Appendix

A. Proofs

Proof of Lemma 1. The closed form optimal solution (p_1^*, p_2^*) of $\max_{p_1,p_2\geq 0} R(p_1,p_2) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(p_1+p_2)$ is derived by the following steps. First, $\lambda_i := \Lambda_i q_i = \Lambda_i \max\{0, 1-\frac{p_i}{b_i}\} = 1 - \frac{\Lambda_i}{b_i}p_i$ because if $\lambda_i = 0$, system shut down (zero revenue). However, we can always have positive revenue by setting $\lambda_i > 0$ for all $i \in \{1,2\}$. Then, we replace the price variables, (p_1, p_2), in unconstrained problem with the demand rate variables, (λ_1, λ_2), yielding,

$$\mathcal{R}^* := \max_{\lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2]} \quad \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right). \tag{8}$$

Next, we derive the optimal solution, $(\lambda_1^*, \lambda_2^*)$, in (8) by using the following KKT conditions:

$$\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right)\frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2}-\frac{b_1}{\Lambda_1}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}+\mu_1=0, \tag{9a}$$

$$\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right)\frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2}-\frac{b_2}{\Lambda_2}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}+\mu_2=0, \tag{9b}$$

$$\lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2], \tag{9c}$$

$$\mu_1,\mu_2\geq 0,$$

$$\mu_1(\lambda_1-\Lambda_1)=0,\ \mu_2(\lambda_2-\Lambda_2)=0. \tag{9d}$$

If $\lambda_1 \neq \Lambda_1$ and $\lambda_2 \neq \Lambda_2$, we get $\beta_1\lambda_1^2 = \beta_2\lambda_2^2$ by stationary conditions (9a) and (9b), and complementary slackness condition (9d). When we plug $\lambda_2 = \sqrt{\frac{\beta_1}{\beta_2}}\lambda_1$ into (8), then $(\lambda_1^*, \lambda_2^*)$ becomes:

$$\lambda_1^* = \frac{1}{2}\frac{b_1+b_2}{\beta_1+\sqrt{\beta_1\beta_2}},\ \lambda_2^* = \frac{1}{2}\frac{b_1+b_2}{\beta_2+\sqrt{\beta_1\beta_2}},$$

where $(\beta_1,\beta_2) = (\frac{b_1}{\Lambda_1}, \frac{b_2}{\Lambda_2})$. The primal feasibility, Eq. (9c), introduces conditions that:

$$\frac{1}{2}\frac{b_1+b_2}{\beta_1+\sqrt{\beta_1\beta_2}} < \Lambda_1 \iff \sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}} < 2\sqrt{\frac{\Lambda_1}{\Lambda_2}},$$

$$\frac{1}{2}\frac{b_1+b_2}{\beta_2+\sqrt{\beta_1\beta_2}} < \Lambda_2 \iff \sqrt{\frac{b_1}{b_2}}-\sqrt{\frac{b_2}{b_1}} < 2\sqrt{\frac{\Lambda_2}{\Lambda_1}}.$$

If $\sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}} \geq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$, primal feasibility does not hold. In this case, $\lambda_1 = \Lambda_1$, then we get $(\lambda_1^*, \lambda_2^*) = \left(\Lambda_1, -\Lambda_1+\sqrt{\Lambda_1(1+\Lambda_2)}\right)$ by plugging in $\lambda_1 = \Lambda_1$ in Eq. (8). Similarly, if $\sqrt{\frac{b_1}{b_2}}-\sqrt{\frac{b_2}{b_1}} \geq 2\sqrt{\frac{\Lambda_2}{\Lambda_1}}$, we get $(\lambda_1^*, \lambda_2^*) = \left(-\Lambda_2+\sqrt{\Lambda_2(1+\Lambda_1)}, \Lambda_2\right)$. (p_1^*, p_2^*) follows since $p_i = b_i - \frac{b_i}{\Lambda_i}\lambda_i,\ \forall i\in\{1,2\}$. □

LEMMA 5. *When $N=2,\ K=1$, $\sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}} > 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$ under linear demand or $\log\frac{\Lambda_2}{\Lambda_1} > \log\frac{\beta_2}{\beta_1}+\frac{\beta_2}{\beta_1}+1$ under exponential demand, then there exists $\alpha^\star > 0$ s.t. $p_1(\alpha) = 0$ for $\forall\alpha\in[0,\alpha^\star]$ under price fairness.*

Proof. According to Lemma 1and Lemma 2, $p_1^* = 0$ and $p_2^* > 0$ under the given conditions. We consider α under $p_1(\alpha) \leq p_2(\alpha)$, which is sufficient condition for $p_1(\alpha) = 0$. i.e., $\{\alpha : p_1(\alpha) = 0\} \subset \{\alpha : p_1(\alpha) \leq p_2(\alpha)\}$. Then, the optimization problem under price fairness in terms of (λ_1, λ_2) is written as follows:

$$\begin{aligned}\max_{\lambda_1,\lambda_2} \quad & f(\lambda_1,\lambda_2) := \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left(F_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)+F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)\\ \text{s.t.} \quad & F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)-F^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)\leq(1-\alpha)F_2^{-1}\left(1-\frac{\lambda_2^*}{\Lambda_2}\right),\\ & \lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2].\end{aligned}$$

The KKT conditions of the above problem are given by

$$\frac{\partial f(\lambda_1,\lambda_2)}{\partial \lambda_1} - \mu_1 + \mu_3 \frac{b_1}{\Lambda_1}\lambda_1 = 0, \tag{10a}$$

$$\frac{\partial f(\lambda_1,\lambda_2)}{\partial \lambda_2} - \mu_2 - \mu_3 \frac{b_2}{\Lambda_2}\lambda_2 = 0, \tag{10b}$$

$$F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right) - F^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right) \le (1-\alpha)\, F_2^{-1}\left(1-\frac{\lambda_2^*}{\Lambda_2}\right),\ \lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2],$$

$$\mu_1(\lambda_1-\Lambda_1)=0,\quad \mu_2(\lambda_2-\Lambda_2)=0,\ \mu_3\left(b_2-\frac{b_2}{\Lambda_2}\lambda_2 - b_1 + \frac{b_1}{\Lambda_1}\lambda_1 - (1-\alpha)(b_2-\frac{b_2}{\Lambda_2}\lambda_2^*)\right)=0, \tag{10c}$$

$$\mu_1,\mu_2,\mu_3\ge 0.$$

If $\mu_3=0$, then the stationary conditions (10a) and (10b) becomes the same as the stationary conditions under $\alpha=0$. This implies $p_1(\alpha)=0$, so we are done. On the other hand, if $\mu_3\ne 0$, then $F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right) - F^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right) = (1-\alpha)\,F_2^{-1}\left(1-\frac{\lambda_2^*}{\Lambda_2}\right)$, by complementary slackness (10c). We want to show that there exists $\alpha^\star>0$, such that $\lambda_1(\alpha^\star)=\Lambda_1$ with $\mu_1=0$, because this is the point where $p_1(\alpha)$ changes from 0 to a positive value.

If we assume such $\alpha^\star=0$, then $\lambda_1(\alpha^\star)$ and $\lambda_2(\alpha^\star)$ satisfy the KKT conditions without fairness criteria. Therefore $\lambda_1(\alpha^\star)$ and $\lambda_2(\alpha^\star)$ should satisfy the stationary conditions without fairness criteria,

$$\frac{\partial f(\lambda_1,\lambda_2)}{\partial\lambda_1}\Big|_{(\Lambda_1,\lambda_2(\alpha^\star))} - \mu_1 = 0 \text{ and } \frac{\partial f(\lambda_1,\lambda_2)}{\partial\lambda_2}\Big|_{(\Lambda_1,\lambda_2(\alpha^\star))} - \mu_2 = 0,$$

where $\mu_1=0$ since $p_1^*=0$ implies $\lambda_1^*=\Lambda_1$ and $\mu_2=0$ since $p_2^*>0$ implies $\lambda_2^*<\Lambda_2$. Therefore, we get $\frac{\partial f(\lambda_1,\lambda_2)}{\partial\lambda_1}\big|_{(\Lambda_1,\lambda_2(\alpha^\star))}=0$ and $\frac{\partial f(\lambda_1,\lambda_2)}{\partial\lambda_2}\big|_{(\Lambda_1,\lambda_2(\alpha^\star))}=0$. This implies $\sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}=2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$ under linear demand and $\log\frac{\Lambda_2}{\Lambda_1}=\log\frac{\beta_2}{\beta_1}+\frac{\beta_2}{\beta_1}+1$ under exponential demand, which contradicts the assumption that $\sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}>2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$ or $\log\frac{\Lambda_2}{\Lambda_1}>\log\frac{\beta_2}{\beta_1}+\frac{\beta_2}{\beta_1}+1$. Therefore, there exists $\alpha^\star>0$ such that $p_1(\alpha)=0$ for all $\alpha\in[0,\alpha^\star]$ under price fairness. □

Proof of Proposition 2. Since $\mathcal{W}(\alpha)=\mathcal{R}(\alpha)+\mathcal{S}(\alpha)$, the derivative of $\mathcal{W}(\alpha)$ with α, (p_1^*,p_2^*), is given by

$$\mathcal{W}'(\alpha) := \frac{d\mathcal{W}(\alpha)}{d\alpha} = \frac{d\mathcal{R}(\alpha)}{d\alpha} + \frac{d\mathcal{S}(\alpha)}{d\alpha} = \frac{\partial\mathcal{R}(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial\mathcal{R}(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha} + \frac{\partial\mathcal{S}(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial\mathcal{S}(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha}.$$

The derivatives of revenue and total surplus with the price at $\alpha=0$ are

$$\frac{\partial\mathcal{R}(\alpha)}{\partial p_i}\Big|_{(p_1^*,p_2^*)} = \frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*} - (p_1^*+p_2^*)\frac{1}{\beta_i}(\frac{\lambda_j^*}{\lambda_1^*+\lambda_2^*})^2,\ \forall i\ne j\in\{1,2\},$$

$$\frac{\partial\mathcal{S}(\alpha)}{\partial p_i}\Big|_{(p_1^*,p_2^*)} = -\frac{1}{2}\frac{\lambda_j^*}{(\lambda_1^*+\lambda_2^*)^2}(\lambda_i^{*2}+2\lambda_1^*\lambda_2^*+\frac{\beta_j}{\beta_i}\lambda_j^{*2}),\ \forall i\ne j\in\{1,2\},$$

where $\beta_i=\frac{b_i}{\Lambda_i}$ $\forall i\in\{1,2\}$. Similarly, the derivatives of $\mathcal{S}_1$ and $\mathcal{S}_2$ are

$$\mathcal{S}_1'(\alpha) := \frac{d\mathcal{S}_1(\alpha)}{d\alpha} = \frac{\partial\mathcal{S}_1(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial\mathcal{S}_1(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha},\quad \mathcal{S}_2'(\alpha) := \frac{d\mathcal{S}_2(\alpha)}{d\alpha} = \frac{\partial\mathcal{S}_2(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial\mathcal{S}_2(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha}.$$

The derivative of surplus with prices at $\alpha=0$, (p_1^*,p_2^*), is as follows,

$$\frac{\partial\mathcal{S}_1(\alpha)}{\partial p_1}\Big|_{(p_1^*,p_2^*)} = -\frac{1}{2\Lambda_1}\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\frac{\lambda_1^*+2\lambda_2^*}{\lambda_1^*+\lambda_2^*},\quad \frac{\partial\mathcal{S}_1(\alpha)}{\partial p_2}\Big|_{(p_1^*,p_2^*)} = -\frac{\Lambda_2}{b_2}\frac{b_1-p_1^*}{2\Lambda_1}\left(\frac{\lambda_1^*}{\lambda_1^*+\lambda_2^*}\right)^2,$$

$$\frac{\partial\mathcal{S}_2(\alpha)}{\partial p_1}\Big|_{(p_1^*,p_2^*)} = -\frac{\Lambda_1}{b_1}\frac{b_2-p_2^*}{2\Lambda_2}\left(\frac{\lambda_2^*}{\lambda_1^*+\lambda_2^*}\right)^2,\quad \frac{\partial\mathcal{S}_2(\alpha)}{\partial p_2}\Big|_{(p_1^*,p_2^*)} = -\frac{1}{2\Lambda_2}\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\frac{2\lambda_1^*+\lambda_2^*}{\lambda_1^*+\lambda_2^*}.$$

The gradients of price with respect to α, $\frac{dp_i}{d\alpha}$, for $\forall i \in \{1,2\}$, are required to obtain $\mathcal{W}'(0)$. Let $\Delta p_1(\alpha) = p_1(\alpha) - p_1^*$ and $\Delta p_2(\alpha) = p_2(\alpha) - p_2^*$. Suppose $p_2^* > p_1^*$ for the proof, then the problem (3) is rewritten as follows,

$$\begin{aligned} \max \quad & R(p_1^* + \Delta p_1, p_2^* + \Delta p_2) \\ \text{s.t.} \quad & \Delta p_1(\alpha) - \Delta p_2(\alpha) \geq \alpha\,(p_2^* - p_1^*)\,. \end{aligned} \tag{11}$$

By using Taylor expansion around (p_1^*, p_2^*), the objective in the above optimization problem is converted to

$$\begin{aligned} R(p_1^* + \Delta p_1, p_2^* + \Delta p_2) = {} & R(p_1^*, p_2^*) + R_1'(p_1^*, p_2^*)\Delta p_1 + R_2'(p_1^*, p_2^*)\Delta p_2 \\ & + \frac{1}{2} R_{11}''(p_1^*, p_2^*)\Delta p_1^2 + R_{12}''(p_1^*, p_2^*)\Delta p_1 \Delta p_2 + \frac{1}{2} R_{22}''(p_1^*, p_2^*)\Delta p_2^2 \\ & + G(\Delta p_1, \Delta p_2), \end{aligned} \tag{12}$$

where $R_i'(p_1^*, p_2^*) := \frac{\partial \mathcal{R}}{\partial p_i}\big|_{(p_1^*, p_2^*)}$, $R_{ii}''(p_1^*, p_2^*) := \frac{\partial^2 \mathcal{R}}{\partial p_i^2}\big|_{(p_1^*, p_2^*)}$, and $R_{ij}''(p_1^*, p_2^*) := \frac{\partial^2 \mathcal{R}}{\partial p_i \partial p_j}\big|_{(p_1^*, p_2^*)}$, $\forall i \neq j \in \{1,2\}$. For convenience, we denote $R_{ij}''(p_1^*, p_2^*)$ as R_{ij}''. $G(\Delta p_1, \Delta p_2)$ corresponds to the remainder term, more specifically,

$$G(\Delta p_1, \Delta p_2) = \frac{1}{6} R_{111}'''(\xi, \eta)\Delta p_1^3 + \frac{1}{2} R_{112}'''(\xi, \eta)\Delta p_1^2 \Delta p_2 + \frac{1}{2} R_{122}'''(\xi, \eta)\Delta p_1 \Delta p_2^2 + \frac{1}{6} R_{222}'''(\xi, \eta)\Delta p_2^3,$$

where $R_{ijk}'''(\xi, \eta) = \frac{\partial^3 \mathcal{R}}{\partial p_i \partial p_j \partial p_k}\big|_{(\xi, \eta)}$ for all $i, j, k \in \{1,2\}$ and $(\xi, \eta) \in [p_1^*, p_1^* + \Delta p_1] \times [p_2^*, p_2^* + \Delta p_2]$.

With the Taylor expansion on R in (12), the optimization problem in (11) is rewritten as

$$\begin{aligned} \max \quad & \frac{1}{2} \begin{pmatrix} \Delta p_1 & \Delta p_2 \end{pmatrix} \begin{pmatrix} R_{11}'' & R_{12}'' \\ R_{12}'' & R_{22}'' \end{pmatrix} \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \end{pmatrix} + G(\Delta p_1, \Delta p_2) \\ \text{s.t.} \quad & \Delta p_1(\alpha) - \Delta p_2(\alpha) \geq \alpha\,(p_2^* - p_1^*)\,. \end{aligned}$$

Dividing the objective and constraints with $\Delta\alpha^2$ and $\Delta\alpha$ respectively, $\Delta\alpha \to 0$ leads to:

$$\begin{aligned} \max_{p_1'(0), p_2'(0)} \quad & \frac{1}{2} \begin{pmatrix} p_1'(0) & p_2'(0) \end{pmatrix} \begin{pmatrix} R_{11}'' & R_{12}'' \\ R_{12}'' & R_{22}'' \end{pmatrix} \begin{pmatrix} p_1'(0) \\ p_2'(0) \end{pmatrix} \\ \text{s.t.} \quad & p_1'(0) - p_2'(0) \geq (p_2^* - p_1^*)\,, \end{aligned} \tag{13}$$

because $p_i'(0) = \lim_{\Delta\alpha \to 0} \frac{\Delta p_i(\alpha)}{\Delta\alpha}$ $\forall i \in \{1,2\}$, $\lim_{\Delta\alpha \to 0} \frac{\alpha}{\Delta\alpha} = 1$, and

$$\begin{aligned} \lim_{\Delta\alpha \to 0} \frac{G(\Delta p_1, \Delta p_2)}{\Delta\alpha^2} = \lim_{\Delta\alpha \to 0} & \frac{1}{6} R_{111}'''(\xi, \eta)(\frac{\Delta p_1}{\Delta\alpha})^3 \Delta\alpha + \frac{1}{2} R_{112}'''(\xi, \eta)(\frac{\Delta p_1}{\Delta\alpha})^2 \frac{\Delta p_2}{\Delta\alpha}\Delta\alpha \\ & + \frac{1}{2} R_{122}'''(\xi, \eta)\frac{\Delta p_1}{\Delta\alpha}(\frac{\Delta p_2}{\Delta\alpha})^2 \Delta\alpha + \frac{1}{6} R_{222}'''(\xi, \eta)(\frac{\Delta p_2}{\Delta\alpha})^3 \Delta\alpha = 0. \end{aligned}$$

(a) Suppose $\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} > 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$, then there exists $\alpha^\star$ s.t. $p_1^*(\alpha) = 0$ for $\alpha \in [0, \alpha^\star]$ by Lemma 5. (13) simplifies to,

$$\begin{aligned} \max_{p_2'(0)} \quad & R_{22}'' \cdot p_2'(0)^2 \\ s.t. \quad & p_2'(0) \leq -p_2^*. \end{aligned} \tag{14}$$

Note that $R_{22}'' = -\frac{2}{\beta_2}(\frac{\Lambda_1}{\Lambda_1 \lambda_2^*})^2 - \frac{2}{\beta_2^2} p_2^* \frac{\Lambda_1^2}{(\Lambda_1 + \lambda_2^*)^3} < 0$. Therefore, the optimal solution in the Problem (14) is $p_2'(0)^* = -p_2^*$. With $p_1'(0) = 0$ and $p_2'(0) = -p_2^*$,

$$\mathcal{W}'(0) = \frac{1}{2} \frac{\lambda_1^*}{(\lambda_1^* + \lambda_2^*)^2} \left({\lambda_2^*}^2 + 2\lambda_1^* \lambda_2^* + \frac{\beta_1}{\beta_2} {\lambda_1^*}^2 \right) p_2^* > 0,$$

$$\mathcal{S}_1'(0) = \frac{1}{2} \frac{\beta_1}{\beta_2} \left(\frac{\lambda_1^*}{\lambda_1^* + \lambda_2^*} \right)^2 p_2^* > 0, \quad \mathcal{S}_2'(0) = \frac{1}{2\Lambda_2} \frac{\lambda_1^* \lambda_2^*}{\lambda_1^* + \lambda_2^*} \frac{2\lambda_1^* + \lambda_2^*}{\lambda_1^* + \lambda_2^*} p_2^* > 0,$$

where $p_2^* = b_2 + \beta_2(\Lambda_1 - \sqrt{\Lambda_1(1 + \Lambda_2)})$, $\lambda_1^* = \Lambda_1$, and $\lambda_2^* = -\Lambda_1 + \sqrt{\Lambda_1(1 + \Lambda_2)}$.

Now, we consider the region where $\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} \le 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$. In this region, as stated in the proof of Lemma 1*(a)*, the relationship $\beta_1 \lambda_1^{*2} = \beta_2 \lambda_2^{*2}$ holds. This observation is useful in proving the following statements.

The value of R''_{11}, R''_{12}, and R''_{22} under this region are

$$R''_{11} = \frac{\partial^2 R}{\partial p_1^2} = -\frac{2}{(\sqrt{\beta_1}+\sqrt{\beta_2})^2}\left(1+\sqrt{\frac{\beta_2}{\beta_1}}\right), \; R''_{12} = \frac{\partial^2 R}{\partial p_1 \partial p_2} = 0,$$

$$R''_{22} = \frac{\partial^2 R}{dp_2^2} = -\frac{2}{(\sqrt{\beta_1}+\sqrt{\beta_2})^2}\left(1+\sqrt{\frac{\beta_1}{\beta_2}}\right).$$

The optimal solution to (13) is

$$p'_1(0) = \frac{R''_{22}}{R''_{11}+R''_{22}}\left(p_2^* - p_1^*\right), \quad p'_2(0) = -\frac{R''_{11}}{R''_{11}+R''_{22}}\left(p_2^* - p_1^*\right).$$

Under $\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} \le 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$, the derivative of revenue w.r.t. p_i is 0, i.e., $\frac{\partial \mathcal{R}(\alpha)}{\partial p_i}\Big|_{(p_1^*,p_2^*)} = 0$, since the solution is in the interior. This implies that $\frac{\partial \mathcal{S}}{\partial p_i}\Big|_{(p_1^*,p_2^*)} = -\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}$. Then, $\mathcal{W}'(0) = \frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\frac{R''_{11}-R''_{22}}{R''_{11}+R''_{22}}\left(p_2^*-p_1^*\right)$. Therefore,

$$\mathcal{W}'(0) \ge 0 \iff R''_{22} - R''_{11} \ge 0 \iff \beta_2 \ge \beta_1 \iff \frac{b_2}{b_1} \ge \frac{\Lambda_2}{\Lambda_1},$$

where the first arrow is implied by the fact that $R''_{11} < 0$, $R''_{22} < 0$, and $p_2^* > p_1^*$.

Under this region, $\mathcal{S}'_1(0) < 0$ since

$$\begin{aligned}
\mathcal{S}'_1(0) &= \frac{\partial \mathcal{S}_1(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial \mathcal{S}_1(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha} \\
&= \frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\left(\frac{\partial \mathcal{S}_1(\alpha)}{\partial p_1}R''_{22} - \frac{\partial \mathcal{S}_1(\alpha)}{\partial p_2}R''_{11}\right) \\
&= -\frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\frac{1}{2\Lambda_1}\frac{\lambda_1^*}{(\lambda_1^*+\lambda_2^*)^2}\left(\lambda_2^*(\lambda_1^*+2\lambda_2^*)R''_{22} - \frac{\beta_1}{\beta_2}\lambda_1^{*2}R''_{11}\right) \\
&= -\frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\frac{1}{2\Lambda_1}\frac{\lambda_1^*\lambda_2^{*2}}{(\lambda_1^*+\lambda_2^*)^2}\left(\left(\sqrt{\frac{\beta_2}{\beta_1}}+2\right)R''_{22} - R''_{11}\right) \\
&= -\frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\frac{1}{2\Lambda_1}\frac{\lambda_1^*\lambda_2^{*2}}{(\lambda_1^*+\lambda_2^*)^2}\frac{2}{(\sqrt{\beta_1}+\sqrt{\beta_2})^2}\left(-\left(\sqrt{\frac{\beta_2}{\beta_1}}+2\right)\left(1+\sqrt{\frac{\beta_1}{\beta_2}}\right) + \left(1+\sqrt{\frac{\beta_2}{\beta_1}}\right)\right) \\
&= \frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\frac{1}{2\Lambda_1}\frac{\lambda_1^*\lambda_2^{*2}}{(\lambda_1^*+\lambda_2^*)^2}\frac{4}{(\sqrt{\beta_1}+\sqrt{\beta_2})^2}\left(1+\sqrt{\frac{\beta_1}{\beta_2}}\right) < 0,
\end{aligned}$$

where the third equality is coming from $\beta_1 \lambda_1^{*2} = \beta_2 \lambda_2^{*2}$ and the last inequality is from $R''_{11} < 0$, $R''_{22} < 0$, and $p_2^* \ge p_1^*$. Similarly, $\mathcal{S}'_2(0) > 0$ since

$$\begin{aligned}
\mathcal{S}'_2(0) &= \frac{\partial \mathcal{S}_2(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial \mathcal{S}_2(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha} \\
&= \frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\left(\frac{\partial \mathcal{S}_2(\alpha)}{\partial p_1}R''_{22} - \frac{\partial \mathcal{S}_2(\alpha)}{\partial p_2}R''_{11}\right) \\
&= \frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\frac{1}{2\Lambda_2}\frac{\lambda_1^{*2}\lambda_2^*}{(\lambda_1^*+\lambda_2^*)^2}\left(\left(\sqrt{\frac{\beta_1}{\beta_2}}+2\right)R''_{11} - R''_{22}\right) \\
&= -\frac{p_2^*-p_1^*}{R''_{11}+R''_{22}}\frac{1}{2\Lambda_1}\frac{\lambda_1^{*2}\lambda_2^*}{(\lambda_1^*+\lambda_2^*)^2}\frac{4}{(\sqrt{\beta_1}+\sqrt{\beta_2})^2}\left(1+\sqrt{\frac{\beta_2}{\beta_1}}\right) > 0.
\end{aligned}$$

Therefore, to satisfy *(b)* $\mathcal{W}'(0) > 0$, $\mathcal{S}'_1(0) < 0$, $\mathcal{S}'_2(0) > 0$, the following conditions should hold

- $p_1^* \neq 0 \iff \sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} \leq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$,
- $p_2^* \geq p_1^* \iff 2\left(\frac{b_2}{b_1} - 1\right)\left(\sqrt{\frac{b_2}{b_1}} + \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right) \geq \left(\frac{b_2}{b_1} + 1\right)\left(\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right)$,
- Given $p_2^* > p_1^*$, $\mathcal{W}'(0) \geq 0 \iff \frac{b_2}{b_1} > \frac{\Lambda_2}{\Lambda_1}$.

Similarly, *(c)* $\mathcal{W}'(0) < 0$, $\mathcal{S}_1'(0) < 0$, $\mathcal{S}_2'(0) > 0$ holds

- $p_1^* \neq 0 \iff \sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} \leq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}$,
- Given $p_2^* > p_1^*$, $\mathcal{W}'(0) < 0 \iff \frac{b_2}{b_1} < \frac{\Lambda_2}{\Lambda_1}$.

The condition of $p_2^* > p_1^*$ is not necessary because $\frac{b_2}{b_1} < \frac{\Lambda_2}{\Lambda_1}$ and $b_2 \geq b_1$ guarantee this condition.

When $p_2^* \leq p_1^*$, all derivations remain identical to the ones mentioned above, except that we switch the index 1 and 2. For instance, from *(c)*, $\mathcal{W}'(0) < 0$, $\mathcal{S}_1'(0) > 0$, $\mathcal{S}_2'(0) < 0$ holds under $p_2^* < p_1^*$.

- $p_2^* \neq 0 \iff \sqrt{\frac{b_1}{b_2}} - \sqrt{\frac{b_2}{b_1}} \leq 2\sqrt{\frac{\Lambda_2}{\Lambda_1}}$,
- Given $p_1^* \geq p_2^*$, $\mathcal{W}'(0) < 0 \iff \frac{b_2}{b_1} > \frac{\Lambda_2}{\Lambda_1}$.

The assumption $b_2 \geq b_1$ implies that $p_2^* \neq 0$, therefore, the first condition holds. Since $p_2^* \leq p_1^* \iff 2\left(\frac{b_2}{b_1} - 1\right)\left(\sqrt{\frac{b_2}{b_1}} + \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right) \leq \left(\frac{b_2}{b_1} + 1\right)\left(\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right)$, the right-hand side $\left(\frac{b_2}{b_1} + 1\right)\left(\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right) > 0$ based on the assumption $b_2 \geq b_1$. This implies that $\frac{b_2}{b_1} > \frac{\Lambda_2}{\Lambda_1}$. Additionally, when $p_1^* = p_2^*$, $\mathcal{S}_1'(0) = 0$, $\mathcal{S}_2'(0) = 0$, and $\mathcal{W}'(0) = 0$. Therefore, the second condition holds. Finally, we establish the final condition *(d)* $\mathcal{W}'(0) \leq 0$, $\mathcal{S}_1'(0) \geq 0$, and $\mathcal{S}_2'(0) \leq 0$. □

Proof of Proposition 3. The optimization problem, with demand rates as decision variables (λ_1, λ_2), under the constraint of α-access fairness, is formulated as follows:

$$\begin{aligned} \mathcal{R}(\alpha) := \max_{\lambda_1, \lambda_2} \quad & g_K(\lambda_1, \lambda_2)\left(F_1^{-1}\left(1 - \frac{\lambda_1}{\Lambda_1}\right) + F_2^{-1}\left(1 - \frac{\lambda_2}{\Lambda_2}\right)\right) \\ \text{s.t.} \quad & g_K(\lambda_1, \lambda_2) \leq (1-\alpha)\, g_K(\lambda_1^*, \lambda_2^*), \\ & \lambda_1 \in [0, \Lambda_1], \quad \lambda_2 \in [0, \Lambda_2], \end{aligned} \tag{15}$$

where $g_K(\lambda_1, \lambda_2) = \frac{\lambda_1\lambda_2(\lambda_2^K - \lambda_1^K)}{\lambda_2^{K+1} - \lambda_1^{K+1}}$, and $(\lambda_1^*, \lambda_2^*)$ represents the optimal solution of problem (15) when $\alpha = 0$, i.e. the absence of fairness criteria.

The objective in (15) is

$$\begin{aligned} \mathcal{R} &:= \Pi_1\lambda_1 p_1 + \Pi_2\lambda_2 p_2 = \Pi_1\lambda_1 F_1^{-1}\left(1 - \frac{\lambda_1}{\Lambda_1}\right) + \Pi_2\lambda_2 F_1^{-1}\left(1 - \frac{\lambda_2}{\Lambda_2}\right) \\ &= g_K(\lambda_1, \lambda_2)\left(F_1^{-1}\left(1 - \frac{\lambda_1}{\Lambda_1}\right) + F_2^{-1}\left(1 - \frac{\lambda_2}{\Lambda_2}\right)\right), \end{aligned}$$

where F_i is the valuation distribution at node $i \in \{1, 2\}$. The last equality is derived from the fact that $g_K(\lambda_1, \lambda_2) = \Pi_1\lambda_1 = \Pi_2\lambda_2$ by (6). The first constraint in (15) follows from

$$\begin{aligned} & |q_1\Pi_1 - q_2\Pi_2| \leq (1-\alpha)\,|q_1^*\Pi_1(p_1^*, p_2^*) - q_2^*\Pi_2(p_1^*, p_2^*)| \\ \iff & \left|q_1\frac{g_K(\lambda_1, \lambda_2)}{\lambda_1} - q_2\frac{g_K(\lambda_1, \lambda_2)}{\lambda_2}\right| \leq (1-\alpha)\left|q_1^*\frac{g_K(\lambda_1^*, \lambda_2^*)}{\lambda_1^*} - q_2^*\frac{g_K(\lambda_1^*, \lambda_2^*)}{\lambda_2^*}\right| \\ \iff & \left|\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\right| g_K(\lambda_1, \lambda_2) \leq (1-\alpha)\left|\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\right| g_K(\lambda_1^*, \lambda_2^*) \\ \iff & g_K(\lambda_1, \lambda_2) \leq (1-\alpha) g_K(\lambda_1^*, \lambda_2^*), \end{aligned}$$

where $q_i^* = \frac{\lambda_i^*}{\Lambda_i}$ for $i \in \{1, 2\}$. The last if and only if condition is from the fact that $\Lambda_1 \neq \Lambda_2$.

