Online Appendix

EC.1. Proofs of Results in Section 3

Proof of Lemma 1 We prove by contradiction. Without loss of generality, we assume that in equilibrium, $\lambda_A > \lambda_B$. For a customer of type x, the utility difference of choosing R versus A is given by (for ease of exposition, we use W_i and p_i to represent $W_i(\lambda_A, \lambda_B, \lambda_R)$ and $p_i(\lambda_A, \lambda_B, \lambda_R)$ for $i = A, B, R$, respectively)

$$u_A(x) - u_R(x) = c(W_R - W_A) - (2 - 2p_A)tx + (1 - p_A)t, \tag{EC.1}$$

which strictly decreases in x. Moreover, $u_A(x) - u_B(x)$ and $u_A(x) - 0$ also strictly decrease in x. Thus, $u_A(x) - \max\{u_R(x), u_B(x), 0\}$ strictly decreases in x. As a result, if a customer of type $\bar{x}$ chooses option A, which implies $u_A(\bar{x}) \geq \max\{u_R(\bar{x}), u_B(\bar{x}), 0\}$, then all customers with $x \in [0, \bar{x})$ will also choose option A. Thus, there exists x_A such that customers choose option A if and only if $x \in [0, x_A]$. Similarly, there exists $x_B \leq x_A$ such that customers choose option B if and only if they have $x \in [1 - x_B, 1]$. For customers over interval $(x_A, 1 - x_B)$, they may choose option R or balk. We next discuss three different cases.

Case 1: $u_A(x_A) > 0$ and $u_B(1 - x_B) > 0$. Since $u_A(x_A) > 0$, customers in the interval $(x_A, x_A + \epsilon)$ must have a positive utility from choosing A, for some sufficiently small $\epsilon > 0$. Since those customers do not choose A, they must have chosen R. Thus, customers at x_A must be indifferent between options A and R. In other words, x_A is the intersection point of the utility curves $u_A(\cdot)$ and $u_R(\cdot)$, and thus,

$$\begin{aligned} u_A(x_A) = u_R(x_A) &\Rightarrow v - tx_A - cW_A = p_A(v - tx_A) + (1 - p_A)(v - t(1 - x_A)) - cW_R \\ &\Rightarrow (1 - p_A)t(1 - 2x_A) = c(W_A - W_R). \end{aligned} \tag{EC.2}$$

Following the same logic, customers of type $x = 1 - x_B$ are indifferent between options B and R,

$$\begin{aligned} u_B(1 - x_B) = u_R(1 - x_B) &\Rightarrow v - tx_B - cW_B = p_A(v - t(1 - x_B)) + (1 - p_A)(v - tx_B) - cW_R \\ &\Rightarrow p_A t(1 - 2x_B) = c(W_B - W_R). \end{aligned} \tag{EC.3}$$

In the CUS mode, we have derived p_A in Lemma 2, and the expected response times are obtained from Gardner et al. (2015). After substituting these expressions, we have $c(W_A - W_R)/(1 - p_A) = \frac{c}{\mu - \Lambda x_A}$ and $c(W_B - W_R)/p_A = \frac{c}{\mu - \Lambda x_B}$. So (EC.2) and (EC.3) are equivalent to

$$t(1 - 2x_A) = \frac{c}{\mu - \Lambda x_A}, \text{ and } t(1 - 2x_B) = \frac{c}{\mu - \Lambda x_B}. \tag{EC.4}$$

In the CUE mode, the expressions for p_A and the expected response times are provided in Lemma EC.3. After substituting these expressions, we have $c(W_A - W_R)/(1 - p_A) = \frac{c\Lambda(2\mu - \Lambda x_A)(x_A + x_B + x_R)}{\mu(\mu - \Lambda x_A)(2\mu + \Lambda(x_B + x_R - x_A))}$ and $c(W_B - W_R)/p_A = \frac{c\Lambda(2\mu - \Lambda x_B)(x_A + x_B + x_R)}{\mu(\mu - \Lambda x_B)(2\mu + \Lambda(x_A + x_R - x_B))}$ where x_R is the proportion of customers who take option R. So (EC.2) and (EC.3) are equivalent to

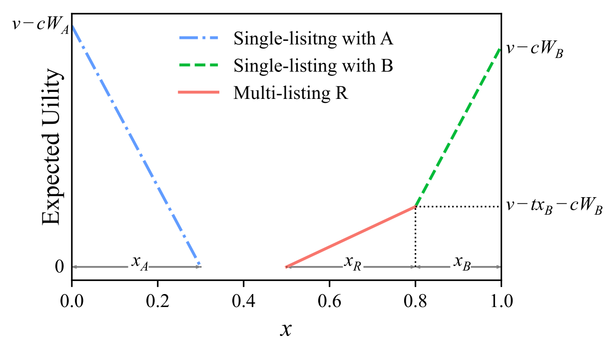
$$t(1 - 2x_A) = \frac{c\Lambda(x_A + x_B + x_R)}{\mu(2\mu - 2\Lambda x_A + \Lambda(x_A + x_B + x_R))}(1 + \frac{\mu}{\mu - \Lambda x_A}), \tag{EC.5}$$

$$t(1-2x_B) = \frac{c\Lambda(x_A+x_B+x_R)}{\mu(2\mu-2\Lambda x_B+\Lambda(x_A+x_B+x_R))}(1+\frac{\mu}{\mu-\Lambda x_B}). \tag{EC.6}$$

Since $\lambda_A > \lambda_B$, we have $x_A > x_B$, and thus $\mu - \Lambda x_A < \mu - \Lambda x_B$. Then, combining equations in (EC.4) in the CUS mode or combining (EC.5) and (EC.6) in CUE mode both lead to contradiction.

Case 2: $u_A(x_A) = 0$ and $u_B(1-x_B) > 0$. As shown in Figure EC.1, x_A is the intersection of the utility curve $u_A(\cdot)$ and $y = 0$, and $1-x_B$ is the intersection of curves $u_R(\cdot)$ and $u_B(\cdot)$. Moreover, $u_R(x) = v - p_B t + (p_B - p_A)tx - cW_R$ strictly increases in x since $p_B > p_A$. Thus, $u_R(\cdot)$ must intersects with $y = 0$ with a unique $x_R \in [x_A, 1-x_B)$ such that a customer chooses R if and only if $x \in [1-x_B-x_R, 1-x_B)$.

Figure EC.1 Utility Curves in Case 2



Since customers with $x = x_A$ have zero utility, we have

$$v - tx_A - cW_A = 0. \tag{EC.7}$$

Since customers with $x = 1-x_B-x_R$ also have zero utility, we have

$$\begin{aligned} 0 &= p_A(v-t(1-x_B-x_R)) + (1-p_A)(v-t(x_B+x_R)) - cW_R = v - tp_A - t(x_B+x_R)(1-2p_A) - cW_R \\ &\geq v - tp_A - t(1-x_A)(1-2p_A) - cW_R. \end{aligned} \tag{EC.8}$$

Combining (EC.7) and (EC.8) gives

$$(1-p_A)t(1-2x_A) \geq c(W_A - W_R). \tag{EC.9}$$

We also know that a customer with $x = 1-x_B$ is indifferent between options B and R, which leads to Equation (EC.3). Since $\lambda_A > \lambda_B$, as we argued in Case 1, we have $x_A > x_B$ and $\mu - \Lambda x_A < \mu - \Lambda x_B$. Then comparing Equations (EC.3) and (EC.9) leads to a contradiction for both CUS and CUE, as discussed in Case 1.

Case 3: $u_A(x_A) > 0$ and $u_B(1-x_B) = 0$. In this case, $1-x_B$ is the intersection of the utility curve $u_B(\cdot)$ and the horizontal line ($y = 0$). Thus, we have $u_B(1-x_B) = v - tx_B - cW_B = 0$. And we also have $u_A(x_A) = v - tx_A - cW_A \geq 0$. Taking the difference gives

$$0 \leq u_A(x_A) - u_B(1-x_B) = t(x_B - x_A) - c(W_A - W_B). \tag{EC.10}$$

On the right-hand side of the above inequality, $t(x_B - x_A) < 0$ and $W_A - W_B > 0$ because $x_A > x_B$ due to $\lambda_A > \lambda_B$. Then it leads to contradiction.

We have thus proved that $\lambda_A > \lambda_B$ is impossible at any equilibrium. ∎

EC.2. Proofs of Results in Section 4

Proof of Proposition 1 Before presenting the proof, we first provide the expressions of $(x_m, \Lambda_1, \Lambda_2, \hat{t}, \kappa)$:
- in the CUS mode, $\Lambda_1 = \Lambda_1^c := 1 - \frac{c}{v-t/2}, \Lambda_2 = \Lambda_2^c := \frac{v-t/2-c}{3v-t-2v^2/t}, \hat{t} = v, \kappa = \frac{1-2\Lambda x_m - c/(v-t/2)}{\Lambda(1-2x_m)}$,

$$x_m = \begin{cases} x_{AR}, & \text{if } t \le v, \text{ or if } \Lambda < \Lambda_2^c \text{ and } t > v, \\ x_s, & \text{if } \Lambda \ge \Lambda_2^c \text{ and } t > v, \end{cases}$$

where $x_{AR} := \frac{t+\Lambda t - \sqrt{(1-\Lambda)^2 t^2 + 8\Lambda c t}}{4\Lambda t}$ and $x_s := \frac{t/2+\Lambda v - \sqrt{(t/2-\Lambda v)^2 + 4\Lambda c t}}{2\Lambda t}$;
- in the CUE mode, $\Lambda_1 = \Lambda_1^e$, which is the root in $\left(\sqrt{1-\frac{2c}{v-t/2}}, 1 - \frac{c}{v-t/2}\right)$ of $v - \frac{t}{2} - \frac{v}{\Lambda_1^e} - c\left(1 - \frac{3}{\Lambda_1^e} + \frac{(1-\Lambda_1^e)t}{(2v-t)(1-\Lambda_1^e) - 2c(2-\Lambda_1^e)}\right) - \frac{2c^2(1-\Lambda_1^e)}{\Lambda_1^e((2v-t)(1-\Lambda_1^e)-2c)} = 0$, $\Lambda_2 = \Lambda_2^e := \frac{t\left(\frac{1}{2} - \frac{v-t/2}{4c} + \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}\right)\left(\frac{v-t/2}{4c} - \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}\right)}{v\left(\frac{v-t/2}{4c} - \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}\right) - c}$,
$\hat{t} = v - \frac{4c^2}{v}$, $\kappa = \frac{\lambda_R^e}{\Lambda(1-2\Lambda x_{AR}^{e'})}$,

$$x_m = \begin{cases} x_{AR}^e, & \text{if } \Lambda < \Lambda_1^e, \\ x_{AR}^{e'}, & \text{if } \Lambda \ge \Lambda_1^e \text{ and } t \le v - \frac{4c^2}{v} \text{ or if } \Lambda_1^e \le \Lambda < \Lambda_2^e \text{ and } t > v - \frac{4c^2}{v}, \\ x_s, & \text{if } \Lambda \ge \Lambda_2^e \text{ and } t > v - \frac{4c^2}{v}, \end{cases}$$

where x_{AR}^e is the root in $\left(0, \frac{1}{2}\right)$ of $t\left(\frac{1}{2} - x_{AR}^e\right) = \frac{2c(1-\Lambda x_{AR}^e)\Lambda}{(1+\Lambda-2\Lambda x_{AR}^e)(1-2\Lambda x_{AR}^e)}$, $(x_{AR}^{e'}, \lambda_R^e)$ is the root of

$$\begin{cases} t\left(\frac{1}{2} - x_{AR}^{e'}\right) = \frac{2c(1-\Lambda x_{AR}^{e'})(2\Lambda x_{AR}^{e'} + \lambda_R^e)}{(1+\lambda_R^e)(1-2\Lambda x_{AR}^{e'})}, \\ v - \frac{t}{2} = \frac{c}{1-\lambda_R^e - 2\Lambda x_{AR}^{e'}} + \frac{c(1-2\Lambda x_{AR}^{e'})}{1+\lambda_R^e}, \end{cases} \quad \text{with } x_{AR}^{e'} \in \left(0, \frac{1}{2}\right), \lambda_R^e \in (0,1).$$

