{T}} + \mathcal{V}(X_{\mathcal{T}}, I_{\mathcal{T}}),$$

where²

$$\mathcal{V}(X_t, I_t) = -\frac{1}{2}\mathcal{A}X_t^2 + \mathcal{B}X_t I_t + \frac{1}{2}C I_t^2. \quad (3)$$

Here, \mathcal{A} , \mathcal{B} , C are the constants from Theorem 1. The corresponding optimal feedback trading rate for $t \in [\mathcal{T}, \infty)$ is given by the third case of (5).

Proof As the other terms are fixed by the dealer's initial holdings, we focus on maximizing the integral

$$-\int_{\mathcal{T}}^{\infty} \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 + \frac{\gamma}{2} X_t^2 \right) dt.$$

The dynamic programming equation for the corresponding value function $\mathcal{V}(X_t, I_t)$ is

$$0 = \inf_{\dot{X}_t} \left\{ -I_t \dot{X}_t - \frac{\nu}{2} \dot{X}_t^2 - \frac{\gamma}{2} X_t^2 + \partial_X \mathcal{V}(X_t, I_t) \dot{X}_t \right. \\ \left. + \partial_I \mathcal{V}(X_t, I_t) (-\beta I_t + \lambda \dot{X}_t) \right\}. \quad (4)$$

With the quadratic ansatz (3), the pointwise maximizer becomes

$$\dot{X}_t = -\frac{\mathcal{A} - \mathcal{B}\lambda}{\nu} X_t - \frac{1 - \mathcal{B} - C\lambda}{\nu} I_t. \quad (5)$$

Plugging this back into the dynamic programming equation (4) and comparing coefficients for the terms proportional to X_t^2 , $X_t I_t$, and I_t^2 , respectively, then leads to the following three algebraic Riccati equations for the constants \mathcal{A} , \mathcal{B} , and C :

$$0 = \frac{(\mathcal{A} - \mathcal{B}\lambda)^2 - \gamma\nu}{2\nu}, \quad (6)$$

$$0 = \frac{(\mathcal{B}\lambda - \mathcal{A})(\mathcal{B} + C\lambda - 1)}{\nu} - \beta\mathcal{B}, \quad (7)$$

$$0 = \frac{(\mathcal{B} + C\lambda - 1)^2}{2\nu} - \beta C. \quad (8)$$

The first Riccati equation (6) gives $\mathcal{A} - \mathcal{B}\lambda = \sqrt{\gamma\nu}$. The second equation (7) yields $\mathcal{B} + C\lambda - 1 = -\beta(\gamma/\nu)^{-1/2}\mathcal{B}$ and the third equation (8) then implies $C = \beta(2\gamma)^{-1}\mathcal{B}^2$. Plugging this and $\mathcal{A} - \mathcal{B}\lambda = \sqrt{\gamma\nu}$ into the second equation (7) in turn shows that \mathcal{B} solves the quadratic equation

$$0 = \sqrt{\frac{\gamma}{\nu}} \left(\mathcal{B} + \frac{\beta\lambda}{2\gamma} \mathcal{B}^2 - 1 \right) + \beta\mathcal{B}.$$

²This is the value function for maximizing the integral term in (2), starting from holdings X_t and transient impact I_t at time t .

The positive solution of this equation (which ensures that a positive initial price impact indeed leads to higher liquidation revenues) is the one provided in Theorem 1. The other two constants are in turn given by $\mathcal{A} = \mathcal{B}\lambda + \sqrt{\gamma v}$ and $C = \frac{\beta}{2\gamma}\mathcal{B}^2$ as asserted. Moreover, in view of (5), the optimal trading rate during the post hedging stage indeed is given by the second case of (5).

Depending on whether or not the dealer wins the client's trade and whether this is a purchase or sale, Lemma 2 yields the following three optimal values for the dealer's post hedging stage:

(i) If there is no trade with the client, then the dealer's inventory remains unchanged and there is no cash transaction with the client. Whence, the dealer's optimal value is

$$P_{\mathcal{T}}^* X_{\mathcal{T}-} + \frac{\kappa}{2} X_{\mathcal{T}-}^2 - \kappa X_0 X_{\mathcal{T}-} + \mathcal{V}(X_{\mathcal{T}-}, I_{\mathcal{T}-}).$$

Recall that this scenario happens with probability $1 - p(\delta)$.

(ii) If the client has purchased X_* shares from the dealer for a price of $P_{\mathcal{T}}^* + \kappa X_{\mathcal{T}-} + I_{\mathcal{T}-} + \delta$ per share, then the dealer's optimal post hedge value is

$$\begin{aligned} & (P_{\mathcal{T}}^* + \kappa X_{\mathcal{T}-} + I_{\mathcal{T}-} + \delta) X_* + P_{\mathcal{T}}^* (X_{\mathcal{T}-} - X_*) + \frac{\kappa}{2} (X_{\mathcal{T}-} - X_*)^2 + \kappa (X_* - X_0) (X_{\mathcal{T}-} - X_*) \\ & + \mathcal{V}(X_{\mathcal{T}-} - X_*, I_{\mathcal{T}-}) \\ & = P_{\mathcal{T}}^* X_{\mathcal{T}-} + I_{\mathcal{T}-} X_* + \delta X_* + \mathcal{V}(X_{\mathcal{T}-} - X_*, I_{\mathcal{T}-}) + \frac{\kappa}{2} (X_{\mathcal{T}-}^2 + 2X_{\mathcal{T}-} X_* - X_*^2 - 2X_0 (X_{\mathcal{T}-} - X_*)). \end{aligned}$$

Recall that this scenario happens with probability $p(\delta)\pi$.

(iii) If the client has sold X_* shares to the dealer for a price of $P_{\mathcal{T}}^* + \kappa X_{\mathcal{T}-} + I_{\mathcal{T}-} - \delta$ per share, then the dealer's optimal post hedge value is

$$\begin{aligned} & - (P_{\mathcal{T}}^* + \kappa X_{\mathcal{T}-} + I_{\mathcal{T}-} - \delta) X_* + P_{\mathcal{T}}^* (X_{\mathcal{T}-} + X_*) + \frac{\kappa}{2} (X_{\mathcal{T}-} + X_*)^2 - \kappa (X_* + X_0) (X_{\mathcal{T}-} + X_*) \\ & + \mathcal{V}(X_{\mathcal{T}-} + X_*, I_{\mathcal{T}-}) \\ & = P_{\mathcal{T}}^* X_{\mathcal{T}-} - I_{\mathcal{T}-} X_* + \delta X_* + \mathcal{V}(X_{\mathcal{T}-} + X_*, I_{\mathcal{T}-}) + \frac{\kappa}{2} (X_{\mathcal{T}-}^2 - 2X_{\mathcal{T}-} X_* - X_*^2 - 2X_0 (X_{\mathcal{T}-} + X_*)). \end{aligned}$$

Recall that this scenario happens with probability $p(\delta)(1 - \pi)$.