The KKT conditions for the problem (15) are given by:

$$\frac{\partial g_K}{\partial \lambda_1}\cdot\left(F_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)+F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)+g_K(\lambda_1,\lambda_2)\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}+\mu_1+\mu_3\frac{\partial g_K}{\partial \lambda_1}=0, \tag{16a}$$

$$\frac{\partial g_K}{\partial \lambda_2}\cdot\left(F_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)+F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)+g_K(\lambda_1,\lambda_2)\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}+\mu_2+\mu_3\frac{\partial g_K}{\partial \lambda_2}=0, \tag{16b}$$

$$g_K(\lambda_1,\lambda_2)\le(1-\alpha)g_K(\lambda_1^*,\lambda_2^*), \tag{16c}$$

$$\lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2],$$

$$\mu_1,\mu_2,\mu_3\ge 0,$$

$$\mu_1(\lambda_1-\Lambda_1)=0,\ \mu_2(\lambda_2-\Lambda_2)=0,\ \mu_3\left(g_K\left(\lambda_1,\lambda_2\right)-(1-\alpha)\,g_K\left(\lambda_1^*,\lambda_2^*\right)\right)=0. \tag{16d}$$

If $\mu_3=0$, the stationary conditions ((16a) and (16b)) take the same forms as those without fair constraints. This implies that $(\lambda_1(\alpha),\lambda_2(\alpha))=(\lambda_1^*,\lambda_2^*)$, which violates the condition $\Lambda_1\neq\Lambda_2$ based on the Proposition 1. Therefore, $\mu_3\neq 0$, and $g_K\left(\lambda_1(\alpha),\lambda_2(\alpha)\right)=(1-\alpha)\,g_K\left(\lambda_1^*,\lambda_2^*\right)$ by complementary slackness (16d).

(a) $p_1>0$ and $p_2>0$: Since $p_1>0$ and $p_2>0$, $\lambda_1\neq\Lambda_1$ and $\lambda_2\neq\Lambda_2$. This implies $\mu_1=\mu_2=0$ by complementary slackness (16d). Under $\mu_3\neq 0$, the stationary conditions ((16a) and (16b)) in KKT conditions become:

$$\frac{\left.\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_2}\right|_{(\lambda_1(\alpha),\lambda_2(\alpha))}}{\left.\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_1}\right|_{(\lambda_1(\alpha),\lambda_2(\alpha))}}=\frac{\left.\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}\right|_{\lambda_2(\alpha)}}{\left.\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}\right|_{\lambda_1(\alpha)}},$$

where $\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_i}=\frac{\lambda_j^{K+1}\left(\lambda_j\left(\lambda_j^K-\lambda_i^K\right)+K\lambda_i^K\left(\lambda_i-\lambda_j\right)\right)}{\left(\lambda_i^{K+1}-\lambda_j^{K+1}\right)^2}$ for $i\neq j\in\{1,2\}$.

With linear demand, the second term becomes:

$$\frac{\left.\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}\right|_{\lambda_2(\alpha)}}{\left.\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}\right|_{\lambda_1(\alpha)}}=\frac{b_2\Lambda_1}{b_1\Lambda_2}.$$

The complementary slackness (16d) in KKT conditions implies that $g_K\left(\lambda_1(\alpha),\lambda_2(\alpha)\right)=(1-\alpha)\,g_K\left(\lambda_1^*,\lambda_2^*\right)$.

When $\alpha=0$ (no fairness), KKT conditions are the same as (16) due to the fact that $\mu_3=0$. Then,

$$\frac{\left.\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_2}\right|_{(\lambda_1^*,\lambda_2^*)}}{\left.\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_1}\right|_{(\lambda_1^*,\lambda_2^*)}}=\frac{\left.\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}\right|_{\lambda_2^*}}{\left.\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}\right|_{\lambda_1^*}}.$$

We can observe that $(\lambda_1(\alpha),\lambda_2(\alpha))=\left((1-\alpha)\,\lambda_1^*,(1-\alpha)\,\lambda_2^*\right)$ satisfies all KKT conditions, which implies that it is the optimal solution under α-access fairness. Therefore, the price under α-access fairness at node $i\in\{1,2\}$ is $p_i(\alpha)=F_i^{-1}\left(1-(1-\alpha)\frac{\lambda_i^*}{\Lambda_i}\right)$. With this optimal solution, normalized surplus at each node and revenue with α can be derived.

The consumer surplus at each node is

$$\mathcal{S}_i(\alpha)=\frac{1}{2}\frac{b_i}{\Lambda_i^2}\lambda_i(\alpha)g_K\left(\lambda_1\left(\alpha\right),\lambda_2\left(\alpha\right)\right)=(1-\alpha)^2\frac{1}{2}\frac{b_i}{\Lambda_i^2}\lambda_i^*(\alpha)g_K\left(\lambda_1^*,\lambda_2^*\right)=(1-\alpha)^2\mathcal{S}_i(0).$$

Therefore, the total consumer surplus is $\mathcal{S}(\alpha) = \Lambda_0 \mathcal{S}_1(\alpha) + \Lambda_1 \mathcal{S}_2(\alpha) = (1-\alpha)^2 \mathcal{S}(0)$ for linear demand.It is obvious that the revenue $\mathcal{R}(\alpha)$ is decreasing function with α. Therefore, $\mathcal{W}(\alpha)$ which is the sum of revenue and total surplus is also decreasing function with α.

Finally, $\Pi_i(\alpha) := \Pi_i(p_1(\alpha), p_2(\alpha)) = \Pi_i^*$ because:

$$\Pi_i(\alpha) = \frac{g_K(\lambda_1(\alpha), \lambda_2(\alpha))}{\lambda_i(\alpha)} = \frac{(1-\alpha) g_K(\lambda_1^*, \lambda_2^*)}{(1-\alpha)\lambda_i^*} = \Pi_i^*.$$

Case (b) Without loss of generality, $p_1^* = 0$: Since $\lambda_1^* = \Lambda_1$, μ_1 does not have to be 0 by complementary slackness (16d). Let $\alpha^\star$ denote the point where $\mu_1 = 0$ and $\lambda_1 = \Lambda_1$ hold simultaneously. $\alpha^\star$ is derived by solving the following equations:

$$\frac{\left.\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_2}\right|_{(\Lambda_1, \lambda_2(\alpha^\star))}}{\left.\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_1}\right|_{(\Lambda_1, \lambda_2(\alpha^\star))}} = \frac{\left.\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}\right|_{\lambda_2(\alpha^\star)}}{\left.\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}\right|_{\Lambda_1}},$$

$$g_K(\Lambda_1, \lambda_2(\alpha^\star)) = (1-\alpha^\star) g_K(\Lambda_1, \lambda_2^*).$$

Under $\alpha \le \alpha^\star$, the dynamic of systems behaves different to that of *(a)*. On the other hand, under $\alpha > \alpha^\star$, the dynamic is the same as *(a)*.

(i) Under $\alpha \le \alpha^\star$, with $\lambda_1(\alpha) = \Lambda_1 = \lambda_1^*$, the consumer surplus at node 1 under linear demand is:

$$\mathcal{S}_1(\alpha) = \frac{1}{2}\frac{b_1}{\Lambda_1}\lambda_1(\alpha) g_K(\lambda_1(\alpha), \lambda_2(\alpha)) = (1-\alpha)\frac{1}{2}\frac{b_1}{\Lambda_1}\lambda_1^* g_K(\lambda_1^*, \lambda_2^*) = (1-\alpha)\mathcal{S}_1(0),$$

where the second equality is from $g_K(\lambda_1(\alpha), \lambda_2(\alpha)) = (1-\alpha) g_K(\lambda_1^*, \lambda_2^*)$, as in *(a)*.

For node 2, $\mathcal{S}_2(\alpha) = (1-\alpha)\frac{1}{2}\frac{b_2}{\Lambda_2}\lambda_2(\alpha) g_K(\Lambda_1, \lambda_2^*) = (1-\alpha)\frac{\lambda_2(\alpha)}{\lambda_2^*}\mathcal{S}_2(0)$, which is a decreasing function with α. This is because

$$\frac{d\mathcal{S}_2(\alpha)}{d\alpha} = -\frac{1}{2}\frac{b_2}{\Lambda_2}\lambda_2(\alpha) g_K(\Lambda_1, \lambda_2^*) + (1-\alpha)\frac{1}{2}\frac{b_2}{\Lambda_2}\frac{d\lambda_2(\alpha)}{d\alpha} g_K(\Lambda_1, \lambda_2^*) < 0.$$

In the above equation, the gradient of $\lambda_2(\alpha)$ with α is negative, i.e., $\frac{d\lambda_2(\alpha)}{d\alpha} < 0$. Since $\lambda_2(\alpha)$ satisfies $g_K(\Lambda_1, \lambda_2(\alpha)) = (1-\alpha) g_K(\Lambda_1, \lambda_2^*)$, its derivative w.r.t. α satisfies:

$$\frac{d g_K(\Lambda_1, \lambda_2(\alpha))}{d\alpha} = \frac{d\lambda_2(\alpha)}{d\alpha}\frac{\partial g_K(\Lambda_1, \lambda_2(\alpha))}{\partial \lambda_2(\alpha)} = -g_K(\Lambda_1, \lambda_2^*) < 0.$$

From the stationary condition (16b), $\frac{\partial g_K(\Lambda_1, \lambda_2(\alpha))}{\partial \lambda_2(\alpha)} > 0$ because $g_K(\lambda_1, \lambda_2)$ and $\left(F_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right) + F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)$ are positive, $\frac{dF^{-1}(1-\frac{\lambda_2}{\Lambda_2})}{d\lambda_2}$ is negative, and $\mu_2 = \mu_3 = 0$ when $(\lambda_1, \lambda_2) = (\Lambda_1, \lambda_2(\alpha))$. Therefore, $\frac{d\lambda_2(\alpha)}{d\alpha} < 0$. The $\mathcal{S}_2(\alpha)$ can be rewritten in terms of price $p_2(\alpha)$ like $(1-\alpha)\frac{1-F(p_2(\alpha))}{1-F(p_2^*)}\mathcal{S}_2(0)$, where $p_2(\alpha)$ is an increasing function with α because $\frac{dp_2(\alpha)}{d\alpha} = -\frac{b_2}{\Lambda_2}\frac{d\lambda_2(\alpha)}{d\alpha} > 0$.

Finally,

$$\Pi_1(\alpha) = \frac{g_K(\lambda_1(\alpha), \lambda_2(\alpha))}{\lambda_1(\alpha)} = \frac{(1-\alpha) g_K(\lambda_1^*, \lambda_2^*)}{\lambda_1^*} = (1-\alpha)\Pi_1^*.$$

$$\Pi_2(\alpha) = \frac{g_K(\lambda_1(\alpha), \lambda_2(\alpha))}{\lambda_2(\alpha)} = \frac{(1-\alpha) g_K(\lambda_1^*, \lambda_2^*)}{\lambda_2(\alpha)} = (1-\alpha)\frac{\lambda_2^*}{\lambda_2(\alpha)}\Pi_2^*.$$

(ii) Under $\alpha > \alpha^\star$, since prices at both nodes have positive values under $\alpha > \alpha^\star$, this case is the same as *(a)*. From KKT conditions, we get $g_K(\lambda_1(\alpha), \lambda_2(\alpha)) = (1-\alpha) g_K(\lambda_1^*, \lambda_2^*) = \frac{1-\alpha}{1-\alpha^\star} g_K(\lambda_1(\alpha^\star), \lambda_2(\alpha^\star))$, which implies $\lambda_i(\alpha) = \frac{1-\alpha}{1-\alpha^\star}\lambda_i(\alpha^\star)$ for all $i \in \{1,2\}$. Then consumer surplus at each node $i \in \{1,2\}$ under linear demand is:

$$\begin{aligned}\mathcal{S}_i(\alpha) &= \frac{1}{2}\frac{b_i}{\Lambda_i}\lambda_i(\alpha) g_K(\lambda_1(\alpha), \lambda_2(\alpha)) \\ &= \frac{1}{2}\frac{b_i}{\Lambda_i}\frac{1-\alpha}{1-\alpha^\star}\lambda_i(\alpha^\star)\frac{1-\alpha}{1-\alpha^\star} g_K(\lambda_1(\alpha^\star), \lambda_2(\alpha^\star)) \\ &= \frac{(1-\alpha)^2}{(1-\alpha^\star)^2}\mathcal{S}_i(\alpha^\star).\end{aligned}$$

□

Proof of Proposition 4. Without loss of generality, suppose that $\beta_1 \geq \beta_2$. For the proof, assume $p_2^* \geq p_1^*$ which implies $\log\frac{\Lambda_2}{\Lambda_1} + \log\frac{\beta_1}{\beta_2} \geq \frac{1}{2}\left(\frac{\beta_2}{\beta_1} - \frac{\beta_1}{\beta_2}\right)$. The optimization problem under price fairness can be rewritten with Taylor expansion shown in (13).

(a)-(i) First, when $\log\frac{\Lambda_2}{\Lambda_1} > \log\frac{\beta_2}{\beta_1} + \frac{\beta_2}{\beta_1} + 1$ where $p_1^* = 0$, we get $\frac{dp_1}{d\alpha}|_{\alpha=0} = 0$ and $p_2'(0)^* = -p_2^*$ by Lemma 5. Then, following the proof of the Proposition 2*(a)*, we get

$$\mathcal{W}'(0) = \frac{\Lambda_1^2\lambda_2^*}{(\Lambda_1+\lambda_2^*)^2}\left(1+\frac{\beta_2}{\beta_1}\right)p_2^* > 0,$$

$$\mathcal{S}_1'(0) = \frac{\beta_2}{\beta_1}\frac{\Lambda_1\lambda_2^*}{(\Lambda_1+\lambda_2^*)^2}p_2^* > 0, \quad \mathcal{S}_2'(0) = \frac{\Lambda_1}{\Lambda_2}\frac{\Lambda_1\lambda_2^*}{(\Lambda_1+\lambda_2^*)^2}p_2^* > 0.$$

(a)-(ii) Next, when $\log\frac{\Lambda_2}{\Lambda_1} \leq \log\frac{\beta_2}{\beta_1} + \frac{\beta_2}{\beta_1} + 1$ and $\log\frac{\Lambda_2}{\Lambda_1} + \log\frac{\beta_1}{\beta_2} \geq \frac{1}{2}\left(\frac{\beta_2}{\beta_1} - \frac{\beta_1}{\beta_2}\right)$, where $p_2^* \geq p_1^* > 0$, $R_{12}'' = 0$. Therefore, following the proof of Proposition 2*(b)*, we get

$$p_1'(0) = \frac{R_{22}''}{R_{11}''+R_{22}''}(p_2^*-p_1^*), \quad p_2'(0) = -\frac{R_{11}''}{R_{11}''+R_{22}''}(p_2^*-p_1^*),$$

where $R_{11}'' = -\frac{\beta_1\lambda_1^*\lambda_2^*}{\lambda_1^+\lambda_2^*}$ and $R_{22}'' = -\frac{\beta_2\lambda_1^*\lambda_2^*}{\lambda_1^+\lambda_2^*}$. Then, the derivative of social welfare at $\alpha = 0$ is

$$\begin{aligned}\mathcal{W}'(0) &= \frac{d\mathcal{R}}{d\alpha}\Big|_{\alpha=0} + \frac{d\mathcal{S}}{d\alpha}\Big|_{\alpha=0} = \frac{d\mathcal{S}}{d\alpha}\Big|_{\alpha=0} \\ &= -\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\frac{\lambda_1^*\lambda_2^*}{(\lambda_1^*+\lambda_2^*)^2}\left(\beta_1\lambda_2^*\frac{dp_1}{d\alpha} + \beta_2\lambda_1^*\frac{dp_2}{d\alpha}\right) \\ &= -\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\frac{\lambda_1^*\lambda_2^*}{(\lambda_1^*+\lambda_2^*)^2}\frac{\beta_1\beta_2}{\beta_1+\beta_2}(\lambda_2^*-\lambda_1^*).\end{aligned}$$

From the above equation, we get $\mathcal{W}'(0) > 0 \iff \lambda_1^* > \lambda_2^* \iff \beta_1 > \beta_2$. The last relationship is coming from the fact that $\beta_1\lambda_2^* = \beta_2\lambda_1^*$ under this region as stated in the proof of Lemma 2*(b)*. For surplus at each node,

$$\begin{aligned}\mathcal{S}_1'(0) &= \frac{\partial\mathcal{S}_1(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial\mathcal{S}_1(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha} \\ &= -\frac{1}{\Lambda_1}\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\left(\beta_1\lambda_2^*\frac{dp_1}{d\alpha} + \beta_2\lambda_1^*\frac{dp_2}{d\alpha}\right) \\ &= -\frac{1}{\Lambda_1}\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\frac{\beta_1\beta_2}{\beta_1+\beta_2}(\lambda_2^*-\lambda_1^*), \\ \mathcal{S}_2'(0) &= \frac{\partial\mathcal{S}_2(\alpha)}{\partial p_1}\frac{dp_1}{d\alpha} + \frac{\partial\mathcal{S}_2(\alpha)}{\partial p_2}\frac{dp_2}{d\alpha} \\ &= -\frac{1}{\Lambda_2}\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\left(\beta_1\lambda_2^*\frac{dp_1}{d\alpha} + \beta_2\lambda_1^*\frac{dp_2}{d\alpha}\right) \\ &= -\frac{1}{\Lambda_2}\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\frac{\beta_1\beta_2}{\beta_1+\beta_2}(\lambda_2^*-\lambda_1^*).\end{aligned}$$

Similar to $\mathcal{W}'(0)$, $\beta_1 > \beta_2 \iff \mathcal{S}_1'(0) > 0 \iff \mathcal{S}_2'(0) > 0$. If $\beta_1 > \beta_2$, the region we consider, always holds.

(a)-(iii) Now, if we consider $p_1^* > p_2^*$, all derivations remain identical to the ones mentioned above, except that we switch index 1 and 2, i.e.,

- $\log \frac{\Lambda_1}{\Lambda_2} > \log \frac{\beta_1}{\beta_2} + \frac{\beta_1}{\beta_2} + 1 \Rightarrow \mathcal{W}'(0) > 0,\ \mathcal{S}_1'(0) > 0,\ \mathcal{S}_2'(0) > 0$
- $\log \frac{\Lambda_1}{\Lambda_2} \le \log \frac{\beta_1}{\beta_2} + \frac{\beta_1}{\beta_2} + 1$, $\log \frac{\Lambda_2}{\Lambda_1} + \log \frac{\beta_1}{\beta_2} < \frac{1}{2}\left(\frac{\beta_2}{\beta_1} - \frac{\beta_1}{\beta_2}\right)$, $\beta_2 > \beta_1 \Rightarrow \mathcal{W}'(0) > 0,\ \mathcal{S}_1'(0) > 0,\ \mathcal{S}_2'(0) > 0$

Therefore, under $\log \frac{\Lambda_1}{\Lambda_2} > \log \frac{\beta_1}{\beta_2} + \frac{\beta_1}{\beta_2} + 1$, where $p_2^* = 0$, we get $\mathcal{W}'(0) > 0,\ \mathcal{S}_1'(0) > 0,\ \mathcal{S}_2'(0) > 0$. On the other hand, the second condition does not hold since we assume that $\beta_1 \ge \beta_2$.

(b) Finally, in the remaining area, we have $\mathcal{W}'(0) \le 0$. Similar to *(a)-(ii)*, the sign of $\mathcal{W}'(0)$ aligns with that of $\mathcal{S}_0'(0)$ and $\mathcal{S}_1'(0)$. □

Proof of Proposition 5. Similar to the proof of Proposition 3, the optimization problem can be formulated as shown in (15). The KKT conditions are the same with those in Proposition 3.

(a) $p_1 > 0$ and $p_2 > 0$: Since $p_1 > 0$ and $p_2 > 0$, $\lambda_1 \ne \Lambda_1$ and $\lambda_2 \ne \Lambda_2$. This implies $\mu_1 = \mu_2 = 0$ by complementary slackness (16d). Under $\mu_3 \ne 0$, the stationary conditions ((16a) and (16b)) in KKT conditions become:

$$\frac{\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_2}\Big|_{(\lambda_1(\alpha),\lambda_2(\alpha))}}{\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_1}\Big|_{(\lambda_1(\alpha),\lambda_2(\alpha))}} = \frac{\frac{dF_2^{-1}(1-\frac{\lambda_2}{\Lambda_2})}{d\lambda_2}\Big|_{\lambda_2(\alpha)}}{\frac{dF_1^{-1}(1-\frac{\lambda_1}{\Lambda_1})}{d\lambda_1}\Big|_{\lambda_1(\alpha)}} = \frac{\beta_1 \lambda_1(\alpha)}{\beta_2 \lambda_2(\alpha)},$$

where $\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_i} = \frac{\lambda_j^{K+1}\left(\lambda_j\left(\lambda_j^K - \lambda_i^K\right) + K\lambda_i^K\left(\lambda_i - \lambda_j\right)\right)}{\left(\lambda_i^{K+1} - \lambda_j^{K+1}\right)^2}$ for $i \ne j \in \{1,2\}$.

The complementary slackness (16d) in KKT conditions implies that $g_K\left(\lambda_1(\alpha), \lambda_2(\alpha)\right) = (1-\alpha)\, g_K\left(\lambda_1^*, \lambda_2^*\right)$.

When $\alpha = 0$ (no fairness), KKT conditions are the same as (16) due to the fact that $\mu_3 = 0$. Then,

$$\frac{\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_2}\Big|_{(\lambda_1^*,\lambda_2^*)}}{\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial \lambda_1}\Big|_{(\lambda_1^*,\lambda_2^*)}} = \frac{\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}\Big|_{\lambda_2^*}}{\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}\Big|_{\lambda_1^*}}.$$

We can observe that $(\lambda_1(\alpha), \lambda_2(\alpha)) = \left((1-\alpha)\lambda_1^*, (1-\alpha)\lambda_2^*\right)$ satisfies all KKT conditions, which implies that it is the optimal solution under α-access fairness. Therefore, the price under α-access fairness at node $i \in \{1,2\}$ is $p_i(\alpha) = F_i^{-1}\left(1 - (1-\alpha)\frac{\lambda_i^*}{\Lambda_i}\right)$. With this optimal solution, normalized surplus at each node and revenue with α can be derived.

Then, the consumer surplus is

$$\mathcal{S}_i(\alpha) = \frac{1}{\Lambda_i \beta_i} g_K\left(\lambda_1(\alpha), \lambda_2(\alpha)\right) = (1-\alpha)\frac{1}{\Lambda_i \beta_i} g_K\left(\lambda_1^*, \lambda_2^*\right) = (1-\alpha)\mathcal{S}_i(0).$$

Therefore, the total consumer surplus is $\mathcal{S}(\alpha) = (1-\alpha)\mathcal{S}(0)$. It is obvious that the revenue $\mathcal{R}(\alpha)$ is decreasing function with α. Therefore, $\mathcal{W}(\alpha)$ which is the sum of revenue and total surplus is also decreasing function with α.

Finally, $\Pi_i(\alpha) := \Pi_i\left(p_1(\alpha), p_2(\alpha)\right) = \Pi_i^*$ because:

$$\Pi_i(\alpha) = \frac{g_K\left(\lambda_1(\alpha), \lambda_2(\alpha)\right)}{\lambda_i(\alpha)} = \frac{(1-\alpha) g_K\left(\lambda_1^*, \lambda_2^*\right)}{(1-\alpha)\lambda_i^*} = \Pi_i^*.$$

Case (b) Without loss of generality, $p_1^*=0$: Since $\lambda_1^*=\Lambda_1$, μ_1 does not have to be 0 by complementary slackness (16d). Let $\alpha^\star$ denote the point where $\mu_1=0$ and $\lambda_1=\Lambda_1$ hold simultaneously. $\alpha^\star$ is derived by solving the following equations:

$$\frac{\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_2}\Big|_{(\Lambda_1,\lambda_2(\alpha^\star))}}{\frac{\partial g_K(\lambda_1,\lambda_2)}{\partial\lambda_1}\Big|_{(\Lambda_1,\lambda_2(\alpha^\star))}}=\frac{\frac{dF_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}\Big|_{\lambda_2(\alpha^\star)}}{\frac{dF_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)}{d\lambda_1}\Big|_{\Lambda_1}},$$
$$g_K\left(\Lambda_1,\lambda_2(\alpha^\star)\right)=(1-\alpha^\star)g_K(\Lambda_1,\lambda_2^*).$$

Under $\alpha\le\alpha^\star$, the dynamic of systems behaves different to that of *(a)*. On the other hand, under $\alpha>\alpha^\star$, the dynamic is the same as *(a)*.

(i) Under $\alpha\le\alpha^\star$, the gradient of $\lambda_2(\alpha)$ with α is negative, i.e., $\frac{d\lambda_2(\alpha)}{d\alpha}<0$. Since $\lambda_2(\alpha)$ satisfies $g_K\left(\Lambda_1,\lambda_2(\alpha)\right)=(1-\alpha)g_K(\Lambda_1,\lambda_2^*)$, its derivative w.r.t. α satisfies:

$$\frac{dg_K\left(\Lambda_1,\lambda_2(\alpha)\right)}{d\alpha}=\frac{d\lambda_2(\alpha)}{d\alpha}\frac{\partial g_K\left(\Lambda_1,\lambda_2(\alpha)\right)}{\partial\lambda_2(\alpha)}=-g_K(\Lambda_1,\lambda_2^*)<0.$$

From the stationary condition (16b), $\frac{\partial g_K(\Lambda_1,\lambda_2(\alpha))}{\partial\lambda_2(\alpha)}>0$ because $\left(F_1^{-1}\left(1-\frac{\lambda_1}{\Lambda_1}\right)+F_2^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)$ and $g_K(\lambda_1,\lambda_2)$ are positive, $\frac{dF^{-1}\left(1-\frac{\lambda_2}{\Lambda_2}\right)}{d\lambda_2}$ is negative, and $\mu_2=\mu_3=0$ when $(\lambda_1,\lambda_2)=(\Lambda_1,\lambda_2(\alpha))$. Therefore, $\frac{d\lambda_2(\alpha)}{d\alpha}<0$. The $\mathcal{S}_2(\alpha)$ can be rewritten in terms of price $p_2(\alpha)$ like $(1-\alpha)\frac{1-F(p_2(\alpha))}{1-F\left(p_2^*\right)}\mathcal{S}_2(0)$, where $p_2(\alpha)$ is an increasing function with α because $\frac{dp_2(\alpha)}{d\alpha}=-\frac{b_2}{\Lambda_2}\frac{d\lambda_2(\alpha)}{d\alpha}>0$.

Under exponential demand, $g_K\left(\lambda_1(\alpha),\lambda_2(\alpha)\right)=(1-\alpha)g_K\left(\lambda_1^*,\lambda_2^*\right)$, which implies $\mathcal{S}_i(\alpha)=(1-\alpha)\mathcal{S}_i(0)$ as in *(a)*. Finally,

$$\Pi_1(\alpha)=\frac{g_K\left(\lambda_1(\alpha),\lambda_2(\alpha)\right)}{\lambda_1(\alpha)}=\frac{(1-\alpha)g_K\left(\lambda_1^*,\lambda_2^*\right)}{\lambda_1^*}=(1-\alpha)\Pi_1^*.$$
$$\Pi_2(\alpha)=\frac{g_K\left(\lambda_1(\alpha),\lambda_2(\alpha)\right)}{\lambda_2(\alpha)}=\frac{(1-\alpha)g_K\left(\lambda_1^*,\lambda_2^*\right)}{\lambda_2(\alpha)}=(1-\alpha)\frac{\lambda_2^*}{\lambda_2(\alpha)}\Pi_2^*.$$

(ii) Under $\alpha>\alpha^\star$, since prices at both nodes have positive values under $\alpha>\alpha^\star$, this case is the same as *(a)*. From KKT conditions, we get $g_K(\lambda_1(\alpha),\lambda_2(\alpha))=(1-\alpha)g_K(\lambda_1^*,\lambda_2^*)=\frac{1-\alpha}{1-\alpha^\star}g_K\left(\lambda_1(\alpha^\star),\lambda_2(\alpha^\star)\right)$, which implies $\lambda_i(\alpha)=\frac{1-\alpha}{1-\alpha^\star}\lambda_i(\alpha^\star)$ for all $i\in\{1,2\}$. Then consumer surplus at each node $i\in\{1,2\}$ under exponential demand is $\mathcal{S}_i(\alpha)=(1-\alpha)\mathcal{S}_i(0)$, as in the previous case. The availability, Π_i is

$$\Pi_i(\alpha)=\frac{g_K\left(\lambda_i(\alpha),\lambda_i(\alpha)\right)}{\lambda_i(\alpha)}=\frac{\frac{1-\alpha}{1-\alpha^\star}g_K\left(\lambda_1(\alpha^\star),\lambda_2(\alpha^\star)\right)}{\frac{1-\alpha}{1-\alpha^\star}\lambda_i(\alpha^\star)}=\Pi_i(\alpha^\star).$$

□

LEMMA 6 **(Optimal Solution under $N=2$, $K=1$, $\alpha=0$, and linear demand)**. *When $N=2$, $K=1$, demand is linear, and $\frac{1}{\mu_1}+\frac{1}{\mu_2}>0$, the optimal solution of the problem* $\max_{p_1,p_2\ge0}R(p_1,p_2)$ *is*

$$(p_1^*,p_2^*)=\begin{cases}\left(0,b_2+\frac{b_2}{\Lambda_2}\frac{-\sqrt{\Lambda_1^2+\left(1+\frac{\Lambda_1}{\mu}\right)\Lambda_1\Lambda_2}+\Lambda_1}{1+\frac{\Lambda_1}{\mu}}\right) & \text{if } \sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}\ge2\sqrt{\frac{\Lambda_1}{\Lambda_2}+\frac{\Lambda_1}{\mu}}\sqrt{\frac{b_1}{b_2}},\\ \left(b_1+\frac{b_1}{\Lambda_1}\frac{-\sqrt{\Lambda_2^2+\left(1+\frac{\Lambda_2}{\mu}\right)\Lambda_1\Lambda_2}+\Lambda_2}{1+\frac{\Lambda_2}{\mu}},0\right) & \text{if } \sqrt{\frac{b_1}{b_2}}-\sqrt{\frac{b_2}{b_1}}\ge2\sqrt{\frac{\Lambda_2}{\Lambda_1}+\frac{\Lambda_2}{\mu}}\sqrt{\frac{b_2}{b_1}},\\ \left(b_1-\mu\sqrt{\beta_1}\left(\sqrt{(\sqrt{\beta_1}+\sqrt{\beta_2})^2+\frac{b_1+b_2}{\mu}}-\sqrt{\beta_1}-\sqrt{\beta_2}\right),\right. & \\ \left.\quad b_2-\mu\sqrt{\beta_2}\left(\sqrt{(\sqrt{\beta_1}+\sqrt{\beta_2})^2+\frac{b_1+b_2}{\mu}}-\sqrt{\beta_1}-\sqrt{\beta_2}\right)\right) & \text{otherwise,}\end{cases}$$

where $\beta_1=\frac{b_1}{\Lambda_1}$, $\beta_2=\frac{b_2}{\Lambda_2}$, *and* $\frac{1}{\mu}=\frac{1}{\mu_1}+\frac{1}{\mu_2}>0$.