Next, we proceed with the proof. By Lemma 1, there exists x_m such that customers over $[0, x_m]$ choose A while $[1-x_m, 1]$ choose B. Again, since $\lambda_A = \lambda_B$, customers who choose R have an equal likelihood of receiving service from Server A and Server B. Thus, all customers have the same utility $u_R(x)$ for choosing R regardless of their parameter x. Therefore, there can be three cases for customers over $(x_m, 1-x_m)$: (1) all choose R, (2) all balk, and (3) choose R with probability κ and balk with probability $1-\kappa$ (equivalent to a portion of customers choosing R). We next show that it must be one of the three cases depending on the system parameters. We next prove for CUS and CUE separately:

CUS mode: Let x_{AR} denote the intersection point of utility curves $u_A(\cdot)$ and $u_R(\cdot)$, so that $u_A(x_{AR}) = u_R(x_{AR})$. Let x_{A0} denote the intersection point of curve $u_A(\cdot)$ and the horizontal line $y = 0$, so that $u_A(x_{A0}) = 0$. It is clear that customers over $[0, x_m]$ will join choose A, where $x_m = \min\{x_{AR}, x_{A0}\}$.

x_{AR} can be solved from (EC.2). By plugging in the expression, we get the following equation,

$$t\left(\frac{1}{2} - x_{AR}\right) - \frac{c}{1-2\Lambda x_{AR}} = 0 \tag{EC.11}$$

As $t > c$, (EC.11) has a unique solution

$$x_{AR} = \frac{t + \Lambda t - \sqrt{(1-\Lambda)^2 t^2 + 8\Lambda c t}}{4\Lambda t} \in [0, 0.5]. \tag{EC.12}$$

When $\Lambda \le \Lambda_1^c := 1 - \frac{c}{v-t/2}$, we have

$$u_R(x_{AR}) = v - \frac{t}{2} - \frac{c}{1-\Lambda} \ge 0, \text{ for all } x \in [0,1]. \tag{EC.13}$$

That implies curve $u_A(\cdot)$ intersects $u_R(\cdot)$ first and thus $x_m = x_{AR} \le x_{A0}$. Moreover, the calculation of $u_R(\cdot)$ assumes that no one has balked, in which case the utility of choosing R remains non-negative. Therefore, customers over $(x_m, 1-x_m)$ must all choose R when $\Lambda \le \Lambda_1^c$.

When $\Lambda > \Lambda_1^c$, the left-hand side of (EC.13) gives a negative value. Thus, not all customers in $(x_m, 1-x_m)$ would choose R. Then they will balk with a certain probability or all balk. In either case, we have $u_A(x_m) = 0$ and $u_R(x_m) \le 0$, which leads to the following conditions.

$$u_A(x_m) = v - tx_m - \frac{c}{1-2\Lambda x_m - \lambda_R} - \frac{c}{1-2\Lambda x_m} = 0, \tag{EC.14}$$

$$u_R(x_m) = v - \frac{t}{2} - \frac{c}{1-2\Lambda x_m - \lambda_R} \le 0. \tag{EC.15}$$

If all customers over $(x_m, 1-x_m)$ balk, then $\lambda_R = 0$. Then (EC.14) and (EC.15) lead to the conditions $\Lambda \ge \Lambda_2^c := \frac{v-t/2-c}{3v-t-2v^2/t}$ and $t > v$, in this case, $\lambda_R = 0$, $x_m = x_s = \frac{t/2+\Lambda v - \sqrt{(t/2-\Lambda v)^2+4\Lambda ct}}{2\Lambda t}$ is the unique solution in $(0, 0.5)$ to $v - tx - \frac{c}{0.5-\Lambda x} = 0$. When $t \le v$ or $t > v$ and $\Lambda \in (\Lambda_1^c, \Lambda_2^c)$, (EC.15) holds with equality for some $\lambda_R > 0$, so we have customers over $(x_m, 1-x_m)$ choosing R with probability $\frac{\lambda_R}{\Lambda-\lambda_A-\lambda_B} = \frac{1-2\Lambda x_m - c/(v-t/2)}{\Lambda(1-2x_m)}$, where $x_m = x_{AR} := \frac{t+\Lambda t - \sqrt{(1-\Lambda)^2t^2+8\Lambda ct}}{4\Lambda t}$ is the unique solution in $(0, 0.5)$ to (EC.11).

CUE mode: Let x_{AR}^e denote the intersection of the utility curves $u_A(\cdot)$ and $u_R(\cdot)$ in the CUE model when nobody balks. In this case, from $u_A(x_{AR}^e) = u_R(x_{AR}^e)$, we have

$$t(\frac{1}{2} - x_{AR}^e) = \frac{2c(1-\Lambda x_{AR}^e)\Lambda}{(1+\lambda_R)(1-2\Lambda x_{AR}^e)}. \tag{EC.16}$$

To have all customers over $(x_{AR}^e, 1-x_{AR}^e)$ choose R, we must have $u_R(x_{AR}^e) \ge 0$ so that $x_m = x_{AR}$ and $\lambda_R = \Lambda(1-2x_{AR}^e)$. Thus, the sufficient and necessary condition for full coverage is

$$u_R(x_{AR}^e) = v - \frac{t}{2} - c\left(\frac{1}{1-\Lambda} + \frac{1-2\Lambda x_{AR}^e}{1+\Lambda(1-2x_{AR}^e)}\right) \ge 0, \tag{EC.17}$$

where x_{AR}^e is the solution to (EC.16).

We next show that (EC.17) holds when $\Lambda \le \Lambda_1^e$ for some $\Lambda_1^e \in (0,1)$. First, (EC.16) can be written as

$$f(x_{AR}^e, \Lambda) := t(\frac{1}{2} - x_{AR}^e) - \left(\frac{c\Lambda}{\left(1+\Lambda(1-2x_{AR}^e)\right)(1-2\Lambda x_{AR}^e)} + \frac{c\Lambda}{1+\Lambda(1-2x_{AR}^e)}\right) = 0.$$

It is easy to verify that for any fixed $\Lambda \in (0,1)$, $\frac{\partial f(x,\Lambda)}{\partial x} < 0$ and $f(0,\Lambda) > 0 > f(0.5,\Lambda)$. Thus, there is a unique solution $x_{AR}^e(\Lambda) \in (0, 0.5)$ such that $f(x_{AR}^e(\Lambda), \Lambda) = 0$. From then on, we will regard the solution to (EC.17), x_{AR}^e, as a function of Λ, which allows us to examine how x_{AR}^e and the associated utility value $u_R(x_{AR}^e)$ changes with Λ.

By invoking the implicit function theorem to the function $f(x_{AR}^e(\Lambda), \Lambda)$, we have

$$\begin{aligned} {x_{AR}^e}'(\Lambda) &= -(\frac{\partial f}{\partial \Lambda})(\frac{\partial f}{\partial x_{AR}^e})^{-1} \\ &= \frac{(1-2x_{AR}^e)(1-2\Lambda x_{AR}^e)(-\frac{1}{\Lambda}) + (1-2x_{AR}^e)(1+\Lambda-2\Lambda x_{AR}^e)(-\frac{x_{AR}^e}{1-\Lambda x_{AR}^e})}{2(1+\Lambda-2\Lambda x_{AR}^e)(1-2\Lambda x_{AR}^e) + (1-2x_{AR}^e)(1+\Lambda-2\Lambda x_{AR}^e)\frac{\Lambda}{1-\Lambda x_{AR}^e} + 2(1-2x_{AR}^e)(1-2\Lambda x_{AR}^e)\Lambda} \\ &< 0. \end{aligned} \tag{EC.18}$$

In deriving the above equation, we used the equality $c = \frac{t(1-2x_{AR}^e)(1+\Lambda-2\Lambda x_{AR}^e)(1-2\Lambda x_{AR}^e)}{4\Lambda(1-\Lambda x_{AR}^e)}$ from (EC.16).

(EC.18) implies that x_{AR}^e strictly decreases with Λ. We next look into how the associated utility $u_R(x_{AR}^e)$ changes with Λ. With slight abuse of notation, we regard u_R as a function of Λ and adopt the notation $u_R(x_{AR}^e(\Lambda), \Lambda)$. Its derivative is given by $\frac{\mathrm{d}u_R(x_{AR}^e(\Lambda),\Lambda)}{\mathrm{d}\Lambda} = \frac{\partial u_R}{\partial \Lambda} + \frac{\partial u_R}{\partial x_{AR}^e}\frac{\mathrm{d}x_{AR}^e}{\mathrm{d}\Lambda}$. It is not difficult to verify that $\frac{\partial u_R}{\partial \Lambda} < 0$ and $\frac{\partial u_R}{\partial x_{AR}^e} > 0$. Thus, we have $\frac{u_R(x_{AR}^e(\Lambda),\Lambda)}{\mathrm{d}\Lambda} < 0$. Furthermore, when $\Lambda = 1 - \frac{c}{v-t/2}$, we have

$$u_R(x_{AR}^e(\Lambda), \Lambda) = v - \frac{t}{2} - c\left(\frac{1}{1-\Lambda} + \frac{1-2\Lambda x_{AR}^e(\Lambda)}{1+\Lambda-2\Lambda x_{AR}^e(\Lambda)}\right) < v - \frac{t}{2} - c\frac{1}{1-\Lambda} = 0.$$

When $\Lambda = \sqrt{1 - \frac{2c}{v-t/2}}$, we have $u_R(x^e_{AR}(\Lambda),\Lambda) > v - \frac{t}{2} - \frac{2c}{1-\Lambda^2} = 0$, where the first inequality follows from

$$u_R(x^e_{AR}(\Lambda),\Lambda) - (v - \frac{t}{2} - \frac{2c}{1-\Lambda^2}) = \frac{2c}{1-\Lambda^2} - \frac{c}{1-\Lambda} - \frac{c(1-2\Lambda x^e_{AR})}{1+\Lambda(1-2x^e_{AR})} = \frac{2c\Lambda^2 x^e_{AR}}{(1+\Lambda)(1+\Lambda(1-2x^e_{AR}))} > 0.$$

Therefore, there must exist a unique $\Lambda^e_1 \in (\sqrt{1-\frac{2c}{v-t/2}}, 1-\frac{c}{v-t/2})$ such that $u_R(x^e_{AR}(\Lambda^e_1),\Lambda^e_1) = 0$. This is because $u_R(x^e_{AR}(1-\frac{c}{v-t/2}), 1-\frac{c}{v-t/2}) < 0 < u_R(x^e_{AR}(\sqrt{1-\frac{2c}{v-t/2}}), \sqrt{1-\frac{2c}{v-t/2}})$. and $u_R(x^e_{AR}(\Lambda),\Lambda)$ strictly decreases in Λ. As a result,

$$u_R(x^e_{AR}(\Lambda),\Lambda) \begin{cases} \geq 0 & \text{if } \Lambda \in [0,\Lambda^e_1], \\ < 0 & \text{otherwise.} \end{cases} \tag{EC.19}$$

So (EC.17) holds and it's full coverage whenever $\Lambda \in [0,\Lambda^e_1]$ for some constant $\Lambda^e_1 \in (\sqrt{1-\frac{2c}{v-t/2}}, 1-\frac{c}{v-t/2})$, and Λ^e_1 is the root of $u_R(x^e_{AR}(\Lambda),\Lambda) = 0$, i.e., the root of $v - \frac{t}{2} - \frac{v}{\Lambda^e_1} - c\left(1 - \frac{3}{\Lambda^e_1} + \frac{(1-\Lambda^e_1)t}{(2v-t)(1-\Lambda^e_1)-2c(2-\Lambda^e_1)}\right) - \frac{2c^2(1-\Lambda^e_1)}{\Lambda^e_1((2v-t)(1-\Lambda^e_1)-2c)} = 0$.