In summary, just before learning whether the trade with the client is realized, the dealer's expected continuation value is³

$$\begin{aligned} & P_{\mathcal{T}}^* X_{\mathcal{T}-} + \frac{\kappa}{2} X_{\mathcal{T}-}^2 - \kappa X_0 X_{\mathcal{T}-} - \frac{\mathcal{A}}{2} X_{\mathcal{T}-}^2 + \mathcal{B} X_{\mathcal{T}-} I_{\mathcal{T}-} + \frac{C}{2} I_{\mathcal{T}-}^2 \\ & + p(\delta) \left(\frac{\kappa}{2} (2(2\pi - 1)(X_{\mathcal{T}-} + X_0) X_* - X_*^2) + (2\pi - 1) I_{\mathcal{T}-} X_* + \delta X_* - \frac{\mathcal{A}}{2} (X_*^2 - 2(2\pi - 1) X_{\mathcal{T}-} X_*) \right. \\ & \quad \left. - \mathcal{B} (2\pi - 1) I_{\mathcal{T}-} X_* \right). \end{aligned} \tag{9}$$

³ Here, we have also plugged in the quadratic form (3) of the execution value function from Lemma 2.

A.1.2. Random Pre-Hedging Stage We now turn to the random pre-hedging stage $(T, T + \tau) = (T, \mathcal{T})$. We write X_T for the dealer's pre-hedge already built up on the deterministic pre-hedging stage $[0, T]$ (which will be determined in the next section), and denote the corresponding permanent and transient price impacts by I_T and κX_T , respectively. Using integration by parts as in Lemma 1, the dealer's P&L from trading on (T, \mathcal{T}) can be written as

$$\begin{aligned} & - \int_T^{\mathcal{T}^-} \left(P_t^* + \kappa(X_t - X_0) + I_t + \frac{\nu}{2} \dot{X}_t \right) dX_t \\ & = P_T^* X_T - P_{\mathcal{T}^-}^* X_{\mathcal{T}^-} + \int_T^{\mathcal{T}} X_t dP_t^* + \frac{\kappa}{2} (X_T^2 - X_{\mathcal{T}^-}^2) + \kappa X_0 (X_{\mathcal{T}^-} - X_T) - \int_T^{\mathcal{T}} \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 \right) dt. \end{aligned}$$

After combining this with the dealer's running inventory costs $\frac{\gamma}{2} X_t^2 dt$ and the continuation value (9), it follows from the martingale property of the unaffected price P_t^* that the dealer's expected P&L on (T, ∞) is

$$\begin{aligned} \mathbb{E}_T \left[P_T^* X_T + \frac{\kappa}{2} X_T^2 - \kappa X_0 X_T - \int_T^{\mathcal{T}} \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 + \frac{\gamma}{2} X_t^2 \right) dt + \mathcal{V}(X_{\mathcal{T}^-}, I_{\mathcal{T}^-}) \right. \\ \left. + p(\delta) \left(\frac{\kappa}{2} (2(2\pi - 1)(X_{\mathcal{T}^-} + X_0) X_* - X_*^2) + (2\pi - 1) I_T X_* + \delta X_* \right. \right. \\ \left. \left. - \frac{\mathcal{A}}{2} (X_*^2 - 2(2\pi - 1) X_{\mathcal{T}^-} X_*) - \mathcal{B} (2\pi - 1) I_T X_* \right) \right]. \end{aligned} \quad (10)$$

As in the post-hedging stage, the random unaffected price P_t^* does not appear in the goal functional (10). However, there still is an external source of randomness: the length $\tau = \mathcal{T} - T$ of the random pre-hedging stage. However, if this random time is independent and exponentially distributed, then the optimal trading rate is still deterministic on $[T, T + \tau)$, up to switching to the optimal post hedging trading rate at the random time $\mathcal{T} = T + \tau$. The corresponding value function also turns out to be linear-quadratic (like in similar problems with deterministic time horizon):

LEMMA 3. *The optimal value for the random pre-hedging problem (10) is*

$$P_T^* X_T + \frac{\kappa}{2} X_T^2 - \kappa X_0 X_T + V(X_T, I_T), \quad (11)$$

where

$$\begin{aligned} V(X_t, I_t) = & -\frac{1}{2} \mathcal{A} X_t^2 + \mathcal{B} X_t I_t + \frac{1}{2} C I_t^2 + p(\delta) (2\pi - 1) X_* \mathcal{D} X_t + p(\delta) (2\pi - 1) X_* \mathcal{E} I_t \\ & + p(\delta) \left(\delta X_* - \frac{\kappa + \mathcal{A}}{2} X_*^2 + \kappa (2\pi - 1) X_* X_0 \right) + p(\delta)^2 (2\pi - 1)^2 X_*^2 \mathcal{F}. \end{aligned}$$

Here, \mathcal{A} , \mathcal{B} , C as well as \mathcal{D} , \mathcal{E} , \mathcal{F} are the constants from Theorem 1. The corresponding optimal feedback trading rate for $t \in (T, T + \tau)$ is given by the second case of (5).

Proof By the memoryless property of the exponential distribution, if the trade with the client has not happened until time $t \geq T$, then the probability that it happens on the infinitesimal interval $[t, t + dt]$ is $\alpha^{-1} dt$. Whence, the dynamic principle (informally) takes the following form here:

$$V(X_t, I_t) = \sup_{\dot{X}_t} \left\{ (1 - \alpha^{-1} dt) \left(-I_t \dot{X}_t dt - \frac{\nu}{2} \dot{X}_t^2 dt - \frac{\gamma}{2} X_t^2 dt + V(X_t + \dot{X}_t dt, I_t - \beta I_t dt + \lambda \dot{X}_t dt) \right) + \alpha^{-1} dt \left(\ell_0 + \ell_X X_t + \ell_I I_t + \mathcal{V}(X_t, I_t) \right) \right\},$$

where

$$\begin{aligned} \ell_0 &= p(\delta) \left(\delta X_* - \frac{\kappa + \mathcal{A}}{2} X_*^2 + \kappa(2\pi - 1) X_* X_0 \right), \\ \ell_X &= p(\delta)(2\pi - 1)(\kappa + \mathcal{A}) X_*, \\ \ell_I &= p(\delta)(2\pi - 1)(1 - \mathcal{B}) X_*. \end{aligned}$$

Here, the first term in the dynamic programming equation corresponds to the case where the trade does not happen and the dealer continues trading. In the second case, the trade happens and the dealer switches to the post hedge value from (10). For infinitesimal dt , Taylor expansion and dropping terms of higher order leads to

$$0 = \sup_{\dot{X}_t} \left\{ -I_t \dot{X}_t - \frac{\nu}{2} \dot{X}_t^2 - \frac{\gamma}{2} X_t^2 - \alpha^{-1} V(X_t, I_t) + \partial_X V(X_t, I_t) \dot{X}_t + \partial_I V(X_t, I_t) (-\beta I_t + \lambda \dot{X}_t) + \left(\ell_0 + \ell_X X_t + \ell_I I_t + \mathcal{V}(X_t, I_t) \right) \alpha^{-1} \right\}.$$

With the linear-quadratic ansatz

$$V(X_t, I_t) = -\frac{1}{2} A X_t^2 + B X_t I_t + \frac{1}{2} C I_t^2 + p(\delta)(2\pi - 1) X_* D X_t + p(\delta)(2\pi - 1) X_* E I_t + \ell_0 + p(\delta)^2 (2\pi - 1)^2 X_*^2 F,$$

the pointwise optimizer becomes

$$\begin{aligned} \dot{X}_t &= \frac{p(\delta)(2\pi - 1) X_* (D + \lambda E)}{\nu} \\ &\quad + \frac{\lambda B - A}{\nu} X_t + \frac{B + \lambda C - 1}{\nu} I_t. \end{aligned} \tag{12}$$