Proof. The closed form optimal solution (p_1^*, p_2^*) of $\max_{p_1,p_2\geq 0} R(p_1,p_2) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2+\frac{\lambda_1\lambda_2}{\mu}}(p_1+p_2)$, where $\frac{1}{\mu} := \frac{1}{\mu_1} + \frac{1}{\mu_2}$, is derived by the following steps. First, $\lambda_i := \Lambda_i q_i = \Lambda_i \max\{0, 1-\frac{p_i}{b_i}\} = 1 - \frac{\Lambda_i}{b_i}p_i$ because if $\lambda_i = 0$, system shut down (zero revenue). However, we can always have positive revenue by setting $\lambda_i > 0$ for all $i \in \{1,2\}$. Then, we replace the price variables, (p_1,p_2), in unconstrained problem with the demand rate variables, (λ_1,λ_2), yielding,

$$\mathcal{R}^* := \max_{\lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2]} \quad \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2+\frac{\lambda_1\lambda_2}{\mu}}\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right). \tag{17}$$

Next, we derive the optimal solution, $(\lambda_1^*,\lambda_2^*)$, in (17) by using the following KKT conditions:

$$\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right)\frac{\lambda_2^2}{(\lambda_1+\lambda_2+\frac{\lambda_1\lambda_2}{\mu})^2}-\frac{b_1}{\Lambda_1}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2+\frac{\lambda_1\lambda_2}{\mu}}+\mu_1=0, \tag{18a}$$

$$\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right)\frac{\lambda_1^2}{(\lambda_1+\lambda_2+\frac{\lambda_1\lambda_2}{\mu})^2}-\frac{b_2}{\Lambda_2}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2+\frac{\lambda_1\lambda_2}{\mu}}+\mu_2=0, \tag{18b}$$

$$\lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2], \tag{18c}$$

$$\mu_1,\mu_2\geq 0,$$

$$\mu_1(\lambda_1-\Lambda_1)=0,\ \mu_2(\lambda_2-\Lambda_2)=0. \tag{18d}$$

If $\lambda_1\neq\Lambda_1$ and $\lambda_2\neq\Lambda_2$, we get $\beta_1\lambda_1^2=\beta_2\lambda_2^2$ by stationary conditions (18a) and (18b), and complementary slackness condition (18d). When we plug $\lambda_2=\sqrt{\frac{\beta_1}{\beta_2}}\lambda_1$ into (17), then $(\lambda_1^*,\lambda_2^*)$ becomes:

$$(\lambda_1^*,\lambda_2^*)=\begin{cases}\left(\frac{1}{2}\frac{b_1+b_2}{\beta_1+\sqrt{\beta_1\beta_2}},\frac{1}{2}\frac{b_1+b_2}{\beta_2+\sqrt{\beta_1\beta_2}}\right), & \text{if } \frac{1}{\mu}=0,\\ \left(\frac{\mu}{\sqrt{\beta_1}}\left(\sqrt{(\sqrt{\beta_1}+\sqrt{\beta_2})^2+\frac{b_1+b_2}{\mu}}-\sqrt{\beta_1}-\sqrt{\beta_2}\right),\right. & \\ \left.\quad\frac{\mu}{\sqrt{\beta_2}}\left(\sqrt{(\sqrt{\beta_1}+\sqrt{\beta_2})^2+\frac{b_1+b_2}{\mu}}-\sqrt{\beta_1}-\sqrt{\beta_2}\right)\right) & \text{otherwise.}\end{cases}$$

where $(\beta_1,\beta_2)=(\frac{b_1}{\Lambda_1},\frac{b_2}{\Lambda_2})$. The primal feasibility, Eq. (18c), introduces conditions that:

$$\frac{\mu}{\sqrt{\beta_1}}\left(\sqrt{(\sqrt{\beta_1}+\sqrt{\beta_2})^2+\frac{b_1+b_2}{\mu}}-\sqrt{\beta_1}-\sqrt{\beta_2}\right)<\Lambda_1 \iff \sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}<2\sqrt{\frac{\Lambda_1}{\Lambda_2}}+\frac{\Lambda_1}{\mu}\sqrt{\frac{b_1}{b_2}},$$

$$\frac{\mu}{\sqrt{\beta_2}}\left(\sqrt{(\sqrt{\beta_1}+\sqrt{\beta_2})^2+\frac{b_1+b_2}{\mu}}-\sqrt{\beta_1}-\sqrt{\beta_2}\right)<\Lambda_2 \iff \sqrt{\frac{b_1}{b_2}}-\sqrt{\frac{b_2}{b_1}}<2\sqrt{\frac{\Lambda_2}{\Lambda_1}}+\frac{\Lambda_2}{\mu}\sqrt{\frac{b_2}{b_1}}.$$

If $\sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}\geq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}+\frac{\Lambda_1}{\mu}\sqrt{\frac{b_1}{b_2}}$, primal feasibility does not hold. In this case, $\lambda_1=\Lambda_1$, then we get

$$(\lambda_1^*,\lambda_2^*)=\left(\Lambda_1,-\frac{\sqrt{\Lambda_1^2+\left(1+\frac{\Lambda_1}{\mu}\right)\Lambda_1\Lambda_2}-\Lambda_1}{1+\frac{\Lambda_1}{\mu}}\right)$$

by plugging in $\lambda_1=\Lambda_1$ in Eq. (17). Similarly, if $\sqrt{\frac{b_1}{b_2}}-\sqrt{\frac{b_2}{b_1}}\geq 2\sqrt{\frac{\Lambda_2}{\Lambda_1}}+\frac{\Lambda_2}{\mu}\sqrt{\frac{b_2}{b_1}}$, we get

$$(\lambda_1^*,\lambda_2^*)=\left(\frac{\sqrt{\Lambda_2^2+\left(1+\frac{\Lambda_2}{\mu}\right)\Lambda_1\Lambda_2}-\Lambda_2}{1+\frac{\Lambda_2}{\mu}},\Lambda_2\right).$$

(p_1^*,p_2^*) can be calculated using $p_i^*=b_i-\frac{b_i}{\Lambda_i}\lambda_i^*,\ \forall i\in\{1,2\}$. □

Proof of Proposition 6. The proof follows similarly to that of Proposition 2.

(a) Suppose $\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} > 2\sqrt{\frac{\Lambda_1}{\Lambda_2}} + \frac{\Lambda_1}{\mu}\sqrt{\frac{b_1}{b_2}}$, then there exists $\alpha^\star$ such that $p_1^*(\alpha) = 0$ for $\alpha \in [0, \alpha^\star]$, following a similar process as in Lemma 5. As in the previous proof, the problem we need to solve can be rewritten as (14). The optimal solution of this problem is $p_2'(0)^* = -p_2^*$. With $p_1'(0) = 0$ and $p_2'(0) = -p_2^*$,

$$\mathcal{S}_1'(0) = p_2'(0)\frac{\mathcal{S}(\alpha)_1}{dp_2}\Big|_{(0,p_2^*)} = p_2^*\frac{\beta_1}{\beta_2}\frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)}{{\lambda_2^*}^2} > 0; \quad \mathcal{S}_2'(0) = p_2'(0)\frac{\mathcal{S}_2(\alpha)}{dp_2}\Big|_{(0,p_2^*)} = p_2^*\frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)^2}{2\lambda_2^*\Lambda_2} > 0,$$

where $g(\lambda_1, \lambda_2, \mu) = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2 + \frac{\lambda_1\lambda_2}{\mu}}$. Therefore, $\mathcal{W}'(0) = \Lambda_1\mathcal{S}_1'(0) + \Lambda_2\mathcal{S}_2'(0) > 0$.

Now, we consider the region where $\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{b_1}{b_2}} \leq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}} + \frac{\Lambda_1}{\mu}\sqrt{\frac{b_1}{b_2}}$. In this region, as stated in the proof of Lemma 6, the relationship $\beta_1{\lambda_1^*}^2 = \beta_2{\lambda_2^*}^2$ holds. Using $\beta_1{\lambda_1^*}^2 = \beta_2{\lambda_2^*}^2$, the derivative of surplus with prices at $\alpha = 0$, (p_1^*, p_2^*), is simplified as follows,

$$\frac{\partial\mathcal{S}_1(\alpha)}{\partial p_1}\Big|_{(p_1^*,p_2^*)} = -\frac{1}{2\Lambda_1}g\left(\lambda_1^*, \lambda_2^*, \mu\right)\left(1 + \frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)}{\lambda_1^*}\right), \quad \frac{\partial\mathcal{S}_1(\alpha)}{\partial p_2}\Big|_{(p_1^*,p_2^*)} = -\frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)^2}{2\lambda_1^*\Lambda_1},$$

$$\frac{\partial\mathcal{S}_2(\alpha)}{\partial p_1}\Big|_{(p_1^*,p_2^*)} = -\frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)^2}{2\lambda_2^*\Lambda_2}, \quad \frac{\partial\mathcal{S}_2(\alpha)}{\partial p_2}\Big|_{(p_1^*,p_2^*)} = -\frac{1}{2\Lambda_2}g\left(\lambda_1^*, \lambda_2^*, \mu\right)\left(1 + \frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)}{\lambda_2^*}\right).$$

The value of $R_{11}'' := \frac{\partial^2 R}{\partial p_1^2}$, $R_{12}'' :== \frac{\partial^2 R}{\partial p_1 \partial p_2}$, and $R_{22}'' := \frac{\partial^2 R}{dp_2^2}$ under this region are

$$R_{11}'' - 2\frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)^2}{\beta_1{\lambda_1^*}^2}\left(1 + \frac{\lambda_1^*}{\lambda_2^*} + \frac{\lambda_1^*}{\mu}\right) < 0; \quad R_{12}'' = 0; \quad R_{22}'' = -2\frac{g\left(\lambda_1^*, \lambda_2^*, \mu\right)^2}{\beta_2{\lambda_2^*}^2}\left(1 + \frac{\lambda_2^*}{\lambda_1^*} + \frac{\lambda_2^*}{\mu}\right).$$

Therefore, the optimal solution to (13) is

$$p_1'(0) = \frac{R_{22}''}{R_{11}'' + R_{22}''}\left(p_2^* - p_1^*\right) > 0, \quad p_2'(0) = -\frac{R_{11}''}{R_{11}'' + R_{22}''}\left(p_2^* - p_1^*\right) < 0,$$

with the assumption $p_2^* > p_1^*$.

The derivative of $\mathcal{S}_1$ and $\mathcal{S}_2$ at $\alpha = 0$ are

$$\begin{aligned}
\mathcal{S}_1'(0) &= p_1'(0)\frac{\partial\mathcal{S}_1}{\partial p_1}\Big|_{(p_1^*,p_2^*)} + p_2'(0)\frac{\partial\mathcal{S}_1}{\partial p_2}\Big|_{(p_1^*,p_2^*)} \\
&= \frac{p_2^* - p_1^*}{R_{11}'' + R_{22}''}\frac{g(\lambda_1^*, \lambda_2^*, \mu)}{2\Lambda_1}\left(-R_{22}''\left(1 + \frac{g(\lambda_1^*, \lambda_2^*, \mu))}{\lambda_1^*}\right) + R_{11}''\frac{g(\lambda_1^*, \lambda_2^*, \mu))}{\lambda_1^*}\right), \\
\mathcal{S}_2'(0) &= p_1'(0)\frac{\partial\mathcal{S}_2}{\partial p_1}\Big|_{(p_1^*,p_2^*)} + p_2'(0)\frac{\partial\mathcal{S}_2}{\partial p_2}\Big|_{(p_1^*,p_2^*)} \\
&= \frac{p_2^* - p_1^*}{R_{11}'' + R_{22}''}\frac{g(\lambda_1^*, \lambda_2^*, \mu)}{2\Lambda_2}\left(-R_{22}''\frac{g(\lambda_1^*, \lambda_2^*, \mu))}{\lambda_2^*} + R_{11}''\left(1 + \frac{g(\lambda_1^*, \lambda_2^*, \mu))}{\lambda_2^*}\right)\right).
\end{aligned}$$

The total surplus is $\mathcal{S}'(0) = \Lambda_1\mathcal{S}_1'(0) + \Lambda_2\mathcal{S}_2'(0) = \frac{p_2^* - p_1^*}{R_{11}'' + R_{22}''}\frac{g(\lambda_1^*, \lambda_2^*, \mu)}{2}\left(1 + \frac{g(\lambda_1^*, \lambda_2^*, \mu)}{\lambda_1^*} + \frac{g(\lambda_1^*, \lambda_2^*, \mu)}{\lambda_2^*}\right)(R_{11}'' - R_{22}'')$.

The derivative of revenue with respect to p_i is 0, i.e., $\frac{\partial\mathcal{R}(\alpha)}{\partial p_i}\Big|_{(p_1^*,p_2^*)} = 0$, since the solution is in the interior of a feasible region. This implies that $\mathcal{W}'(0) = \mathcal{S}'(0)$. Then,

$$\mathcal{W}'(0) \geq 0 \iff R_{22}'' - R_{11}'' \geq 0 \iff \left(\sqrt{\beta_2} - \sqrt{\beta_1}\right)\left(\sqrt{\beta_2} + \sqrt{\beta_1} + \frac{\sqrt{\beta_1}}{\mu}\lambda_1^*\right) \geq 0 \iff \beta_2 \geq \beta_1 \iff \frac{b_2}{b_1} \geq \frac{\Lambda_2}{\Lambda_1},$$

where the first implication follows from the fact that $R_{11}'' < 0$, $R_{22}'' < 0$, and $p_2^* > p_1^*$. The second implication arises from the condition $\sqrt{\beta_1}\lambda_1^* = \sqrt{\beta_2}\lambda_2^*$. The condition $R_{22}'' - R_{11}'' \geq 0$ implies $\mathcal{S}_2'(0) > 0$ because

$$\mathcal{S}_2'(0) = \underbrace{\frac{p_2^* - p_1^*}{R_{11}'' + R_{22}''}\frac{g(\lambda_1^*, \lambda_2^*, \mu)}{2\Lambda_2}}_{<0}\left(\underbrace{(R_{11}'' - R_{22}'')}_{\leq 0}\underbrace{\frac{g(\lambda_1^*, \lambda_2^*, \mu)}{\lambda_2^*}}_{\geq 0} + \underbrace{R_{11}''}_{<0}\right) > 0.$$

For $\mathcal{S}_1'(0)$, it is

$$\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\left(-R_{22}''\left(1+\frac{g\left(\lambda_1^*,\lambda_2^*,\mu\right)}{\lambda_1^*}\right)+R_{11}''\frac{g\left(\lambda_1^*,\lambda_2^*,\mu\right)}{\lambda_1^*}\right)$$
$$=\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{2g\left(\lambda_1^*,\lambda_2^*,\mu\right)^2}{\beta_2{\lambda_2^*}^2}\left(1+\sqrt{\frac{\beta_1}{\beta_2}}+\left(\frac{\beta_1}{\beta_2}-1\right)\frac{g\left(\lambda_1^*,\lambda_2^*,\mu\right)}{\lambda_2^*}+\frac{\lambda_2^*}{\mu}\left(1+g\left(\lambda_1^*,\lambda_2^*,\mu\right)\left(\sqrt{\frac{\beta_1}{\beta_2}}-1\right)\right)\right)$$
$$\leq\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{2g\left(\lambda_1^*,\lambda_2^*,\mu\right)^2}{\beta_2{\lambda_2^*}^2}\left(1+\sqrt{\frac{\beta_1}{\beta_2}}+\left(\frac{\beta_1}{\beta_2}-1\right)\frac{g\left(\lambda_1^*,\lambda_2^*,\infty\right)}{\lambda_2^*}+\frac{\lambda_2^*}{\mu}\left(1+g\left(\lambda_1^*,\lambda_2^*,\infty\right)\left(\sqrt{\frac{\beta_1}{\beta_2}}-1\right)\right)\right)$$
$$=\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{2g\left(\lambda_1^*,\lambda_2^*,\mu\right)^2}{\beta_2{\lambda_2^*}^2}\left(2\sqrt{\frac{\beta_1}{\beta_2}}+\frac{\lambda_2^*}{\mu}\frac{2\sqrt{\beta_1}}{\sqrt{\beta_1}+\sqrt{\beta_2}}\right)<0.$$

The first inequality holds because $g\left(\lambda_1^*,\lambda_2^*,\mu\right)=\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*+\frac{\lambda_1^*\lambda_2^*}{\mu}}<\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}=g\left(\lambda_1^*,\lambda_2^*,\infty\right)$, $\frac{\beta_1}{\beta_2}\leq 1$, and $\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}<0$.

Similarly, $\mathcal{W}'(0)<0 \iff R_{22}''-R_{11}''<0 \iff \beta_2<\beta_1 \iff \frac{b_2}{b_1}<\frac{\Lambda_2}{\Lambda_1}$. The condition $R_{22}''-R_{11}''<0$ implies $\mathcal{S}_1'(0)<0$ because

$$\mathcal{S}_1'\left(0\right)=\underbrace{\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{g(\lambda_1^*,\lambda_2^*,\mu)}{2\Lambda_1}}_{<0}\left(\underbrace{(R_{11}''-R_{22}'')}_{\geq 0}\underbrace{\frac{g(\lambda_1^*,\lambda_2^*,\mu)}{\lambda_2^*}}_{\geq 0}+\underbrace{(-R_{22}'')}_{>0}\right)<0.$$

For $\mathcal{S}_2'(0)$, it is

$$\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\left(-R_{22}''\frac{g\left(\lambda_1^*,\lambda_2^*,\mu\right)}{\lambda_2^*}+R_{11}''\left(1+\frac{g\left(\lambda_1^*,\lambda_2^*,\mu\right)}{\lambda_2^*}\right)\right)$$
$$=-\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{2g\left(\lambda_1^*,\lambda_2^*,\mu\right)^2}{\beta_1{\lambda_1^*}^2}\left(1+\sqrt{\frac{\beta_2}{\beta_1}}+\left(\frac{\beta_2}{\beta_1}-1\right)\frac{g\left(\lambda_1^*,\lambda_2^*,\mu\right)}{\lambda_1^*}+\frac{\lambda_1^*}{\mu}\left(1+g\left(\lambda_1^*,\lambda_2^*,\mu\right)\left(\sqrt{\frac{\beta_2}{\beta_1}}-1\right)\right)\right)$$
$$>-\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{2g\left(\lambda_1^*,\lambda_2^*,\mu\right)^2}{\beta_1{\lambda_1^*}^2}\left(1+\sqrt{\frac{\beta_2}{\beta_1}}+\left(\frac{\beta_2}{\beta_1}-1\right)\frac{g\left(\lambda_1^*,\lambda_2^*,\infty\right)}{\lambda_1^*}+\frac{\lambda_1^*}{\mu}\left(1+g\left(\lambda_1^*,\lambda_2^*,\infty\right)\left(\sqrt{\frac{\beta_2}{\beta_1}}-1\right)\right)\right)$$
$$=-\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}\frac{2g\left(\lambda_1^*,\lambda_2^*,\mu\right)^2}{\beta_1{\lambda_1^*}^2}\left(2\sqrt{\frac{\beta_2}{\beta_1}}+\frac{\lambda_1^*}{\mu}\frac{2\sqrt{\beta_2}}{\sqrt{\beta_1}+\sqrt{\beta_2}}\right)>0.$$

The first inequality holds because $g\left(\lambda_1^*,\lambda_2^*,\mu\right)=\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*+\frac{\lambda_1^*\lambda_2^*}{\mu}}<\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}=g\left(\lambda_1^*,\lambda_2^*,\infty\right)$, $\frac{\beta_2}{\beta_1}<1$, and $-\frac{p_2^*-p_1^*}{R_{11}''+R_{22}''}>0$.

Therefore, to satisfy *(b)* $\mathcal{W}'(0)\geq 0$, $\mathcal{S}_1'(0)<0$, $\mathcal{S}_2'(0)\geq 0$, the following conditions should hold

- $p_1^*\neq 0 \iff \sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}\leq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}+\frac{\Lambda_1}{\mu}\sqrt{\frac{b_1}{b_2}}$,
- $p_2^*>p_1^* \iff b_2-b_1>\mu\left(\sqrt{\left(\sqrt{\beta_1}+\sqrt{\beta_2}\right)^2+\frac{b_1+b_2}{\mu}}-\left(\sqrt{\beta_1}+\sqrt{\beta_2}\right)\right)\left(\sqrt{\beta_2}-\sqrt{\beta_1}\right)$,
- Given $p_2^*\geq p_1^*$, $\mathcal{W}'(0)\geq 0 \iff \frac{b_2}{b_1}\geq\frac{\Lambda_2}{\Lambda_1}$.

With the last condition, the second condition can be rewritten as

$$\frac{\Lambda_1}{\mu}\frac{\frac{b_2}{b_1}+\frac{b_1}{b_2}-2}{\frac{\Lambda_1}{\Lambda_2}\sqrt{\frac{b_1}{b_2}}-\frac{b_1}{b_2}\sqrt{\frac{\Lambda_1}{\Lambda_2}}}+2\left(\frac{b_2}{b_1}-1\right)\left(\sqrt{\frac{b_2}{b_1}}+\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right)>\left(\frac{b_2}{b_1}+1\right)\left(\sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right).$$

Similarly, *(c)* $\mathcal{W}'(0)<0$, $\mathcal{S}_1'(0)\leq 0$, $\mathcal{S}_2'(0)\geq 0$ holds

- $p_1^*\neq 0 \iff \sqrt{\frac{b_2}{b_1}}-\sqrt{\frac{b_1}{b_2}}\leq 2\sqrt{\frac{\Lambda_1}{\Lambda_2}}+\frac{\Lambda_1}{\mu}\sqrt{\frac{b_1}{b_2}}$,

- Given $p_2^* \geq p_1^*$, $\mathcal{W}'(0) < 0 \iff \frac{b_2}{b_1} < \frac{\Lambda_2}{\Lambda_1}$.

The condition of $p_2^* > p_1^*$ is not necessary because $\frac{b_2}{b_1} < \frac{\Lambda_2}{\Lambda_1}$ ($\beta_2 < \beta_1$) and $b_2 \geq b_1$ guarantee this condition. More specifically,

$$\underbrace{b_2 - b_1}_{\geq 0} + \underbrace{\mu}_{>0} \underbrace{\left(\sqrt{\left(\sqrt{\beta_1} + \sqrt{\beta_2}\right)^2 + \frac{b_1 + b_2}{\mu}} - \left(\sqrt{\beta_1} + \sqrt{\beta_2}\right)\right)}_{>0} \underbrace{\left(\sqrt{\beta_1} - \sqrt{\beta_2}\right)}_{>0} > 0$$

implies $p_2^* - p_1^* > 0$.

When $p_2^* < p_1^*$, all derivations remain identical to the ones mentioned above, except that we switch the index 1 and 2. In other words, the region similar to *(c)*, the condition $\mathcal{W}'(0) < 0$, $\mathcal{S}_1'(0) > 0$, $\mathcal{S}_2'(0) < 0$ holds when

- $p_2^* \neq 0 \iff \sqrt{\frac{b_1}{b_2}} - \sqrt{\frac{b_2}{b_1}} \leq 2\sqrt{\frac{\Lambda_2}{\Lambda_1}} + \frac{\Lambda_2}{\mu}\sqrt{\frac{b_2}{b_1}}$,
- $p_2^* \leq p_1^* \iff b_2 - b_1 \leq \mu \left(\sqrt{\left(\sqrt{\beta_1} + \sqrt{\beta_2}\right)^2 + \frac{b_1 + b_2}{\mu}} - \left(\sqrt{\beta_1} + \sqrt{\beta_2}\right)\right) \left(\sqrt{\beta_2} - \sqrt{\beta_1}\right)$,
- Given $p_1^* > p_2^*$, $\mathcal{W}'(0) < 0 \iff \frac{b_2}{b_1} > \frac{\Lambda_2}{\Lambda_1}$.

Similarly, with the last condition, the second condition can be rewritten as,

$$\frac{\Lambda_1}{\mu} \frac{\frac{b_2}{b_1} + \frac{b_1}{b_2} - 2}{\frac{\Lambda_1}{\Lambda_2}\sqrt{\frac{b_1}{b_2}} - \frac{b_1}{b_2}\sqrt{\frac{\Lambda_1}{\Lambda_2}}} + 2\left(\frac{b_2}{b_1} - 1\right)\left(\sqrt{\frac{b_2}{b_1}} + \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right) \leq \left(\frac{b_2}{b_1} + 1\right)\left(\sqrt{\frac{b_2}{b_1}} - \sqrt{\frac{\Lambda_2}{\Lambda_1}}\right).$$

The assumption $b_2 \geq b_1$ implies the first condition because it implies $\sqrt{\frac{b_1}{b_2}} - \sqrt{\frac{b_2}{b_1}} \leq 0$. The second condition implies the third condition because of the assumption $b_2 \geq b_1$ and $\sqrt{\left(\sqrt{\beta_1} + \sqrt{\beta_2}\right)^2 + \frac{b_1 + b_2}{\mu}} - \left(\sqrt{\beta_1} + \sqrt{\beta_2}\right) > 0$. Therefore, the second condition is the only condition that this region requires. Additionally, we consider the case $p_1^* = p_2^*$ as well, it enables $\mathcal{S}_1'(0) = 0$, $\mathcal{S}_2'(0) = 0$, and $\mathcal{W}'(0) = 0$. Finally, we establish the final condition *(d)* $\mathcal{W}'(0) \leq 0$, $\mathcal{S}_1'(0) \geq 0$, and $\mathcal{S}_2'(0) \leq 0$.

The region similar to *(b)* does not exist when $p_2^* < p_1^*$ because $p_2^* < p_1^*$ implies $\frac{b_2}{\Lambda_2} > \frac{b_1}{\Lambda_1}$, which always yields $\mathcal{W}'(0) < 0$, not $\mathcal{W}'(0) > 0$. The region similar to *(a)* also does not exist because $b_2 \geq b_1$ contradicts $\sqrt{\frac{b_1}{b_2}} - \sqrt{\frac{b_2}{b_1}} > 2\sqrt{\frac{\Lambda_2}{\Lambda_1}} + \frac{\Lambda_2}{\mu}\sqrt{\frac{b_2}{b_1}}$. □

Proof of Proposition 7. The optimization problem under the constraint of α-access fairness with 2 nodes and 1 unit is formulated as follows. We set demand rates as decision variables (λ_1, λ_2).

$$\begin{aligned} \mathcal{R}(\alpha) := \max_{\lambda_1, \lambda_2} \quad & g(\lambda_1, \lambda_2, \mu)\left(b_1\left(1 - \frac{\lambda_1}{\Lambda_1}\right) + b_2\left(1 - \frac{\lambda_2}{\Lambda_2}\right)\right) \\ \text{s.t.} \quad & g(\lambda_1, \lambda_2, \mu) \leq (1 - \alpha)\, g(\lambda_1^*, \lambda_2^*, \mu), \\ & \lambda_1 \in [0, \Lambda_1], \\ & \lambda_2 \in [0, \Lambda_2], \end{aligned} \tag{19}$$

where $g(\lambda_1, \lambda_2, \mu) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 + \frac{\lambda_1 \lambda_2}{\mu}}$ with $\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2}$. $(\lambda_1^*, \lambda_2^*)$ represents the optimal solution of problem (19) when $\alpha = 0$, i.e. the absence of fairness criteria.

The access fairness can be written as $g(\lambda_1, \lambda_2, \mu) \leq (1 - \alpha)\, g(\lambda_1^*, \lambda_2^*, \mu)$ because

$$\begin{aligned} & |q_1 \Pi_1 - q_2 \Pi_2| \leq (1 - \alpha)\, |q_1^* \Pi_1(p_1^*, p_2^*) - q_2^* \Pi_2(p_1^*, p_2^*)| \\ \iff & \left| q_1 \frac{g(\lambda_1, \lambda_2, \mu)}{\lambda_1} - q_2 \frac{g(\lambda_1, \lambda_2, \mu)}{\lambda_2} \right| \leq (1 - \alpha) \left| q_1^* \frac{g(\lambda_1^*, \lambda_2^*, \mu)}{\lambda_1^*} - q_2^* \frac{g(\lambda_1^*, \lambda_2^*, \mu)}{\lambda_2^*} \right| \\ \iff & \left| \frac{1}{\Lambda_1} - \frac{1}{\Lambda_2} \right| g(\lambda_1, \lambda_2, \mu) \leq (1 - \alpha) \left| \frac{1}{\Lambda_1} - \frac{1}{\Lambda_2} \right| g(\lambda_1^*, \lambda_2^*, \mu) \\ \iff & g(\lambda_1, \lambda_2, \mu) \leq (1 - \alpha) g(\lambda_1^*, \lambda_2^*, \mu). \end{aligned}$$

The KKT conditions for the problem (19) are given by:

$$-\frac{\partial g}{\partial \lambda_1}\left(b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)+b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)-g(\lambda_1,\lambda_2,\mu)\frac{d}{d\lambda_1}\left(b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\right)+\nu_1+\nu_3\frac{\partial g}{\partial \lambda_1}=0, \tag{20a}$$

$$-\frac{\partial g}{\partial \lambda_2}\left(b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)+b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)-g(\lambda_1,\lambda_2,\mu)\frac{d}{d\lambda_2}\left(b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)+\nu_2+\nu_3\frac{\partial g}{\partial \lambda_2}=0, \tag{20b}$$

$$g(\lambda_1,\lambda_2,\mu_1,\mu_2)\le(1-\alpha)g(\lambda_1^*,\lambda_2^*,\mu), \tag{20c}$$

$$\lambda_1\in[0,\Lambda_1],\ \lambda_2\in[0,\Lambda_2],$$

$$\nu_1,\nu_2,\nu_3\ge 0,$$

$$\nu_1(\lambda_1-\Lambda_1)=0,\ \nu_2(\lambda_2-\Lambda_2)=0,\ \nu_3\left(g\left(\lambda_1,\lambda_2,\mu_1,\mu_2\right)-(1-\alpha)\,g\left(\lambda_1^*,\lambda_2^*,\mu\right)\right)=0. \tag{20d}$$

We do not consider the dual variable of $\lambda_1\ge 0$ and $\lambda_2\ge 0$ because either of λ_1 or λ_2 cannot be 0 since it incurs 0 revenue which is not an optimal solution.

If $\nu_3=0$, the stationary conditions (20a and 20b) take the same forms as those without fair constraints. This implies that $(\lambda_1(\alpha),\lambda_2(\alpha))=(\lambda_1^*,\lambda_2^*)$, which violates the feasibility condition (20c). Therefore, $\nu_3\neq 0$, and $g\left(\lambda_1(\alpha),\lambda_2(\alpha),\mu\right)=(1-\alpha)\,g\left(\lambda_1^*,\lambda_2^*,\mu\right)$ by complementary slackness (20d).