When $\Lambda > \Lambda^e_1$, (EC.17) does not hold. That means some customers will balk. Let $x^{e'}_{AR}$, λ^e_R denote the intersection of the utility curves $u_A(\cdot)$ and $u_R(\cdot)$ and the arrival rate of customers choosing R at the equilibrium in this case. From $u_A(x^{e'}_{AR}) = u_R(x^{e'}_{AR})$, we have

$$t(\frac{1}{2} - x^{e'}_{AR}) \equiv \frac{2c(1-\Lambda x^{e'}_{AR})(2\Lambda x^{e'}_{AR} + \lambda^e_R)}{(1+\lambda^e_R)(1-2\Lambda x^{e'}_{AR})}. \tag{EC.20}$$

Let all other parameters including Λ be fixed, we consider the solution $x^{e'}_{AR}$ to the above equation as a function of λ^e_R. Then (EC.20) implies that

$$\frac{\partial \lambda^e_R}{\partial x^{e'}_{AR}} = \frac{-t - \frac{c\Lambda}{2(\frac{1}{2}-\Lambda x^{e'}_{AR})^2} - \frac{2c\Lambda}{1+\lambda^e_R}}{\frac{2c(1-\Lambda x^{e'}_{AR})}{(1+\lambda^e_R)^2}} < -\Lambda\frac{(1+\lambda^e_R)^2}{(1-2\Lambda x^{e'}_{AR})^2(1-\Lambda x^{e'}_{AR})} - \Lambda\frac{1+\lambda^e_R}{1-\Lambda x^{e'}_{AR}} < -2\Lambda < 0. \tag{EC.21}$$

Moreover,

$$\begin{aligned} \frac{\partial u_R}{\partial x^{e'}_{AR}} &= -c\frac{2\Lambda + \frac{\partial \lambda^e_R}{\partial x^{e'}_{AR}}}{(1-2\Lambda x^{e'}_{AR} - \lambda^e_R)^2} + c\frac{\Lambda(1+\lambda^e_R) + (\mu - \Lambda x^{e'}_{AR})\frac{\partial \lambda^e_R}{\partial x^{e'}_{AR}}}{\mu(1+\lambda^e_R)^2} \\ &= -2\Lambda c\left(\frac{1}{(1-2\Lambda x^{e'}_{AR} - \lambda^e_R)^2} - \frac{1}{1+\lambda^e_R}\right) - \frac{\partial \lambda^e_R}{\partial x^{e'}_{AR}} c\left(\frac{1}{(1-2\Lambda x^{e'}_{AR} - \lambda^e_R)^2} - \frac{1-2\Lambda x^{e'}_{AR}}{(1+\lambda^e_R)^2}\right) \\ &> -c(2\Lambda + \frac{\partial \lambda^e_R}{\partial x^{e'}_{AR}})\left(\frac{1}{(1-2\Lambda x^{e'}_{AR} - \lambda^e_R)^2} - \frac{1}{1+\lambda^e_R}\right) > 0, \end{aligned}$$

where $2\Lambda + \frac{\partial \lambda^e_R}{\partial x^{e'}_{AR}} < 0$ due to (EC.21). We also have $\frac{\partial u_R}{\partial \lambda^e_R} < 0$ as the expected waiting time for option R increases in λ^e_R. Therefore, we have

$$\frac{\mathrm{d}u_R}{\mathrm{d}\lambda^e_R} = \frac{\partial u_R}{\partial x^{e'}_{AR}}\frac{\mathrm{d}x^{e'}_{AR}}{\mathrm{d}\lambda^e_R} + \frac{\partial u_R}{\partial \lambda^e_R} < 0.$$

For a given $\Lambda > \Lambda^e_1$, then for $\overline{\lambda}_R$ that satisfy $\overline{\lambda}_R = \Lambda(1-2x^{e'}_{AR}(\overline{\lambda}_R))$, $u_R(x^{e'}_{AR}(\overline{\lambda}_R),\overline{\lambda}_R) < 0$ as result of (EC.19). Therefore, when $\Lambda > \Lambda^e_1$, there is a $\lambda^e_R \in (0,\overline{\lambda}_R)$ such that $u_R(x^{e'}_{AR}(\lambda^e_R),\lambda^e_R) = 0$ if and only if $u_R(x^{e'}_{AR}(0),0) > 0$, which is equivalent to the following set of equations

$$t(\frac{1}{2} - x^{e'}_{AR}(0)) = \frac{c(1-\Lambda x^{e'}_{AR}(0))\Lambda x^{e'}_{AR}}{1/2(1/2 - \Lambda x^{e'}_{AR}(0))} = c\left(\frac{1}{1-2\Lambda x^{e'}_{AR}(0)} - 1 + 2\Lambda x^{e'}_{AR}(0)\right), \tag{EC.22}$$

$$u_R(x^{e'}_{AR}(0),0) = v - \frac{t}{2} - c\left(\frac{1}{1-2\Lambda x^{e'}_{AR}(0)} + 1 - 2\Lambda x^{e'}_{AR}(0)\right) > 0. \tag{EC.23}$$

The second inequality is equivalent to

$$x_{AR}^{e'}(0) < \frac{\frac{1}{2} - \frac{v-t/2}{4c} + \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}}{\Lambda}. \tag{EC.24}$$

Since the LHS of (EC.22) deceases in $x_{AR}^{e'}(0)$ and the RHS increases in $x_{AR}^{e'}(0)$, (EC.24) is equivalent to the following inequality in (EC.22),

$$\begin{aligned} & t(\tfrac{1}{2} - \frac{\frac{1}{2} - \frac{v-t/2}{4c} + \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}}{\Lambda}) < c\left(\frac{1}{\frac{v-t/2}{2c} - \sqrt{(\frac{v-t/2}{2c})^2 - 1}} - \frac{v-t/2}{2c} + \sqrt{(\frac{v-t/2}{2c})^2 - 1}\right) \\ \Leftrightarrow \quad & \frac{\frac{1}{2} - \frac{v-t/2}{4c} + \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}}{\Lambda} > \frac{1}{2} - \frac{c(1+\frac{v-t/2}{2c} - \sqrt{(\frac{v-t/2}{2c})^2 - 1})(1-\frac{v-t/2}{2c} + \sqrt{(\frac{v-t/2}{2c})^2 - 1})}{t(\frac{v-t/2}{2c} - \sqrt{(\frac{v-t/2}{2c})^2 - 1})} \end{aligned} \tag{EC.25}$$

The RHS is negative if and only if $t \le v - \frac{4c^2}{v}$. Thus, if $t \le v - \frac{4c^2}{v}$, then (EC.22) and (EC.23) always hold. If $t > v - \frac{4c^2}{v}$, then (EC.25) holds when

$$\Lambda < \Lambda_2^e := \frac{t(\frac{1}{2} - \frac{v-t/2}{4c} + \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}})(\frac{v-t/2}{4c} - \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}})}{v(\frac{v-t/2}{4c} - \sqrt{(\frac{v-t/2}{4c})^2 - \frac{1}{4}}) - c}. \tag{EC.26}$$

So if $t \le v - \frac{4c^2}{v}$ and $\Lambda > \Lambda_1^e$ or $t > v - \frac{4c^2}{v}$ and $\Lambda_1^e \le \Lambda < \Lambda_2^e$, then the customers over $(x_{AR}^{e'}, 1 - x_{AR}^{e'})$ choose R with probability $\frac{\lambda_R^e}{\Lambda(1-2x_{AR}^{e'})}$. $(x_{AR}^{e'}, \lambda_R^e)$ is the root of

$$\begin{cases} t\left(\frac{1}{2} - x_{AR}^{e'}\right) = \frac{2c(1-\Lambda x_{AR}^{e'})(2\Lambda x_{AR}^{e'} + \lambda_R^e)}{(1+\lambda_R^e)(1-2\Lambda x_{AR}^{e'})}, \\ v - \frac{t}{2} = \frac{c}{1-\lambda_R^e - 2\Lambda x_{AR}^{e'}} + \frac{c(1-2\Lambda x_{AR}^{e'})}{1+\lambda_R^e}, \end{cases} \quad \text{with } x_{AR}^{e'} \in \left(0, \frac{1}{2}\right), \lambda_R^e \in (0,1). \tag{EC.27}$$

If $t > v - \frac{4c^2}{v}$ and $\Lambda \ge \Lambda_2^e$, then (EC.22) and (EC.23) do not have a solution $\lambda_R^e > 0$. That implies that no customers would join queue R. $x_m = x_s$ in this case. ■

Proof of Proposition 2 Similar to the analysis in Lemma 1, there must exists x_s such that customers over $[0, x_s]$ choose A while $[1 - x_s, 1]$ choose B in equilibrium. When $u_A^s(0.5) = v - \frac{t}{2} - \frac{c}{0.5-\Lambda/2} \ge 0$, full coverage is possible, i.e., when $\Lambda \le \Lambda_s := 1 - \frac{2c}{v-t/2}$, $x_s = 0.5$.