Plugging this back into the dynamic programming equation and comparing coefficients for the terms proportional to X_t^2 , $X_t I_t$, I_t^2 , X_t , I_t , and independent of these state variables in turn leads to six equations for the six constants A, B, C and D, E, F , respectively:

$$0 = \frac{(A - \lambda B)^2 - \nu(\alpha^{-1}(\mathcal{A} - A) + \gamma)}{2\nu},$$

$$0 = \frac{(A - \lambda B)(1 - B - \lambda C)}{\nu} - \beta B + \alpha^{-1}(\mathcal{B} - B),$$

$$0 = \frac{(1 - B - \lambda C)^2 + \alpha^{-1}\nu(C - C) - 2\nu\beta C}{2\nu},$$

as well as

$$0 = \alpha^{-1}(\kappa + \mathcal{A} - D) - \frac{(A - \lambda B)(D + \lambda E)}{\nu}, \quad (13)$$

$$0 = \frac{(B + \lambda C - 1)(D + \lambda E)}{\nu} \quad (14)$$

$$- (\alpha^{-1} + \beta)E + \alpha^{-1}(1 - \mathcal{B}), \quad (15)$$

$$0 = \frac{(D + \lambda E)^2}{2\nu} - \alpha^{-1}F. \quad (16)$$

Note that the first equations for A , B , and C decouple from the other three equations. Moreover, in view of the Riccati equations (6)-(8) for \mathcal{A} , \mathcal{B} , and \mathcal{C} , the equations for A , B , and C are also solved by $A = \mathcal{A}$, $B = \mathcal{B}$, $C = \mathcal{C}$, which have the explicit expressions from Theorem 1.

The linear equations (13)-(15) in turn have the solutions $D = \mathcal{D}$ and $E = \mathcal{E}$ with the explicit expressions provided in Theorem 1. After plugging these into (16), we then obtain the explicit formula from Theorem 1 for the last coefficient $F = \mathcal{F}$.

Finally, in view of (12), the optimal trading rate on the random pre-hedging stage indeed is given by the second case of (5).

The dealer's optimal pre-hedge X_t during the random pre-hedging stage and the corresponding transient price impact I_t solve the inhomogeneous linear ODE

$$\begin{bmatrix} \dot{X}_t \\ \dot{I}_t \end{bmatrix} = - \underbrace{\begin{bmatrix} c_X & c_I \\ \lambda c_X & \lambda c_I + \beta \end{bmatrix}}_{=M} \begin{bmatrix} X_t \\ I_t \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda \end{bmatrix} (\mathcal{D} + \lambda \mathcal{E}) \frac{p(\delta)X_*}{\nu} (2\pi - 1).$$

with initial conditions X_T and I_T . Via variation of constants, the explicit solution is given by

$$\begin{aligned} \begin{bmatrix} X_t \\ I_t \end{bmatrix} &= e^{-M(t-T)} \begin{bmatrix} X_T \\ I_T \end{bmatrix} + (\mathcal{D} + \lambda \mathcal{E}) \frac{p(\delta)X_*}{\nu} (2\pi - 1) \int_T^t e^{-M(t-s)} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} ds \\ &= \begin{bmatrix} e^{-\frac{1}{2}(\Lambda+\omega)(t-T)} \left(\frac{\Lambda+2c_X-\omega}{2\Lambda} X_T + \frac{c_I}{\Lambda} I_T + \left(\frac{\Lambda-2c_X+\omega}{2\Lambda} X_T - \frac{c_I}{\Lambda} I_T \right) e^{\Lambda(t-T)} \right) \\ e^{-\frac{1}{2}(\Lambda+\omega)(t-T)} \left(\frac{c_X\lambda}{\Lambda} X_T + \frac{\Lambda-2c_X+\omega}{2\Lambda} I_T + \left(-\frac{c_X\lambda}{\Lambda} X_T + \frac{\Lambda+2c_X-\omega}{2\Lambda} I_T \right) e^{\Lambda(t-T)} \right) \end{bmatrix} \quad (17) \\ &\quad + (\mathcal{D} + \lambda \mathcal{E}) \frac{p(\delta)X_*}{\nu} (2\pi - 1) \begin{bmatrix} \frac{1}{c_X} - e^{\frac{1}{2}\omega(T-t)} \left(\frac{2c_X-\omega}{c_X\Lambda} \sinh\left(\frac{1}{2}\Lambda(T-t)\right) + \frac{1}{c_X} \cosh\left(\frac{1}{2}\Lambda(T-t)\right) \right) \\ \frac{2\lambda}{\Lambda} e^{\frac{1}{2}\omega(T-t)} \sinh\left(\frac{1}{2}\Lambda(T-t)\right) \end{bmatrix}, \end{aligned}$$

where $\omega = c_X + \lambda c_I + \beta$ and $\Lambda = \sqrt{\omega^2 - 4\beta c_X}$.

A.1.3. Deterministic Pre-Hedging Stage We now turn to the deterministic pre-hedging stage $[0, T]$ of the model, where the dealer starts trading from initial holdings X_0 and zero initial impact $I_0 = 0$. Using integration by parts as in Lemma 1, the dealer's P&L from trading on $[0, T]$ can be written as

$$\begin{aligned} & - \int_0^T \left(P_t^* + \kappa(X_t - X_0) + I_t + \frac{\nu}{2} \dot{X}_t \right) dX_t \\ & = P_0^* X_0 - P_T^* X_T + \int_0^T X_t dP_t^* + \frac{\kappa}{2} (X_0^2 - X_T^2) + \kappa X_0 (X_T - X_0) - \int_0^T \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 \right) dt. \end{aligned}$$

After combining this with the dealer's running inventory costs $\frac{\gamma}{2} X_t^2 dt$ and the continuation value (11), it follows from the martingale property of the unaffected price P_t^* that the dealer's expected P&L on $[0, \infty)$ is

$$P_0^* X_0 - \frac{\kappa}{2} X_0^2 + \mathbb{E} \left[- \int_0^T \left(I_t \dot{X}_t + \frac{\nu}{2} \dot{X}_t^2 + \frac{\gamma}{2} X_t^2 \right) dt + V(X_T, I_T) \right]. \quad (18)$$

In particular, all external randomness has disappeared from the argument, so we can drop the expectation and focus on deterministic trading strategies without loss of generality. In view of the quadratic form of the continuation value $V(X_T, I_T)$ from Lemma 3, the deterministic stage of the pre-hedging problem is another linear-quadratic control problem that can be solved in closed form:⁴

LEMMA 4. *The optimal value for the deterministic pre-hedging problem (18) is*

$$P_0^* X_0 - \frac{\kappa}{2} X_0^2 + V_0(X_0, 0),$$

where⁵

$$\begin{aligned} V_0(t, X_t, I_t) = & -\frac{1}{2} \mathcal{A} X_t^2 + \mathcal{B} X_t I_t + \frac{1}{2} C I_t^2 + p(\delta)(2\pi - 1) X_* D(t) X_t + p(\delta)(2\pi - 1) X_* E(t) I_t \\ & + p(\delta) \left(\delta X_* - \frac{\kappa + \mathcal{A}}{2} X_*^2 + \kappa(2\pi - 1) X_* X_0 \right) + p(\delta)^2 (2\pi - 1)^2 X_*^2 F(t), \end{aligned}$$

and the deterministic functions $D(t)$, $E(t)$, $F(t)$ solve the ODEs from Theorem 1. The corresponding optimal trading rate for $t \in [0, T]$ is given by the first case of (5).