(a) $p_1^*>0$ and $p_2^*>0$: Since $p_1>0$ and $p_2>0$, $\lambda_1\neq\Lambda_1$ and $\lambda_2\neq\Lambda_2$. This implies $\nu_1=\nu_2=0$ by complementary slackness (20d). If we suppose $\lambda_1(\alpha)\neq\Lambda_1$ and $\lambda_2(\alpha)\neq\Lambda_2$, then the stationary conditions (16a and 16b) in KKT conditions become,

$$\frac{\frac{\partial g(\lambda_1,\lambda_2)}{\partial\lambda_2}\Big|_{(\lambda_1(\alpha),\lambda_2(\alpha))}}{\frac{\partial g(\lambda_1,\lambda_2,\mu)}{\partial\lambda_1}\Big|_{(\lambda_1(\alpha),\lambda_2(\alpha))}}=\frac{\frac{d}{d\lambda_2}\left(b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)\Big|_{\lambda_2(\alpha)}}{\frac{d}{d\lambda_1}\left(b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\right)\Big|_{\lambda_1(\alpha)}}\Leftrightarrow \beta_1\lambda_1(\alpha)^2=\beta_2\lambda_2(\alpha)^2$$

where $\beta_i=\frac{b_i}{\Lambda_i}$. Since $\nu_3\neq 0$, by complementary slackness (20d), $g\left(\lambda_1(\alpha),\lambda_2(\alpha),\mu\right)=(1-\alpha)\,g\left(\lambda_1^*,\lambda_2^*,\mu\right)$. If we plug $\lambda_2(\alpha)=\sqrt{\frac{\beta_1}{\beta_2}}\lambda_1(\alpha)$ and $\lambda_2^*=\sqrt{\frac{\beta_1}{\beta_2}}\lambda_1^*$ in the aforementioned equation, we get

$$(\lambda_1(\alpha),\lambda_2(\alpha))=\left((1-\alpha)\,\frac{\left(\sqrt{\beta_1}+\sqrt{\beta_2}\right)\lambda_1^*}{\sqrt{\beta_1}+\sqrt{\beta_2}+\alpha\sqrt{\beta_1}\frac{\lambda_1^*}{\mu}},(1-\alpha)\,\frac{\left(\sqrt{\beta_1}+\sqrt{\beta_2}\right)\lambda_2^*}{\sqrt{\beta_1}+\sqrt{\beta_2}+\alpha\sqrt{\beta_2}\frac{\lambda_2^*}{\mu}}\right).$$

The access at each node $i\in\{1,2\}$ is given by

$$\frac{1}{\Lambda_i}g\left(\lambda_1(\alpha),\lambda_2(\alpha),\mu\right)=\frac{1}{\Lambda_i}(1-\alpha)g\left(\lambda_1^*,\lambda_2^*,\mu\right),$$

which is decreasing with respect to α.

The consumer surplus at each node $i\in\{1,2\}$ is

$$\begin{aligned}\mathcal{S}_i(\alpha)&=\frac{1}{2}\frac{b_i}{\Lambda_i^2}\lambda_i(\alpha)g\left(\lambda_1\left(\alpha\right),\lambda_2\left(\alpha\right),\mu\right)\\&=(1-\alpha)^2\frac{1}{2}\frac{b_i}{\Lambda_i^2}\lambda_i^*g\left(\lambda_1^*,\lambda_2^*,\mu\right)\frac{\sqrt{\beta_1}+\sqrt{\beta_2}}{\sqrt{\beta_1}+\sqrt{\beta_2}+\alpha\sqrt{\beta_i}\frac{\lambda_i^*}{\mu}}\\&=(1-\alpha)^2\frac{\sqrt{\beta_1}+\sqrt{\beta_2}}{\sqrt{\beta_1}+\sqrt{\beta_2}+\alpha\sqrt{\beta_i}\frac{\lambda_i^*}{\mu}}\mathcal{S}_i(0).\end{aligned}$$

This is a decreasing function with respect to α because

$$\frac{d}{d\alpha}\left(\frac{\sqrt{\beta_1}+\sqrt{\beta_2}}{\sqrt{\beta_1}+\sqrt{\beta_2}+\alpha\sqrt{\beta_i}\frac{\lambda_i^*}{\mu}}\right)=-\frac{\left(\sqrt{\beta_1}+\sqrt{\beta_2}\right)\sqrt{\beta_i}\frac{\lambda_i^*}{\mu}}{\left(\sqrt{\beta_1}+\sqrt{\beta_2}+\alpha\sqrt{\beta_i}\frac{\lambda_i^*}{\mu}\right)^2}<0 \tag{21}$$

implies $\frac{d}{d\alpha}\mathcal{S}_i(\alpha) < 0$.

To show that $\lambda_i(\alpha) \neq \Lambda_i$, we proceed by contradiction. Assume, without loss of generality, that there exists an α such that $\lambda_1(\alpha) = \Lambda_1$. Given that $p_1^* > 0$, it follows that $\lambda_1^* < \Lambda_1$. Therefore, there exists some $\bar{\alpha} \in [0,1]$ such that $\lambda_1(\alpha) < \Lambda_1$ for $\alpha \in [0, \bar{\alpha})$ and $\lambda_1(\bar{\alpha}) = \Lambda_1$. However, since (21) implies that $\lambda_1(\alpha)$ is a strictly decreasing function with respect to α, it is impossible for $\lambda_1(\alpha)$ to reach Λ_1. This contradiction implies that $\lambda_1(\alpha) \neq \Lambda_1$.

(b) Without loss of generality $p_1^* = 0$: Since $\lambda_1^* = \Lambda_1$, ν_1 does not have to be 0 by complementary slackness (20d). Let $\alpha^\star$ denote the point where $\nu_1 = 0$ and $\lambda_1 = \Lambda_1$ hold simultaneously. $\alpha^\star$ is derived by solving the following equations:

$$\frac{\frac{\partial g(\lambda_1,\lambda_2)}{\partial \lambda_2}\big|_{(\Lambda_1,\lambda_2(\alpha^\star))}}{\frac{\partial g(\lambda_1,\lambda_2)}{\partial \lambda_1}\big|_{(\Lambda_1,\lambda_2(\alpha^\star))}} = \frac{\frac{d}{d\lambda_2}\left(b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right)\Big|_{\lambda_2(\alpha^\star)}}{\frac{d}{d\lambda_1}\left(b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\right)\Big|_{\Lambda_1}}, \quad g\left(\Lambda_1, \lambda_2(\alpha^\star)\right) = (1-\alpha^\star) g(\Lambda_1, \lambda_2^*).$$

Under $\alpha \leq \alpha^\star$, the dynamic of systems behaves different to that of *(a)*. On the other hand, under $\alpha > \alpha^\star$, the dynamic is the same as *(a)*.

(i) Under $\alpha \leq \alpha^\star$, with $\lambda_1(\alpha) = \Lambda_1 = \lambda_1^*$, the normalized surplus at node 1 is decreasing in α because

$$\mathcal{S}_1(\alpha) = \frac{1}{2}\frac{b_1}{\Lambda_1}\lambda_1(\alpha) g\left(\lambda_1(\alpha), \lambda_2(\alpha)\right) = (1-\alpha)\frac{1}{2}\frac{b_1}{\Lambda_1}\lambda_1^* g\left(\lambda_1^*, \lambda_2^*\right) = (1-\alpha)\mathcal{S}_1(0),$$

where the second equality is from $g(\lambda_1(\alpha), \lambda_2(\alpha)) = (1-\alpha) g(\lambda_1^*, \lambda_2^*)$, as in *(a)*.

For node 2, $\mathcal{S}_2(\alpha) = (1-\alpha)\frac{1}{2}\frac{b_2}{\Lambda_2}\lambda_2(\alpha) g\left(\Lambda_1, \lambda_2^*\right) = (1-\alpha)\frac{\lambda_2(\alpha)}{\lambda_2^*}\mathcal{S}_2(0)$, which is a decreasing function of α. This is because

$$\frac{d\mathcal{S}_2(\alpha)}{d\alpha} = -\frac{1}{2}\frac{b_2}{\Lambda_2}\lambda_2(\alpha) g\left(\Lambda_1, \lambda_2^*\right) + (1-\alpha)\frac{1}{2}\frac{b_2}{\Lambda_2}\frac{d\lambda_2(\alpha)}{d\alpha} g\left(\Lambda_1, \lambda_2^*\right) < 0.$$

In the above equation, the gradient of $\lambda_2(\alpha)$ with α is negative, i.e., $\frac{d\lambda_2(\alpha)}{d\alpha} < 0$. Since $\lambda_2(\alpha)$ satisfies $g\left(\Lambda_1, \lambda_2(\alpha)\right) = (1-\alpha) g(\Lambda_1, \lambda_2^*)$, its derivative with respect to α satisfies

$$\frac{dg\left(\Lambda_1, \lambda_2(\alpha)\right)}{d\alpha} = \frac{d\lambda_2(\alpha)}{d\alpha}\frac{\partial g\left(\Lambda_1, \lambda_2(\alpha)\right)}{\partial \lambda_2(\alpha)} = -g(\Lambda_1, \lambda_2^*) < 0.$$

Therefore, $\frac{d\lambda_2(\alpha)}{d\alpha} < 0$ because of $\frac{\partial g(\Lambda_1, \lambda_2(\alpha))}{\partial \lambda_2(\alpha)} = \frac{\Lambda_1^2 \mu^2}{(\lambda_2(\alpha)\mu + \Lambda_1(\lambda_2(\alpha)+\mu))^2} > 0$.

(ii) Under $\alpha > \alpha^\star$, since prices at both nodes have positive values under $\alpha > \alpha^\star$, we can view the point $\alpha^\star$ as the point where $\alpha = 0$ in *(a)*. Therefore, $\mathcal{S}_i(\alpha)$ is decreasing in α.

Proof of Lemma 3. The optimization problem with 2 nodes is formulated as follows:

$$\begin{aligned}
\max_{\lambda_1,\lambda_2,\psi_1,\psi_2} \quad & \left(\lambda_1\left(b_1 - \frac{b_1}{\Lambda_1}\lambda_1\right) - c_1\psi_1\right)(1-\pi(0,K)) + \left(\lambda_2\left(b_2 - \frac{b_2}{\Lambda_2}\lambda_2\right) - c_2\psi_2\right)(1-\pi(K,0)) \\
\text{s.t.} \quad & (\lambda_1+\psi_1)\,\pi(k, K-k) = (\lambda_2+\psi_2)\,\pi(k-1, K-k+1), \quad \forall k \in \{1,2,\ldots,K\}, \\
& \sum_{k=0}^{K} \pi(k, K-k) = 1, \\
& \lambda_1 \in [0, \Lambda_1], \quad \lambda_2 \in [0, \Lambda_2], \quad \psi_1 \geq 0, \quad \psi_2 \geq 0.
\end{aligned}$$

Let the optimal solution of the above problem be denoted as $(\lambda_1^*, \lambda_2^*, \psi_1^*, \psi_2^*)$, with $\psi_1^* > 0$ and $\psi_2^* > 0$. Assume the corresponding steady-state probability is $\pi(k, K-k)^*$ for all $k \in \{0,1,\ldots,K\}$. Without loss of generality, consider $\lambda_2^*\psi_1^* > \lambda_1^*\psi_2^*$. Define $\psi_1^\star = \frac{\lambda_2^*\psi_1^* - \lambda_1^*\psi_2^*}{\lambda_2^* + \psi_2^*} > 0$ and $\psi_2^\star = 0$. $(\lambda_1^*, \lambda_2^*, \psi_1^\star, \psi_2^\star)$ constitutes a feasible solution of problem, maintaining the same steady-state probability $\pi(k, K-k)^*$. The objective value, however, is larger because

$$\begin{aligned}
&\left(\lambda_1^*\left(b_1 - \frac{b_1}{\Lambda_1}\lambda_1^*\right) - c_1\psi_1^\star\right)(1-\pi(0,K)^*) + \left(\lambda_2^*\left(b_2 - \frac{b_2}{\Lambda_2}\lambda_2^*\right) - c_2\psi_2^\star\right)(1-\pi(K,0)^*)\\
&- \left(\lambda_1^*\left(b_1 - \frac{b_1}{\Lambda_1}\lambda_1^*\right) - c_1\psi_1^*\right)(1-\pi(0,K)^*) + \left(\lambda_2^*\left(b_2 - \frac{b_2}{\Lambda_2}\lambda_2^*\right) - c_2\psi_2^*\right)(1-\pi(K,0)^*)\\
&= c_1\psi_1^*(1-\pi(0,K)^*) + c_2\psi_2^*(1-\pi(K,0)^*) - c_1\psi_1^\star(1-\pi(0,K)^*) - c_2\psi_2^\star(1-\pi(K,0)^*)\\
&= c_1\frac{\psi_1^*\psi_2^* + \lambda_1^*\psi_2^*}{\lambda_2^* + \psi_2^*}(1-\pi(0,K)^*) + c_2\psi_2^*(1-\pi(K,0)^*) > 0.
\end{aligned}$$

Thus, either ψ_1^* or ψ_2^* must be zero. If c_1 and c_2 are sufficiently large, for instance, infinite, then both ψ_1^* and ψ_2^* are 0. □

Proof of Lemma 4 The optimization problem is formulated as follows:

$$\begin{aligned}
\max_{\lambda_1,\lambda_2,\psi_1,\psi_2}\quad & \lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\Pi_1 + \lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\Pi_2 - c_1\psi_1\Pi_1 - c_2\psi_2\Pi_2\\
\text{s.t.}\quad & (\lambda_1+\psi_1)\Pi_1 = (\lambda_2+\psi_2)\Pi_2, && \text{(22a)}\\
& \Pi_1 + \Pi_2 = 1, && \text{(22b)}\\
& \Pi_1, \Pi_2 \in [0,1], \quad \lambda_1 \in [0,\Lambda_1], \quad \lambda_2 \in [0,\Lambda_2], \quad \psi_1,\psi_2 \geq 0.
\end{aligned}$$

Based on Lemma 3, without loss of generality, let $\psi_2^* = 0$. Then, from (22a) and (22b), we obtain $(\Pi_1, \Pi_2) = \left(\frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2}, \frac{\lambda_1+\psi_1}{\lambda_1+\psi_1+\lambda_2}\right)$. Thus, the optimization problem can be rewritten as

$$\begin{aligned}
\max_{\lambda_1,\lambda_2,\psi_1}\quad & \lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2} + \lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\frac{\lambda_1+\psi_1}{\lambda_1+\psi_1+\lambda_2} - c_1\psi_1\frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2} && \text{(23)}\\
\text{s.t.}\quad & \lambda_1\in[0,\Lambda_1], \quad \lambda_2\in[0,\Lambda_2], \quad \psi_1 \geq 0.
\end{aligned}$$

For simplicity, let us denote $D = \lambda_1+\lambda_2+\psi_1$ and $N = \lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\lambda_2 + \lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)(\lambda_1+\psi_1) - c_1\psi_1\lambda_2$. The KKT conditions for problem (23) are as follows:

$$\begin{aligned}
\frac{1}{D^2}\left[D\left(b_1\left(1-\frac{2\lambda_1}{\Lambda_1}\right)\lambda_2 + \lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\right) - N\right] - \nu_1 + \nu_2 = 0, && \text{(24a)}\\
\frac{1}{D^2}\left[D\left(\lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right) + b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)(\lambda_1+\psi_1) - \frac{\lambda_2 b_2}{\Lambda_2}(\lambda_1+\psi_1) - c_1\psi_1\right) - N\right] - \nu_3 + \nu_4 = 0, && \text{(24b)}\\
\frac{1}{D^2}\left[D\left(\lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right) - c_1\lambda_2\right) - N\right] - \nu_5 = 0, && \text{(24c)}\\
\lambda_1\in[0,\Lambda_1], \lambda_2\in[0,\Lambda_2], \psi_1 \geq 0, && \text{(24d)}\\
\nu_1,\nu_2,\nu_3,\nu_4,\nu_5 \geq 0,\\
\nu_1\lambda_1 = 0, \nu_2(\lambda_1-\Lambda_1) = 0, \nu_3\lambda_2 = 0, \nu_4(\lambda_2-\Lambda_2) = 0, \nu_5\psi_1 = 0. && \text{(24e)}
\end{aligned}$$

First, when $\psi_1^* = 0$, the optimal solution is the same as the scenario without repositioning. When $\psi_1^* > 0$, by the complementary slackness (24e), we have $\nu_5 = 0$. Then, from (24c), at least one of the following conditions must be satisfied:

$$\left(\lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right) - \lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right) - c_1(\lambda_1+\lambda_2)\right)\lambda_2 = 0 \quad \text{or} \quad D = \infty.$$

Equivalently, these conditions can be expressed as:

$$\lambda_2 b_2 \left(1 - \frac{\lambda_2}{\Lambda_2}\right) - \lambda_1 b_1 \left(1 - \frac{\lambda_1}{\Lambda_1}\right) = c_1(\lambda_1 + \lambda_2), \tag{25a}$$

$$\lambda_2 = 0, \quad \text{or} \tag{25b}$$

$$\lambda_1 + \lambda_2 + \psi_1 = \infty. \tag{25c}$$

First, suppose that only (25a) holds, i.e., $\lambda_1 + \lambda_2 + \psi_1 < \infty$ and $\lambda_2 > 0$.

- If $\lambda_1 \in (0, \Lambda_1)$, then by the complementary slackness conditions in (24e), we have $\nu_1 = 0$ and $\nu_2 = 0$. Under these conditions, the system of equations derived from (24a) and (24c) becomes:

$$b_1 \left(1 - \frac{2\lambda_1}{\Lambda_1}\right) \lambda_2 + \lambda_2 b_2 \left(1 - \frac{\lambda_2}{\Lambda_2}\right) = \frac{N}{D},$$
$$\lambda_2 b_2 \left(1 - \frac{\lambda_2}{\Lambda_2}\right) - c_1 \lambda_2 = \frac{N}{D}.$$

  Solving this system yields the optimal solution $\lambda_1^* = \Lambda_1 \frac{b_1 + c_1}{2b_1}$. Substituting this expression into (25a) leads to

$$\lambda_2^* = \Lambda_2 \frac{b_2 - c_1}{2b_2} \pm \frac{1}{2b_2} \sqrt{\frac{\Lambda_2 \left(\Lambda_2 b_1 (b_2 - c_1)^2 - \Lambda_1 b_2 (b_1 + c_1)^2\right)}{b_1}}.$$

  In addition, we can verify that $\lambda_2^* \neq 0$ and $\lambda_2^* \neq \Lambda_2$ when $\lambda_1^* = \Lambda_1 \cdot \frac{b_1 + c_1}{2b_1}$, based on the condition in (25a). This implies $\nu_3 = \nu_4 = 0$ by the complementary slackness conditions. Substituting $(\lambda_1^*, \lambda_2^*)$ into (24b) yields $\psi_1^* = -\Lambda_1 \frac{b_1 + c_1}{2b_1} < 0$, which contradicts the primal feasibility condition $\psi_1^* \geq 0$ (24d). Hence, the condition in (25a) cannot hold.

- If $\lambda_1^* = 0$, then by (25a), we have $\lambda_2^* = \Lambda_2 \frac{b_2 - c_1}{b_2}$. If $\lambda_2^* = 0$, then the objective value is zero. Therefore, we assume $b_2 > c_1$, which implies $\lambda_2^* \in (0, \Lambda_2)$ and hence $\nu_3 = \nu_4 = 0$ by complementary slackness. Substituting into (24b), we obtain $\psi_1^* = -\frac{\Lambda_1 (b_1 + c_1)^2}{4b_1 (b_2 - c_1)}$, which is strictly negative. This violates the primal feasibility condition $\psi_1^* \geq 0$ in (24d), and therefore this case cannot be part of the optimal solution.

- If $\lambda_1^* = \Lambda_1$, then (25a) becomes

$$\lambda_2 b_2 \left(1 - \frac{\lambda_2}{\Lambda_2}\right) - c_1 \lambda_2 = c_1 \Lambda_1.$$

  It is evident that $\lambda_2^* \neq 0$ and $\lambda_2^* \neq \Lambda_2$, which implies $\nu_3 = \nu_4 = 0$ by complementary slackness. The corresponding closed-form solution is given by $\lambda_2^* = \Lambda_2 \frac{b_2 - c_1}{2b_2} \pm \frac{1}{2b_2} \sqrt{\frac{\Lambda_2 (\Lambda_2 (b_2 - c_1)^2 - 4\Lambda_1 b_2 c_1)}{b_1}}$. Then, from (24b), we obtain $\psi_1^* = -\Lambda_1 \pm \frac{2\Lambda_1 \Lambda_2 c_1}{\sqrt{\Lambda_2^2 (b_2 - c_1)^2 - 4\Lambda_1 \Lambda_2 b_2 c_1}}$. This expression yields an objective value of $c_1 \Lambda_1$. To ensure that both λ_2^* and ψ_1^* are real-valued, the following condition must be satisfied:

$$\frac{\Lambda_2 (b_2 - c_1)^2}{4b_2} > c_1 \Lambda_1. \tag{26}$$

  We will revisit this condition later to demonstrate that the corresponding solution yields a suboptimal objective value.

Next, if $\lambda_2 = 0$ (25b), then the objective value is 0, which is strictly less than the objective value in the no-repositioning case. Therefore, this case cannot be optimal.

Lastly, if $\lambda_1 + \lambda_2 + \psi_1 = \infty$ as in (25c), then $\psi_1 = \infty$ because each λ_i is bounded above by Λ_i for any $i \in \{1, 2\}$. In this case, the objective simplifies to $\lambda_2 b_2 \left(1 - \frac{\lambda_2}{\Lambda_2}\right) - c_1 \lambda_2$. Under the condition $b_2 \geq c_1$, the optimal value is achieved at $\lambda_2^* = \Lambda_2 \cdot \frac{b_2 - c_1}{2b_2}$ (with any feasible λ_1), yielding an objective value of $\frac{\Lambda_2(b_2-c_1)^2}{4b_2}$. Otherwise, if $b_2 < c_1$, we have $\lambda_2^* = 0$.

Comparing the objective value $\frac{\Lambda_2(b_2-c_1)^2}{4b_2}$ with the value under the alternative solution $\lambda_1^* = \Lambda_1$, which yields $c_1\Lambda_1$, we find that $\frac{\Lambda_2(b_2-c_1)^2}{4b_2} > c_1\Lambda_1$ due to condition (26).

Finally, to derive the condition under which repositioning occurs, we compare the optimal objective value with repositioning to that without repositioning. When $\psi_1^* = 0$, Lemma 1 implies that the objective value is ${(b_1+b_2)^2}/{\left(\sqrt{\frac{b_1}{\Lambda_1}}+\sqrt{\frac{b_2}{\Lambda_2}}\right)^2}$. Repositioning is therefore beneficial when

$$\frac{\Lambda_2(b_2-c_1)^2}{4b_2} > \frac{(b_1+b_2)^2}{\left(\sqrt{\frac{b_1}{\Lambda_1}}+\sqrt{\frac{b_2}{\Lambda_2}}\right)^2} \Leftrightarrow \sqrt{\frac{\Lambda_2}{b_2}}(b_2-c_1) > \frac{2(b_1+b_2)}{\sqrt{\frac{b_1}{\Lambda_1}}+\sqrt{\frac{b_2}{\Lambda_2}}}.$$

We note that this condition implies $b_2 > c_1$. □

Proof of Proposition 8. We begin by formulating the optimization problem as follows:

$$\begin{aligned} \max_{\lambda_1,\lambda_2,\psi_1} \quad & \lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2} + \lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\frac{\lambda_1+\psi_1}{\lambda_1+\psi_1+\lambda_2} - c_1\psi_1\frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2} \\ \text{s.t.} \quad & \frac{\lambda_1}{\Lambda_1}\frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2} - \frac{\lambda_2}{\Lambda_2}\frac{\lambda_1+\psi_1}{\lambda_1+\psi_1+\lambda_2} \le (1-\alpha)\left(\frac{1}{\Lambda_1}-\frac{1}{\Lambda_2}\right)\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}, \\ & \psi_1 \ge 0, \quad \lambda_1 \in [0,\Lambda_1], \quad \lambda_2 \in [0,\Lambda_2]. \end{aligned} \tag{27}$$

This problem can be reformulated as follows by substituting $\psi_1 = \frac{\lambda_1+\lambda_2}{\lambda_2 - r_1} r_1$:

$$\begin{aligned} \max_{\lambda_1,\lambda_2,r_1} \quad & \lambda_1 b_1\left(1-\frac{\lambda_1}{\Lambda_1}\right)\frac{\lambda_2-r_1}{\lambda_1+\lambda_2} + \lambda_2 b_2\left(1-\frac{\lambda_2}{\Lambda_2}\right)\frac{\lambda_1+r_1}{\lambda_1+\lambda_2} - c_1 r_1 \\ \text{s.t.} \quad & \frac{\lambda_1}{\Lambda_1}\frac{\lambda_2-r_1}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\Lambda_2}\frac{\lambda_1+r_1}{\lambda_1+\lambda_2} \le (1-\alpha)\left(\frac{1}{\Lambda_1}-\frac{1}{\Lambda_2}\right)\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}, \\ & 0 \le r_1 \le \lambda_2, \quad \lambda_1 \in [0,\Lambda_1], \quad \lambda_2 \in [0,\Lambda_2]. \end{aligned} \tag{28}$$

If we substitute $r_1 = \psi_1 \frac{\lambda_2}{\lambda_1+\psi_1+\lambda_2}$ into (28), we can recover (27). Then, the KKT conditions are,

$$\frac{\lambda_2-r_1}{(\lambda_1+\lambda_2)^2}\left(\lambda_2\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right)-\lambda_1\frac{b_1}{\Lambda_1}(\lambda_1+\lambda_2)+\nu_0\lambda_2\left(\frac{1}{\Lambda_1}-\frac{1}{\Lambda_2}\right)\right)+\nu_1-\nu_2=0, \tag{29a}$$

$$\frac{\lambda_1+r_1}{(\lambda_1+\lambda_2)^2}\left(\lambda_1\left(b_1+b_2-\frac{b_1}{\Lambda_1}\lambda_1-\frac{b_2}{\Lambda_2}\lambda_2\right)-\lambda_2\frac{b_2}{\Lambda_2}(\lambda_1+\lambda_2)+\nu_0\lambda_1\left(\frac{1}{\Lambda_1}-\frac{1}{\Lambda_2}\right)\right)+\nu_3-\nu_4+\nu_6=0, \tag{29b}$$

$$\frac{1}{\lambda_1+\lambda_2}\left(-\lambda_1\left(b_1-\frac{b_1}{\Lambda_1}\lambda_1\right)+\lambda_2\left(b_2-\frac{b_2}{\Lambda_2}\lambda_2\right)-c_1(\lambda_1+\lambda_2)-\nu_0\left(\frac{\lambda_1}{\Lambda_1}+\frac{\lambda_2}{\Lambda_2}\right)\right)+\nu_5-\nu_6=0, \tag{29c}$$

$$\begin{aligned} \frac{\lambda_1}{\Lambda_1}\frac{\lambda_2-r_1}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\Lambda_2}\frac{\lambda_1+r_1}{\lambda_1+\lambda_2} &\le (1-\alpha)\left(\frac{1}{\Lambda_1}-\frac{1}{\Lambda_2}\right)\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}, \\ 0 \le r_1 \le \lambda_2, \quad \lambda_1 &\in [0,\Lambda_1], \quad \lambda_2 \in [0,\Lambda_2], \\ \nu_0\left(\frac{\lambda_1}{\Lambda_1}\frac{\lambda_2-r_1}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\Lambda_2}\frac{\lambda_1+r_1}{\lambda_1+\lambda_2} - (1-\alpha)\left(\frac{1}{\Lambda_1}-\frac{1}{\Lambda_2}\right)\frac{\lambda_1^*\lambda_2^*}{\lambda_1^*+\lambda_2^*}\right) &= 0, \\ \nu_1\lambda_1 = 0, (\Lambda_1-\lambda_1)\nu_2 = 0, \nu_3\lambda_2 = 0, \nu_4(\Lambda_2-\lambda_2) = 0, \nu_5 r_1 = 0, \nu_6(\lambda_1 - r_1) &= 0, \\ \nu_0,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6 &\ge 0. \end{aligned}$$

(*a*) Assuming $\alpha \in [0,\alpha^\star]$, we set $r_1 = 0$ (hence, $\nu_6 = 0$), which aligns the dynamics with the access fairness condition in Proposition 3. By the complementary slackness conditions, we have $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 0$, since $\lambda_1(\alpha) = (1-\alpha)\lambda_1^*$ and $\lambda_2(\alpha) = (1-\alpha)\lambda_2^*$. Note that at the threshold point $\alpha^\star$, r_1 becomes nonzero, so $\alpha^\star$

is the smallest value of α such that $\nu_5 = 0$. Substituting $\lambda_1(\alpha) = (1-\alpha)\lambda_1^*$, $\lambda_2(\alpha) = (1-\alpha)\lambda_2^*$, $r_1 = 0$, and $\nu_5 = 0$ into the stationarity conditions (29b) and (29c), we obtain:

$$\alpha^\star = \frac{\Lambda_2 - \Lambda_1}{b_1 + b_2} \cdot \frac{1}{\sqrt{b_2\Lambda_2} + \sqrt{b_1\Lambda_1}} \left((b_1 + c_1)\sqrt{\frac{b_2}{\Lambda_2}} + (c_1 - b_2)\sqrt{\frac{b_1}{\Lambda_1}} \right).$$

Note that the closed-form solution for λ_i^* is $\lambda_i^* = {(b_1+b_2)}/{2}\left(\frac{b_i}{\Lambda_i} + \sqrt{\frac{b_1 b_2}{\Lambda_1\Lambda_2}}\right)$, for all $i \in \{1,2\}$, as stated in Lemma 1.

Under $\alpha \in (\alpha^\star, 1]$, let us assume that $\lambda_1(\alpha) = (1-\alpha^\star)\lambda_1^*$ and $\lambda_2(\alpha) = (1-\alpha^\star)\lambda_2^*$. These λ_1 and λ_2 satisfy the stationarity conditions (29a)–(29c) when $\nu_6 = 0$. Then, we have

$$r_1(\alpha) = (\alpha - \alpha^\star)\frac{\lambda_1^*\lambda_2^*}{\frac{\lambda_1^*}{\Lambda_1} + \frac{\lambda_2^*}{\Lambda_2}} = (\alpha - \alpha^\star)\frac{1}{2}\frac{b_1 + b_2}{\sqrt{\frac{b_1}{\Lambda_1}} + \sqrt{\frac{b_2}{\Lambda_2}}}\left(\frac{1}{\sqrt{b_1\Lambda_1}} + \frac{1}{\sqrt{b_2\Lambda_2}}\right)^{-1}.$$

Therefore, $\Pi_1(\alpha)$ decreases and $\Pi_2(\alpha)$ increases with a stationary $\lambda_1(\alpha) = (1-\alpha^\star)\lambda_1^*$ and $\lambda_2(\alpha) = (1-\alpha^\star)\lambda_2^*$. This leads to decreases in $\mathcal{A}_1(\alpha)$ and $\mathcal{S}_1(\alpha)$, whereas increases in $\mathcal{A}_2(\alpha)$ and $\mathcal{S}_2(\alpha)$ since $\mathcal{A}_1(\alpha) = \frac{\lambda_1(\alpha^\star)}{\Lambda_1}\frac{\lambda_2(\alpha^\star) - r_1(\alpha)}{\lambda_1(\alpha^\star)+\lambda_2(\alpha^\star)}$, $\mathcal{A}_2(\alpha) = \frac{\lambda_2(\alpha^\star)}{\Lambda_2}\frac{\lambda_1(\alpha^\star) + r_1(\alpha)}{\lambda_1(\alpha^\star)+\lambda_2(\alpha^\star)}$, and $\mathcal{S}_i(\alpha) = \frac{b_i}{2\Lambda_i}\lambda_i(\alpha^\star)\mathcal{A}_i(\alpha)$.

(*b*) With two nodes, if $\psi_1^* > 0$, then $\Pi_1^* = 0$. Consequently, there exists a threshold $\alpha^\star$ such that $\Pi_1(\alpha) = 0$ for $\alpha \in [0, \alpha^\star]$ and $\Pi_1(\alpha) > 0$ for $\alpha \in (\alpha^\star, 1]$.

For any $\alpha \in [0, \alpha^\star]$, $\mathcal{A}_1(\alpha)$ and $\mathcal{S}_1(\alpha)$ remain unchanged because $\Pi_1(\alpha) = 0$ (indicating that $r_1 = \lambda_2$). $\mathcal{A}_2(\alpha)$ and $\mathcal{S}_2(\alpha)$ decrease since $\lambda_2(\alpha) = (1-\alpha)\lambda_2^*$ is adjusted to maintain access fairness.