When $x_s < 0.5$, it must be be the intersection of $u_A^s(\cdot)$ and the horizontal line $y = 0$. This implies

$$v - tx_s - \frac{c}{0.5 - \Lambda x_s} = 0, \tag{EC.28}$$

(EC.28) have a unique solution $x_s = \frac{t/2 + \Lambda v - \sqrt{(t/2 - \Lambda v)^2 + 4\Lambda ct}}{2\Lambda t} \in (0, 0.5]$. ■

Proof of Proposition 3 Denote λ_p as the arrival rate of the pooling system at equilibrium. The utility functions under CUS and CUE are given by $u^p(x) = v - \frac{t}{2} - \frac{c}{1-\lambda_p}$, $u^p(x) = v - \frac{t}{2} - \frac{2c}{1-\lambda_p^2}$. Both functions are decreasing with respect to λ_p, Thus, full coverage is possible when $\Lambda < \Lambda_p^c := 1 - \frac{c}{v-t/2}$ and $\Lambda < \Lambda_p^e := \sqrt{1 - \frac{2c}{v-t/2}}$. Otherwise, the total throughput is limited to Λ_p^c and Λ_p^e. ■

COROLLARY EC.1 **(Throughput and Social Welfare Under Multi-listing).**

(i) In the CUS mode, the equilibrium throughput of a multi-listing system is given by

$$Th_m = \begin{cases} \Lambda, & \text{if } \Lambda < \Lambda_1^c, \\ 1 - \frac{c}{v-t/2}, & \text{if } \Lambda \geq \Lambda_1^c \text{ and } t \leq v, \text{ or } \Lambda_1^c \leq \Lambda < \Lambda_2^c \text{ and } t > v, \\ 2\Lambda x_s, & \text{if } \Lambda \geq \Lambda_2^c \text{ and } t > v. \end{cases}$$

Social welfare in equilibrium is given by

$$SW_m = \begin{cases} \Lambda(v - \frac{t}{2} - \frac{c}{1-\Lambda} + t(x_{AR})^2), & \text{if } \Lambda < \Lambda_1^c, \\ \Lambda t(x_{AR})^2, & \text{if } \Lambda \geq \Lambda_1^c \text{ if } t \leq v, \text{ or } \Lambda_1^c \leq \Lambda < \Lambda_2^c \text{ if } t > v, \\ \Lambda t {x_s}^2, & \text{if } \Lambda \geq \Lambda_2^c \text{ and } t > v. \end{cases}$$

(ii) In the CUE mode, the equilibrium throughput of a multi-listing system is given by

$$Th_m^e = \begin{cases} \Lambda, & \text{if } \Lambda < \Lambda_1^e, \\ 2\Lambda x_{AR}^{e'} + \lambda_R^e, & \text{if } \Lambda \geq \Lambda_1^e \text{ and } t \leq v - \frac{4c^2}{v}, \text{ or } \Lambda_1^e \leq \Lambda < \Lambda_2^e \text{ and } t > v - \frac{4c^2}{v}, \\ 2\Lambda x_s, & \text{if } \Lambda \geq \Lambda_2^e \text{ and } t > v - \frac{4c^2}{v}. \end{cases}$$

Social welfare in equilibrium is given by $SW_m^e =$

$$\begin{cases} \Lambda\left(v - \frac{t}{2} - \frac{c}{1-\Lambda} - \frac{c(1-2\Lambda x_{AR}^e)}{1+\Lambda-2\Lambda x_{AR}^e} + t(x_{AR}^e)^2\right), & \text{if } \Lambda \leq \Lambda_1^e, \\ \Lambda t(x_{AR}^{e'})^2, & \text{if } \Lambda \geq \Lambda_1^e \text{ and } t \leq v - \frac{4c^2}{v}, \text{ or } \Lambda_1^e \leq \Lambda < \Lambda_2^e \text{ and } t > v - \frac{4c^2}{v}, \\ \Lambda t(x_s)^2, & \text{if } \Lambda > \Lambda_2^e \text{ and } t > v - \frac{4c^2}{v}. \end{cases}$$

COROLLARY EC.2 **(Throughput and Social Welfare Under Single-listing).** *The equilibrium throughput of a single-listing system is given by*

$$Th_s = \begin{cases} \Lambda, & \text{if } \Lambda \leq \Lambda_s, \\ 2\Lambda x_s, & \text{if } \Lambda > \Lambda_s. \end{cases}$$

Social welfare in equilibrium is given by

$$SW_s = \begin{cases} \Lambda(v - \frac{t}{4} - \frac{2c}{1-\Lambda}), & \text{if } \Lambda \leq \Lambda_s, \\ \Lambda t {x_s}^2, & \text{if } \Lambda > \Lambda_s. \end{cases}$$

COROLLARY EC.3 **(Throughput and Social Welfare Under Pooling).**

(i) In the CUS mode, the equilibrium throughput of a pooling system is given by $Th_p = \min\{\Lambda, \Lambda_p^c\}$. *Social welfare in equilibrium is given by*

$$SW_p = \begin{cases} \Lambda\left(v - \frac{t}{2} - \frac{c}{1-\Lambda}\right), & \text{if } \Lambda \leq \Lambda_p^c, \\ 0, & \text{if } \Lambda > \Lambda_p^c. \end{cases}$$

(ii) In the CUE mode, the equilibrium throughput of a pooling system is given by $Th_p^e = \min\{\Lambda, \Lambda_p^e\}$. *Social welfare in equilibrium is given by*

$$SW_p^e = \begin{cases} \Lambda\left(v - \frac{t}{2} - \frac{2c}{1-\Lambda^2}\right), & \text{if } \Lambda \leq \Lambda_p^e, \\ 0, & \text{if } \Lambda > \Lambda_p^e. \end{cases}$$

EC.3. Proofs of Results in Section 5

Table EC.1 Throughput Under CUS

Λ	$[0, \Lambda_s]$	$[\Lambda_s, \Lambda_1^c]$	$[\Lambda_1^c, \Lambda_2^c]$	$[\Lambda_2^c, \infty]$
Pooling	Λ		$1 - \frac{c}{v-t/2}$	
Single	Λ	$2\Lambda x_s$		
Multi	Λ		$1 - \frac{c}{v-t/2}$	$1 - \frac{c}{v-t/2}$, if $t \leq v$; $2\Lambda x_s$, if $t > v$

Proof of Theorem 1 From Corollary EC.1, Corollary EC.2 and Corollary EC.3, it is straightforward that $Th_m = Th_p = Th_s$ when $\Lambda \le \Lambda_s$, and $Th_m = Th_p > Th_s$ when $\Lambda < \Lambda_p^c = \Lambda_1^c$. When $\Lambda > \Lambda_p^c$, we first compare the throughput rates between a pooling system and a single-listing system, namely $Th_p(\Lambda) = 1 - \frac{c}{v-t/2}$ and $Th_s(\Lambda) = 2\Lambda x_s = 1 - \frac{c}{v/2 - tx_s/2}$, the final equality is derived from (EC.28).

Note that $Th_p(\Lambda) \equiv 1 - \frac{c}{v-t/2}$ does not change with Λ; while $Th_s(\Lambda) = 1 - \frac{2c}{v-tx_s}$ strictly increases in Λ as $\frac{\mathrm{d}x_s}{\mathrm{d}\Lambda} = -\frac{cx_s}{c\Lambda + t(1/2-\Lambda x_s)^2} < 0$. As a result, either the two curves $Th_p(\Lambda)$ and $Th_s(\Lambda)$ never intersect, in which case $Th_p > Th_s$ for all $\Lambda > \Lambda_p$, or Th_p and Th_s intersect once, in which case $Th_p > Th_s$ before the intersection and $Th_p < Th_s$ after the intersection.

When $t \le v$, we have $v - t/2 \ge v/2 > v/2 - tx_s/2$, so $Th_p(\Lambda) = 1 - \frac{c}{v-t/2} > 1 - \frac{2c}{v-tx_s} = Th_s(\Lambda)$, for all $\Lambda > \Lambda_s$. So the two curves never intersect. When $t > v$, we find that the two curves intersect at $\Lambda_2^c := \frac{v-t/2-c}{3v-t-2v^2/t}$ as $Th_p(\Lambda_2^c) = Th_s(\Lambda_2^c)$. Thus, $Th_p(\Lambda) > Th_s(\Lambda)$ when $\Lambda \in (\Lambda_s, \Lambda_2^c)$, and $Th_p(\Lambda) < Th_s(\Lambda)$ when $\Lambda > \Lambda_2^c$.

The comparison between $Th_m(\Lambda)$ and $Th_p(\Lambda)$ (or $Th_s(\Lambda)$) follows immediately as $Th_m(\Lambda)$ is the maximum of $Th_p(\Lambda)$ and $Th_s(\Lambda)$ by its expression. ■

Table EC.2 Throughput Under CUE

Λ	$[0, \Lambda_s]$	$[\Lambda_s, \Lambda_p^e]$	$[\Lambda_p^e, \Lambda_1^e]$	$[\Lambda_1^e, \Lambda_2^e]$	$[\Lambda_2^e, \infty]$
Pooling	Λ		$\sqrt{1 - \frac{2c}{v-t/2}}$		
Single	Λ	$2\Lambda x_s$			
Multi	Λ			$2\Lambda x_{AR}^{e'} + \lambda_R$	$2\Lambda x_{AR}^{e'} + \lambda_R$ if $t < v - \frac{4c^2}{v}$, $2\Lambda x_s$ if $t \ge v - \frac{4c^2}{v}$

Proof of Theorem 2 For $\Lambda \le \Lambda_p^e$, the comparison is straightforward. When $\Lambda > \Lambda_p^e$, we first compare Th_p^e and Th_s^e. Similar to the CUS case, when $\Lambda > \Lambda_p^e$, $Th_p^e(\Lambda) \equiv \sqrt{1 - \frac{2c}{v-t/2}}$ while $Th_s^e(\Lambda) = 2\Lambda x_s = 1 - \frac{2c}{v-tx_s}$ strictly increases in Λ, and either the two curves never intersect or intersect only once.

When $t \le 2v - \frac{v^2}{v-c}$, we have

$$Th_p^e(\Lambda) = \sqrt{1 - \frac{2c}{v-t/2}} \ge \sqrt{1 - \frac{4c(v-c)}{v^2}} = 1 - \frac{2c}{v} > 1 - \frac{2c}{v-tx_s} = Th_s^e(\Lambda), \text{ for all } \Lambda. \tag{EC.29}$$

In (EC.29), the first inequality follows from $t \le 2v - \frac{v^2}{v-c}$ and the second inequality follows from $x_s > 0$.

When $t > 2v - \frac{v^2}{v-c}$, we find that the two curves intersect at $\Lambda_{sp}^e := \frac{\sqrt{\frac{1}{4} - \frac{c}{2v-t}}}{\frac{v}{t} - \frac{c}{\frac{t}{2} - t\sqrt{\frac{1}{4} - \frac{c}{2v-t}}}}$ as $Th_p^e(\Lambda_{sp}^e) = Th_s^e(\Lambda_{sp}^e)$. Thus, $Th_p^e(\Lambda) > Th_s^e(\Lambda)$ when $\Lambda \in (\Lambda_s, \Lambda_{sp}^e)$, and $Th_p^e(\Lambda) < Th_s^e(\Lambda)$ when $\Lambda > \Lambda_{sp}^e$.

When $\Lambda_p^e < \Lambda \le \Lambda_1^e$, only the multi-listing system achieves full coverage, and thus, maximum throughput.