⁴ There is a vast literature on solutions of such linear-quadratic control problems with quadratic terminal conditions in much more general settings. For example, Nutz, Webster, and Zhao (2023) study a model where the terminal condition is derived from a block trade in a closing auction for exogenously given stochastic client flow. In the present paper, the probability of the client trade is an output of the model, that is naturally intertwined with pre- and post-hedging.

⁵ This is the value function for maximizing the integral term in (18), starting from holdings X_0 and transient impact $I_0 = 0$ at time 0.

Proof The dynamic programming equation for (18) is

$$0 = \inf_{\dot{X}_t} \left\{ -I_t \dot{X}_t - \frac{\nu}{2} \dot{X}_t^2 - \frac{\gamma}{2} X_t^2 + \partial_t V(t, X_t, I_t) + \partial_X V_0(t, X_t, I_t) \dot{X}_t + \partial_I V_0(t, X_t, I_t) (-\beta I_t + \lambda \dot{X}_t) \right\},$$

which is complemented by the terminal condition $V_0(T, X_T, I_T) = V(X_T, I_T)$. With the linear-quadratic ansatz

$$\begin{aligned} V_0(t, X_t, I_t) = & -\frac{1}{2} A(t) X_t^2 + B(t) X_t I_t + \frac{1}{2} C(t) I_t^2 + p(\delta)(2\pi - 1) X_* D(t) X_t + p(\delta)(2\pi - 1) X_* E(t) I_t \\ & + p(\delta) \left(\delta X_* - \frac{\kappa + \mathcal{A}}{2} X_*^2 + \kappa(2\pi - 1) X_* X_0 \right) + p(\delta)^2 (2\pi - 1)^2 X_*^2 F(t), \end{aligned}$$

the pointwise optimizer becomes

$$\dot{X}_t = \frac{p(\delta)(2\pi - 1) X_* (D(t) + \lambda E(t))}{\nu} + \frac{\lambda B(t) - A(t)}{\nu} X_t + \frac{B(t) + \lambda C(t) - 1}{\nu} I_t. \quad (19)$$

Plugging this back into the dynamic programming equation and comparing coefficients for the terms proportional to X_t^2 , $X_t I_t$, I_t^2 , X_t , I_t , and independent of these state variables in turn leads to six Riccati ODEs for the six deterministic functions $A_0(t)$, $B_0(t)$, $C_0(t)$, $D_0(t)$, $E_0(t)$, $F_0(t)$.

The ODEs for the first three functions once again decouple from the other equations:

$$\begin{aligned} 0 &= -\dot{A}(t) + \frac{1}{\nu} (\lambda B(t) - A(t))^2 - \gamma, \\ 0 &= \dot{B}(t) - \beta B(t) + \frac{1}{\nu} (\lambda B(t) - A(t))(B(t) + \lambda C(t) - 1), \\ 0 &= \dot{C}(t) + \frac{1}{\nu} (B(t) + \lambda C(t) - 1)^2 - 2\beta C(t), \end{aligned}$$

with terminal conditions $A(T) = \mathcal{A}$, $B(T) = \mathcal{B}$, and $C(T) = \mathcal{C}$. In view of the algebraic Riccati equations (6)-(8) for \mathcal{A} , \mathcal{B} , \mathcal{C} , these equations can still be solved with the same constant values $A(t) = \mathcal{A}$, $B(t) = \mathcal{B}$, and $C(t) = \mathcal{C}$ as in the subsequent stages of the model. The ODEs for the other three coefficient functions in turn simplify to the equations from Theorem 1.

Finally, in view of (19), the optimal trading rate during the deterministic pre-hedging stage indeed is of the form asserted in the first case of (5).

The solution of the linear system for $D(t)$ and $E(t)$ is given by the matrix exponential

$$\begin{bmatrix} D(t) \\ E(t) \end{bmatrix} = \exp \left(\begin{bmatrix} c_x & \lambda c_x \\ c_I & \lambda c_I + \beta \end{bmatrix} (t - T) \right) \begin{bmatrix} \mathcal{D} \\ \mathcal{E} \end{bmatrix}. \quad (20)$$

By diagonalizing the argument of the matrix exponential, this leads to the explicit formulas

$$D(t) = \frac{e^{\frac{1}{2}\omega(t-T)} \left(\sinh\left(\frac{1}{2}\Lambda(t-T)\right) (c_X(\mathcal{D} + 2\lambda\mathcal{E}) - \mathcal{D}(\beta + \lambda c_I)) + \mathcal{D}\Lambda \cosh\left(\frac{1}{2}\Lambda(t-T)\right) \right)}{\Lambda},$$

$$E(t) = \frac{e^{\frac{1}{2}\omega(t-T)} \left(\sinh\left(\frac{1}{2}\Lambda(t-T)\right) (2c_I\mathcal{D} + c_I\lambda\mathcal{E} - c_X\mathcal{E} + \beta\mathcal{E}) + \mathcal{E}\Lambda \cosh\left(\frac{1}{2}\Lambda(t-T)\right) \right)}{\Lambda},$$

where $\omega = c_X + \lambda c_I + \beta$ and $\Lambda = \sqrt{\omega^2 - 4\beta c_X}$. The function $F(t)$ can in turn be computed by a direct integration:

$$\begin{aligned} F(t) &= \mathcal{F} + \int_t^T \frac{(D(s) + \lambda E(s))^2}{2\nu} ds \\ &= \mathcal{F} + \frac{1}{2\nu} \int_t^T e^{-\omega(T-s)} \left(\mu \cosh\left(\frac{1}{2}\Lambda(T-s)\right) - \sigma \sinh\left(\frac{1}{2}\Lambda(T-s)\right) \right)^2 ds \\ &= \mathcal{F} + \frac{1}{4\nu\omega(\omega^2 - \Lambda^2)} \left(e^{-\omega(T-t)} \left((\Lambda^2 - \omega^2)(\mu^2 - \sigma^2) + \omega \left(\Lambda(\mu^2 + \sigma^2) - 2\mu\sigma\omega \right) \sinh(\Lambda(t-T)) \right) \right. \\ &\quad \left. - \omega e^{-\omega(T-t)} \left(\omega(\mu^2 + \sigma^2) - 2\Lambda\mu\sigma \right) \cosh(\Lambda(t-T)) + \left(\Lambda^2(\sigma^2 - \mu^2) - 2\Lambda\mu\sigma\omega + 2\mu^2\omega^2 \right) \right), \end{aligned} \quad (21)$$

where $\mu = \mathcal{D} + \lambda\mathcal{E}$ and $\sigma = \frac{(\omega - 2\beta)\mathcal{D} + \lambda\omega\mathcal{E}}{\Lambda}$.