At the threshold $\alpha^\star$, as $\Pi_1(\alpha)$ increases from 0, it follows that $0 < r_1(\alpha) < \lambda_2$, resulting in $\nu_5 = \nu_6 = 0$. Additionally, since $0 < \lambda_2(\alpha) < \Lambda_2$, we have $\nu_3 = \nu_4$. Based on the stationarity conditions derived from equations (29b) and (29c), we can express $\alpha^\star$ as:

$$\alpha^\star = 1 - 2\frac{\sqrt{b_2\Lambda_2(c_1\Lambda_1 + 2b_2\Lambda_2 - c_1\Lambda_2)} - b_2}{(b_2 - c_1)\Lambda_2}.$$

For $\alpha \in (\alpha^\star, 1]$, $\lambda_1(\alpha) = (1-\alpha^\star)\lambda_1^*$ and $\lambda_2(\alpha) = (1-\alpha^\star)\lambda_2^*$ satisfy the KKT conditions. Consequently, only $r_1(\alpha)$ increases, leading to an increase in $\Pi_1(\alpha)$ and a decrease in $\Pi_2(\alpha)$. As a result, both $\mathcal{A}_1(\alpha)$ and $\mathcal{S}_1(\alpha)$ increase, while $\mathcal{A}_2(\alpha)$ and $\mathcal{S}_2(\alpha)$ decrease. □

Proof of Proposition 9. The optimization problem under price fairness with multiple nodes can be rewritten using the Taylor expansion, the same as the optimization problem (13). For simplicity, we define $g_K(\lambda_1, \lambda_2) := \frac{\lambda_1\lambda_2(\lambda_2^K - \lambda_1^K)}{\lambda_2^{K+1} - \lambda_1^{K+1}}$. Then, $\mathcal{S}_i = \frac{b_i - p_i}{2\Lambda_i} g_K(\lambda_1, \lambda_2)$ and $\mathcal{W} = \frac{1}{2} g_K(\lambda_1, \lambda_2)(b_1 + b_2 + p_1 + p_2)$.

(a) When $R_1' = 0$, same as the proof of Proposition 2 (*a*), we know $p_2'(0)^* = -p_2^*$. Then, with $\beta_i = \frac{b_i}{\Lambda_i}$,

$$\mathcal{W}'(0) = \frac{g_K(\Lambda_1, \lambda_2^*)}{2p_2^*}\left(p_2^* + (\beta_1\Lambda_1 + \beta_2\lambda_2^*)\right) > 0,$$

$$\mathcal{S}_1'(0) = \frac{1}{2}\beta_1 g_K(\Lambda_1, \lambda_2^*) > 0, \quad \mathcal{S}_2'(0) = \frac{1}{2}\beta_2 g_K(\Lambda_1, \lambda_2^*) > 0.$$

(b) When $p_1^* \neq 0$, similar to the the Proposition 2 *(b)*, we obtain

$$p_1'(0) = \frac{R_{12}'' + R_{22}''}{R_{11}'' + 2R_{12}'' + R_{22}''}(p_2^* - p_1^*), \quad p_2'(0) = -\frac{R_{11}'' + R_{12}''}{R_{11}'' + 2R_{12}'' + R_{22}''}(p_2^* - p_1^*).$$

The derivative of social welfare at $\alpha = 0$ is

$$\begin{aligned}\mathcal{W}'(0) &= \frac{d\mathcal{R}}{d\alpha}\Big|_{\alpha=0} + \frac{d\mathcal{S}}{d\alpha}\Big|_{\alpha=0} = \frac{d\mathcal{S}}{d\alpha}\Big|_{\alpha=0} \\ &= -\frac{1}{2}\frac{b_1 + b_2}{p_1^* + p_2^*} g_K(\lambda_1^*, \lambda_2^*)\left(\frac{dp_1}{d\alpha}|_{\alpha=0} + \frac{dp_2}{d\alpha}|_{\alpha=0}\right) \\ &= -\frac{1}{2}\frac{b_1 + b_2}{p_1^* + p_2^*} g_K(\lambda_1^*, \lambda_2^*)\frac{R_{22}'' - R_{11}''}{R_{11}'' + 2R_{12}'' + R_{22}''}(p_2^* - p_1^*),\end{aligned}$$

where the second equality is derived by the fact that $\frac{\partial R}{\partial p_1} = \frac{\partial R}{\partial p_2} = 0$ since the solution is in the interior of a feasible region. Therefore, the sign of $\mathcal{W}'(0)$ is equal to the sign of $R''_{22} - R''_{11}$ since $R''_{11} + 2R''_{12} + R''_{22} < 0$ to ensure that problem (13) is bounded.

For surplus at each node $i \in \{1,2\}$,

$$\mathcal{S}'_1(0) = -\frac{p_2^* - p_1^*}{R''_{11} + 2R''_{12} + R''_{22}} \frac{g_K(\lambda_1^*, \lambda_2^*)}{2\Lambda_1} \left(R''_{12} + R''_{22} + \frac{b_1 - p_1^*}{p_1^* + p_2^*}(R''_{22} - R''_{11}) \right),$$

$$\mathcal{S}'_2(0) = -\frac{p_2^* - p_1^*}{R''_{11} + 2R''_{12} + R''_{22}} \frac{g_K(\lambda_1^*, \lambda_2^*)}{2\Lambda_2} \left(\frac{b_2 - p_2^*}{p_1^* + p_2^*}(R''_{22} - R''_{11}) - (R''_{11} + R''_{12}) \right).$$

The sign of surplus also depends on the sign of $R''_{22} - R''_{11} \geq 0$.

- If $R''_{22} - R''_{11} \geq 0$, $R''_{11} + 2R''_{12} + R''_{22} < 0$ implies $R''_{11} + R''_{12} < 0$. Therefore, $\mathcal{S}'_2(0) > 0$.
- If $R''_{22} - R''_{11} < 0$, $R''_{11} + 2R''_{12} + R''_{22} < 0$ implies $R''_{12} + R''_{22} < 0$. Therefore, $\mathcal{S}'_1(0) < 0$. □

Proof of Proposition 10. The optimization problem under price fairness is rewritten with Taylor expansion same as the optimization problem (13).

(a) When $R'_1 = 0$, following the proof of Proposition 2*(a)*, we get $p'_2(0)^* = -p_2^*$. Then,

$$\begin{aligned}
\mathcal{W}'(0) &= \frac{dp_2}{d\alpha}\Big|_{\alpha=0} \cdot \frac{dW}{dp_2}\Big|_{p_2=p_2^*} \\
&= (-p_2^*) \cdot \left(g(\Lambda_1, \lambda_2^*) + \frac{\partial g}{\partial p_2}|_{p_2=p_2^*} \cdot \left(p_2^* + \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right) \\
&= \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) g_K(\Lambda_1, \lambda_2^*),
\end{aligned}$$

where the last equality is from $g_K(\Lambda_1, \lambda_2^*) + \frac{\partial g}{\partial p_2}|_{p_2=p_2^*} p_2^* = 0$ due to $p_2^* = \arg\max_{p_2 \geq 0} g_K(\Lambda_1, \lambda_2) p_2$. Here, $g_K(\lambda_1, \lambda_2) = \frac{\lambda_1 \lambda_2 (\lambda_2^K - \lambda_1^K)}{\lambda_2^{K+1} - \lambda_1^{K+1}}$. The surplus at each node is

$$\mathcal{S}'_1(0) = \frac{g_K(\Lambda_1, \lambda_2^*)}{\beta_1 \Lambda_1} > 0, \quad \mathcal{S}'_2(0) = \frac{g_K(\Lambda_1, \lambda_2^*)}{\beta_2 \Lambda_2} > 0.$$

(b) When $p_1^* \neq 0$, similar to the proof of Proposition 2*(b)*, we obtain

$$p'_1(0) = \frac{R''_{12} + R''_{22}}{R''_{11} + 2R''_{12} + R''_{22}}(p_2^* - p_1^*), \quad p'_2(0) = -\frac{R''_{11} + R''_{12}}{R''_{11} + 2R''_{12} + R''_{22}}(p_2^* - p_1^*).$$

The derivative of social welfare at $\alpha = 0$ is

$$\begin{aligned}
\mathcal{W}'(0) &= \frac{dp_1}{d\alpha}\Big|_{\alpha=0} \cdot \frac{\partial W}{\partial p_1}\Big|_{p_1=p_1^*} + \frac{dp_2}{d\alpha}\Big|_{\alpha=0} \cdot \frac{\partial W}{\partial p_2}\Big|_{p_2=p_2^*} \\
&= \frac{dp_1}{d\alpha}\Big|_{\alpha=0} + \left(g_K(\lambda_1^*, \lambda_2^*) + \frac{\partial g}{\partial p_1}\Big|_{p_1=p_1^*} \left(p_1^* + p_2^* + \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right) \\
&\quad + \frac{dp_2}{d\alpha}\Big|_{\alpha=0} \left(g_K(\lambda_1^*, \lambda_2^*) + \frac{\partial g}{\partial p_2}\Big|_{p_2=p_2^*} \left(p_1^* + p_2^* + \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right) \\
&= -\frac{g_K(\lambda_1^*, \lambda_2^*)}{p_1^* + p_2^*} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \left(\frac{dp_1}{d\alpha}\Big|_{\alpha=0} + \frac{dp_2}{d\alpha}\Big|_{\alpha=0} \right) \\
&= -\frac{g_K(\lambda_1^*, \lambda_2^*)}{p_1^* + p_2^*} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \frac{R''_{22} - R''_{11}}{R''_{11} + 2R''_{12} + R''_{22}} (p_2^* - p_1^*),
\end{aligned}$$

where the second last equality is from the stationary condition in KKT conditions optimal solution, i.e., $\frac{\partial R}{\partial p_i}|_{p_i=p_i^*} = 0 \iff g_K(\lambda_1^*, \lambda_2^*) + \frac{\partial g_K}{\partial p_i}(p_1^* + p_2^*) = 0$ for any $i \in \{1,2\}$. Therefore, the sign of $\mathcal{W}'(0)$ is equal to the sign of $R''_{22} - R''_{11}$ due to $R''_{11} + 2R''_{12} + R''_{22} < 0$ similar to the proof of Proposition 9.

For surplus at each node $i \in \{1,2\}$,

$$\begin{aligned}
\mathcal{S}_1'(0) &= \frac{dp_1}{d\alpha}_{\alpha=0} \cdot \frac{\partial S_1}{\partial p_1}\Big|_{p_1=p_1^*} + \frac{dp_2}{d\alpha}_{\alpha=0} \cdot \frac{\partial S_2}{\partial p_2}\Big|_{p_2=p_2^*} \\
&= \left(\frac{1}{\Lambda_1\beta_1}\frac{\partial g_K}{\partial p_1}\Big|_{p_1=p_1^*}\right)\cdot \frac{dp_1}{d\alpha}_{\alpha=0} + \left(\frac{1}{\Lambda_1\beta_1}\frac{\partial g_K}{\partial p_2}\Big|_{p_2=p_2^*}\right)\cdot \frac{dp_2}{d\alpha}_{\alpha=0} \\
&= -\frac{1}{\Lambda_1\beta_1}\frac{g_K(\lambda_1^*,\lambda_2^*)}{p_1^*+p_2^*}\left(\frac{dp_1}{d\alpha}+\frac{dp_2}{d\alpha}\right) \\
&= -\frac{1}{\Lambda_1\beta_1}\frac{g_K(\lambda_1^*,\lambda_2^*)}{p_1^*+p_2^*}\frac{R_{22}''-R_{11}''}{R_{11}''+2R_{12}''+R_{22}''}(p_2^*-p_1^*).
\end{aligned}$$

Similarly, we obtain $\mathcal{S}_2'(0) = -\frac{1}{\Lambda_2\beta_2}\frac{g_K(\lambda_1^*,\lambda_2^*)}{p_1^*+p_2^*}\frac{R_{22}''-R_{11}''}{R_{11}''+2R_{12}''+R_{22}''}(p_2^*-p_1^*)$. Therefore, the sign of social welfare and surplus at both locations at $\alpha=0$ are determined by the sign of $R_{22}''-R_{11}''$. □

Proof of Proposition 11. The original problem with K units under origin-based access fairness is written as follows:

$$\begin{aligned}
\mathcal{R}_K(\alpha) := \max_q \quad & \sum_{(i,j)\in\mathcal{A}} \Pi_i\Lambda_{ij}q_{ij}p_{ij} \\
\text{s.t.} \quad & \sum_{j\in[N]} \Pi_i\Lambda_{ij}q_{ij} = \sum_{j\in N}\Pi_j\Lambda_{ji}q_{ji}, \quad \forall i\in[N], \\
& \left|\sum_{k\in[N]}\Pi_iT_{ik}q_{ik} - \sum_{k\in[N]}\Pi_jT_{jk}q_{jk}\right| \le (1-\alpha)M^*, \quad \forall i\in[N], \\
& q_{ij}\in[0,1], \quad \forall (i,j)\in\mathcal{A},
\end{aligned} \tag{30}$$

where $M^* = \max_i \sum_{k\in[N]}\Pi_i^*T_{ik}q_{ik}^* - \min_j\sum_{k\in[N]}\Pi_j^*T_{jk}q_{jk}^*$. Note that Π_i depends on K. Let $q_K^*(\alpha)$ denote the optimal solution to (30).

The approximated problem under access fairness is written as follows:

$$\begin{aligned}
\hat{\mathcal{R}}(\alpha) := \max_q \quad & \sum_{(i,j)\in\mathcal{A}} \Lambda_{ij}q_{ij}p_{ij} \\
\text{s.t.} \quad & \sum_{j\in[N]} \Lambda_{ij}q_{ij} = \sum_{k\in[N]}\Lambda_{ji}q_{ji}, \quad \forall i\in[N], \\
& \frac{K}{N+K-1}\left|\sum_{k\in[N]}T_{ik}q_{ik} - \sum_{k\in[N]}T_{jk}q_{jk}\right| \le (1-\alpha)M^*, \quad \forall i\in[N], \\
& \sum_{(i,j)\in\mathcal{A}}\frac{\Lambda_{ij}q_{ij}}{\mu_{ij}} \le K, \\
& q_{ij}\in[0,1], \quad \forall (i,j)\in\mathcal{A}.
\end{aligned} \tag{31}$$

Let $\hat{q}(\alpha)$ denote the optimal solution. The objective function in (31) is concave because the distribution of valuation, $F(p)$, is regular (Hartline 2013). The constraints are affine with respect to q, therefore, this problem is convex.

First, $\mathcal{R}_K(\alpha) \le \hat{\mathcal{R}}(\alpha)$, i.e., $\mathcal{R}_K(q_K^*(\alpha)) \le \hat{\mathcal{R}}(\hat{q}(\alpha))$, is established as follows. Assume $q_{ij}(\alpha) = \Pi_i^* q_{K,ij}^*(\alpha)$, then the following bound holds:

$$\begin{aligned}
\mathcal{R}_K(q_K^*(\alpha)) &= \sum_{i,j}\Pi_i\Lambda_{ij}q_{K,ij}^*(\alpha)F^{-1}\left(1-q_{K,ij}^*(\alpha)\right) \\
&\le \sum_{i,j}\Lambda_{ij}\Pi_i^*q_{K,ij}^*(\alpha)F^{-1}\left(1-\Pi_i^*q_{K,ij}^*(\alpha)\right) \\
&= \hat{\mathcal{R}}_K(q(\alpha)),
\end{aligned}$$

where inequality follows from the fact that $F^{-1}(\cdot)$ is non-decreasing and $\Pi_i \le 1$.

To complete the proof, it suffices to show that $q_{ij}(\alpha)$ is a feasible solution to the problem (31). (i) The balance equation holds because $q(\alpha)$ holds the standard flow balance equation (1):

$$\sum_{j\in[N]} \Lambda_{ij} q_{ij}(\alpha) = \sum_{j\in[N]} \Lambda_{ji} q_{ji}(\alpha) \iff \sum_{j\in[N]} \Lambda_{ij} \Pi_i^* q_{K,ij}^*(\alpha) = \sum_{j\in[N]} \Lambda_{ji} \Pi_j^* q_{K,ji}^*(\alpha).$$

(ii) The fairness constraint holds because:

$$\begin{aligned}\frac{K}{N+K-1}\left|\sum_{k\in[N]} T_{ik} q_{ik}(\alpha) - \sum_{k\in[N]} T_{ik} q_{ik}(\alpha)\right| &= \frac{K}{N+K-1}\left|\sum_{k\in[N]} T_{ik}\Pi_i^* q_{ik}^*(\alpha) - \sum_{k\in[N]} T_{ik}\Pi_j^* q_{jk}^*(\alpha)\right| \\ &\le \frac{K}{N+K-1}(1-\alpha)M^* \le (1-\alpha)M^*.\end{aligned}$$

(iii) The constraint on overall workload holds due to Little's law,

$$\sum_{(i,j)\in\mathcal{A}} \frac{\Lambda_{ij} q_{ij}}{\mu_{ij}} = \sum_{(i,j)\in\mathcal{A}} \Lambda_{ij} q_{ij}^* \Pi_i^* \frac{1}{\mu_{ij}} = \mathbb{E}\left[\#\text{ of units in transit queues}\right] \le K.$$

(iv) $q_{ij}(\alpha) \in [0,1]$ because $q_{ij}(\alpha) = \Pi_i^* q_{ij}^*(\alpha) \in [0, q_{i,j}^*] \in [0,1]$.

Next, the inequality $\mathcal{R}_K(\hat{q}(\alpha)) \le \mathcal{R}_K(\alpha)$, i.e. $\mathcal{R}_K(\hat{q}(\alpha)) \le \mathcal{R}_K(q_K^*(\alpha))$ is proven by that $\hat{q}(\alpha)$ is a feasible solution for the problem (30).

(i) The balance equation holds,

$$\hat{\Pi}_i \sum_{j\in[N]} \Lambda_{ij}\hat{q}_{ij}(\alpha) - \hat{\Pi}_j \sum_{j\in[N]} \Lambda_{ji}\hat{q}_{ji}(\alpha) = \hat{\Pi}\left(\sum_{j\in[N]} \Lambda_{ij}\hat{q}_{ij}(\alpha) - \sum_{j\in[N]} \Lambda_{ji}\hat{q}_{ji}(\alpha)\right) = 0.$$

In the above equation, $\hat{\Pi}_i = \hat{\Pi}(K) = \frac{K}{N+K-1+\sum_{(i,j)\mathcal{A}} \frac{\Lambda_{ij}\hat{q}_{ij}}{\mu_{ij}}\left(1-\hat{\Pi}(K-1)\right)}$ for all $i \in [N]$, where $\hat{\Pi}(0) = 0$, because state-independent policy $\hat{\mathbf{q}}$ satisfies the balanced demand flows, i.e., $\sum_{j\in[N]} \Lambda_{ij}\hat{q}_{ij} = \sum_{j\in[N]} \Lambda_{ji}\hat{q}_{ji}$ for all $(i,j) \in \mathcal{A}$ (Benjaafar et al. 2023).

(ii) The fairness constraint holds because:

$$\begin{aligned}\left|\sum_{k\in[N]} \hat{\Pi}_i T_{ik}\hat{q}_{ik}(\alpha) - \sum_{k\in[N]} \hat{\Pi}_j T_{jk}\hat{q}_{jk}(\alpha)\right| &= \hat{\Pi}\left|\sum_{k\in[N]} T_{ik}\hat{q}_{ik}(\alpha) - \sum_{k\in[N]} T_{jk}\hat{q}_{jk}(\alpha)\right| \\ &\le \hat{\Pi}\cdot\frac{N+K-1}{K}\cdot(1-\alpha)M^* \\ &\le (1-\alpha)M^*.\end{aligned}$$

The last inequality holds because $\hat{\Pi} = \frac{K}{N+K-1+\sum_{(i,j)\mathcal{A}} \frac{\Lambda_{ij}\hat{q}_{ij}}{\mu_{ij}}\left(1-\hat{\Pi}(K-1)\right)} \le \frac{K}{N+K-1}$.

(iii) The last inequality is the same between (30) and (31), therefore, it holds.

Lastly, the inequality, $\frac{K}{N+K-1+\frac{\Lambda}{\mu}}\hat{\mathcal{R}}_K(\alpha) \le \mathcal{R}_K(\hat{q}(\alpha))$ is proven as follows:

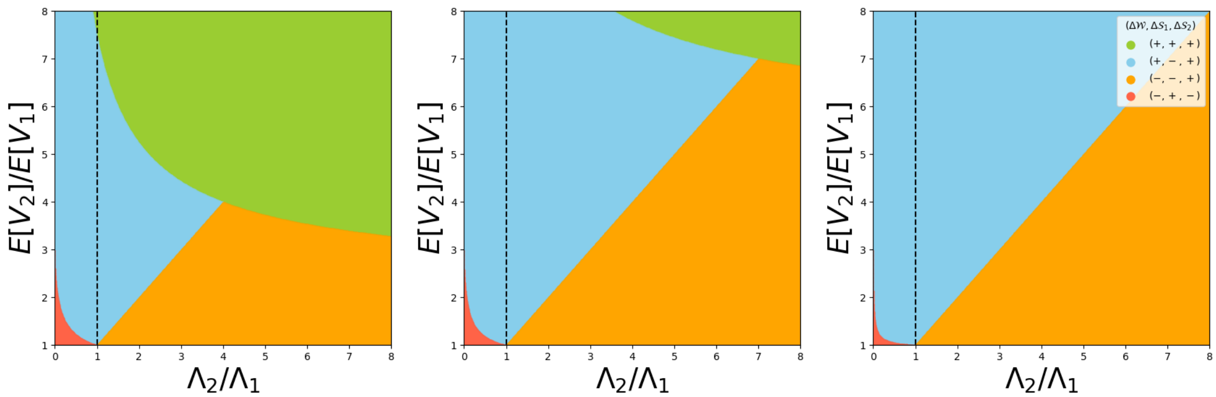
$$\begin{aligned}\frac{\mathcal{R}_K(\hat{q}(\alpha))}{\hat{\mathcal{R}}(\hat{q}(\alpha))} &= \frac{\sum_{i,j}\hat{\Pi}_i\Lambda_{ij}\hat{q}_{ij}(\alpha)F^{-1}(1-\hat{q}_{ij}(\alpha))}{\sum_{i,j}\Lambda_{ij}\hat{q}_{ij}(\alpha)F^{-1}(1-\hat{q}_{ij}(\alpha))} = \hat{\Pi}\cdot\frac{\sum_{i,j}\Lambda_{ij}\hat{q}_{ij}(\alpha)F^{-1}(1-\hat{q}_{ij}(\alpha))}{\sum_{i,j}\Lambda_{ij}\hat{q}_{ij}(\alpha)F^{-1}(1-\hat{q}_{ij}(\alpha))} \\ &= \hat{\Pi} = \frac{K}{N+K-1+\sum_{(i,j)\mathcal{A}}\frac{\Lambda_{ij}\hat{q}_{ij}}{\mu_{ij}}\left(1-\hat{\Pi}(K-1)\right)} \ge \frac{K}{N+K-1+\frac{\Lambda}{\mu}}.\end{aligned}$$

The last inequality holds because $0 \le \hat{q}_{ij}\left(1-\hat{\Pi}(k)\right) \le 1$ for any $k \in [K]$. □

B. Additional Results

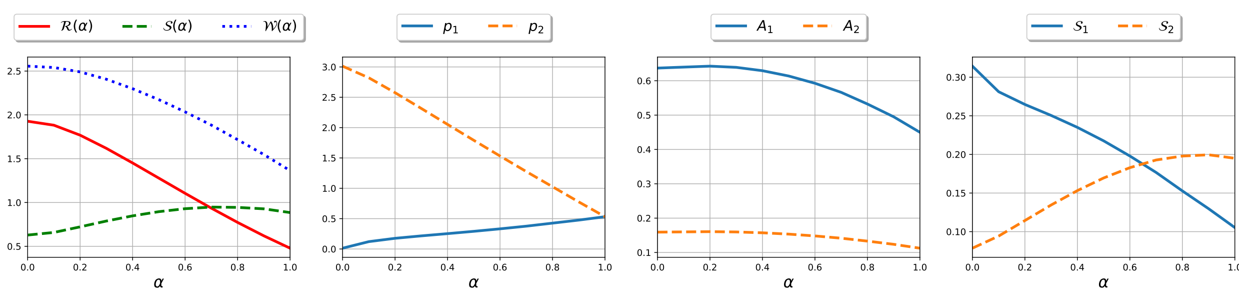
B.1. Non-zero Travel Time

Figure 1 Impact of Price Fairness under Linear Demand with different μ

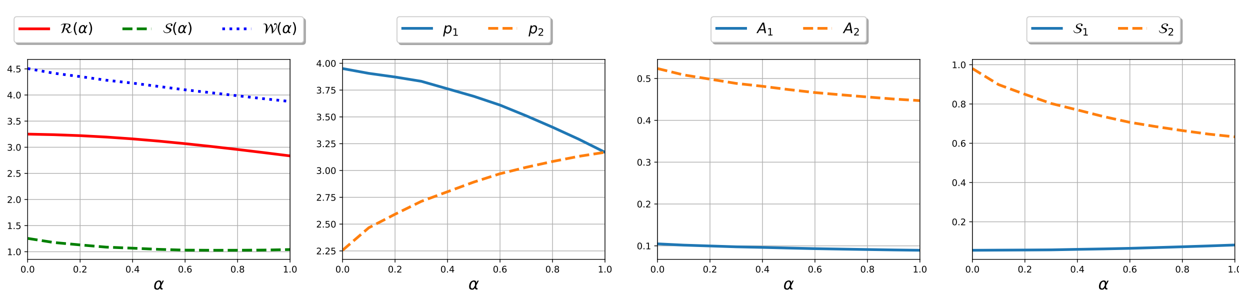


(a) $\mu = \Lambda_1$. (b) $\mu = 4\Lambda_1$. (c) $\mu = 100\Lambda_1$.

Figure 2 Impact of Price Fairness under Linear Demand with Positive Travel Time with 5 units

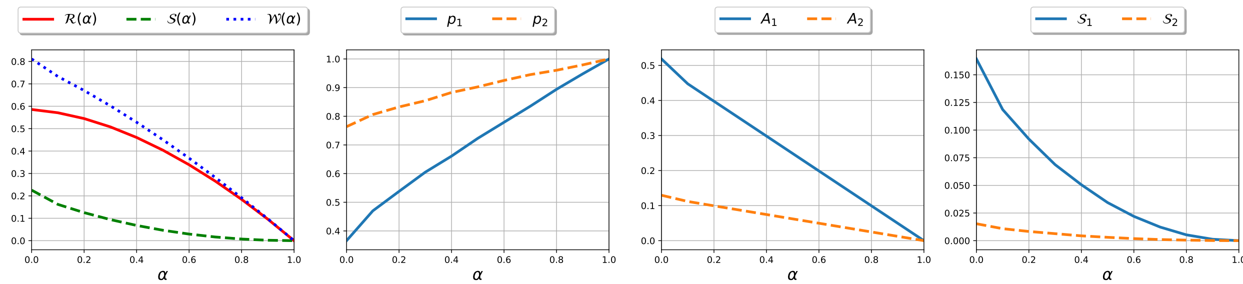


(a) $(\Lambda_1, \Lambda_2) = (1, 4)$, $(b_1, b_2) = (1, 4)$, $\mu = 1/4$, and $K = 5$

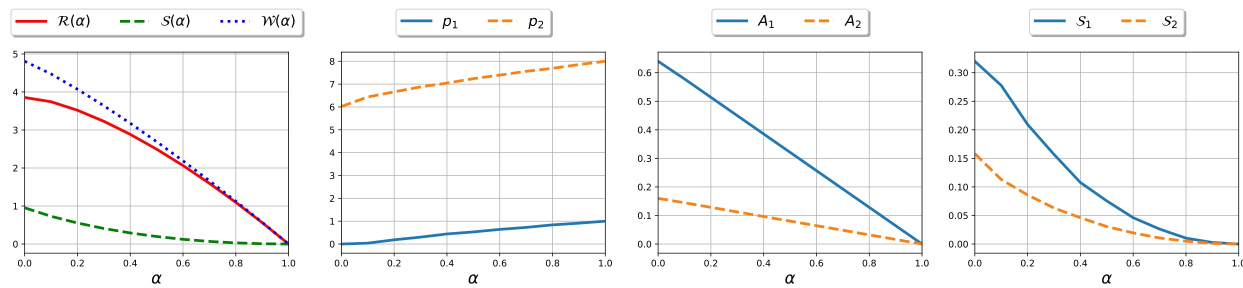


(b) $(\Lambda_1, \Lambda_2) = (5, 1)$, $(b_1, b_2) = (5, 6)$, $\mu = 1/4$, and $K = 5$

Figure 3 Impact of Access Fairness under Linear Demand with Positive Travel Time with 5 units



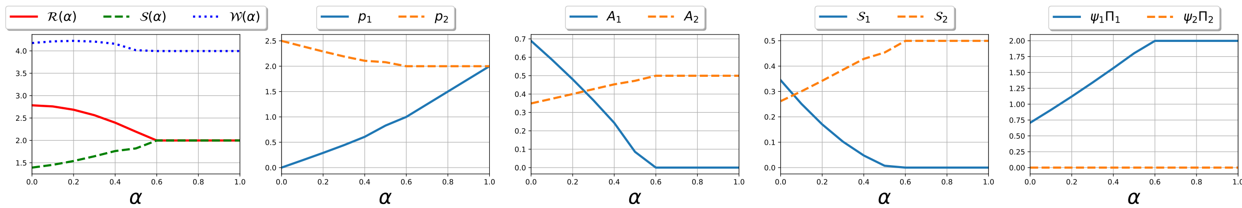
(a) $(\Lambda_1, \Lambda_2) = (1, 4)$, $(b_1, b_2) = (1, 1)$, $\mu = 1/4$, and $K = 5$



(b) $(\Lambda_1, \Lambda_2) = (1, 4)$, $(b_1, b_2) = (1, 8)$, $\mu = 1/4$, and $K = 5$

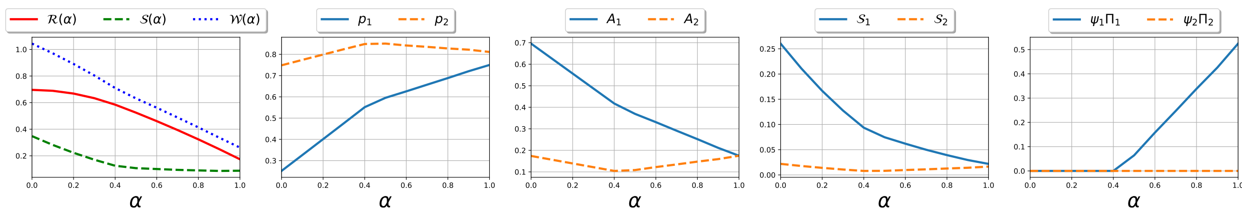
B.2. Vehicle Repositioning with Multiple Units

Figure 4 Impact of Price Fairness with Repositioning

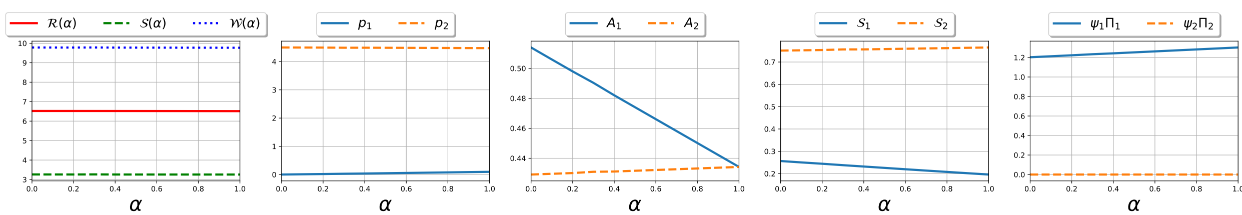


Note. $(\Lambda_1, \Lambda_2) = (1, 4)$, $(b_1, b_2) = (1, 4)$, $(c_1, c_2) = (1, 1)$, and $K = 5$.