When $\Lambda > \Lambda_1^e$ and $t \le v - \frac{4c^2}{v}$, or when $\Lambda \in (\Lambda_1^e, \Lambda_2^e)$ and $t > v - \frac{4c^2}{v}$, $Th_m^e = 2\Lambda x_{AR}^{e'} + \lambda_R^e$, where $x_{AR}^{e'}$ and λ_R^e are characterized by (EC.27) in the proof of Proposition 1. By adding up the two equations in (EC.27), we have

$$\begin{aligned}
v - tx_{AR}^{e'} &= c\left(\frac{1}{1-2\Lambda x_{AR}^{e'} - \lambda_R^e} + \frac{1-2\Lambda x_{AR}^{e'}}{1+\lambda_R^e} + \frac{2(1-\Lambda x_{AR}^{e'})(2\Lambda x_{AR}^{e'} + \lambda_R^e)}{(1+\lambda_R^e)(1-2\Lambda x_{AR}^{e'})}\right) \\
&= c\left(\frac{1}{1-2\Lambda x_{AR}^{e'} - \lambda_R^e} + \frac{1+2\lambda_R^e - 2\Lambda x_{AR}^{e'}\lambda_R^e}{(1+\lambda_R^e)(1-2\Lambda x_{AR}^{e'})}\right) \\
&= c\left(\frac{1}{1-2\Lambda x_{AR}^{e'} - \lambda_R^e} + \frac{1}{1-2\Lambda x_{AR}^{e'}} + \frac{\lambda_R^e}{1+\lambda_R^e}\right)
\end{aligned}$$

$$
\begin{aligned}
&= c\left(\frac{2}{1-2\Lambda x^{e'}_{AR}-\lambda^e_R}+\frac{\lambda^e_R}{1+\lambda^e_R}-\frac{\lambda^e_R}{(1-2\Lambda x^{e'}_{AR})(1-2\Lambda x^{e'}_{AR}-\lambda^e_R)}\right)\\
&< \frac{c}{\frac{1}{2}-\Lambda x^{e'}_{AR}-\lambda^e_R/2}.
\end{aligned}
$$

We next prove $x^{e'}_{AR} < x_s$. Recall that $x^{e'}_{AR}$ is the intersection of $u_A(\cdot)$ and $u_R(\cdot)$, i.e., $u_A(x^{e'}_{AR}) = u_R(x^{e'}_{AR}) = 0$. We then consider the expected utility when a customer with parameter $x^{e'}_{AR}$ choosing A in a single-listing system. We have

$$
u^s_A(x^{e'}_{AR}) = v - tx^{e'}_{AR} - \frac{2c}{1-2\Lambda x^{e'}_{AR}} > v - \frac{t}{2} - \frac{c}{1-2\Lambda x^{e'}_{AR}} - \frac{c}{1-2\Lambda x^{e'}_{AR}-\lambda^e_R} = u_A(x^{e'}_{AR}) = 0.
$$

In the same time, we have $u^s_A(x_s) = v - tx_s - \frac{c}{0.5-\Lambda x_s} = 0$. We then deduce that $x^{e'}_{AR} < x_s$ as $v - tx - \frac{c}{0.5-\Lambda x}$ is strictly decreasing in x. With $x^{e'}_{AR} < x_s$, we have

$$
\frac{c}{1/2-\Lambda x_s} = v - tx_s < v - tx^{e'}_{AR} < \frac{c}{1/2-\Lambda x^{e'}_{AR}-\lambda^e_R/2} \Rightarrow Th^e_s(\Lambda) = 2\Lambda x_s < 2\Lambda x^{e'}_{AR} + \lambda^e_R = Th^e_m(\Lambda).
$$

Finally, we prove that $Th^e_m(\Lambda) > Th^e_p(\Lambda)$ when $\Lambda > \Lambda^e_1$. Note that $Th^e_m(\Lambda)$ have different expressions depending on the values of Λ and t. We discuss the two cases separately.

When $t < v - \frac{4c^2}{v}$ and $\Lambda > \Lambda^e_1$, or when $t \ge v - \frac{4c^2}{v}$ and $\Lambda \in (\Lambda^e_1, \Lambda^e_2)$, we have $Th^e_m(\Lambda) = 2\Lambda x^{e'}_{AR} + \lambda^e_R$. In this case, we have $Th^e_m(\Lambda) = 2\Lambda x^{e'}_{AR} + \lambda^e_R > \sqrt{1-\frac{2c}{v-t/2}} = Th^e_p(\Lambda)$ due to the following inequality,

$$
\begin{aligned}
v-\frac{t}{2} &= c\left(\frac{1}{1-2\Lambda x^{e'}_{AR}-\lambda^e_R}+\frac{1-2\Lambda x^{e'}_{AR}}{1+\lambda^e_R}\right)\\
&= c\left(\frac{1}{1-2\Lambda x^{e'}_{AR}-\lambda^e_R}+\frac{1}{1+2\Lambda x^{e'}_{AR}+\lambda^e_R}-\frac{2\Lambda x^{e'}_{AR}(\lambda^e_R+2\Lambda x^{e'}_{AR})}{(1+\lambda^e_R)(1+2\Lambda x^{e'}_{AR}+\lambda^e_R)}\right)\\
&< c\left(\frac{1}{1-2\Lambda x^{e'}_{AR}-\lambda^e_R}+\frac{1}{1+2\Lambda x^{e'}_{AR}+\lambda^e_R}\right)\\
&= \frac{c}{\frac{1}{2}-(2\Lambda x^{e'}_{AR}+\lambda^e_R)^2/2}.
\end{aligned}
$$

When $t \ge v - \frac{4c^2}{v}$ and $\Lambda > \Lambda^e_2$, we have $Th^e_m(\Lambda) = 2\Lambda x_s$. In this case, no one chooses meaning the utility for R is negative even if $\lambda_R = 0$. Specifically, $u_R = v - \frac{t}{2} - c(\frac{1}{1-2\Lambda x} + 1 - 2\Lambda x) < 0$. This implies

$$
v-\frac{t}{2}-\frac{2c}{1-(2\Lambda x)^2} = v-\frac{t}{2}-c(\frac{1}{1-2\Lambda x}+\frac{1}{1+2\Lambda x}) < v-\frac{t}{2}-c(\frac{1}{1-2\Lambda x}+1-2\Lambda x) < 0.
$$

Therefore, in this case, $Th^e_m(\Lambda) = 2\Lambda x > \sqrt{1-\frac{2c}{v-t/2}} = Th^e_p$.

■

Proof of Proposition 4 From Corollary EC.1, when $\Lambda \le \Lambda^e_1$, both CUS and CUE have no customer balking, resulting in equal throughput. When $\Lambda^e_1 \le \Lambda \le \Lambda^c_1$, CUS has full coverage while CUE has balking, thus CUS has higher thourghput. When $\Lambda > \Lambda^c_1$ and the multi-listing queue is non-empty for CUE, players in the multi-listing queue has non-negative utility, therefore $v - \frac{t}{2} - \frac{c}{1-2\Lambda x^{e'}_{AR}-\lambda^e_R} > 0$, thus $2\Lambda x^{e'}_{AR} + \lambda^e_R < 1 - \frac{c}{v-t/2}$, implying less throughput than CUS. When $\Lambda > \Lambda^c_1$ and the multi-listing queue is empty for CUE, then CUE has throughput $2\Lambda x_s$, also resulting in lower or equal throughput compared to CUS. ■

Table EC.3 Social Welfare Under CUS

Λ	$[0,\Lambda_s]$	$[\Lambda_s,\Lambda_1^c]$	$[\Lambda_1^c,\Lambda_2^c]$	$[\Lambda_2^c,\infty]$
Pooling	$\Lambda(v-\frac{t}{2}-\frac{c}{1-\Lambda})$		0	
Single	$\Lambda(v-\frac{t}{4}-\frac{2c}{1-\Lambda})$	$\Lambda t{x_s}^2$		
Multi	$\Lambda(v-\frac{t}{2}-\frac{c}{1-\Lambda}+t(x_{AR})^2)$		$\Lambda t(x_{AR})^2$	$\Lambda t(x_{AR})^2$if $t\le v$; $\Lambda t{x_s}^2$ if $t>v$

Proof of Theorem 3 When $\Lambda<\Lambda_s$, it is straightforward to show that $SW_m(\Lambda)$ dominates both $SW_s(\Lambda)$ and $SW_p(\Lambda)$ by comparing those expressions in Corollary EC.1, Corollary EC.2 and Corollary EC.3, and noting

$$SW_m(\Lambda)-SW_s(\Lambda)=\Lambda\left(\frac{c}{1-\Lambda}-t(\frac{1}{2}-x_{AR})(\frac{1}{2}+x_{AR})\right)\ge\Lambda\left(\frac{c}{1-\Lambda}-t(\frac{1}{2}-x_{AR})\right)=\Lambda(\frac{c}{1-\Lambda}-\frac{c}{1-2\Lambda x_{AR}})>0.$$

We next compare $SW_s(\Lambda)$ and $SW_p(\Lambda)$ when $\Lambda<\Lambda_s$.

(i) When $t\le 4c$, for all $\Lambda<\Lambda_s$, we have $SW_s=\Lambda(v-\frac{t}{2}-\frac{2c}{1-\Lambda}+\frac{t}{4})<\Lambda(v-\frac{t}{2}-\frac{2c}{1-\Lambda}+c)\le\Lambda(v-\frac{t}{2}-\frac{c}{1-\Lambda})=SW_p$, where the first inequality follows from $t\le 4c$, and the second inequality follows from $\frac{c}{1-\Lambda}>c$.

(ii) When $4c<t\le v$ and $\Lambda<\Lambda_s$, solving $SW_s(\Lambda)<SW_p(\Lambda)$ shows that it is equivalent to $\Lambda>1-4c/t$. Thus, when $\Lambda<1-4c/t$, $SW_s(\Lambda)>SW_p(\Lambda)$ and when $\Lambda>1-4c/t$, $SW_s(\Lambda)<SW_p(\Lambda)$.

(iii) When $t>v$ and $\Lambda<\Lambda_s$, we note that customers of type 0.5 in a pooling system have a positive utility by joining either queue due to $\Lambda<\Lambda_s$. This implies that $u_s=v-\frac{t}{2}-\frac{2c}{1-\Lambda}>0$, which implies that $\frac{c}{1-\Lambda}<\frac{v}{2}-\frac{t}{4}<\frac{t}{4}$. Consequently, $SW_s(\Lambda)=\Lambda(v-\frac{t}{2}-\frac{2c}{1-\Lambda}+\frac{t}{4})>\Lambda(v-\frac{t}{2}-\frac{c}{1-\Lambda})=SW_p(\Lambda)$.

(iv) In addition, when $4c>v$, we must have $v>t$ since $\frac{v}{2}>v-2c>\frac{t}{2}$, the second inequality is due to Assumption 1-(i), thus $\Lambda>0>1-4c/t$ always holds, so we must have $SW_s(\Lambda)<SW_p(\Lambda)$ in this case.

When $\Lambda>\Lambda_2^c$ and $t>v$, we have $SW_m(\Lambda)=\Lambda tx_s^2$, in which case $SW_m(\Lambda)=SW_s(\Lambda)$.

When $\Lambda>\Lambda_1^c$ and $t\le v$, or when $\Lambda_1^c<\Lambda\le\Lambda_2^c$ and $t>v$, $SW_m(\Lambda)=\Lambda t(x_{AR})^2$. In the proof of Theorem 2, we have proved $x_{AR}^{e'}<x_s$; following a similar argument, we can prove $x_{AR}<x_s$. As a result, $SW_m(\Lambda)=\Lambda t(x_{AR})^2<\Lambda tx_s^2=SW_s(\Lambda)$. ■

Table EC.4 Social Welfare Under CUE

Λ	$[0,\Lambda_s]$	$[\Lambda_s,\Lambda_p^e]$	$[\Lambda_p^e,\Lambda_1^e]$	$[\Lambda_1^e,\Lambda_2^e]$	$[\Lambda_2^e,\infty]$
Pooling	$\Lambda(v-\frac{t}{2}-\frac{2c}{1-\Lambda^2})$		0		
Single	$\Lambda(v-\frac{t}{4}-\frac{2c}{1-\Lambda})$	$\Lambda t{x_s}^2$			
Multi	$\Lambda(v-\frac{t}{2}-\frac{c}{1-\Lambda}-\frac{c(1-2\Lambda x_{AR}^e)}{1+\Lambda-2\Lambda x_{AR}^e}+t(x_{AR}^e)^2)$			$\Lambda t(x_{AR}^{e'})^2$	$\Lambda t{x_{AR}^{e'}}^2$if $t\le v-\frac{4c^2}{v}$; $\Lambda t{x_s}^2$ if $t>v-\frac{4c^2}{v}$