Like for the random pre-hedging stage, fully explicit formulas for the optimal pre-hedge and the corresponding transient price impact can be derived by solving the inhomogeneous ODE that these terms satisfy using variations of constants. To wit, the pre-hedge path over the fixed horizon can be expressed as:

$$\begin{aligned} X_t &= e^{-\frac{1}{2}\omega t} \left(\cosh(\Lambda t/2) + \frac{\omega - 2c_X}{\Lambda} \sinh(\Lambda t/2) \right) X_0 + \frac{(2\pi - 1)p(\delta)X_*}{2\nu e^{\frac{1}{2}\omega T} \Lambda^2} \\ &\quad \times \left(h(t, T, \Lambda) + h(t, T, -\Lambda) - 2 \frac{\sinh(\omega t/2)}{\omega} (\rho_1 \cosh(\Lambda(t-T)/2) + \rho_2 \sinh(\Lambda(t-T)/2)) \right), \end{aligned} \quad (22)$$

where $\rho_1 = (2\eta^2 - \Lambda^2 - \omega\eta)(\mathcal{D} + \mathcal{E}\lambda) + \beta(\omega - 2\eta)(\mathcal{D} - \mathcal{E}\lambda)$, $\rho_2 = \Lambda(\eta - \omega)(\mathcal{D} + \mathcal{E}\lambda) + \Lambda\beta(\mathcal{D} - \mathcal{E}\lambda)$, $\eta = \lambda c_I + c_X$ and

$$\begin{aligned} h(t, T, \Lambda) &= e^{-\frac{1}{2}\Lambda T} \sinh((\omega + \Lambda)t/2) \\ &\quad \times \frac{(2\eta - \omega + \Lambda)((\eta + \Lambda)(\mathcal{D} + \mathcal{E}\lambda) - \beta(\mathcal{D} - \mathcal{E}\lambda))}{\omega + \Lambda}. \end{aligned}$$

For the corresponding transient impact, we obtain

$$I_t = \lambda \frac{e^{-\frac{1}{2}t(\Lambda+\omega)} (1 - e^{\Lambda t}) c_X}{\Lambda} X_0 + \lambda \frac{(2\pi - 1)p(\delta)X_*}{4\nu\omega\Lambda^2} \quad (23)$$

$$\times \left(((\mathcal{D} + \mathcal{E}\lambda)(\Lambda + \omega) - 2\beta\mathcal{D}) \left(\Lambda e^{\frac{1}{2}(\omega+\Lambda)(t-T)} - (\Lambda - \omega)e^{-\frac{1}{2}(\omega-\Lambda)(t-T)-\omega T} - \omega e^{-\frac{1}{2}(\omega+\Lambda)(t+T)} \right) \right.$$

$$\left. + ((\mathcal{D} + \mathcal{E}\lambda)(\Lambda - \omega) + 2\beta\mathcal{D}) \left(\Lambda e^{\frac{1}{2}(\omega-\Lambda)(t-T)} - (\Lambda + \omega)e^{-\frac{1}{2}(\omega+\Lambda)(t-T)-\omega T} + \omega e^{-\frac{1}{2}(\omega-\Lambda)(t+T)} \right) \right).$$

A.2. Proof of Corollary 1

In the Almgren-Chriss model without transient price impact ($\lambda = 0$), all constants and functions related to transient impact vanish:

$$\mathcal{B} = \mathcal{C} = \mathcal{E} = 0, \quad E(t) = 0.$$

The remaining constants in turn reduce to

$$\mathcal{A} = \sqrt{\gamma\nu}, \quad \mathcal{D} = \frac{\kappa + \sqrt{\gamma\nu}}{1 + \alpha\sqrt{\gamma/\nu}}, \quad \mathcal{F} = \frac{\alpha}{2\nu} \left(\frac{\kappa + \sqrt{\gamma\nu}}{1 + \alpha\sqrt{\gamma/\nu}} \right)^2.$$

The functions $D(t)$ and $F(t)$ from Theorem 1 then simplify to the expressions from Corollary 1. Without transient impact, the optimal holdings (5) satisfy scalar linear ODEs, which have the solutions (7)-(9) reported in Corollary 1.

A.3. Proof of Theorem 2

When the client's trade probability is price sensitive, the continuation value from the post-hedge period (9) remains unchanged, up to replacing the trade probability $p(\delta)$ with the random variable $p(\delta, P_T)$ that depends on the market price P_T at the trading time. Together with the P&L on the pre-hedge stage, it then follows that the goal functional of the dealer is

$$P_0 X_0 - \frac{\kappa}{2} X_0^2 - \frac{\mathcal{A}}{2} X_{T-}^2 - \int_0^T \left(\frac{\gamma}{2} X_t^2 + \frac{\nu}{2} \dot{X}_t^2 \right) dt$$

$$+ p(\delta, P_T) \left(\underbrace{\kappa X_0 X_* + \delta X_*}_{=\ell_0(\delta)} - \frac{\mathcal{A} + \kappa}{2} X_*^2 + \underbrace{(\mathcal{A} + \kappa) X_* X_{T-}}_{=\ell_X} \right).$$

When the client's trade probability is of the affine linear form (22), the terminal condition of the value function for the above problem becomes a quadratic function of X_{T-} and P_T^* , so it is natural

to guess that this also holds at earlier times. When the unaffected price process P_t^* is a Brownian motion, there are indeed no further state variables and the quadratic ansatz

$$V(t, X_t, P_t^*) = -\frac{A(t)}{2}X_t^2 + BX_tP_t^* + \frac{C(t)}{2}(P_t^*)^2 + \left((\phi - \varsigma_1(\delta - \delta_0) + \varsigma_2P_0)\ell_X - \varsigma_2\kappa\ell_0(\delta) \right) D(t)X_t \\ - \varsigma_2\ell_0(\delta)E(t)P_t^* + P_0X_0 - \frac{\kappa}{2}X_0^2 + \left(\phi - \varsigma_1(\delta - \delta_0) + \varsigma_2P_0 \right) \ell_0(\delta) + F(t)$$

leads to the dynamic programming equation

$$0 = \sup_{\dot{X}_t} \left\{ -\frac{\gamma}{2}X_t^2 - \frac{\nu}{2}\dot{X}_t^2 - A(t)X_t\dot{X}_t + B(t)P_t^*\dot{X}_t + D(t)\dot{X}_t + \frac{C(t)}{2}\sigma^2 \right. \\ \left. - \frac{A'(t)}{2}X_t^2 + B'(t)X_tP_t^* + \frac{C'(t)}{2}(P_t^*)^2 + D'(t)X_t + E'(t)P_t^* + F'(t) \right\},$$

with the terminal conditions

$$-\frac{A(T)}{2} = -\frac{\mathcal{A}}{2} - \varsigma_1\kappa\ell_X, \quad B(T) = -\varsigma_2\ell_X, \quad \frac{C(T)}{2} = 0, \quad D(T) = 1, \quad E(T) = 1, \quad F(T) = 0.$$

The pointwise maximizer of the DPE is

$$\dot{X}_t = \frac{((\phi - \varsigma_1(\delta - \delta_0) + \varsigma_2P_0)\ell_X - \varsigma_2\kappa\ell_0(\delta))D(t) - \frac{A(t)}{\nu}X_t + \frac{B(t)}{\nu}P_t^*}{\nu}.$$

Plugging this back into the DPE and comparing coefficients for the terms proportional to X_t^2 , $X_tP_t^*$, $(P_t^*)^2$, X_t , P_t^* and independent of these state variables in turn leads to six ODEs for the six functions $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $F(t)$, which are complemented by the above terminal conditions. The ODE for $A(t)$ is autonomous:

$$0 = A(t)^2 - (A'(t) + \gamma)\nu, \quad A(T) = \sqrt{\gamma\nu} + 2\varsigma_2\kappa\ell_X.$$

(Here we have used that $\mathcal{A} = \sqrt{\gamma\nu}$.) Unlike in the version of the model the the trade probability only depends on the spread, this can no longer be solved with a constant function, but is still independent from all the other functions and has the explicit solution

$$A(t) = \sqrt{\gamma\nu} \times \frac{1 + \frac{\varsigma_2\kappa\ell_X}{\varsigma_2\kappa\ell_X + \sqrt{\gamma\nu}} e^{-2(T-t)\sqrt{\frac{\gamma}{\nu}}}}{1 - \frac{\varsigma_2\kappa\ell_X}{\varsigma_2\kappa\ell_X + \sqrt{\gamma\nu}} e^{-2(T-t)\sqrt{\frac{\gamma}{\nu}}}},$$

which smoothly interpolates between the terminal condition $A(T) = \sqrt{\gamma\nu} + 2\varsigma_2\kappa\ell_X > \sqrt{\gamma\nu}$ and the long run stationary point $\sqrt{\gamma\nu}$. This moderates the pre hedging close to the trading time T somewhat by making the coefficient of X_t in the optimal trading rate more negative.