Figure 5 Impact of Access Fairness with Repositioning



(a) $(\Lambda_1, \Lambda_2) = (1, 4)$, $(b_1, b_2) = (1, 1)$, $(c_1, c_2) = (1, 1)$, and $K = 5$

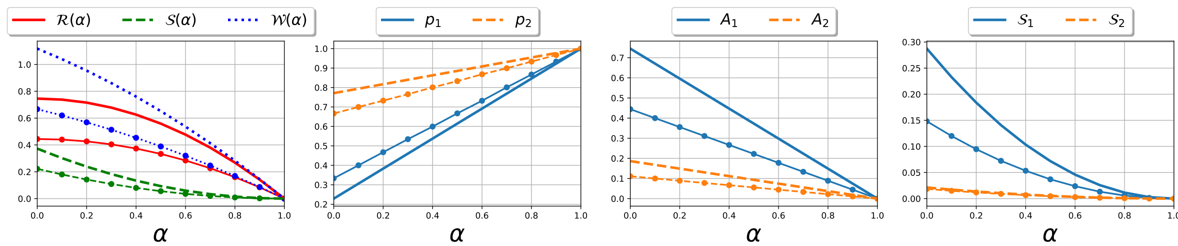


(b) $(\Lambda_1, \Lambda_2) = (1, 4)$, $(b_1, b_2) = (1, 8)$, $(c_1, c_2) = (1, 1)$, and $K = 5$

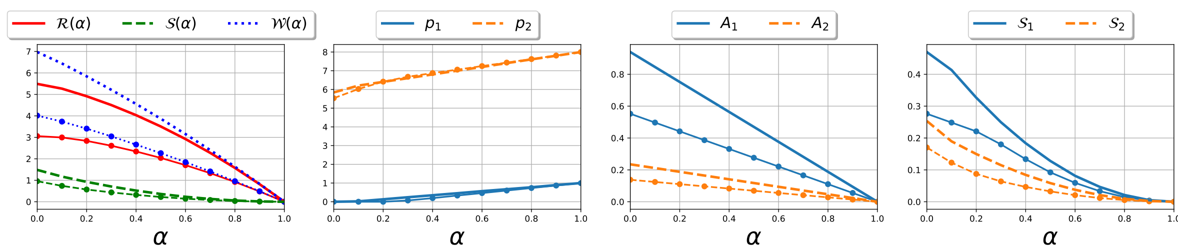
Note. The plots (a) and (b) correspond to those in Figure 10.

B.3. Access Fairness with $N = 2$ and any K

Figure 6 Impact of Access Fairness under Linear Demand



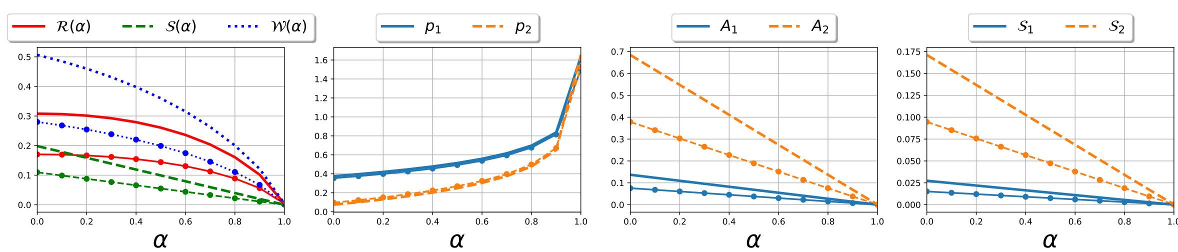
(a) $(\Lambda_1, \Lambda_2) = (1, 4)$ and $(b_1, b_2) = (1, 1)$.



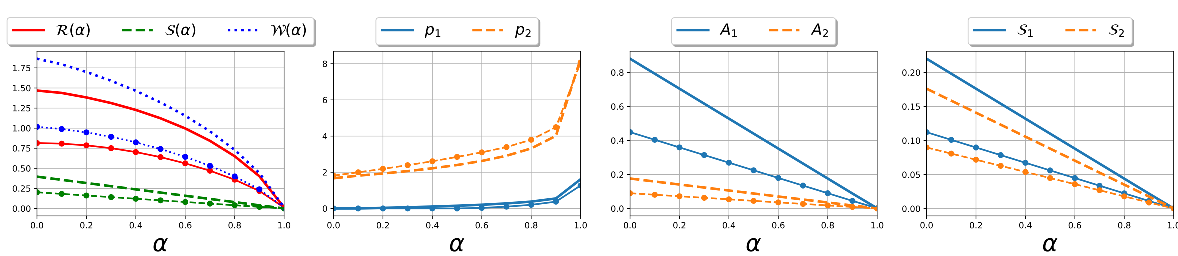
(b) $(\Lambda_1, \Lambda_2) = (1, 4)$ and $(b_1, b_2) = (1, 8)$.

Note. Lines without and with markers correspond to $K = 1$ and $K = 10$. The plots (a) and (b) correspond to Proposition 3.

Figure 7 Impact of Access Fairness under Exponential Demand



(a) $(\Lambda_1, \Lambda_2) = (5, 1)$ and $(\beta_1, \beta_2) = (5, 4)$.



(b) $(\Lambda_1, \Lambda_2) = (1, 5)$ and $(\beta_1, \beta_2) = (4, 1)$.

Note. Lines without and with markers correspond to $K = 1$ and $K = 10$. The plots (a) and (b) correspond to Proposition 5.

B.4. Simplification and Analytical Intractability for $N = 3$

To better understand the impact of fairness criteria in a network with $N = 3$ nodes, we first attempted an analytical approach by deriving a closed-form solution for the system without fairness constraints. However, the resulting expressions proved to be analytically intractable due to their complexity. This limitation highlighted the necessity of alternative methodologies – such as the heuristic-based approach detailed in Section 5 – to effectively analyze systems involving multiple nodes.

Specifically, we examined the complete graph with three nodes, aiming to determine the optimal set of decision variables,

$$(\lambda_{12}^*, \lambda_{21}^*, \lambda_{13}^*, \lambda_{31}^*, \lambda_{23}^*, \lambda_{32}^*)$$

that maximize the total revenue $\mathcal{R}$, defined as follows:

$$\begin{aligned}\mathcal{R} = &\frac{1}{(\lambda_{12}+\lambda_{13}+\lambda_{21})(\lambda_{12}\lambda_{31}+\lambda_{13}\lambda_{31}+\lambda_{13}\lambda_{32})+(\lambda_{12}+\lambda_{13}+\lambda_{31})(\lambda_{12}\lambda_{23}+\lambda_{13}\lambda_{21}+\lambda_{13}\lambda_{23})} \\ &\times\Bigg\{ [\lambda_{21}(\lambda_{12}\lambda_{31}+\lambda_{13}\lambda_{31}+\lambda_{13}\lambda_{32})+\lambda_{31}(\lambda_{12}\lambda_{23}+\lambda_{13}\lambda_{21}+\lambda_{13}\lambda_{23})] \\ &\quad\times\left[\lambda_{12}b_{12}\left(1-\frac{\lambda_{12}}{\Lambda_{12}}\right)+\lambda_{13}b_{13}\left(1-\frac{\lambda_{13}}{\Lambda_{13}}\right)\right] \\ &\quad+(\lambda_{12}+\lambda_{13})(\lambda_{12}\lambda_{31}+\lambda_{13}\lambda_{31}+\lambda_{13}\lambda_{32})\left[\lambda_{21}b_{21}\left(1-\frac{\lambda_{21}}{\Lambda_{21}}\right)+\lambda_{23}b_{23}\left(1-\frac{\lambda_{23}}{\Lambda_{23}}\right)\right] \\ &\quad+(\lambda_{12}+\lambda_{13})(\lambda_{12}\lambda_{23}+\lambda_{13}\lambda_{21}+\lambda_{13}\lambda_{23})\left[\lambda_{31}b_{31}\left(1-\frac{\lambda_{31}}{\Lambda_{31}}\right)+\lambda_{32}b_{32}\left(1-\frac{\lambda_{32}}{\Lambda_{32}}\right)\right]\Bigg\}.\end{aligned}$$

We encountered significant challenges in deriving a closed-form solution using the KKT conditions, even in the absence of fairness constraints. To make analytical progress, we simplified the problem by setting $\Lambda_{13} = \Lambda_{31} = 0$. This simplification allows us to derive a closed-form solution in the no-fairness setting. Given that our fairness analysis builds upon the closed-form solution derived in the no-fairness setting, it is important to note that the resulting expression is sufficiently lengthy to offer limited scope for meaningful analysis. The closed-form solution is presented below:

$$\lambda_{32}^* = \sqrt{\frac{b_{12}\Lambda_{32}}{\Lambda_{12}+b_{32}}}\lambda_{12}^*, \quad \lambda_{21}^* = \frac{\Lambda_{21}}{b_{21}}\left(\frac{b_{12}+b_{21}}{2}-\frac{b_{12}}{\Lambda_{12}}\lambda_{12}^*\right), \quad \lambda_{23}^* = \frac{\Lambda_{23}}{b_{23}}\left(\frac{b_{23}+b_{32}}{2}-\frac{b_{32}}{\Lambda_{32}}\lambda_{32}^*\right),$$

where $\lambda_{12}^* = \Big(\Lambda_{12}\Lambda_{32}(-\sqrt{S}(\Lambda_{12}\Lambda_{23}b_{21}b_{32} - \Lambda_{12}\Lambda_{32}b_{21}b_{23} + \Lambda_{21}\Lambda_{32}b_{12}b_{23})(\Lambda_{12}\Lambda_{23}b_{21}b_{23}b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + \Lambda_{12}\Lambda_{23}b_{21}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{21}\Lambda_{32}b_{12}^2b_{23} + \Lambda_{21}\Lambda_{32}b_{12}b_{21}b_{23}) + (2\Lambda_{12}\Lambda_{21}\Lambda_{23}b_{12}b_{21}b_{23}^2b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + 2\Lambda_{12}\Lambda_{21}\Lambda_{23}b_{12}b_{21}b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + 2\Lambda_{12}\Lambda_{21}\Lambda_{23}b_{21}^2b_{23}^2b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + 2\Lambda_{12}\Lambda_{21}\Lambda_{23}b_{21}^2b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{12}\Lambda_{23}^2b_{21}^2b_{23}^2b_{32} + 2\Lambda_{12}\Lambda_{23}^2b_{21}^2b_{23}b_{32}^2 + \Lambda_{12}\Lambda_{23}^2b_{21}^2b_{32}^3 + \Lambda_{21}^2\Lambda_{32}b_{12}^3b_{23}^2 + 2\Lambda_{21}^2\Lambda_{32}b_{12}^2b_{21}b_{23}^2 + \Lambda_{21}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^2)(\Lambda_{12}^2\Lambda_{23}^2b_{21}^2b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{12}^2\Lambda_{23}^2b_{21}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) - \Lambda_{12}^2\Lambda_{23}\Lambda_{32}b_{21}^2b_{23}^2b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) - \Lambda_{12}^2\Lambda_{23}\Lambda_{32}b_{21}^2b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}b_{23}b_{32} + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}b_{21}^2b_{23}b_{32} + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}b_{21}b_{23}^2b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}b_{21}b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) - \Lambda_{12}\Lambda_{21}\Lambda_{32}^2b_{12}^2b_{21}b_{23}^2 - \Lambda_{12}\Lambda_{21}\Lambda_{32}^2b_{12}b_{21}^2b_{23}^2 + \Lambda_{21}^2\Lambda_{32}^2b_{12}^3b_{23}^2 + \Lambda_{21}^2\Lambda_{32}^2b_{12}^2b_{21}b_{23}^2))\Big)/\Big(2(\Lambda_{12}\Lambda_{23}b_{21}b_{32} - \Lambda_{12}\Lambda_{32}b_{21}b_{23} + \Lambda_{21}\Lambda_{32}b_{12}b_{23})(\Lambda_{12}\Lambda_{23}b_{21}b_{23}b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{12}\Lambda_{23}b_{21}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{21}\Lambda_{32}b_{12}^2b_{23} + \Lambda_{21}\Lambda_{32}b_{12}b_{21}b_{23})(\Lambda_{12}^2\Lambda_{23}^2b_{21}^2b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) + \Lambda_{12}^2\Lambda_{23}^2b_{21}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) - \Lambda_{12}^2\Lambda_{23}\Lambda_{32}b_{21}^2b_{23}^2b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})}) -$

$$\Lambda_{12}^2\Lambda_{23}\Lambda_{32}b_{21}^2b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}b_{23}b_{32} + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}b_{21}^2b_{23}b_{32} +$$

$$\Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}b_{21}b_{23}^2b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + \Lambda_{12}\Lambda_{21}\Lambda_{23}\Lambda_{32}b_{12}b_{21}b_{23}b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - \Lambda_{12}\Lambda_{21}\Lambda_{32}^2b_{12}^2b_{21}b_{23}^2 -$$

$$\Lambda_{12}\Lambda_{21}\Lambda_{32}^2b_{12}b_{21}^2b_{23}^2 + \Lambda_{21}^2\Lambda_{32}^2b_{12}^3b_{23}^2 + \Lambda_{21}^2\Lambda_{32}^2b_{12}^2b_{21}b_{23}^2)\Big)$$

$$S = b_{21}b_{23}\Big(- 2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{12}^3b_{21}b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{12}^3b_{21}b_{23}b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{12}^2b_{21}^2b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{12}^2b_{21}^2b_{23}b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{12}b_{21}^3b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{12}b_{21}^3b_{23}b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{21}^4b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}^2b_{21}^4b_{23}b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{12}^3b_{21}b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{12}^3b_{21}b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}^2b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}^2b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{12}b_{21}^3b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{12}b_{21}^3b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{21}^4b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}^2\Lambda_{21}^2\Lambda_{23}\Lambda_{32}b_{21}^4b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}^2b_{21}^2b_{23}^2b_{32}^2 -$$

$$2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}^2b_{21}^2b_{23}b_{32}^3 - \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}^2b_{21}^2b_{32}^4 - 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^3b_{23}^2b_{32}^2 - 4\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^3b_{23}b_{32}^3 -$$

$$2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^3b_{32}^4 + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^2b_{23}^3b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^2b_{23}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^2b_{23}b_{32}^4\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{12}b_{21}^2b_{32}^5\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^4b_{23}^2b_{32}^2 -$$

$$2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^4b_{23}b_{32}^3 - \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^4b_{32}^4 + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^3b_{23}^3b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^3b_{23}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^3b_{23}b_{32}^4\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^3b_{21}^3b_{32}^5\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}^2b_{23}^3b_{32} +$$

$$2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}^2b_{23}^2b_{32}^2 + \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}^2b_{23}b_{32}^3 + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^3b_{23}^3b_{32} + 4\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^3b_{23}^2b_{32}^2 +$$

$$2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^3b_{23}b_{32}^3 + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^4b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^3b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}b_{32}^4\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^4b_{23}^3b_{32} +$$

$$2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^4b_{23}^2b_{32}^2 + \Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^4b_{23}b_{32}^3 + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^3b_{23}^4b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^3b_{23}^3b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^3b_{23}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}^2\Lambda_{21}\Lambda_{23}^2\Lambda_{32}b_{21}^3b_{23}b_{32}^4\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$\Lambda_{12}^2\Lambda_{23}^3\Lambda_{32}b_{21}^3b_{23}^4b_{32} + 4\Lambda_{12}^2\Lambda_{23}^3\Lambda_{32}b_{21}^3b_{23}^3b_{32}^2 + 6\Lambda_{12}^2\Lambda_{23}^3\Lambda_{32}b_{21}^3b_{23}^2b_{32}^3 + 4\Lambda_{12}^2\Lambda_{23}^3\Lambda_{32}b_{21}^3b_{23}b_{32}^4 + \Lambda_{12}^2\Lambda_{23}^3\Lambda_{32}b_{21}^3b_{32}^5 - \Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^5b_{23}^2b_{32} -$$

$$4\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^4b_{21}b_{23}^2b_{32} + 2\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^4b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^4b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$6\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^3b_{21}^2b_{23}^2b_{32} + 6\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^3b_{21}b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^3b_{21}b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$4\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}^3b_{23}^2b_{32} + 6\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}^2b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 6\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}^2b_{21}^2b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}b_{21}^4b_{23}^2b_{32} + 2\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}b_{21}^3b_{23}^3b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 2\Lambda_{12}\Lambda_{21}^3\Lambda_{23}\Lambda_{32}b_{12}b_{21}^3b_{23}^2b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} +$$

$$\Lambda_{12}\Lambda_{21}^3\Lambda_{32}^2b_{12}^5b_{23}^3 + 4\Lambda_{12}\Lambda_{21}^3\Lambda_{32}^2b_{12}^4b_{21}b_{23}^3 + 6\Lambda_{12}\Lambda_{21}^3\Lambda_{32}^2b_{12}^3b_{21}^2b_{23}^3 + 4\Lambda_{12}\Lambda_{21}^3\Lambda_{32}^2b_{12}^2b_{21}^3b_{23}^3 + \Lambda_{12}\Lambda_{21}^3\Lambda_{32}^2b_{12}b_{21}^4b_{23}^3 +$$

$$4\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^3b_{21}b_{23}^3b_{32} + 8\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^3b_{21}b_{23}^2b_{32}^2 + 4\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^3b_{21}b_{23}b_{32}^3 + 8\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}^2b_{23}^3b_{32} +$$

$$16\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}^2b_{23}^2b_{32}^2 + 8\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}^2b_{23}b_{32}^3 - 2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}b_{23}^4b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$6\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}b_{23}^3b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 6\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}b_{23}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}^2b_{21}b_{23}b_{32}^4\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + 4\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^3b_{23}^3b_{32} + 8\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^3b_{23}^2b_{32}^2 +$$

$$4\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^3b_{23}b_{32}^3 - 2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^4b_{32}\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 6\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^3b_{32}^2\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} -$$

$$6\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}^2b_{32}^3\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} - 2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}^2\Lambda_{32}b_{12}b_{21}^2b_{23}b_{32}^4\sqrt{\Lambda_{32}b_{12}/(\Lambda_{12}b_{32})} + \Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}^3b_{21}b_{23}^4 +$$

$$2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}^3b_{21}b_{23}^3b_{32} + \Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}^3b_{21}b_{23}^2b_{32}^2 + 2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}^2b_{21}^2b_{23}^4 + 4\Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}^2b_{21}^2b_{23}^3b_{32} +$$

$$2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}^2b_{21}^2b_{23}^2b_{32}^2 + \Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}b_{21}^3b_{23}^4 + 2\Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}b_{21}^3b_{23}^3b_{32} + \Lambda_{12}\Lambda_{21}^2\Lambda_{23}\Lambda_{32}^2b_{12}b_{21}^3b_{23}^2b_{32}^2 -$$

$$\Lambda_{12}\Lambda_{21}\Lambda_{23}^3\Lambda_{32}b_{12}b_{21}^2b_{23}^4b_{32} - 4\Lambda_{12}\Lambda_{21}\Lambda_{23}^3\Lambda_{32}b_{12}b_{21}^2b_{23}^3b_{32}^2 - 6\Lambda_{12}\Lambda_{21}\Lambda_{23}^3\Lambda_{32}b_{12}b_{21}^2b_{23}^2b_{32}^3 - 4\Lambda_{12}\Lambda_{21}\Lambda_{23}^3\Lambda_{32}b_{12}b_{21}^2b_{23}b_{32}^4 -$$

$$\Lambda_{12}\Lambda_{21}\Lambda_{23}^3\Lambda_{32}b_{12}b_{21}^2b_{32}^5 - \Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^4b_{23}^4 - 2\Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^4b_{23}^3b_{32} - \Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^4b_{23}^2b_{32}^2 - 2\Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^3b_{21}b_{23}^4 -$$

$$4\Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^3b_{21}b_{23}^3b_{32} - 2\Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^3b_{21}b_{23}^2b_{32}^2 - \Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^2b_{21}^2b_{23}^4 - 2\Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^2b_{21}^2b_{23}^3b_{32} - \Lambda_{21}^3\Lambda_{23}\Lambda_{32}^2b_{12}^2b_{21}^2b_{23}^2b_{32}^2\Big).$$

However, under the additional assumptions that $\Lambda_{13} = \Lambda_{31} = 0$ and that market sizes and the demand function are symmetric with respect to location 2, Proposition 12 shows that this symmetric three-node problem can be reduced to the $N = 2$ case.

PROPOSITION 12 **(Symmetric three-location reduction)**. *Let $N = 3$ and suppose $\Lambda_{13} = \Lambda_{31} = 0$, $\Lambda_{12} = \Lambda_{32}$, $\Lambda_{21} = \Lambda_{23}$, $b_{12} = b_{32}$, and $b_{21} = b_{23}$. Then the optimization problem with either price fairness or access fairness is equivalent to the two-node optimization problem with parameters $(\Lambda_{12}, 2\Lambda_{21})$ and (b_{12}, b_{21}).*

Proof. Under this structure, the revenue simplifies to

$$\mathcal{R} = \frac{\lambda_{12}\lambda_{32}}{\lambda_{12}\lambda_{32} + \lambda_{21}\lambda_{32} + \lambda_{23}\lambda_{12}} \left(\left(b_{12} + b_{21} - \frac{b_{12}}{\Lambda_{12}}\lambda_{12} - \frac{b_{21}}{\Lambda_{21}}\lambda_{21} \right) \lambda_{21} + \left(b_{23} + b_{32} - \frac{b_{23}}{\Lambda_{23}}\lambda_{23} - \frac{b_{32}}{\Lambda_{32}}\lambda_{32} \right) \lambda_{23} \right).$$

Given the network symmetry, we can set $\lambda_{12} = \lambda_{32}$ and $\lambda_{21} = \lambda_{23}$, yielding a further simplification

$$\begin{aligned}\mathcal{R} &= \frac{\lambda_{12}^2}{\lambda_{12}^2 + 2\lambda_{21}\lambda_{12}} \left(\left(b_{12} + b_{21} - \frac{b_{12}}{\Lambda_{12}}\lambda_{12} - \frac{b_{21}}{\Lambda_{21}}\lambda_{21} \right) \lambda_{21} + \left(b_{12} + b_{21} - \frac{b_{12}}{\Lambda_{12}}\lambda_{12} - \frac{b_{21}}{\Lambda_{21}}\lambda_{21} \right) \lambda_{21} \right) \\ &= \frac{2\lambda_{12}\lambda_{21}}{\lambda_{12} + 2\lambda_{21}} \left(b_{12} + b_{21} - \frac{b_{12}}{\Lambda_{12}}\lambda_{12} - \frac{b_{21}}{\Lambda_{21}}\lambda_{21} \right).\end{aligned}$$

Replacing λ_{ij} with its quantile representation $\lambda_{ij} = \Lambda_{ij} q_{ij}$, we obtain

$$\mathcal{R} = \frac{\Lambda_{12} q_{12} \left(2\Lambda_{21} q_{21} \right)}{\Lambda_{12} q_{12} + 2\Lambda_{21} q_{21}} \left(b_{12} + b_{21} - b_{12} q_{12} - b_{21} q_{21} \right).$$

(Price Fairness) In this setting, trip-based price fairness and origin-based price fairness coincide.

Trip-based price fairness aims to minimize the maximum price difference across trips. Under the symmetry assumptions, this reduces to

$$|p_{12} - p_{21}| = |p_{12} - p_{23}| = |p_{32} - p_{21}| = |p_{32} - p_{23}| = \left| b_{12} \left(1 - \frac{\lambda_{12}}{\Lambda_{12}} \right) - b_{21} \left(1 - \frac{\lambda_{21}}{\Lambda_{21}} \right) \right| = |b_{12}(1 - q_{12}) - b_{21}(1 - q_{21})|.$$

On the other hand, origin-based price fairness minimizes the maximum difference of the average prices of trips that share the same origin across nodes,

$$|\bar{p}_1 - \bar{p}_2| = |\bar{p}_2 - \bar{p}_3| = \left| p_{12} - \frac{\Lambda_{21} p_{21} + \Lambda_{23} p_{23}}{\Lambda_{21} + \Lambda_{23}} \right| = |p_{12} - p_{21}|.$$

The last equality follows from $\Lambda_{21} = \Lambda_{23}$ and $p_{21} = p_{23}$ by symmetry. Hence, trip-based and origin-based price fairness are equivalent.

Then, the optimization problem under price fairness is written as,

$$\begin{aligned}\mathcal{R}(\alpha) := \max_{q_1, q_2} \quad & \frac{\Lambda_{12} q_{12} \left(2\Lambda_{21} q_{21} \right)}{\Lambda_{12} q_{12} + 2\Lambda_{21} q_{21}} \left(b_{12} + b_{21} - b_{12} q_{12} - b_{21} q_{21} \right) \\ \text{s.t.} \quad & |b_{12}(1 - q_{12}) - b_{21}(1 - q_{21})| \leq (1 - \alpha)|p_{12}^* - p_{21}^*|, \\ & q_1, q_2 \in [0, 1].\end{aligned}$$

Notice that this problem is equivalent to the two-node case with parameters $(\Lambda_1, \Lambda_2) = (\Lambda_{12}, 2\Lambda_{21})$ and $(b_1, b_2) = (b_{12}, b_{21})$. Therefore, the optimal quantiles $q_{ij}(\alpha)$ coincide with those in the two-node case. More specifically, let $\tilde{q}(\alpha)$ denote the optimal quantile under α in the two-node case. Then $q_{12}(\alpha) = q_{32}(\alpha)$ coincide with $\tilde{q}_1(\alpha)$, and $q_{21}(\alpha) = q_{23}(\alpha)$ coincide with $\tilde{q}_2(\alpha)$.

To specify surplus, let Π_i denote the availability at node i. By symmetry, we have $\Pi_1 = \Pi_3$, and the balance equations yield

$$2\Pi_1 + \Pi_2 = 1 \text{ and } \Pi_1\lambda_{12} = \Pi_2\lambda_{21}.$$

Solving yields

$$\Pi_1 = \Pi_3 = \frac{\lambda_{21}}{\lambda_{12}+2\lambda_{21}}, \qquad \Pi_2 = \frac{\lambda_{12}}{\lambda_{12}+2\lambda_{21}}.$$

Then, surplus of each trip is,

$$\mathcal{S}_{12}(\alpha) = \mathcal{S}_{32}(\alpha) = \frac{b_{12}\lambda_{12}^2}{2\Lambda_{12}}\Pi_1 = \frac{b_{12}\Lambda_{12}q_{12}(\alpha)^2}{2}\frac{\Lambda_{21}q_{21}(\alpha)}{\Lambda_{12}q_{12}(\alpha)+2\Lambda_{21}q_{21}(\alpha)} = \frac{b_1\Lambda_1\tilde{q}_1(\alpha)^2}{4}\frac{\Lambda_2\tilde{q}_2(\alpha)}{\Lambda_1\tilde{q}_1(\alpha)+\Lambda_2\tilde{q}_2(\alpha)} = \frac{1}{2}\tilde{\mathcal{S}}_1(\alpha)$$

Similarly, $\mathcal{S}_{21}(\alpha) = \mathcal{S}_{23}(\alpha) = \frac{1}{2}\tilde{\mathcal{S}}_2(\alpha)$. Therefore, the total surplus in the three-node case is distinct from the two-node case, since

$$\mathcal{S}(\alpha) = \Lambda_{12}\mathcal{S}_{12}(\alpha) + \Lambda_{32}\mathcal{S}_{32}(\alpha) + \Lambda_{21}\mathcal{S}_{21}(\alpha) + \Lambda_{23}\mathcal{S}_{23}(\alpha) = 2\Lambda_{12}\mathcal{S}_{12}(\alpha) + 2\Lambda_{21}\mathcal{S}_{21}(\alpha) = \Lambda_1\tilde{\mathcal{S}}_1(\alpha) + \frac{\Lambda_2}{2}\tilde{\mathcal{S}}_2(\alpha),$$

which is less than $\tilde{\mathcal{S}}(\alpha) = \Lambda_1\tilde{\mathcal{S}}_1(\alpha) + \Lambda_2\tilde{\mathcal{S}}_2(\alpha)$.

(Access Fairness) Similarly, trip-based access fairness and origin-based access fairness coincide. The access of each trip is

$$A_{12} = A_{32} = \frac{\lambda_{12}}{\Lambda_{12}}\Pi_1 = \frac{1}{\Lambda_{12}}\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+2\lambda_{21}}, \quad A_{21} = A_{23} = \frac{\lambda_{21}}{\Lambda_{21}}\Pi_2 = \frac{1}{\Lambda_{21}}\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+2\lambda_{21}}.$$

Trip-based access fairness then minimizes

$$|A_{12}-A_{21}| = |A_{12}-A_{23}| = |A_{32}-A_{21}| = |A_{32}-A_{23}| = \left|\frac{1}{\Lambda_{12}} - \frac{1}{\Lambda_{21}}\right|\frac{\lambda_{12}\lambda_{21}}{\lambda_{12}+2\lambda_{21}} = \left|\frac{1}{\Lambda_{12}} - \frac{1}{\Lambda_{21}}\right|\frac{\Lambda_{12}\Lambda_{21}q_{12}q_{21}}{\Lambda_{12}q_{12}+2\Lambda_{21}q_{21}}.$$

Origin-based access fairness minimizes the differences in average access across nodes,

$$|\bar{A}_1 - \bar{A}_2| = |\bar{A}_2 - \bar{A}_3| = \left|A_{12} - \frac{\Lambda_{21}A_{21}+\Lambda_{23}A_{23}}{\Lambda_{21}+\Lambda_{23}}\right| = |A_{12}-A_{21}|.$$

The last inequality follows from $\Lambda_{21} = \Lambda_{23}$ and $A_{21} = A_{23}$. Therefore, under the stated symmetry conditions, trip-based and origin-based access fairness yield the same fairness criterion.

The optimization problem under access fairness is written as,

$$\begin{aligned} \mathcal{R}(\alpha) := \max_{q_1,q_2} \quad & \frac{\Lambda_{12}q_{12}\left(2\Lambda_{21}q_{21}\right)}{\Lambda_{12}q_{12}+2\Lambda_{21}q_{21}}(b_{12}+b_{21}-b_{12}q_{12}-b_{21}q_{21}) \\ \text{s.t.} \quad & \left|\frac{1}{\Lambda_{12}} - \frac{1}{\Lambda_{21}}\right|\frac{\Lambda_{12}\Lambda_{21}q_{12}q_{21}}{\Lambda_{12}q_{12}+2\Lambda_{21}q_{21}} \le (1-\alpha)|A_{12}^* - A_{21}^*|, \\ & q_1, q_2 \in [0,1]. \end{aligned}$$

Similar to price fairness, this problem is equivalent to the access fairness optimization problem with parameters $(\Lambda_{12}, 2\Lambda_{21})$ and (b_{12}, b_{21}). Therefore, the optimal quantiles $q_{ij}(\alpha)$ coincide with those in the two-node case, just as in the price fairness setting. It implies that, as in the two-node case, access changes linearly with α. Additionally, the three-node symmetric setting yields the same revenue and prices under any α, but not the same total surplus, as observed in the price fairness case. □

B.5. Multiple Units and Nodes

We consider scenarios where the system has more than two nodes and multiple units. The primary emphasis of this section lies in exploring whether there are fairness not seen in situations with two nodes that manifest in scenarios with 3 nodes. Moreover, we explore the difference between trip-based fairness (comparing all pairs of nodes) and origin-based fairness (each node does a weighted average over destinations).