Proof of Theorem 4 When $\Lambda<\Lambda_s$, it is straightforward to show that $SW_m^e(\Lambda)>SW_p^e(\Lambda)$ by noting

$$\frac{2c}{1-\Lambda^2}=\frac{c}{1-\Lambda}+\frac{c}{1+\Lambda}>\frac{c}{1-\Lambda}+\frac{c(1-2\Lambda x_{AR}^e)}{1+\Lambda-2\Lambda x_{AR}^e}.$$

We can also show that $SW_m^e(\Lambda)>SW_s(\Lambda)$ when $\Lambda<\Lambda_s$:

$$\begin{aligned}SW_m^e(\Lambda)-SW_s(\Lambda)\ &=\Lambda\left(\frac{c}{1-\Lambda}-\frac{c(1-2\Lambda x_{AR}^e)}{1+\Lambda-2\Lambda x_{AR}^e}+t(x_{AR}^e)^2-\frac{t}{4}\right)\\ &=\Lambda\left(\frac{c}{1-\Lambda}-\frac{c(1-2\Lambda x_{AR}^e)}{1+\Lambda-2\Lambda x_{AR}^e}-t(\frac{1}{2}-x_{AR}^e)(\frac{1}{2}+x_{AR}^e)\right)\end{aligned}$$

$$
\begin{aligned}
&= \Lambda\left(\frac{c}{1-\Lambda} - \frac{c(1-2\Lambda x^e_{AR})}{1+\Lambda-2\Lambda x^e_{AR}} - \frac{c\Lambda(1-\Lambda x^e_{AR})(\frac{1}{2}+x^e_{AR})}{(1+\Lambda-2\Lambda x^e_{AR})(\frac{1}{2}-\Lambda x^e_{AR})}\right)\\
&= \Lambda\left(\frac{c}{1-\Lambda} - c\frac{2(\frac{1}{2}-\Lambda x^e_{AR})^2+\Lambda(1-\Lambda x^e_{AR})(\frac{1}{2}+x^e_{AR})}{(1+\Lambda-2\Lambda x^e_{AR})(\frac{1}{2}-\Lambda x^e_{AR})}\right)\\
&= \Lambda\left(\frac{c}{1-\Lambda} - c\frac{\frac{1}{2}(1+\Lambda-2\Lambda x^e_{AR})-\Lambda^2 x^e_{AR}(\frac{1}{2}-x^e_{AR})}{(1+\Lambda-2\Lambda x^e_{AR})(\frac{1}{2}-\Lambda x^e_{AR})}\right)\\
&\geq \Lambda\left(\frac{c}{1-\Lambda} - \frac{c}{1-2\Lambda x^e_{AR}}\right) > 0,
\end{aligned}
$$

where the last inequality follows from $2\Lambda x^e_{AR} \leq \Lambda < \Lambda_s < 1$.

To compare $SW_s(\Lambda)$ and $SW_p(\Lambda)$ when $\Lambda < \Lambda_s$, we calculate their difference as

$$SW_s(\Lambda) - SW^e_p(\Lambda) = \Lambda(\frac{t}{4} + \frac{2c}{1-\Lambda^2} - \frac{2c}{1-\Lambda}) = \Lambda(\frac{t}{4} - \frac{2c\Lambda}{1-\Lambda^2}).$$

The RHS of the above equation has a root $\frac{\sqrt{16c^2+t^2}-4c}{t}$. By monotonicity, $SW_s(\Lambda) > SW^e_p(\Lambda)$ if $\Lambda < \frac{\sqrt{16c^2+t^2}-4c}{t}$, and $SW_s(\Lambda) < SW^e_p(\Lambda)$ otherwise. Therefore, when $t \geq \frac{3v-3c-\sqrt{v^2-2vc+9c^2}}{2}$, we have $\Lambda_s = 1 - \frac{2c}{v-t/2} \leq \frac{\sqrt{16c^2+t^2}-4c}{t}$, and thus $SW_s(\Lambda) > SW^e_p(\Lambda)$ for all $\Lambda < \Lambda_s \leq \frac{\sqrt{16c^2+t^2}-4c}{t}$; when $t < \frac{3v-3c-\sqrt{v^2-2vc+9c^2}}{2}$, $\frac{\sqrt{16c^2+t^2}-4c}{t} \in (0, \Lambda_s)$. So $SW_s(\Lambda) > SW_p(\Lambda)$ when $\Lambda \in (0, \frac{\sqrt{16c^2+t^2}-4c}{t})$ and $SW_s(\Lambda) < SW_p(\Lambda)$ when $\Lambda \in (\frac{\sqrt{16c^2+t^2}-4c}{t}, \Lambda_s)$.

When $\Lambda > \Lambda^e_1$, $SW^e_p \equiv 0$ and thus is always the smallest among the three. The proof of $SW^e_m(\Lambda) < SW^e_s(\Lambda)$ when $\Lambda > \Lambda^e_1$ follows the same argument as the CUS case by noting $x^{e'}_{AR} < x_s$. ■

Proof of Proposition 5 Recall that x_{AR} and $x^{e'}_{AR}$ denote the values of x at which the curves $u_A(x)$ and $u_R(x)$ intersect under the CUS model and CUE models, respectively. The value $x^{e'}_{AR}$ must satisfy equation (EC.20), which then implies

$$t(\frac{1}{2} - x^{e'}_{AR}) - \frac{c}{1-2\Lambda x^{e'}_{AR}} = -c\frac{1-2\Lambda x^{e'}_{AR} - \lambda_R}{1+\lambda_R} < 0. \tag{EC.30}$$

While (EC.11) implies $t\left(\frac{1}{2} - x_{AR}\right) - \frac{c}{1-2\Lambda x_{AR}} = 0$, since $t\left(\frac{1}{2} - x\right) - \frac{c}{1-2\Lambda x}$ is decreasing in x, we must have $x^{e'}_{AR} > x_{AR}$. Therefore, Corollary EC.1 implies that CUE results in higher or equal social welfare than CUS when $\Lambda > \Lambda^c_1$. ■

EC.4. Proofs of Results in Section 6

Proof of Proposition 6 To prove this result, it suffices to show the structure of the optimal solution to Problem (4) for any fixed arrival-rate vector $(\lambda_A, \lambda_B, \lambda_R)$ (including the optimal arrival-rate vector). Let $(x^*_A, x^*_B, x^*_R) = (\lambda_A/\Lambda, \lambda_B/\Lambda, \lambda_R/\Lambda)$. We next solve the following optimization problem for given (x^*_A, x^*_B, x^*_R):

$$
\begin{aligned}
\max_{G_A(\cdot), G_B(\cdot), G_R(\cdot)} \quad & \int_0^1 (G_A(x)u_A(x) + G_B(x)u_B(x) + G_R(x)u_R(x))\,\mathrm{d}x \qquad \text{(EC.31)}\\
& 0 \leq G_A(x) \leq 1,\ 0 \leq G_B(x) \leq 1,\ 0 \leq G_R(x) \leq 1, \quad \forall x \in [0,1]\\
& G_A(x) + G_B(x) + G_R(x) \leq 1, \quad \forall x \in [0,1]\\
& x^*_A = \int_0^1 G_A(x)\mathrm{d}x,\ x^*_B = \int_0^1 G_B(x)\mathrm{d}x,\ x^*_R = \int_0^1 G_R(x)\mathrm{d}x\\
& u_A(x) = v - tx - cW_A\\
& u_B(x) = v - t(1-x) - cW_B\\
& u_R(x) = v - p_A tx - (1-p_A)t(1-x) - cW_R.
\end{aligned}
$$

Note that for fixed (x_A^*, x_B^*, x_R^*), W_A, W_B, W_R, p_A are all fixed. The objective function is

$$\begin{aligned}
&\int_0^1 \left(G_A(x)u_A(x) + G_B(x)u_B(x) + G_R(x)u_R(x)\right) \mathrm{d}x \\
=\ &\int_0^1 (v - tx - cW_A)\, G_A(x)\mathrm{d}x + \int_0^1 (v - t(1-x) - cW_B)\, G_B(x)\mathrm{d}x \\
&+ \int_0^1 (v + (1-2p_A)tx - (1-p_A)t - cW_R)\, G_R(x)\mathrm{d}x \\
=\ &\underbrace{\int_0^1 \left((v - cW_A)\, G_A(x) + (v - t - cW_B)\, G_B(x) + (v - (1-p_A)t - cW_R)\, G_R(x)\right) \mathrm{d}x}_{\text{a fixed number since } x_A^*, x_B^*, x_R^* \text{are fixed}} \\
&+ \int_0^1 t\left(-xG_A(x) + xG_B(x) + (1-2p_A)xG_R(x)\right) \mathrm{d}x.
\end{aligned}$$

With fixed x_A^*, x_B^* and x_R^*, maximizing the objective function is equivalent to maximizing

$$\int_0^1 \left(-xG_A(x) + xG_B(x) + (1-2p_A)xG_R(x)\right) \mathrm{d}x.$$

To proceed, we first prove the following lemma:

LEMMA EC.1. *For $C \in [0,1]$, the optimization problem*

$$\begin{aligned}
&\max_{F(\cdot)} \int_0^1 xF(x)dx \\
&\text{s.t. } 0 \le F(x) \le 1,\ \forall x \in [0,1] \\
&\int_0^1 F(x)dx = C
\end{aligned} \tag{EC.32}$$

has the following optimal solution:

$$F(x) = \begin{cases} 0, & x \in [0, 1-C) \\ 1, & x \in [1-C, 1], \end{cases} \tag{EC.33}$$

Proof of Lemma EC.1 Suppose $F^*(\cdot)$ is the optimal solution, we prove by contradiction that $\int_{1-c}^1 F^*(x)\mathrm{d}x = C$.

If $\int_{1-c}^1 F^*(x)\mathrm{d}x < C$, then $\int_0^{1-C} F^*(x)\mathrm{d}x = C - \int_{1-C}^1 F^*(x)\mathrm{d}x > 0$. It implies that there must exist an $t_1 < 1-C$ and $\epsilon_1 > 0$ and $\Delta t_1 > 0$ such that $F^*(x) \ge \epsilon_1$, $\forall x \in [t_1, t_1 + \Delta t_1]$ with $t_1 + \Delta t_1 < 1 - c$. Also, since $\int_{1-C}^1 F^*(x)\mathrm{d}x < C$, there must exist $t_2 > 1 - C$ and $\epsilon_2 > 0$ and $\Delta t_2 > 0$ such that $F^*(x) \le 1 - \epsilon_2$, $\forall x \in [t_2, t_2 + \Delta t_2]$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, $\Delta t = \min\{\Delta t_1, \Delta t_2\}$, define

$$F^{**}(x) = \begin{cases} F^*(x) - \epsilon, & \forall x \in [t_1, t_1 + \Delta t], \\ F^*(x) + \epsilon, & \forall x \in [t_2, t_2 + \Delta t], \\ F^*(x), & \text{otherwise}. \end{cases}$$

We can see that we will still have $\int_0^1 F^{**}(x)\mathrm{d}x = C$ and

$$\begin{aligned}
&\int_0^1 xF^{**}(x)\mathrm{d}x - \int_0^1 xF^*(x)\mathrm{d}x \\
=\ &\int_{t_1}^{t_1+\Delta t} x(F^*(x) - \epsilon)\mathrm{d}x - \int_{t_1}^{t_1+\Delta t} xF^*(x)\mathrm{d}x + \int_{t_2}^{t_2+\Delta t} x(F^*(x) + \epsilon)\mathrm{d}x - \int_{t_2}^{t_2+\Delta t} xF^*(x)\mathrm{d}x \\
=\ &-\epsilon \int_{t_1}^{t_1+\Delta t} x\mathrm{d}x + \epsilon \int_{t_2}^{t_2+\Delta t} x\mathrm{d}x > 0.
\end{aligned}$$

The last inequality holds because $t_1 < 1-C < t_2$. So $F^*(\cdot)$ cannot be optimal, $\int_{1-C}^1 F^*(x)\mathrm{d}x < C$ does not hold, we must have $\int_{1-C}^1 F^*(x)\mathrm{d}x \geq C$ and since $\int_{1-C}^1 F^*(x)\mathrm{d}x \leq \int_0^1 F^*(x)\mathrm{d}x = C$, we can have $\int_{1-C}^1 F^*(x)\mathrm{d}x = C$, so (EC.33) is the optimal solution. ∎

To find the solution that maximizes $\int_0^1 (-xG_A(x) + xG_B(x) + (1-2p_A)xG_R(x))\,\mathrm{d}x$, we first relax the constraint $G_A(x) + G_B(x) + G_R(x) \leq 1,\ \forall x \in [0,1]$. As we will later see, this constraint will be satisfied even after relaxation. We further suppose that W.O.L.G, $p_A \leq \frac{1}{2}$ given (x_A^*, x_B^*, x_R^*). Thus, $x \geq (1-2p_A)x \geq 0 \geq -x$.