With the function $A(t)$ at hand, the function $B(t)$ solves a linear ODE:

$$0 = B'(t) - \frac{1}{\nu} A(t)B(t).$$

Again, the solution depends on X_* now, but not on δ . By variation of constants, it is given by

$$B(t) = \exp\left(-\int_t^T \frac{1}{\nu} A(s) ds\right) B(T) = -\frac{\varsigma_2 \ell_X \sqrt{\gamma \nu} e^{-(T-t)\sqrt{\frac{\gamma}{\nu}}}}{\varsigma_2 \kappa \ell_X (1 - e^{-2(T-t)\sqrt{\frac{\gamma}{\nu}}}) + \sqrt{\gamma \nu}}.$$

With $B(t)$ at hand, the function $C(t)$ can be computed by a direct integration:

$$C'(t) + \frac{1}{\nu} B(t)^2 = 0 \quad \rightsquigarrow \quad C(t) = \frac{1}{\nu} \int_t^T B(s)^2 ds = \frac{\varsigma_2^2 \ell_X^2}{2\varsigma_2 \kappa \ell_X + \sqrt{\gamma \nu} + \sqrt{\gamma \nu} \coth\left(\sqrt{\frac{\gamma}{\nu}}(T-t)\right)}.$$

The functions $D(t)$ and $E(t)$ also solve linear ODEs,

$$0 = D'(t) - \frac{1}{\nu} A(t)D(t), \quad 0 = E'(t) + \frac{1}{\nu} B(t)D(t).$$

By variation of constants,

$$D(t) = \exp\left(-\int_t^T \frac{1}{\nu} A(s) ds\right) = \frac{\sqrt{\gamma \nu} e^{-(T-t)\sqrt{\frac{\gamma}{\nu}}}}{b_1 \kappa \ell_X (1 - e^{-2(T-t)\sqrt{\frac{\gamma}{\nu}}}) + \sqrt{\gamma \nu}},$$

and $E(t) = \exp\left(\int_t^T \frac{1}{\nu} B(s) ds\right)$, so that

$$E(t) = \exp\left(\sqrt{\frac{b_1 \ell_X}{\kappa(b_1 \kappa \ell_X + \sqrt{\gamma \nu})}} \left(\tanh^{-1}\left(e^{\sqrt{\frac{\gamma}{\nu}}(t-T)} \sqrt{\frac{b_1 \kappa \ell_X}{b_1 \kappa \ell_X + \sqrt{\gamma \nu}}}\right) - \tanh^{-1}\left(\sqrt{\frac{b_1 \kappa \ell_X}{b_1 \kappa \ell_X + \sqrt{\gamma \nu}}}\right) \right)\right).$$

Finally, the function $F(t)$ can be computed by a direct integration:

$$F(t) = \frac{((\phi - \varsigma_1(\delta - \delta_0) + \varsigma_2 P_0)\ell_X - \varsigma_2 \kappa \ell_0(\delta))^2}{4\varsigma_2 \kappa \ell_X + 2\sqrt{\gamma \nu} + 2\sqrt{\gamma \nu} \coth\left(\sqrt{\frac{\gamma}{\nu}}(T-t)\right)} + 2\sqrt{\frac{\nu}{\gamma}} \log\left(1 + \tanh\left(\sqrt{\frac{\gamma}{\nu}}(T-t)\right)\right) \\ + \frac{\sigma^2 \varsigma_2 \ell_X}{18\kappa} \left(\frac{\nu(\varsigma_2 \kappa \ell_X - \sqrt{\gamma \nu}) \log(\gamma \nu)}{(\varsigma_2 \kappa \ell_X)^2 - \gamma \nu} - \frac{2\varsigma_2 \kappa \ell_X \log\left(1 - \tanh\left(\sqrt{\frac{\gamma}{\nu}}(T-t)\right)\right)}{\varsigma_2 \kappa \ell_X \sqrt{\frac{\gamma}{\nu}} + \gamma} \right. \\ \left. - \frac{2\nu(2\varsigma_2 \kappa \ell_X + \sqrt{\gamma \nu}) \log\left(\sqrt{\gamma \nu} + (2\varsigma_2 \kappa \ell_X + \sqrt{\gamma \nu}) \tanh\left(\sqrt{\frac{\gamma}{\nu}}(T-t)\right)\right)}{2(\varsigma_2 \kappa \ell_X)^2 + 3\varsigma_2 \kappa \ell_X \sqrt{\gamma \nu} + \gamma \nu} \right).$$

Appendix B: Proofs for Discrete Model

To solve for the Nash equilibrium with pre hedging, we fix the optimal trade size and spread quoted by dealer B and look for the optimal response of dealer A. By symmetry of the model, the Nash equilibrium is in turn determined by the consistency condition that both dealers use the same feedback functions of their initial inventories. Crucially, each dealer does not observe the other's initial inventory, but only knows its distribution. Like in many other models with asymmetric information, we focus on linear policies, which turn out to be compatible with the uniform distributions we have assumed for the dealers' initial inventories:

$$\Delta Y_0 = a - bY_0, \quad \delta_B = c - dY_0.$$

The P&L of dealer A, adjusted for inventory costs, then is ⁶

$$\begin{aligned} & - \left(P_0^* + \frac{I}{2}(\Delta X_0 + \Delta Y_0) \right) \Delta X_0 - \frac{\gamma}{2}(X_0 + \Delta X_0)^2 \\ & + \mathbf{1}_{\text{trade}} \mathbf{1}_{\delta < \delta_B} \left((P_{\mathcal{T}}^* + \delta + (I - B)(\Delta X_0 + \Delta Y_0)) X_* \right. \\ & \quad \left. + \left(P_{\mathcal{T}}^* + \frac{I - 2B}{2}(\Delta X_0 + \Delta Y_0) + \frac{I}{2}(X_* - X_0 - Y_0) \right) (X_0 + \Delta X_0 - X_*) \right) \\ & + \mathbf{1}_{\text{trade}} \mathbf{1}_{\delta > \delta_B} \left(P_{\mathcal{T}}^* + \frac{I - 2B}{2}(\Delta X_0 + \Delta Y_0) + \frac{I}{2}(X_* - X_0 - Y_0) \right) (X_0 + \Delta X_0) \\ & + (1 - \mathbf{1}_{\text{trade}}) \left(P_{\mathcal{T}}^* + \frac{I - 2B}{2}(\Delta X_0 + \Delta Y_0) - \frac{I}{2}(X_0 + Y_0) \right) (X_0 + \Delta X_0). \end{aligned} \quad (24)$$

Here, we use the notation $I = \kappa + \lambda$ and $B = \beta\lambda$ to simplify exposition in this part of the appendix.⁷

We now compute the expectation of this expression from the point of view of dealer A. To this end, we first use that the probability ϕ that the client trades with one of the dealers is independent from all other random quantities in the model, so that we can substitute $\mathbf{1}_{\text{trade}}$ by its expectation ϕ throughout in dealer A's expected P&L.