Even under this restricted setting, finding all six optimal prices is computationally challenging due to the non-convex nature of the problem, even without fairness constraints (Banerjee et al. 2022). Section B.4 also demonstrates that deriving a closed-form solution—even under simplifying assumptions—remains complex when $N = 3$. Therefore, we rely on numerical analysis in this section.

Linear Demand. We conduct experiments on a small network comprising 3 nodes and 10 units under linear demand. To find the optimal solution, we utilize *Couenne*, a solver designed to find global optima for non-convex problems, within *Pyomo*, using an initial value obtained from running *Ipopt* (a solver that provides local optima) 10 times. The details of the experiments and the method for handling $q = \max\left(0, 1 - \frac{p_i}{b_i}\right)$ in price fairness are explained in Appendix C.

Figure 8 illustrates both trip-based and origin-based price fairness. Unlike the two-node scenario, Figure 8 presents a case where the system abandons certain markets (origin-destination pairs). For instance, under origin-based price fairness, beyond a certain threshold of α (in this case, 0.4), the system withdraws from the market for the trip from node 3 to node 2. In other words, the price for this trip exceeds the maximum support, $b_{3,2}$, leading to zero surplus. To achieve origin-based price fairness, the system strategically drops this market, and, in fact, the gap between the average prices at locations 1 and 2 increases. Consequently, origin-based price fairness may not be suitable in such situations, as it carries the potential risk of strategic market abandonment.

Figure 9 depicts the results under both trip-based and origin-based access fairness. Similar to the two-node scenario, both total surplus and social welfare decrease as α increases. However, in contrast to the two-node scenario, some trips experience surplus gains. The loss of affordability for one trip can benefit other trips that share the same origin, as vehicles are rerouted to those trips. A significant drawback of trip-based access fairness emerges beyond a certain threshold of α (e.g., $\alpha \geq 0.4$ in Figure 9), where revenue, consumer surplus, and social welfare decline sharply toward zero, resulting in a system shutdown. As Proposition 13 shows, perfect trip-based access fairness leads to a system shutdown.

Proposition 13 **(Perfect Trip-Based Access Fairness Results in System Shutdown)**. *If there exists $i \in [N]$ such that $\sum_{j \neq i} \Lambda_{ij} \neq \sum_{j \neq i} \Lambda_{ji}$, then $\mathcal{R}(1) = \mathcal{S}(1) = \mathcal{W}(1) = 0$.*

Proof. When perfect trip-based access fairness holds, $\Pi_i q_{ij} = A$ for all i and j due to the fairness constraint. The balance equations imply that

$$\sum_{k \neq i} \Lambda_{ik} \Pi_i q_{ik} = \sum_{k \neq i} \Lambda_{ki} \Pi_k q_{ki} \Rightarrow \left(\sum_{k \neq i} \Lambda_{ik} - \sum_{k \neq i} \Lambda_{ki} \right) A = 0, \quad \forall i \in [N].$$

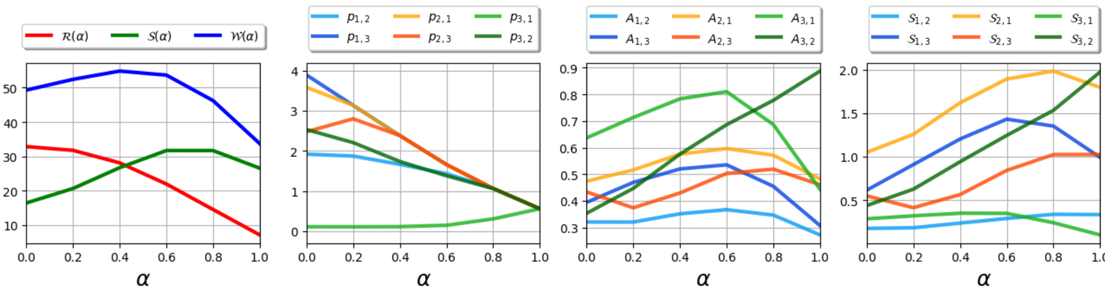
If there exists $i \in [N]$ s.t. $\sum_{j \neq i} \Lambda_{ij} \neq \sum_{j \neq i} \Lambda_{ji}$, A should be 0. Then,

$$\mathcal{R} = \sum_{i,j} \Pi_i \Lambda_{ij} q_{ij} p_{ij} = A \sum_{i,j} \Lambda_{ij} p_{ij} = 0,$$

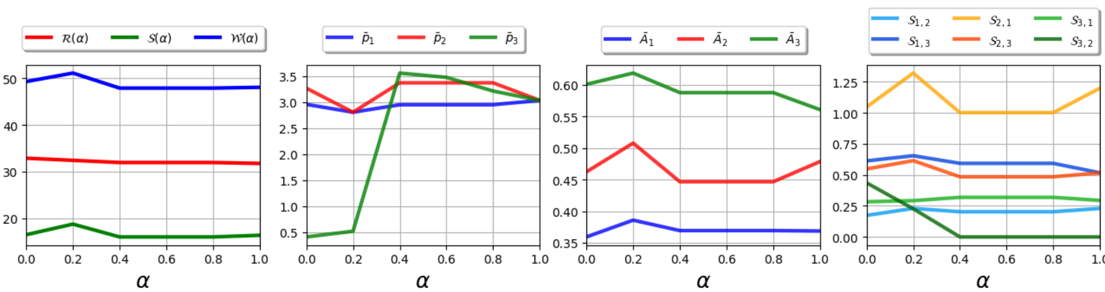
$$\mathcal{S} = \sum_{i,j} \Pi_i \mathbb{E}\left[(V_{ij} - p_{ij})^+\right] = \sum_{i,j} \Pi_i q_{ij} \mathbb{E}\left[V_{ij} - p_{ij} | V_{ij} \geq p_{ij}\right] = A \sum_{i,j} \mathbb{E}\left[V_{ij} - p_{ij} | V_{ij} \geq p_{ij}\right] = 0.$$

□

Figure 8 Impact of Price Fairness with 3 Nodes and 10 Units



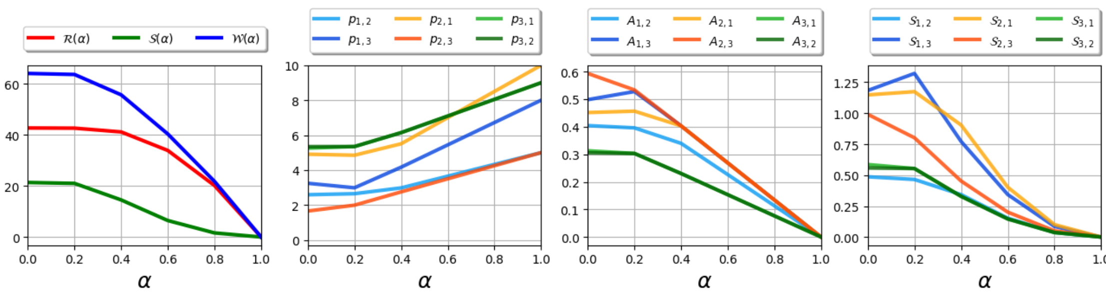
(a) Trip-based Price Fairness



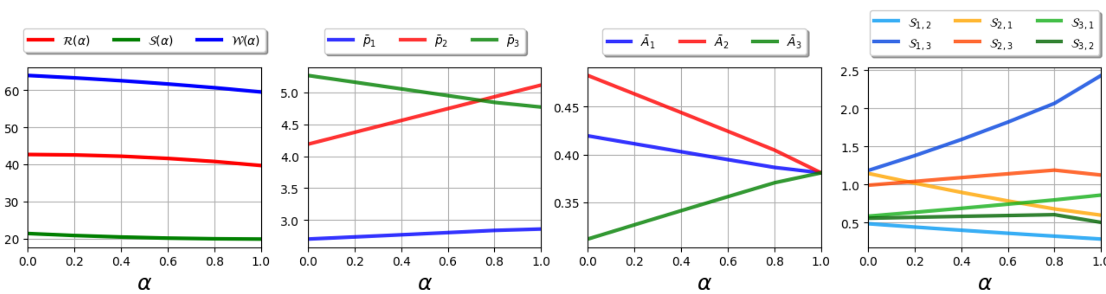
(b) Origin-based Price Fairness

Note. Parameters: $(\Lambda_{1,2}, \Lambda_{1,3}, \Lambda_{2,1}, \Lambda_{2,3}, \Lambda_{3,1}, \Lambda_{3,2}) = (9, 10, 5, 2, 7, 1)$ and $(b_{1,2}, b_{1,3}, b_{2,1}, b_{2,3}, b_{3,1}, b_{3,2}) = (3, 7, 8, 5, 1, 5)$. For all $i \in [3]$, $\bar{p}_i := \frac{\sum j \neq i \Lambda_{ij} p_{ij}}{\sum_{j \neq i} \Lambda_{ij}}$ and $\bar{A}_i := \frac{\sum j \neq i \Lambda_{ij} A_{ij}}{\sum_{j \neq i} \Lambda_{ij}}$.

Figure 9 Impact of Access Fairness with 3 Nodes and 10 Units



(a) Trip-based Access Fairness



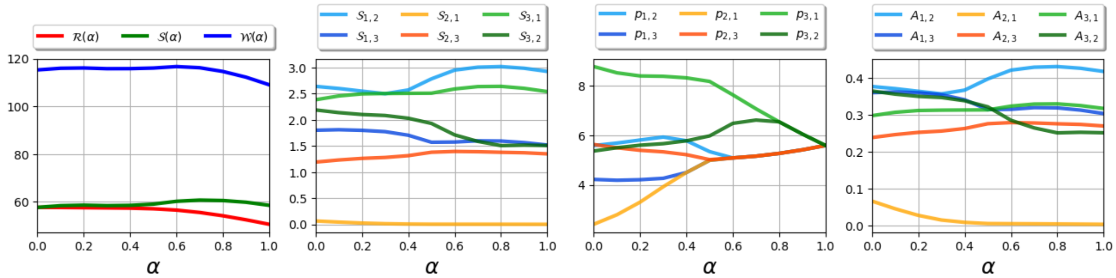
(b) Origin-based Access Fairness

Note. $(\Lambda_{1,2},\Lambda_{1,3},\Lambda_{2,1},\Lambda_{2,3},\Lambda_{3,1},\Lambda_{3,2}) = (10,2,7,2,6,1)$ and $(b_{1,2},b_{1,3},b_{2,1},b_{2,3},b_{3,1},b_{3,2}) = (5,8,10,5,9,9)$.

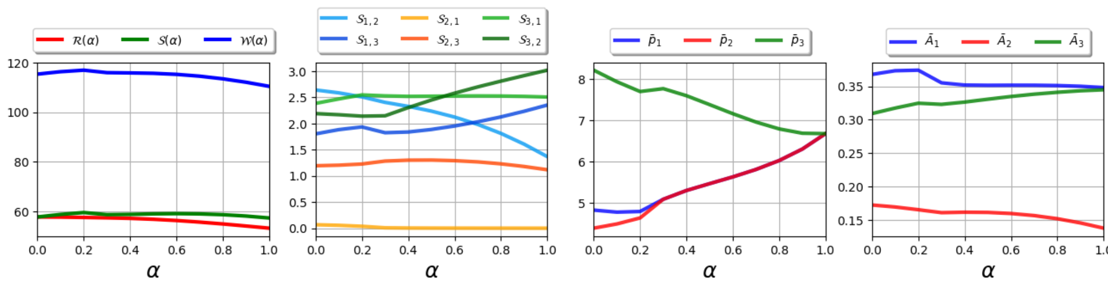
Exponential Demand. Figure 10 illustrates both trip-based and origin-based price fairness. In contrast to the scenario with only 2 nodes, Figure 10 reveals examples where the system gives up on certain markets (origin-destination pairs). For instance, for the trip-based price fairness, beyond a certain point of α, the price for the trip $(2,1)$ (representing the trip from node 2 to node 1) experiences a significant increase. This increase is indicated by the yellow line in Figure 10, leading to a decrease in its accessibility to consumers. Additionally, depending on whether it is trip-based or origin-based price fairness, the change in surplus for each trip varies.

Figure 11 depicts trip-based and origin-based access fairness for the same experimental setup. Similar to the case with 2 nodes, both total surplus and social welfare decline as α increases. In contrast to the scenario with 2 nodes, some routes experience surplus gains. The loss of affordability for one trip can lead to advantages for other trips, achieved by rerouting vehicles to those routes. A drawback of trip-based access fairness becomes evident beyond a specific α (e.g. $\alpha \geq 0.3$ in Figure 11), where revenue, consumer surplus, and social welfare experience a sharp decline towards zero (system shutdown).

Figure 10 Impact of Price Fairness with 3 Nodes and 10 Units



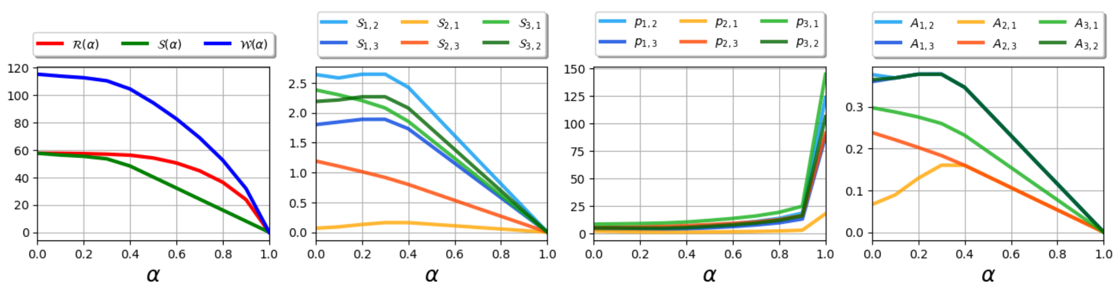
(a) Trip-based Price Fairness



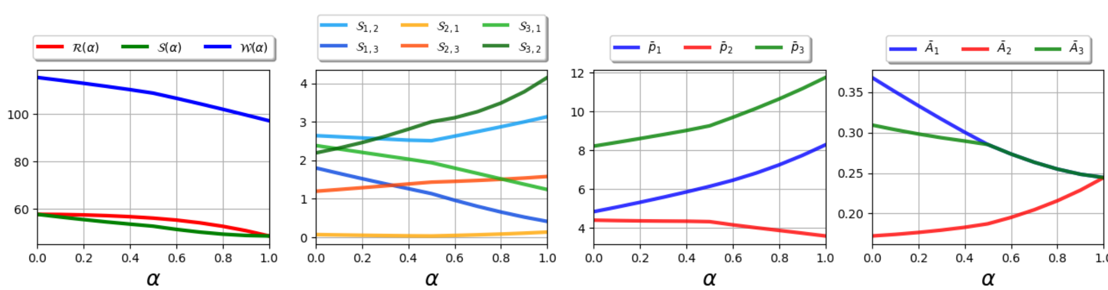
(b) Origin-based Price Fairness

Note. $(\Lambda_{1,2}, \Lambda_{1,3}, \Lambda_{2,1}, \Lambda_{2,3}, \Lambda_{3,1}, \Lambda_{3,2}) = (4, 5, 5, 8, 10, 2)$ and $(\beta_{1,2}, \beta_{1,3}, \beta_{2,1}, \beta_{2,3}, \beta_{3,1}, \beta_{3,2}) = (\frac{1}{7}, \frac{1}{5}, 1, \frac{1}{5}, \frac{1}{8}, \frac{1}{6})$.

Figure 11 Impact of Access Fairness with 3 Nodes and 10 Units



(a) Trip-based Access Fairness

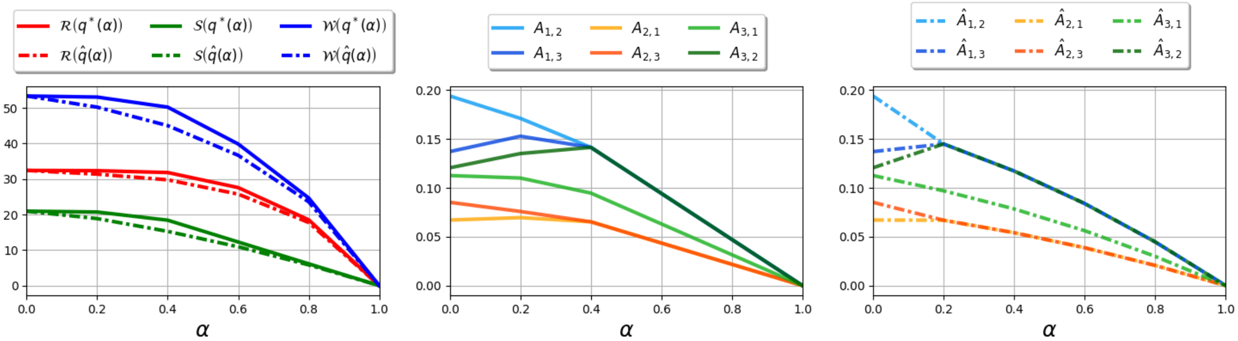


(b) Origin-based Access Fairness

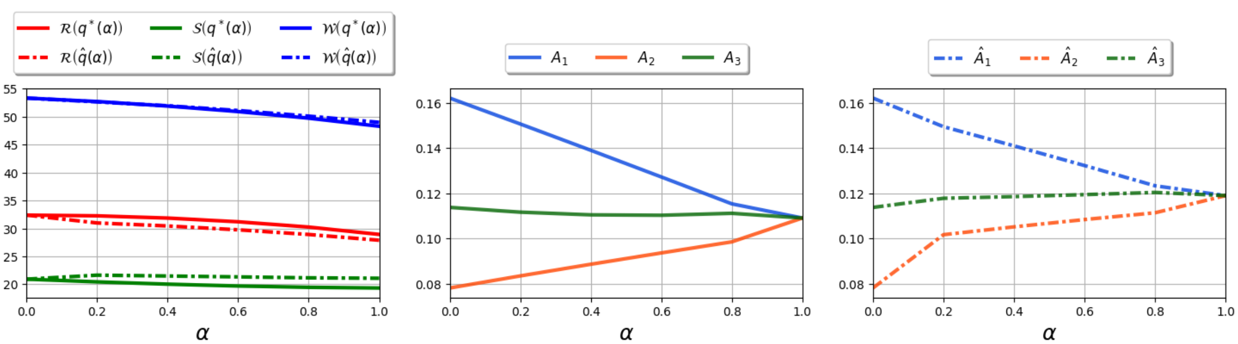
Note. Parameters are the same with Figure 10.

B.6. Access Fairness Heuristics

Figure 12 Comparison between Original and Approximated Framework under Access Fairness



(a) Trip-based Access Fairness



(b) Origin-based Access Fairness

Note. We use an exponential demand model, i.e., $\lambda_{ij} = \Lambda_{ij}\exp(-\beta_{ij}p_{ij})$. The access $\hat{A}_{ij}$ from node i to node j is calculated based on $\hat{q}(\alpha)$, given by $\hat{A}_{ij} = \hat{q}_{ij}(\alpha)\Pi_i(\hat{q}_{ij}(\alpha))$ for all $(i,j) \in \mathcal{A}$. The average access values are defined as $A_i := \frac{\sum_{(i,j)\in\mathcal{A}} \Lambda_{ij} A_{ij}}{\sum_{(i,j)\in\mathcal{A}} \Lambda_{ij}}$ and $\hat{A}_i := \frac{\sum_{(i,j)\in\mathcal{A}} \Lambda_{ij} \hat{A}ij}{\sum_{(i,j)\in\mathcal{A}} \Lambda_{ij}}$ for all $i \in [3]$. The parameters used are $(\Lambda_{1,2}, \Lambda_{1,3}, \Lambda_{2,1}, \Lambda_{2,3}, \Lambda_{3,1}, \Lambda_{3,2}) = (4, 5, 5, 8, 10, 2)$, $(\beta_{1,2}, \beta_{1,3}, \beta_{2,1}, \beta_{2,3}, \beta_{3,1}, \beta_{3,2}) = (1/3, 1/5, 1/4, 1/5, 1/8, 1/6)$, and $(\mu_{1,2}, \mu_{1,3}, \mu_{2,1}, \mu_{2,3}, \mu_{3,1}, \mu_{3,2}) = (1, 2, 3, 1, 1, 1)$.

C. Experimental Details

We use *Ipopt* and *Couenne* to obtain the optimal solutions. The settings for each solver are the same as their default configurations, except that for *Couenne*, we set `time_limit=7200` (2 hours).

When there are more than 2 units under linear demand, we need to consider the case that the price exceeds the support, i.e., $p_{ij} > b_{ij}$. To reflect this case, we use the following optimization problem with 3 nodes K units with an arbitrary large number $M \geq 0$.

$$
\begin{aligned}
\max_{\mathbf{q},\mathbf{p},\mathbf{z}\geq 0} \quad & \sum_{i\in[3]}\sum_{j\in[3]\setminus\{i\}} \Lambda_{ij} q_{ij} p_{ij} \Pi_i \\
\text{subject to} \quad & \pi(\mathbf{x}) \sum_{j\in[3]\setminus\{i\}} \Lambda_{ij} q_{ij} = \sum_{j\in[3]\setminus\{i\}} \Lambda_{ji} q_{ji} \pi(\mathbf{x}-e_i+e_j) \quad \forall \mathbf{x}\in\{\mathbf{x} \mid \mathbf{x}\in\mathbb{S}_{3,K}, \mathbf{x}_i \geq 1\}, \quad \forall i\in[3], \\
& \sum_{\mathbf{x}\in\mathbb{S}_{3,K}} \pi(\mathbf{x}) = 1, \\
& \Pi_i = \sum_{\mathbf{x}\in\{\mathbf{x}\mid\mathbf{x}\in\mathbb{S}_{3,K},\mathbf{x}_i\geq 1\}} \pi(\mathbf{x}), \quad \forall i\in[3], \\
& q_{ij} \geq 0, \quad \forall i,j\in[3], i\neq j, \\
& q_{ij} \leq 1, \quad \forall i,j\in[3], i\neq j, \\
& q_{ij} = \left(1-\frac{p_{ij}}{b_{ij}}\right) z_{ij}, \quad \forall i,j\in[3], i\neq j, \\
& 1-\frac{p_{ij}}{b_{ij}} \geq z_{ij}, \quad \forall i,j\in[3], i\neq j, \\
& 1-\frac{p_{ij}}{b_{ij}} \geq -M\cdot(1-z_{ij}), \quad \forall i,j\in[3], i\neq j, \\
& p_{ij} \geq 0, \quad \forall i,j\in[3], i\neq j, \\
& z_{ij} \in \{0,1\}, \quad \forall i,j\in[3], i\neq j, \\
& p_{ij} - p_{kl} \leq (1-\alpha)\left(\max_{i,j} p^*_{ij} - \min_{i,j} p^*_{ij}\right), \quad \forall i,j,k,l\in[3], i\neq j, k\neq l.
\end{aligned}
$$

The above problem addresses trip-based price fairness. For origin-based price fairness, the last constraint can be replaced as follows:

$$
\frac{\sum_{i\neq k}\Lambda_{ik}p_{ik}}{\sum_{i\neq k}\Lambda_{ik}} - \frac{\sum_{j\neq k}\Lambda_{jk}p_{jk}}{\sum_{j\neq k}\Lambda_{jk}} \leq (1-\alpha)\left(\max_i \frac{\sum_{i\neq k}\Lambda_{ik}p^*_{ik}}{\sum_{i\neq k}\Lambda_{ik}} - \min_i \frac{\sum_{i\neq k}\Lambda_{ik}p^*_{ik}}{\sum_{i\neq k}\Lambda_{ik}}\right), \quad \forall i,j\in[3], i\neq j.
$$

D. Price Fairness Heuristic Algorithm

Algorithm 1 (Trip-based) α-Price Fair Algorithm

Input: $\mathcal{A}$, $\{p^*_{ij}\}_{(i,j)\in\mathcal{A}}$, $\{d_{ij}\}_{(i,j)\in\mathcal{A}}$, $\{F_{ij}\}_{(i,j)\in\mathcal{A}}$, $\{\Lambda_{ij}\}_{(i,j)\in\mathcal{A}}$, $\{\mu_{ij}\}_{(i,j)\in\mathcal{A}}$, K, α, `grid`, `step`

Output: $\mathbf{p}^{\texttt{ALG}}$, $R^{\texttt{ALG}}$

1: $(\underline{i},\underline{j}),(\bar{i},\bar{j}) \leftarrow \left(\arg\min_{(i,j)\in\mathcal{A}} \frac{p^*_{ij}}{d_{ij}},\ \arg\max_{(i,j)\in\mathcal{A}} \frac{p^*_{ij}}{d_{ij}}\right)$

2: $\left(\underline{p},\ \bar{p}, R^{\texttt{ALG}}\right) \leftarrow \left(\frac{p^*_{\underline{ij}}}{d_{\underline{ij}}},\ \frac{p^*_{\bar{i}\bar{j}}}{d_{\bar{i}\bar{j}}}, 0\right)$

3: **for** β in $[1,\ldots,\alpha+\texttt{grid},\alpha]$ **do**

4: $\quad W \leftarrow (1-\beta)\left(\bar{p}-\underline{p}\right)$

5: $\quad (\texttt{bottom},\ \texttt{top}) \leftarrow \left(\max(0,\underline{p}-\frac{W}{2}),\ \max(0,\underline{p}-\frac{W}{2})+W\right)$

6: $\quad$ **while** $\texttt{top} \le \bar{p}+\frac{W}{2}$ **do**

7: $\quad\quad$ **for** (i,j) **in** $\mathcal{A}$ **do**

8: $\quad\quad\quad$ **if** $\frac{p^*_{ij}}{d_{ij}} < \texttt{bottom}$ **then**

9: $\quad\quad\quad\quad p_{ij} \leftarrow \texttt{bottom}\cdot d_{ij}$

10: $\quad\quad\quad$ **else if** $\frac{p^*_{ij}}{d_{ij}} > \texttt{top}$ **then**

11: $\quad\quad\quad\quad p_{ij} \leftarrow \texttt{top}\cdot d_{ij}$

12: $\quad\quad\quad$ **else**

13: $\quad\quad\quad\quad p_{ij} \leftarrow p^*_{ij}$

14: $\quad\quad$ **if** $\texttt{bottom} < \underline{p}$ **then** ▷ Consider a feasible solution under a tight condition

15: $\quad\quad\quad p_{\underline{i},\underline{j}} \leftarrow \texttt{bottom}\cdot d_{\underline{i},\underline{j}}$

16: $\quad\quad$ **else if** $\texttt{top} > \bar{p}$ **then**

17: $\quad\quad\quad p_{\bar{i},\bar{j}} \leftarrow \texttt{top}\cdot d_{\bar{i},\bar{j}}$

18: $\quad\quad$ **if** $R_K(\mathbf{p}) > R^{\texttt{ALG}}$ **and** $\sum_{(i,j)\in\mathcal{A}} \frac{\Lambda_{ij}\bar{F}_{ij}(q_{ij})}{\mu_{ij}}\Pi_i(p) \le K$ **then**

19: $\quad\quad\quad (\mathbf{p}^{\texttt{ALG}},\ R^{\texttt{ALG}}) \leftarrow (\mathbf{p},\ R_K(\mathbf{p}))$

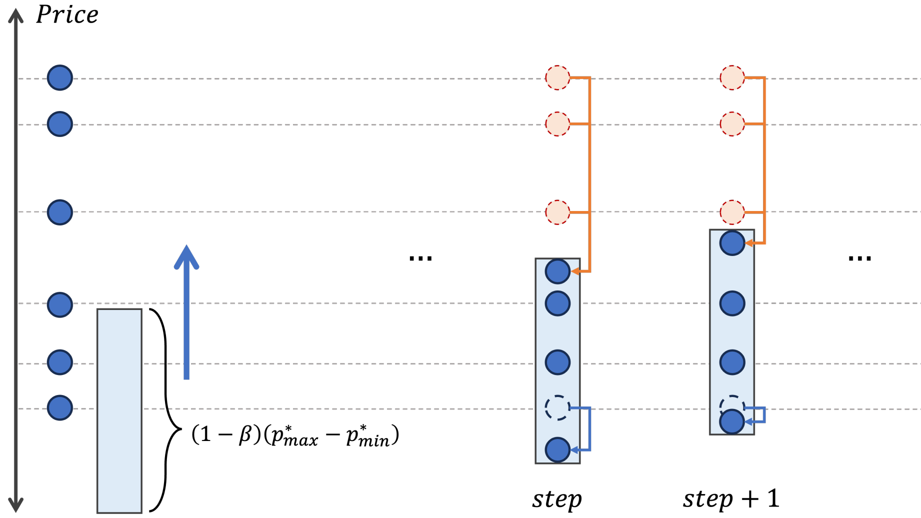
20: $\quad\quad (\texttt{bottom},\ \texttt{top}) \leftarrow (\texttt{bottom}+\texttt{step},\ \texttt{top}+\texttt{step})$

Figure 13 provides an overview of Algorithm 1. The algorithm introduces a sliding window, W, that adjust the original solution within the window. We explore several sizes of windows, all of which are equal to or smaller than the fairness gap, i.e., $(1-\beta)\left(\max_{i,j}\frac{p^*_{ij}}{d_{ij}} - \min_{i,j}\frac{p^*_{ij}}{d_{ij}}\right)$ where $\beta \in \{\alpha, \alpha+\texttt{grid}, ..., 1\}$ (lines 4-5). The window is initially centered around the minimum price (line 5) and then gradually shifts towards the maximum price using a specific size (lines 6 and 20). The feasible solution exceeds the `top` (or falls below the `bottom`) of the window, it is readjusted to align with the `top` (or `bottom`) of the window (lines 8-11).

If the prices are not tightly bound by the window, we adjust the maximum (or minimum) price of the feasible solution is adjusted to the `top` (or `bottom`) of the window (lines 14-17). In the final step, we select the solution that yields the highest objective value among feasible solutions (lines 18-20).

Within the algorithm, the determination of the objective value for the modified price, $R_K(\mathbf{p})$, entails the calculation of steady-state probabilities. For the computation of steady-state probabilities, we use dynamic

Figure 13 Algorithm for Price Fairness



Note. Each blue circle represents the price of a trip. A sliding window of size $(1-\beta)(p^*_{\max}-p^*_{\min})$ is employed with $d_{ij}=1$, where $\beta \in \{\alpha, \alpha+\texttt{grid}, \alpha+2\cdot\texttt{grid}, \ldots, 1\}$. At each step, the algorithm adjusts prices to fit precisely within the specified window. For instance, the algorithm shifts dashed orange circles (prices outside the window) to the window boundary and repositions dashed blue circles (prices closest to the boundary without a price) to the boundary.

programming with a time complexity of $\mathcal{O}(N^2K)$. Further details on this approach can be found in Section D.1.