We maximize $\int_0^1 -xG_A(x)\mathrm{d}x$ and $\int_0^1 xG_B(x) + (1-2p_A)xG_R(x)\mathrm{d}x$, separately.

$\int_0^1 (xG_B(x) + (1-2p_A)xG_R(x))\,\mathrm{d}x = \int_0^1 (1-2p_A)x(G_R(x) + G_B(x))\,\mathrm{d}x + \int_0^1 2p_A x G_B(x)\mathrm{d}x$. Maximizing the first term in this equation is equivalent to maximizing $\int_0^1 x\,(G_R(x) + G_B(x))\,\mathrm{d}x$ with constraint $\int_0^1 (G_R(x) + G_B(x))\mathrm{d}x = x_B^* + x_R^*$. According to Lemma EC.1, the optimal solution is

$$G_R(x) + G_B(x) = \begin{cases} 0, & x \in [0, 1-x_B^*-x_R^*), \\ 1, & x \in [1-x_B^*-x_R^*, 1]. \end{cases}$$

Maximizing the second term is equivalent to maximizing $\int_0^1 xG_B(x)\mathrm{d}x$ with constraint $\int_0^1 G_B(x) = x_B^*$. According to Lemma EC.1, the optimal solution is

$$G_B(x) = \begin{cases} 0, & x \in [0, 1-x_B^*), \\ 1, & x \in [1-x_B^*, 1]. \end{cases}$$

Subtracting $G_B(x)$ from $G_R(x) + G_B(x)$ gives

$$G_R(x) = \begin{cases} 1, & x \in [1-x_B^*-x_R^*, 1-x_B^*], \\ 0, & \text{otherwise.} \end{cases}$$

For $\int_0^1 -xG_A(x)\mathrm{d}x$, similar to Lemma EC.1, we can also prove that the optimal solution to maximizing $\int_0^1 -xG_A(x)\mathrm{d}x$ with constraint $\int_0^1 G_A(x) = x_A^*$ is

$$G_A(x) = \begin{cases} 1, & x \in [0, x_A^*], \\ 0, & x \in (x_A^*, 1]. \end{cases}$$

Since $x_A^* + x_B^* + x_R^* \leq 1$, the constraint $G_A(x) + G_B(x) + G_R(x) \leq 1,\ \forall x \in [0,1]$ is indeed satisfied. Hence, the above (G_A, G_B, G_R) is optimal.

This implies that an optimal solution to Problem (4) must have the structure specified by Proposition 6.

∎

Proof of Lemma 2 Gardner et al. (2015) consider a system with 2 servers, where the service times are exponentially distributed with rate μ_1 and μ_2. There are three class of jobs: A, B, R where type A (type B) customers queue only for server A(server B), and type R customers queue for both servers. The arrival rate for type c_i is λ_{c_i}. Write the state of the system as $(c_1, ..., c_{n-1}, c_n)$, which means that there are n jobs in the system, and c_i is the class of the i-th coming job in the system.

LEMMA EC.2 **(Gardner et al. (2015)).** *The limiting probability of being in state $(c_1, \ldots, c_{n-1}, c_n)$ depends on c_1, as follows:*

$$\pi_{(A,\ldots,c_{n-1},c_n)} = C_{\mathbb{W}} \left(\frac{\lambda_A}{\mu_1}\right)^{a_0} \left(\frac{\lambda_A}{\mu_1+\mu_2}\right)^{a_1} \left(\frac{\lambda_B}{\mu_1+\mu_2}\right)^{b_1} \left(\frac{\lambda_R}{\mu_1+\mu_2}\right)^{r}$$

$$\pi_{(B,\ldots,c_{n-1},c_n)} = C_{\mathbb{W}} \left(\frac{\lambda_B}{\mu_2}\right)^{b_0} \left(\frac{\lambda_A}{\mu_1+\mu_2}\right)^{a_1} \left(\frac{\lambda_B}{\mu_1+\mu_2}\right)^{b_1} \left(\frac{\lambda_R}{\mu_1+\mu_2}\right)^{r}$$

$$\pi_{(R,\ldots,c_{n-1},c_n)} = C_{\mathbb{W}} \left(\frac{\lambda_A}{\mu_1+\mu_2}\right)^{a_1} \left(\frac{\lambda_B}{\mu_1+\mu_2}\right)^{b_1} \left(\frac{\lambda_R}{\mu_1+\mu_2}\right)^{r},$$

where a_0 is the number of class A jobs before the first class B or R job, b_0 is the number of class B jobs before the first class A or R job, a_1 (respectively, b_1) is the number of class A (class B) jobs after the first job of class R or $B(A)$, r is the total number of class R jobs, and

$$C_{\mathbb{W}} = \left(1 - \frac{\lambda_A}{\mu_1}\right)\left(1 - \frac{\lambda_B}{\mu_2}\right)\left(1 - \frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right)$$

is a normalizing constant.

Based on their results, we can first calculate the probability of each server serving a type-R customer.

When $c_1 = A$, server A must be serving type A customer, server B is serving type R customer if the first job in the queue $(c_1, \ldots, c_{n-1}, c_n)$ that is not A is R. It has total steady probability base on Lemma EC.2:

$$C_{\mathbb{W}} \sum_{a_0=1}^{\infty} \left(\frac{\lambda_A}{\mu_1}\right)^{a_0} \left(\frac{\lambda_R}{\mu_1+\mu_2}\right) \sum_{k=0}^{\infty} \left(\frac{\lambda_A+\lambda_B+\lambda_R}{\mu_1+\mu_2}\right)^{k} = C_{\mathbb{W}} \left(\frac{\lambda_A}{\mu_1-\lambda_A}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right)$$

When $c_1 = B$, similarly we will have the total steady probability for server A serving type R and server B serving B customer is

$$C_{\mathbb{W}} \sum_{b_0=1}^{\infty} \left(\frac{\lambda_B}{\mu_2}\right)^{b_0} \left(\frac{\lambda_R}{\mu_1+\mu_2}\right) \sum_{k=0}^{\infty} \left(\frac{\lambda_A+\lambda_B+\lambda_R}{\mu_1+\mu_2}\right)^{k} = C_{\mathbb{W}} \left(\frac{\lambda_B}{\mu_2-\lambda_B}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right).$$

When $c_1 = R$, the total steady probability for both servers serving type R is

$$C_{\mathbb{W}} \left(\frac{\lambda_R}{\mu_1+\mu_2}\right) \sum_{k=0}^{\infty} \left(\frac{\lambda_A+\lambda_B+\lambda_R}{\mu_1+\mu_2}\right)^{k} = C_{\mathbb{W}} \frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}.$$

So server A serves type R customer in steady state with probability

$$C_{\mathbb{W}} \left(\frac{\lambda_B}{\mu_2-\lambda_B}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right) + C_{\mathbb{W}} \frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B} = C_{\mathbb{W}} \left(\frac{\mu_2}{\mu_2-\lambda_B}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right).$$

Server B serves type R customer in steady state with probability

$$C_{\mathbb{W}} \left(\frac{\lambda_A}{\mu_1-\lambda_A}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right) + C_{\mathbb{W}} \frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B} = C_{\mathbb{W}} \left(\frac{\mu_1}{\mu_1-\lambda_A}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right).$$

So type R customer will be served by server A with probability

$$p_A = \frac{C_{\mathbb{W}} \left(\frac{\mu_2}{\mu_2-\lambda_B}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right)\mu_1}{C_{\mathbb{W}} \left(\frac{\mu_2}{\mu_2-\lambda_B}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right)\mu_1 + C_{\mathbb{W}} \left(\frac{\mu_1}{\mu_1-\lambda_A}\right)\left(\frac{\lambda_R}{\mu_1+\mu_2-\lambda_A-\lambda_B}\right)\mu_2} = \frac{\mu_1-\lambda_A}{\mu_1+\mu_2-\lambda_A-\lambda_B}.$$

■

Proof of Proposition 7 Under the CUS mode, the social optimum is achieved by plugging in the expressions into Problem (5):

$$\begin{aligned}
\max_{x_A,x_B,x_R} S(x_A,x_B,x_R) = \; & \Lambda x_A\left[v-\tfrac{tx_A}{2}-\tfrac{c}{0.5-\Lambda x_A}-\tfrac{c}{1-\Lambda(x_A+x_B+x_R)}+\tfrac{c}{1-\Lambda(x_A+x_B)}\right]+ \\
& \Lambda x_B\left[v-\tfrac{tx_B}{2}-\tfrac{c}{0.5-\Lambda x_B}-\tfrac{c}{1-\Lambda(x_A+x_B+x_R)}+\tfrac{c}{1-\Lambda(x_A+x_B)}\right]+ \\
& \Lambda x_R\left[v-\tfrac{t}{2}+t\tfrac{\Lambda(x_A-x_B)(1/2-x_B-x_R/2)}{1-\Lambda(x_A+x_B)}-\tfrac{c}{1-\Lambda(x_A+x_B+x_R)}\right] \\
s.t. \quad & 0\le x_B\le x_A\le 0.5 \\
& 0\le x_R\le 1-x_A-x_B \\
& \Lambda x_A<0.5 \\
& \Lambda(x_A+x_B+x_R)<1
\end{aligned} \tag{EC.34}$$

(i) After variable substitution $x_A=y+z$, $x_B=y-z$ and $x_A+x_B+x_R=w$ in (EC.34), the problem becomes

$$\begin{aligned}
\max_{y,z,w} S(y,z,w) = \; & wv-t(y^2+z^2)-\tfrac{cw}{1-\Lambda w}-\tfrac{cy(1-2\Lambda y)+2c\Lambda z^2}{(0.5-\Lambda y)^2-\Lambda^2z^2}+\tfrac{cy}{0.5-\Lambda y}-\tfrac{t}{2}(w-2y)+\tfrac{(w-2y)(1-w+2z)t\Lambda z}{1-2\Lambda y} \\
s.t. \quad & 0\le y\le 0.5, \\
& \Lambda(y+z)<0.5, \\
& 0\le z\le y\le \tfrac{w}{2}\le\tfrac{1}{2}.
\end{aligned} \tag{EC.35}$$

So we have

$$\frac{\partial S(y,z,w)}{\partial y}=t(1-2y)+\frac{2c}{(1-2\Lambda y)^2}-c\frac{(0.5-\Lambda y)^2+\Lambda^2z^2}{\left((0.5-\Lambda y)^2-\Lambda^2z^2\right)^2}-\frac{(1-w\Lambda)(1-w+2z)t\Lambda z}{2(0.5-\Lambda y)^2}. \tag{EC.36}$$

At the same time, when $y=0$, we must have $z=0$ and therefore $\frac{\partial S(y,z,w)}{\partial y}|_{y=0}=t+2c>0$, so we must have $y>0$ in optimal solution, i.e., $2y=x_A^*+x_B^*>0$, some customers are single-listed.