Next, we take into account that the unaffected price P^* is a martingale, and also independent from all other random quantities in the model. Whence, we can replace its value $P_{\mathcal{T}}^*$ at the trading time with the client with its initial level P_0^* in expectation. As a consequence, the unaffected price

⁶ Here, both dealers interact through the spreads they quote and also through the common permanent and transient impact of their trades on the market price. As is standard for linear impact models, trades are settled at the average between pre and post trade prices (Obizhaeva and Wang 2013).

⁷ To avoid confusion with the impact process I_t , the ODE $B(t)$, or dealer B, we revert to standard notation in the main text.

only appears in the expectation of the P&L (24) through a term $P_0^* X_0$ that affects neither the optimal trade size or spread.

The only random quantity left to evaluate is the initial inventory Y_0 of dealer B, which appears both directly and indirectly through the trades and spread of dealer B. As Y_0 is uniformly distributed on $[-\zeta, \zeta]$, all expectations that appear in dealer A's expected P&L can be computed in closed form. To wit, we have

$$\mathbb{E}[\mathbf{1}_{\delta < \delta_B}] = \mathbb{P}\left[Y_0 < \frac{c - \delta}{d}\right] = \frac{c - \delta + d\zeta}{2d\zeta}, \quad \mathbb{E}[\mathbf{1}_{\delta > \delta_B}] = 1 - \mathbb{P}\left[Y_0 < \frac{c - \delta}{d}\right] = \frac{\delta - c + d\zeta}{2d\zeta},$$

and

$$\begin{aligned} \mathbb{E}[Y_0] &= 0, \\ \mathbb{E}[\mathbf{1}_{\delta < \delta_B} Y_0] &= \mathbb{E}\left[\mathbf{1}_{Y_0 < \frac{c - \delta}{d}} Y_0\right] = \frac{1}{2\zeta} \int_{-\zeta}^{\frac{c - \delta}{d}} x dx = \frac{1}{4\zeta} \left(\left(\frac{c - \delta}{d} \right)^2 - \zeta^2 \right), \\ \mathbb{E}[\mathbf{1}_{\delta > \delta_B} Y_0] &= \mathbb{E}\left[\mathbf{1}_{Y_0 > \frac{c - \delta}{d}} Y_0\right] = \frac{1}{2\zeta} \int_{\frac{c - \delta}{d}}^{\zeta} x dx = \frac{1}{4\zeta} \left(\zeta^2 - \left(\frac{c - \delta}{d} \right)^2 \right). \end{aligned}$$

With these formulas at hand, we can now finish the computation of the expectation of dealer A's P&L (24). From the first term, we obtain

$$-\frac{Ia}{2} \Delta X_0 - \frac{I}{2} \Delta X_0^2 - \frac{\gamma}{2} (X_0 + \Delta X_0)^2.$$

From the second term, we obtain

$$\begin{aligned} &\phi \mathbb{E}[\mathbf{1}_{\delta < \delta_B}] \left((\delta + (I - B)(\Delta X_0 + a)) X_* + \left(\frac{I - 2B}{2} (\Delta X_0 + a) + \frac{I}{2} (X_* - X_0) \right) (X_0 + \Delta X_0 - X_*) \right) \\ &- \phi \mathbb{E}[\mathbf{1}_{\delta < \delta_B} Y_0] \left((I - B) b X_* + \left(\frac{I - 2B}{2} b + \frac{I}{2} \right) (X_0 + \Delta X_0 - X_*) \right) \\ &= \phi \frac{c - \delta + d\zeta}{2d\zeta} \left((\delta + (I - B)(\Delta X_0 + a)) X_* + \left(\frac{I - 2B}{2} (\Delta X_0 + a) + \frac{I}{2} (X_* - X_0) \right) (X_0 + \Delta X_0 - X_*) \right) \\ &- \phi \frac{1}{4\zeta} \left(\left(\frac{c - \delta}{d} \right)^2 - \zeta^2 \right) \left((I - B) b X_* + \left(\frac{I - 2B}{2} b + \frac{I}{2} \right) (X_0 + \Delta X_0 - X_*) \right). \end{aligned}$$

The other two terms in (24) can be computed in the same way. From the third, we obtain

$$\begin{aligned} &\phi \mathbb{E}[\mathbf{1}_{\delta > \delta_B}] \left(\frac{I - 2B}{2} (\Delta X_0 + a) + \frac{I}{2} (X_* - X_0) \right) (X_0 + \Delta X_0) - \phi \mathbb{E}[\mathbf{1}_{\delta > \delta_B} Y_0] \left(\frac{I - 2B}{2} b + \frac{I}{2} \right) (X_0 + \Delta X_0) \\ &= \phi \frac{\delta - c + d\zeta}{2d\zeta} \left(\frac{I - 2B}{2} (\Delta X_0 + a) + \frac{I}{2} (X_* - X_0) \right) (X_0 + \Delta X_0) \\ &- \phi \frac{1}{4\zeta} \left(\zeta^2 - \left(\frac{c - \delta}{d} \right)^2 \right) \left(\frac{I - 2B}{2} b + \frac{I}{2} \right) (X_0 + \Delta X_0). \end{aligned}$$

Finally, from the fourth term in (24), we obtain

$$(1 - \phi) \left(\frac{I - 2B}{2} (\Delta X_0 + a) - \frac{I}{2} X_0 \right) (X_0 + \Delta X_0).$$

After these preparations, the expected P&L of dealer A turns out to be a quadratic function of their trade size ΔX_0 . The optimizer is

$$\Delta X_0 = -\frac{B + \gamma}{2B + \gamma} X_0 + \frac{IX_* \phi (c + 3d\zeta - \delta) - 4aBd\zeta}{4d\zeta(2B + \gamma)}. \quad (25)$$

After plugging (25) back into dealer A's expected P&L, we then obtain the value that dealer A needs to maximize over their spread δ . This again turns out to be a quadratic function, whose maximizer is

$$\delta_A = \frac{4B\zeta(d(-aI + 4d\zeta + 2IX_*) + 2(b-1)cI + 4cd) + 4\gamma\zeta(d(-aI + 2d\zeta + IX_*) + (b-1)cI + 2cd) - I^2 X_* \phi (c + 3d\zeta)}{4\zeta(2B + \gamma)((b-1)I + 4d) - I^2 X_* \phi} - \frac{4Bd\zeta I}{4\zeta(2B + \gamma)((b-1)I + 4d) - I^2 X_* \phi} X_0$$