In the context of origin-based price fairness, additional conditions are required to assess whether the disparity between the maximum and minimum weighted average prices falls within the designated price threshold. To elaborate further, adjustments are necessary in the line 16 as presented:

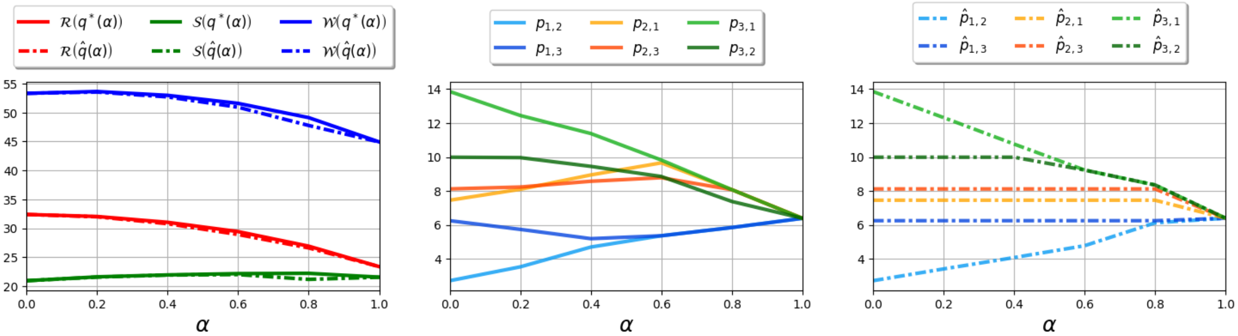
$$\textbf{if } R_K(\mathbf{p}) > R^{ALG} \textbf{ and } \sum_{(i,j)\in\mathcal{A}} \frac{\Lambda_{ij}\bar{F}_{ij}(q_{ij})}{\mu_{ij}}\Pi_i(p) \leq K$$

$$\textbf{and } \max_i\left(\sum_j T_{ij}\frac{p_{ij}}{d_{ij}}\right) - \min_i\left(\sum_j T_{ij}\frac{p_{ij}}{d_{ij}}\right) \leq (1-\beta)M^* \textbf{ then,}$$

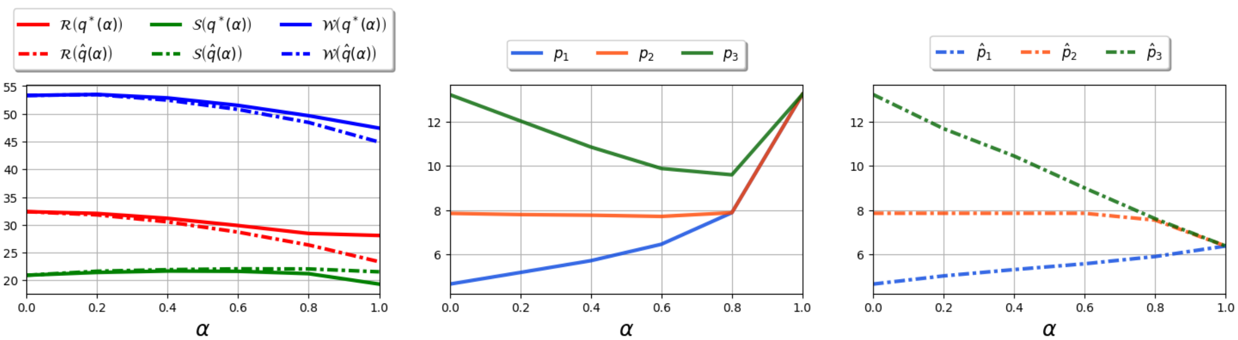
where $M^* := \max_i\left(\sum_j T_{ij}\frac{p^*_{ij}}{d_{ij}}\right) - \min_i\left(\sum_j T_{ij}\frac{p^*_{ij}}{d_{ij}}\right)$.

Figure 14 illustrates the algorithm's performance (both for trip and origin) with 3 nodes and 5 units. Figure 14a shows that our algorithm struggles to account for price fluctuations within the range of maximum and minimum prices, resulting in a disparity of total surplus between the optimal scenario and the heuristic approach. However, both the trip-based and origin-based pricing algorithms effectively approximate the revenue, particularly for small values of α.

Figure 14 Comparison between Original and Approximated Framework under Price Fairness



(a) Trip-based Price Fairness



(b) Origin-based Price Fairness

Note. Demand function and parameters are the same as Figure 12.

D.1. Modified Buzen Algorithm in Closed Queueing Networks

As mentioned in Banerjee et al. (2022), the system is modeled as a closed migration process (Kelly and Yudovina 2014). The steady-state probability of a state $\mathbf{x} \in \mathbb{S}_{N,K}$ for a given $\mathbf{q}$ is expressed as:

$$\pi(\mathbf{x}) = \frac{1}{Z_{N,K}(\mathbf{q})} \left[\prod_{i\in[N]} \left(\frac{w_i(\mathbf{q})}{\sum_{k\in[N]\setminus\{i\}} \Lambda_{ik} q_{ik}} \right)^{x_i} \right] \left[\prod_{(i,j)\in\mathcal{A}} \left(\frac{w_{ij}(\mathbf{q})}{\mu_{ij}} \right)^{y_{ij}} \frac{1}{y_{ij}!} \right],$$

where $w_i(\mathbf{q})$ and $w_{ij}(\mathbf{q})$ are the invariant distributions corresponding to the $N^2 \times N$ routing probability matrix. This matrix is defined by $P_{i,ij} = \frac{\Lambda_{ij} q_{ij}}{\sum_{k\in[N]\setminus\{i\}} \Lambda_{ik} q_{ik}}$ and $P_{ij,j} = 1$ for all $(i,j) \in \mathcal{A}$. We remark that the invariant distribution, w, can be computed based on the invariant distribution without travel time, $\tilde{w}$. Specifically, $w_i(\mathbf{q}) = \frac{1}{2}\tilde{w}_i(\mathbf{q})$ and $w_{ij}(\mathbf{q}) = \frac{1}{2}\tilde{w}_i(\mathbf{q}) \frac{\Lambda_{ij} q_{ij}}{\sum_{k\in[N]\setminus\{i\}} \Lambda_{ik} q_{ik}}$ (Banerjee et al. 2022). For simplicity, let $\lambda_i = \sum_{k\in[N]\setminus\{i\}} \Lambda_{ik} q_{ik}$, $w_i(\mathbf{q}) = w_i$ for all $i \in [N]$ (as well as $w_{ij}(\mathbf{q})$), and $Z_{N,K}(\mathbf{q}) = Z_{N,K}$.

To calculate metrics such as revenue, total surplus, and social welfare, we first need to determine $\Pi_i(\mathbf{x})$ for all $i \in [N]$, which is given by

$$\Pi_i(\mathbf{x}) = \sum_{\mathbf{x} \mid x_i \geq 1} \pi(\mathbf{x}) = \frac{Z_{N,K-1}}{Z_{N,K}} \frac{w_i}{\lambda_i}.$$

Here, $Z_{N,K}(\mathbf{q})$ is the normalizing constant that ensures the sum of steady-state probabilities equals 1, i.e., $Z_{N,K}(\mathbf{q}) = \sum_{\mathbf{x}\in\mathbb{S}_{N,K}} \pi(\mathbf{x})$. However, computing $Z_{N,K}(\mathbf{q})$ requires evaluating $\binom{K+N^2-1}{K}$ possible states, which is computationally expensive.

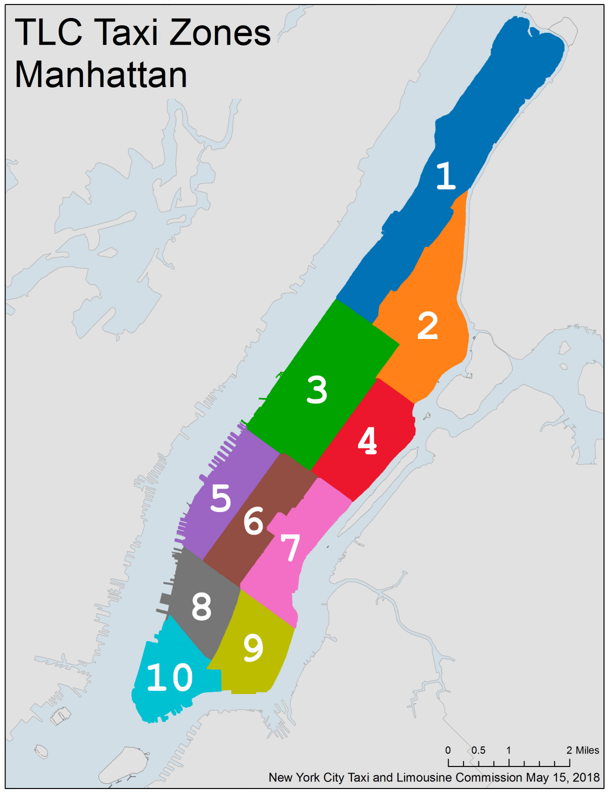
To address this, we use a modified version of the Buzen Algorithm (Buzen 1973), which incorporates travel time into the algorithm originally designed for negligible travel time. We use dynamic programming using the fact that

$$\begin{aligned}
Z_{N,K} &= \sum_{\mathbf{x}\in\mathbb{S}_{N,K}, x_N=0} \pi(\mathbf{x}) + \sum_{\mathbf{x}\in\mathbb{S}_{N,K}, x_N\geq 1} \pi(\mathbf{x}) \\
&= \sum_{k=0}^{K} \sum_{\mathbf{x}\in\mathbb{S}_{N,K}, \sum_{i=1}^{N-1} x_i = K-k} \pi(\mathbf{x}) + \frac{w_N}{\lambda_N} \sum_{\mathbf{x}\in\mathbb{S}_{N,K}, x_N\geq 1} \left[\prod_{i\in[N]} \left(\frac{w_i}{\lambda_i} \right)^{x_i - 1} \right] \left[\prod_{(i,j)\in\mathcal{A}} \left(\frac{w_{ij}}{\mu_{ij}} \right)^{y_{ij}} \frac{1}{y_{ij}!} \right] \\
&= \sum_{k=0}^{K} \left(\sum_{\sum_{i=1}^{N-1} y_{iN} + y_{Ni} = k} \left(\frac{w_{iN}}{\mu_{iN}} \right)^{y_{iN}} \frac{1}{y_{iN}!} \left(\frac{w_{Ni}}{\mu_{Ni}} \right)^{y_{Ni}} \frac{1}{y_{Ni}!} \right) Z_{N-1,K-k} + \frac{w_N}{\lambda_N} Z_{N,K-1} \\
&= \sum_{k=0}^{K} \frac{1}{k!} \left(\sum_{i=1}^{N-1} \left(\frac{w_{iN}}{\mu_{iN}} + \frac{w_{Ni}}{\mu_{Ni}} \right) \right)^{k} Z_{N-1,K-k} + \frac{w_N}{\lambda_N} Z_{N,K-1},
\end{aligned}$$

where we initialize $Z_{1,k} = \left(\frac{w_1}{\lambda_1} \right)^k$ for $k \in [K]$ and $Z_{n,0} = 1$ for $n \in [N]$. Then, we can compute $Z_{N,K}$ in $\mathcal{O}(NK^2)$.

E. Case Study

Figure 15 Neighborhoods of Manhattan Mapped to 10 Zones



Note. 10 neighborhoods are constructed based on taxi zones.

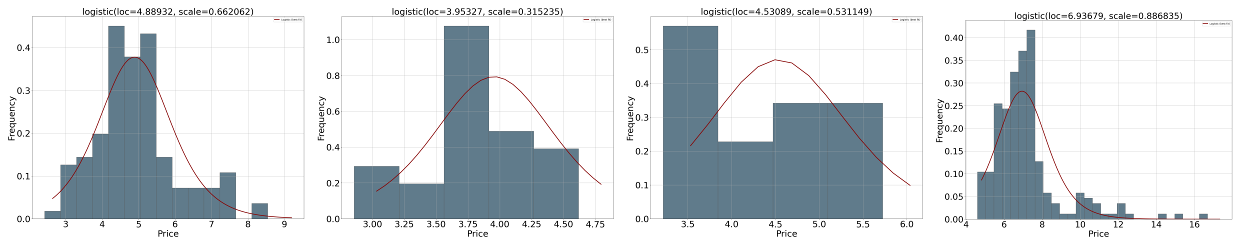
E.1. Details

Table 1 presents the average residual sum of squares for uniform, exponential, and logistic demand relative to the empirical distribution, highlighting that logistic demand has the lowest RSS. Figure 16 shows the empirical distribution of prices and the fitted logistic distribution for a randomly selected set of 4 trips. It is evident that the price distribution reveals a poor fit for the uniform and exponential distributions.

Table 1 The Average RSS of Each Demand Function

Demand Function	Uniform	Exponential	Logistic
RSS	0.1828	0.1893	0.0403

Figure 16 Logistic Demand Estimation (Price per Miles)



Trip (1,6) Trip (2,10) Trip (9,1) Trip (10,5)

Note. The bins represent the empirical distribution of prices in the dataset, while the red curve illustrates the estimated logistic distribution.

The following tables present the estimated parameters of network. Each value within the tables corresponds to the respective value from node i (row) to node j (column).

Table 2 The Number of Trips via Uber (U_{ij})

	1	2	3	4	5	6	7	8	9	10
1	-	216	120	66	56	80	31	16	11	26
2	236	-	130	152	37	76	68	16	18	20
3	133	72	-	279	148	359	65	35	7	52
4	34	101	238	-	91	529	169	45	17	97
5	39	20	66	84	-	397	99	109	17	118
6	43	23	73	134	238	-	168	137	38	142
7	24	17	54	188	121	520	-	109	52	105
8	13	5	17	80	137	318	110	-	38	161
9	7	14	19	52	61	179	104	101	-	106
10	17	11	28	45	148	244	97	143	44	-

Note. Each value corresponds to the number of trips undertaken by Uber between 7:00 AM and 9:00 AM on March 3, 2023.

Table 3 Estimated Market Size (Λ_{ij})

	1	2	3	4	5	6	7	8	9	10
1	-	357	189	113	79	127	47	26	16	37
2	389	-	235	271	55	125	97	20	31	29
3	268	177	-	623	274	685	129	62	12	91
4	65	284	526	-	182	1108	384	77	40	180
5	70	39	151	150	-	692	175	173	27	194
6	75	38	154	351	451	-	413	245	63	262
7	40	39	105	406	206	932	-	179	94	178
8	20	7	36	116	215	511	182	-	62	225
9	14	28	26	86	96	276	176	144	-	151
10	20	13	39	72	200	355	149	197	76	-

Note. Each value corresponds to the number of trips undertaken by Uber between 7:00 AM and 9:00 AM on March 3, 2023.

Table 4 **Estimated k_{ij}**

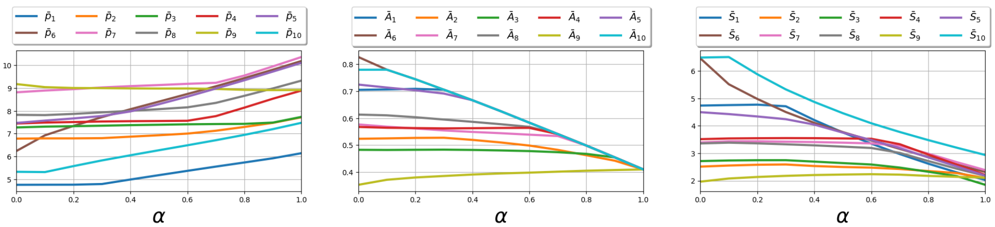
	1	2	3	4	5	6	7	8	9	10
1	-	297.00	331.10	1779.63	12627.13	1611.61	232620.86	2864746.28	12989702.72	290554.83
2	219.83	-	1664.15	166.08	10394.42	1008.56	33889.19	1156.98	40034.44	279481.01
3	75.96	0.20	-	3287.73	382.53	625.02	18465.05	2348.84	339.67	3804.07
4	1125.65	100.67	2330.11	-	758.02	349.14	215.04	643.57	597.39	6939.68
5	2870.71	1510.94	100.37	9426.91	-	481.50	1978.04	634.49	1366.90	3210.78
6	3264.35	1739.99	702.07	482.07	333.29	-	515.19	518.42	279.20	927.66
7	4460.56	209.24	3272.19	328.79	7066.89	346.40	-	1769.82	144.82	355.57
8	21527.60	48263314.95	1400.02	1026.71	1793.69	633.41	1073.84	-	1536.10	185.77
9	5066.28	9808.57	593.63	873.74	5085.45	1530.41	250.28	259.04	-	87.74
10	1263.55	12738.92	25668.42	2062.02	2494.80	582.00	255.49	185.45	342.35	-

Table 5 **Estimated β_{ij}**

	1	2	3	4	5	6	7	8	9	10
1	-	0.84	0.94	1.31	2.16	1.51	2.69	3.33	4.13	2.98
2	0.76	-	0.98	0.61	1.73	1.11	2.01	1.48	2.66	3.17
3	0.70	-0.21	-	0.90	0.76	0.76	1.36	1.24	0.93	1.58
4	1.18	0.52	0.86	-	0.92	0.68	0.72	1.02	1.30	2.02
5	1.77	1.36	0.53	1.11	-	0.58	0.79	0.76	0.91	1.21
6	1.68	1.17	0.73	0.65	0.54	-	0.56	0.69	0.67	1.03
7	1.96	0.86	0.99	0.62	0.94	0.54	-	0.84	0.62	1.14
8	2.00	2.93	1.06	0.93	0.88	0.76	0.81	-	0.66	0.53
9	1.88	2.04	1.03	1.04	1.06	0.92	0.66	0.48	-	0.59
10	1.56	1.94	1.65	1.43	1.13	0.97	0.92	0.52	0.71	-

E.2. Additional Results (7 - 9 AM)

Figure 17 The Change of Weighted Average of Price/mile, Access, and Surplus



Note. $\bar{p}_i := \frac{\sum_{j\neq i} \Lambda_{ij} p_{ij}/d_{ij}}{\sum_{j\neq i} \Lambda_{ij}}$, $\bar{A}_i := \frac{\sum_{j\neq i} \Lambda_{ij} A_{ij}}{\sum_{j\neq i} \Lambda_{ij}}$, and $\bar{S}_i := \frac{\sum_{j\neq i} \Lambda_{ij} S_{ij}}{\sum_{j\neq i} \Lambda_{ij}}$ for all $i \in [10]$.

Figure 18 The Price/mile Changes across 10 Neighborhoods under Price Fairness

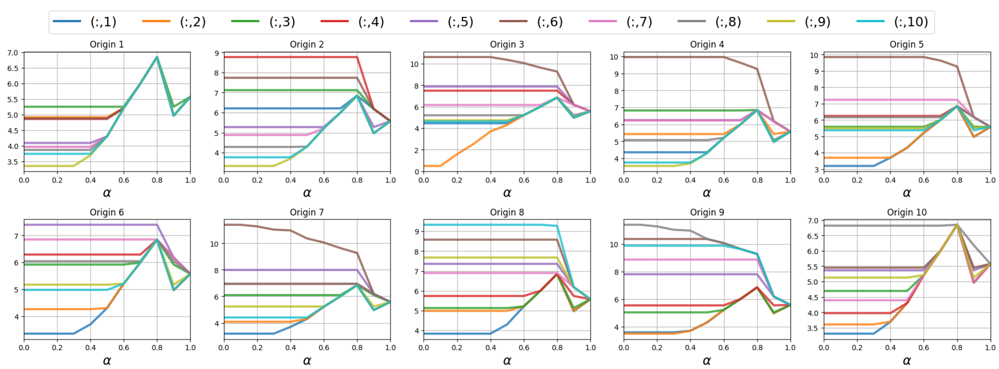
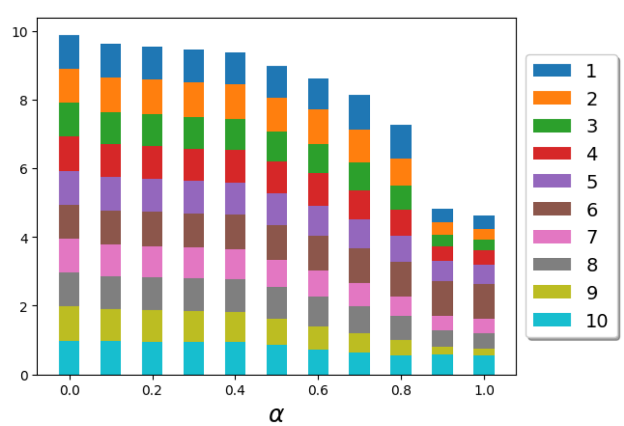


Figure 19 The Availability of Each Location



Note. The x-axis corresponds to α, while the y-axis corresponds to availability. Each color represents a distinct region, and the length of each bar indicates the corresponding availability, Π_i, obtained from the approximated problem.

E.3. Another Time Frame (6 − 8 PM)

We maintain the same setting as described in Section 5, but adjust the time frame. Specifically, we analyze trips with pick-up times between 6 PM and 8 PM on March 3, 2023, during which there are $K = 3680$ units. The optimal value we obtained have less than 5% error.

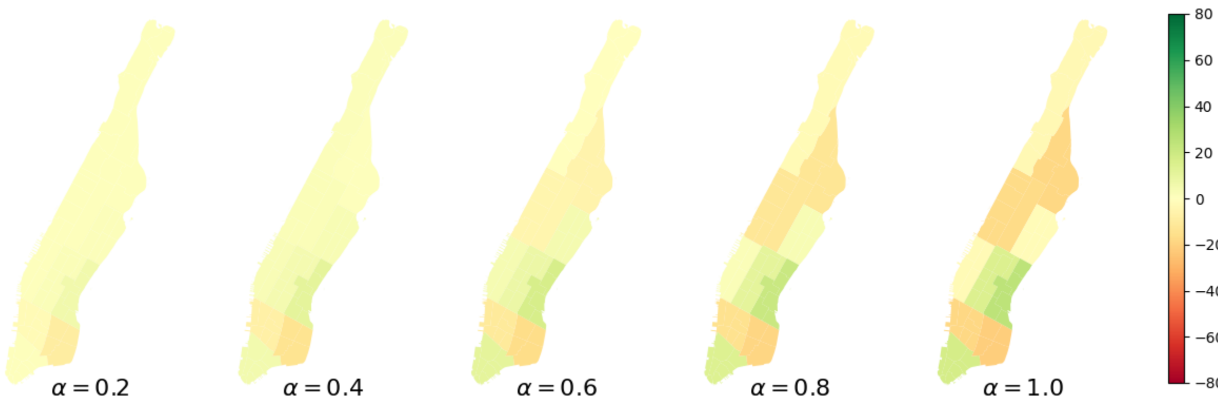
E.3.1. Access Fairness Table 6 presents the effects of access fairness on revenue, total surplus, and social welfare for different α values under origin-based access fairness. Revenue and social welfare consistently decrease as α increases, consistent with the findings in Section 5. For total surplus, there is a slight initial increase followed by a decrease.

Table 6 Revenue, Total Surplus, and Social Welfare Changes under Origin-based Access Fairness (%)

α	0.2	0.4	0.6	0.8	1.0
$\hat{\mathcal{R}}(\alpha)/\hat{\mathcal{R}}(0)$	99.88	99.53	99.14	98.64	97.63
$\hat{\mathcal{S}}(\alpha)/\hat{\mathcal{S}}(0)$	100.24	101.07	101.14	100.21	98.92
$\hat{\mathcal{W}}(\alpha)/\hat{\mathcal{W}}(0)$	99.98	99.95	99.69	99.07	97.99

However, since the surplus is calculated based on prices from heuristics, it is challenging to definitively conclude that it increases, particularly given that the optimal values obtained have an error margin of less than 5%. Therefore, we infer that all metrics exhibit only minor changes during this time frame. This subtle variation can be attributed to the relatively small initial access disparity; in other words, origin-based access is relatively uniform even without fairness criteria. Specifically, the access disparity, defined as the gap between maximum and minimum access, is approximately 0.2, compared to approximately 0.5 in Section 5.

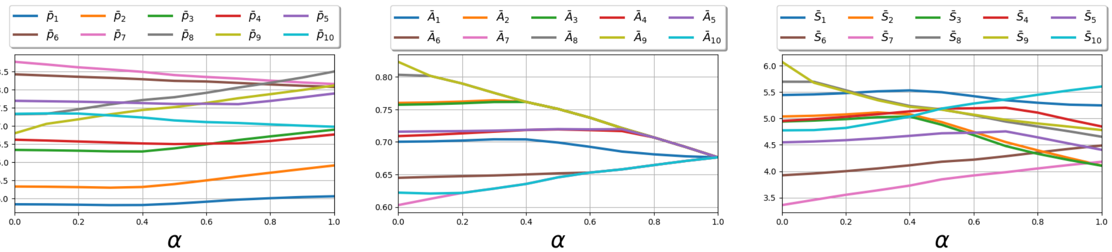
Figure 20 The Surplus Change across 10 Neighborhoods under Access Fairness (%)



Note. The scale corresponds to $\frac{\mathcal{S}_i(\alpha)-\mathcal{S}_i(0)}{\mathcal{S}_i(0)} \times 100$ for all locations $i \in [10]$.

Figure 20 illustrates the change in weighted average surplus at each location with respect to that without fairness criteria. With a little access fairness, the majority of locations experience a small surplus gain, except for a few zones including the Lower East Side (location 9) and SoHo (location 8), which initially had the highest access, as shown in Figure 21.

Figure 21 The Change of Weighted Average of Price/mile, Access, and Surplus



Note. $\bar{p}_i := \frac{\sum_{j\neq i}\Lambda_{ij}p_{ij}/d_{ij}}{\sum_{j\neq i}\Lambda_{ij}}$, $\bar{A}_i := \frac{\sum_{j\neq i}\Lambda_{ij}A_{ij}}{\sum_{j\neq i}\Lambda_{ij}}$, and $\bar{S}_i := \frac{\sum_{j\neq i}\Lambda_{ij}S_{ij}}{\sum_{j\neq i}\Lambda_{ij}}$ for all $i \in [10]$.

This aligns with the expectation that reducing access is associated with increasing prices for trips originating from the Lower East Side and SoHo, leading to surplus loss. Conversely, with higher α, midtown (locations 6 and 7) and the Financial District (location 10) notably experience surplus enhancement due to their initially lower access.

E.3.2. Price Fairness Table 7 shows the influence of trip-based price fairness on revenue, total surplus, and social welfare with α. As α increases, revenue and social welfare decrease, but consumer surplus experiences a decrease and then increase at $\alpha = 0.8$.

Table 7 Revenue, Total Surplus, and Social Welfare Changes under Trip-based Price Fairness (%)

α	0.2	0.4	0.6	0.8	1.0
$\hat{\mathcal{R}}(\alpha)/\hat{\mathcal{R}}(0)$	98.64	93.01	83.19	68.72	58.55
$\hat{\mathcal{S}}(\alpha)/\hat{\mathcal{S}}(0)$	97.84	89.78	79.39	73.18	90.78
$\hat{\mathcal{W}}(\alpha)/\hat{\mathcal{W}}(0)$	98.42	92.12	82.14	69.95	67.41

In Figure 22, trips originating from Upper Manhattan (locations 1 and 2) show a significant surplus reduction, attributed to their initial lower prices that are subsequently raised by price fairness. In contrast, trips originating from midtown (locations 6 and 7) and Lower Manhattan excluding the Financial District (locations 8 and 9) show surplus gains. This pattern can be linked to the changes of prices of individual trips with α in Figure 23. Trips originating from Upper Manhattan experience an overall increase in prices, whereas those from midtown predominantly experience price decreases. Additionally, beyond $\alpha = 0.8$, prices tend to decrease as shows in Figure 23, which causes the total surplus to increase after this point.

Moreover, the change in surplus is not solely due to a change in affordability, instead, the decreased availability further contributes to surplus changes. The Upper Manhattan area experiences a particularly noticeable decline in availability, resulting in a decrease in surplus. In contrast, the availability of the Lower Manhattan area remains relatively consistent throughout the initial values, resulting in surplus gains. The changes in availability across each location are illustrated in Figure 24.

Figure 22 The Surplus Change across 10 **Neighborhoods under Price Fairness**

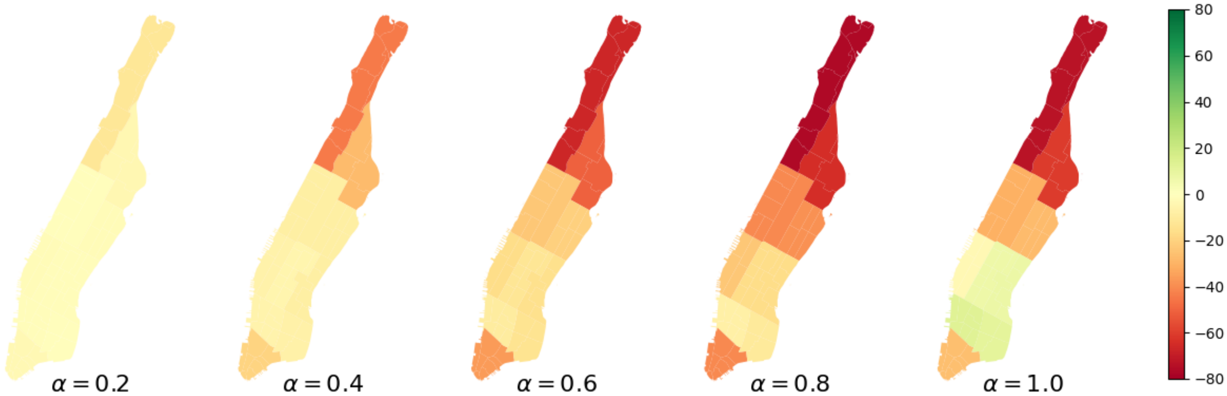


Figure 23 The Price/mile Changes across 10 Neighborhoods under Price Fairness

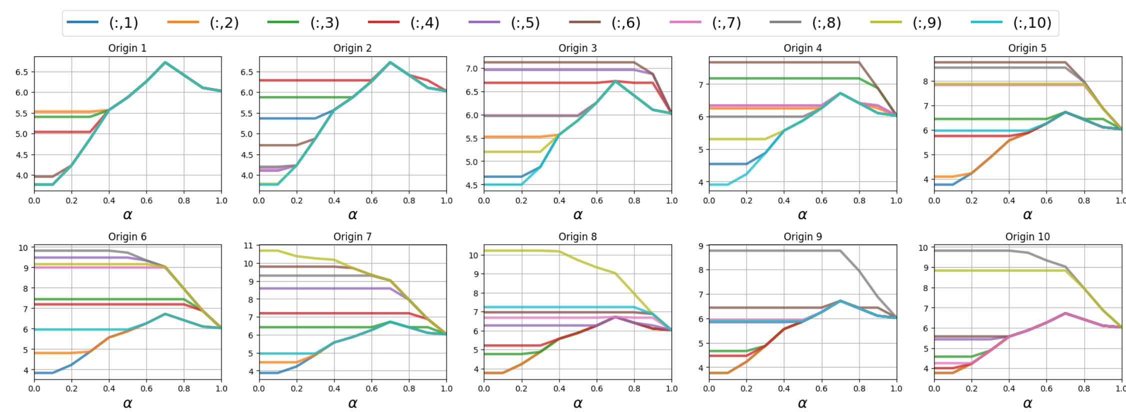
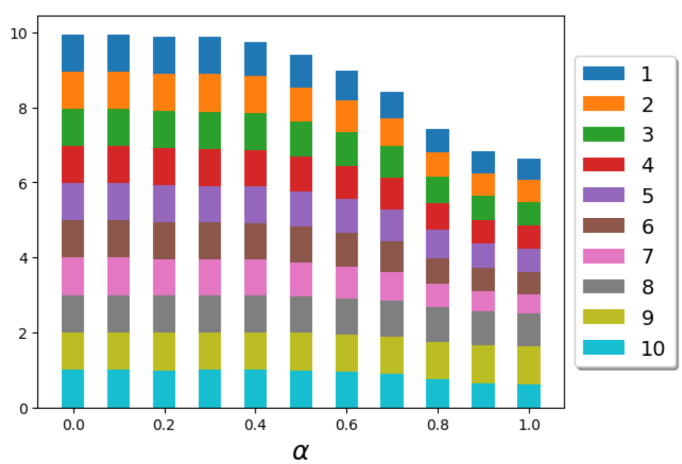


Figure 24 The Availability of Each Location



Note. The x-axis corresponds to α, while the y-axis corresponds to availability. Each color represents a distinct region, and the length of each bar indicates the corresponding availability, Π_i, obtained from the approximated problem.