This is indeed of the same form as the spread $\delta_B = c - dY_0$ we have postulated for dealer B. For consistency with a symmetric Nash equilibrium, both spreads need to be the same feedback functions of the dealers' initial inventories. This requires

$$c = \frac{4B\zeta(d(-aI + 4d\zeta + 2IX_*) + 2(b-1)cI + 4cd) + 4\gamma\zeta(d(-aI + 2d\zeta + IX_*) + (b-1)cI + 2cd) - I^2 X_* \phi (c + 3d\zeta)}{4\zeta(2B + \gamma)((b-1)I + 4d) - I^2 X_* \phi},$$

$$d = \frac{4Bd\zeta I}{4\zeta(2B + \gamma)((b-1)I + 4d) - I^2 X_* \phi}.$$

The second equation has the explicit solution

$$d = \frac{I(4\zeta((3-2b)B - b\gamma + \gamma) + IX_* \phi)}{16\zeta(2B + \gamma)}.$$

After plugging this into the first equation, we obtain the explicit solution

$$c = \frac{I(-8a(B + \gamma) + 4B(-2b\zeta + 3\zeta + 4X_*) + 4\gamma(-b\zeta + \zeta + 2X_*) - 5IX_* \phi)}{16(2B + \gamma)}.$$

After inserting c , d as well as the optimal spread $\delta = c - dX_0$, the trade size (25) of dealer A needs to be the same feedback function of the initial inventory as the one we have assumed for dealer B (i.e., $\Delta Y_0 = a - bY_0$). This leads to another two consistency conditions that need to be satisfied in symmetric Nash equilibrium:

$$a = \frac{3IX_* \phi - 4aB}{4(2B + \gamma)}, \quad b = \frac{4B\zeta + 4\gamma\zeta - IX_* \phi}{8B\zeta + 4\gamma\zeta}.$$

As a consequence,

$$a = \frac{3IX_* \phi}{4(3B + \gamma)}, \quad b = \frac{4\zeta(B + \gamma) - IX_* \phi}{4\zeta(2B + \gamma)}. \quad (26)$$

With these explicit formulas for a and b , we can in turn also express the constants c and d in terms of the model parameters:

$$c = \frac{I(4(3B + \gamma)(B(\zeta + 2X_*) + \gamma X_*) - IX_*\phi(9B + 5\gamma))}{8(2B + \gamma)(3B + \gamma)}, \quad d = \frac{4B\zeta I + I^2 X_*\phi}{8\zeta(2B + \gamma)}. \quad (27)$$

As a result, the spread paid by the client is minimum of two independent uniform random variables on the interval $[c - d\zeta, c + d\zeta]$, which has expectation

$$\frac{2(c - d\zeta) + (c + d\zeta)}{3} = c - \frac{d\zeta}{3} = \frac{IX_*}{12} \left(6 - \frac{I\phi(15B + 8\gamma)}{(2B + \gamma)(3B + \gamma)} \right) + \frac{BI}{3(2B + \gamma)}\zeta$$

The additional impact paid for by the client is

$$(I - B)(\Delta X_0 + \Delta Y_0) = (I - B)(2a - b(X_0 + Y_0)).$$

As X_0 and Y_0 are independent and uniformly distributed on $[-\zeta, \zeta]$, the expected impact cost to the client in turn is

$$(I - B)2a = (I - B)\frac{3IX_*\phi}{2(3B + \gamma)}.$$

Together with the expected spread, this leads to all-in expected execution cost per share of

$$\frac{I(-36B^2(\phi - 1) + 3B(\gamma(10 - 6\phi) + 7I\phi) + 2\gamma(3\gamma + 5I\phi))}{12(2B + \gamma)(3B + \gamma)} X_* + \frac{BI}{3(2B + \gamma)}\zeta.$$

Now suppose pre-hedging is not possible in the model from the previous section. This means that $\Delta X_0 = \Delta Y_0 = 0$ and both dealers simply liquidate their positions (including, potentially, the trade with the client) at time $t = 1$. Their only optimization in turn is to choose which spread to quote.

Without pre-hedging, the P&L of dealer (A) simplifies to

$$\begin{aligned} & -\frac{\gamma}{2}X_0^2 + \mathbf{1}_{\text{trade}}\mathbf{1}_{\delta < \delta_B} \left((P_{\mathcal{T}}^* + \delta)X_* + \left(P_{\mathcal{T}}^* - \frac{I}{2}(X_0 + Y_0 - X_*) \right) (X_0 - X_*) \right) \\ & + \mathbf{1}_{\text{trade}}\mathbf{1}_{\delta > \delta_B} \left(\left(P_{\mathcal{T}}^* - \frac{I}{2}(X_0 + Y_0 - X_*) \right) X_0 \right) + (1 - \mathbf{1}_{\text{trade}}) \left(\left(P_{\mathcal{T}}^* - \frac{I}{2}(X_0 + Y_0) \right) X_0 \right). \end{aligned}$$

After taking expectations and plugging in the ansatz $\delta_2 = c - dY_0$, the calculations from the previous sections yield

$$\begin{aligned} & P_0^*X_0 - \frac{\gamma}{2}X_0^2 + \phi\mathbb{P}[\delta < \delta_B] \left(\delta X_* - \frac{I}{2}(X_0 - X_*)^2 \right) - \frac{I}{2}(X_0 - X_*)\phi\mathbb{E}[1_{\delta < \delta_B}Y_0] \\ & + \phi\mathbb{P}[\delta > \delta_B] \left(\frac{I}{2}X_*X_0 - \frac{I}{2}X_0^2 \right) - \frac{I}{2}X_0\phi\mathbb{E}[1_{\delta > \delta_B}Y_0] - (1 - \phi)\frac{I}{2}X_0^2 \\ & = P_0^*X_0 - \frac{\gamma}{2}X_0^2 + \phi\frac{c - \delta + d\zeta}{2d\zeta} \left(\delta X_* - \frac{I}{2}(X_0 - X_*)^2 \right) - \phi\frac{I}{2}(X_0 - X_*)\frac{1}{4\zeta} \left(\left(\frac{c - \delta}{d} \right)^2 - \zeta^2 \right) \\ & + \phi\frac{\delta - c + d\zeta}{2d\zeta} \left(\frac{I}{2}X_*X_0 - \frac{I}{2}X_0^2 \right) - \phi\frac{I}{2}X_0\frac{1}{4\zeta} \left(\zeta^2 - \left(\frac{c - \delta}{d} \right)^2 \right) - (1 - \phi)\frac{I}{2}X_0^2. \end{aligned}$$

The optimal spread quoted by dealer A in turn is

$$\delta_A = \frac{2cd - cI + 2d^2\zeta + dIX_*}{4d - I} - \frac{dI}{4d - I}X_0.$$

To make this consistent with the spread $\delta_B = c - dY_0$ quoted by dealer B, we therefore need

$$c = \frac{I}{2}(\zeta + X_*), \quad d = \frac{I}{2}.$$

The spread cost paid by the client in turn is the minimum of two uniform random variables on the interval $[c - d\zeta, c + d\zeta]$, which has expectation

$$c - \frac{d\zeta}{3} = \frac{I}{6}(2\zeta + 3X_*).$$

As a consequence, the difference between the all-in expected transaction costs with and without pre-hedging is

$$\frac{IX_*\phi(-36B^2 + 3B(7I - 6\gamma) + 10\gamma I) - 4\zeta I(B + \gamma)(3B + \gamma)}{12(2B + \gamma)(3B + \gamma)}.$$

Pre hedging is in turn beneficial to the client if

$$\frac{4(B + \gamma)(3B + \gamma)}{X(21B + 10\gamma)} + \frac{18B(2B + \gamma)}{21B + 10\gamma} > I, \quad \text{where } X = X_*\phi/\zeta.$$

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