(ii) and (iii) We first prove that when $x_R^*>0$ and $x_A^*+x_B^*+x_R^*<1$, we will have $x_A^*>x_B^*$. After variable substitution $x_A=y+z$, $x_B=y-z$ in (EC.34), the social optimal problem becomes

$$\begin{aligned}
\max_{y,z,x_R} S(y,z,x_R) = \; & (2y+x_R)v-t(y^2+z^2)-\tfrac{c(2y+x_R)}{1-\Lambda(2y+x_R)}-\tfrac{cy(1-2\Lambda y)+2c\Lambda z^2}{(0.5-\Lambda y)^2-\Lambda^2z^2}+\tfrac{cy}{0.5-\Lambda y}-\tfrac{t}{2}x_R+\tfrac{x_Rt\Lambda z(1-2y-x_R+2z)}{1-2\Lambda y} \\
s.t. \quad & 0\le y\le 0.5, \\
& \Lambda(y+z)<0.5, \\
& 0\le z\le y, \\
& 0\le x_R\le 1-2y.
\end{aligned} \tag{EC.37}$$

From (EC.37), we have

$$\frac{\partial S(y,z,x_R)}{\partial z}=-2tz-\frac{2c(0.5-\Lambda y)\Lambda z}{\left((0.5-\Lambda y)^2-\Lambda^2z^2\right)^2}+\frac{2t\Lambda z x_R}{0.5-\Lambda y}+\frac{x_R(1-2y-x_R)\Lambda t}{1-2\Lambda y}.$$

So when $x_R^*>0$ and $x_A^*+x_B^*+x_R^*<1$, suppose y^* is the optimal solution, we must have $1-2y^*-x_R^*>0$, thus $\frac{\partial S(y^*,z,x_R^*)}{\partial z}|_{z=0}=\frac{x_R^*(1-2y^*-x_R^*)\Lambda t}{1-\Lambda y^*}>0$, therefore, we must have $z^*>0$ in the optimal solution, that is $x_A^*>x_B^*$.

We then prove that when $x_A^* + x_B^* + x_R^* = 1$, we must have $x_A^* = x_B^*$. When $x_A^* + x_B^* + x_R^* = 1$, after substituting $x_A = y + z$ and $x_B = y - z$ and $x_R = 1 - 2y$ in (EC.34), we have

$$\begin{aligned}
\max_{y,z} S(y,z) = \ & -t(y^2+z^2) - \frac{c(0.5-\Lambda y)}{\Lambda\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)} + \frac{cy}{0.5-\Lambda y} + ty + \frac{(1-2y)\Lambda t z^2}{0.5-\Lambda y} \\
s.t. \qquad & 0 \le y \le 0.5, \\
& \Lambda(y+z) < 0.5, \\
& 0 \le z \le y.
\end{aligned}$$

So we have $\frac{\partial S(y,z)}{\partial y} = t(1-2y) + \frac{2c}{(1-2\Lambda y)^2} - c\frac{(0.5-\Lambda y)^2+\Lambda^2 z^2}{\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2} - \frac{(1-\Lambda)t\Lambda z^2}{(0.5-\Lambda y)^2}$. It is a decreasing function of y. At the same time, when $y = 0$ and $y = 0.5$, we must have $z = 0$ and therefore

$$\frac{\partial S(y,z)}{\partial y}|_{y=0} = t + 2c > 0 \text{ and } \frac{\partial S(y,z)}{\partial y}|_{y=0.5} = -\frac{c}{(1-2\Lambda)^2} < 0.$$

So if y^* and z^* are the optimal solution, we must have

$$t(1-2y^*) + \frac{2c}{(1-2\Lambda y^*)^2} - c\frac{(0.5-\Lambda y^*)^2+\Lambda^2 z^{*2}}{\big((0.5-\Lambda y^*)^2-\Lambda^2 z^{*2}\big)^2} - \frac{(1-\Lambda)t\Lambda z^{*2}}{(0.5-\Lambda y^*)^2} = 0 \tag{EC.38}$$

so for all y and z that satisfies (EC.38), we must have

$$t = \frac{c\frac{(0.5-\Lambda y)^2+\Lambda^2 z^2}{\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2} - \frac{2c}{(1-2\Lambda y)^2}}{(1-2y) - \frac{(1-\Lambda)\Lambda z^2}{(0.5-\Lambda y)^2}} = \frac{c\left((0.5-\Lambda y)^4 + 4(0.5-\Lambda y)^2\Lambda^2 z^2 - \Lambda^4 z^4\right)}{2\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2\big((1-2y)(0.5-\Lambda y)^2-(1-\Lambda)\Lambda z^2\big)}, \tag{EC.39}$$

so we must have $\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2\big((1-2y)(0.5-\Lambda y)^2-(1-\Lambda)\Lambda z^2\big) > 0$.

At the same time, we have

$$\frac{\partial S(y,z)}{\partial z^2} = t\frac{\Lambda - 0.5 - \Lambda y}{0.5-\Lambda y} - \frac{c\Lambda(0.5-\Lambda y)}{\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2},$$

which is a decreasing function of z^2.

If $\Lambda - 0.5 - \Lambda y^* \le 0$, $\frac{\partial S(y^*,z)}{\partial z^2} < 0$, then we must have $z^* = 0$. If $\Lambda - 0.5 - \Lambda y^* > 0$, then for all the y and z that satisfy (EC.38), by plugging in EC.39, we have

$$\begin{aligned}
\frac{\partial S(y,z)}{\partial z^2} = & \frac{c}{2\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2}\left(\frac{\Lambda-0.5-\Lambda y}{0.5-\Lambda y} \times \frac{(0.5-\Lambda y)^4+4(0.5-\Lambda y)^2\Lambda^2 z^2-\Lambda^4 z^4}{(1-2y)(0.5-\Lambda y)^2-(1-\Lambda)\Lambda z^2} - 2\Lambda(0.5-\Lambda y)\right) \\
= & \frac{c}{2\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2(0.5-\Lambda y)\big((1-2y)(0.5-\Lambda y)^2-(1-\Lambda)\Lambda z^2\big)} \times \\
& \big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)\big((3\Lambda y-\Lambda-0.5)(0.5-\Lambda y)^2+(\Lambda-0.5-\Lambda y)\Lambda^2 z^2\big) \\
= & \frac{c\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)}{2\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2(0.5-\Lambda y)\big((1-2y)(0.5-\Lambda y)^2-(1-\Lambda)\Lambda z^2\big)} \times \\
& \big(-2(0.5-\Lambda y)(0.5-\Lambda y)^2-(\Lambda-0.5-\Lambda y)((0.5-\Lambda y)^2-\Lambda^2 z^2)\big) < 0.
\end{aligned}$$

The last inequality holds because $\big((0.5-\Lambda y)^2-\Lambda^2 z^2\big)^2\big((1-2y)(0.5-\Lambda y)^2-(1-\Lambda)\Lambda z^2\big) > 0$ from (EC.39). Therefore, $z^* = 0$ in this case.

We have thus proved that when $x_A^* + x_B^* + x_R^* = 1$, we must have a symmetric optimum, i.e., $x_A^* - x_B^* = z^* = 0$.

We next prove that when $\Lambda \le 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$, the social optimum is full coverage without balking, while $\Lambda \ge 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$, the social optimum has balking. From problem (EC.35), we have

$$\frac{\partial S(y,z,w)}{\partial w} = v - \frac{t}{2} - \frac{c}{(1-\Lambda w)^2} + \frac{(1+2y+2z-2w)t\Lambda z}{1-2\Lambda y} \tag{EC.40}$$

It is a strictly decreasing function of w. If $w=1$, we have $z=0$, then $\frac{\partial S(y,z,w)}{\partial w}=v-\frac{t}{2}-\frac{c}{(1-\Lambda)^2}$, to have $w=1$, we must have $\frac{\partial S(y,z,w)}{\partial w}=v-\frac{t}{2}-\frac{c}{(1-\Lambda)^2}\geq 0$, that is $\Lambda\leq 1-\sqrt{\frac{c}{v-\frac{t}{2}}}$.

So when $\Lambda\leq 1-\sqrt{\frac{c}{v-\frac{t}{2}}}$, the social optimum has no balking, when $\Lambda>1-\sqrt{\frac{c}{v-\frac{t}{2}}}$, the social optimum has balking.

Finally, we prove that when $t\leq v$, we must have $x_R^*>0$. From (EC.37), we have

$$\frac{\partial S(y,z,x_R)}{\partial x_R}|_{x_R=0}=v-\frac{t}{2}-\frac{c}{(1-2\Lambda y)^2}+\frac{(1-2y+2z)\Lambda tz}{1-2\Lambda y}\geq v-\frac{t}{2}-\frac{c}{(1-2\Lambda y)^2}>\frac{t}{2}-ty-\frac{c}{(1-2\Lambda y)^2}\geq 0$$

the last inequality will be proved in the following (EC.41), and $v-\frac{t}{2}>\frac{t}{2}-ty$ is because $t\leq v$ and $y>0$. ■

Proof of Theorem 5 (i) We have proved that when $\Lambda\leq 1-\sqrt{\frac{c}{v-\frac{t}{2}}}$, both the social optimum and equilibrium have the same throughput (full coverage). So customers under-choose multi-listing in equilibrium, $q_R^{eq}<x_R^*$ is equivalent to proving $2x_m=q_A^{eq}+q_B^{eq}>x_A^*+x_B^*$ (which is also the proof for (iii)).

Proposition 1 showed that x_m is either x_{AR} or x_s. We first prove $x_A^*+x_B^*<2x_{AR}$. From (EC.36), we know that $\frac{\partial S(y,z,w)}{\partial y}$ is a strictly decreasing function of y. In the meantime, when $y=0$, we must have $z=0$ and therefore $\frac{\partial S(y,z,w)}{\partial y}|_{y=0}=t+2c>0$. We have $y\leq 0.5$ and $y<\frac{0.5}{\Lambda}-z$, when y get close to $\min\{0.5,\frac{0.5}{\Lambda}-z\}$, we have $\frac{\partial S(y,z,w)}{\partial y}<0$. So if y^* and z^* are the optimal solution, we must have either $2y^*<w$ and

$$t(1-2y^*)+\frac{2c}{(1-2\Lambda y^*)^2}-c\frac{(0.5-\Lambda y^*)^2+\Lambda^2z^{*2}}{\left((0.5-\Lambda y^*)^2-\Lambda^2z^{*2}\right)^2}-\frac{(1-w\Lambda)(1-w+2z)t\Lambda z^*}{2(0.5-\Lambda y^*)^2}=0,$$

or $2y^*=w$ and

$$t(1-2y^*)+\frac{2c}{(1-2\Lambda y^*)^2}-c\frac{(0.5-\Lambda y^*)^2+\Lambda^2z^{*2}}{\left((0.5-\Lambda y^*)^2-\Lambda^2z^{*2}\right)^2}-\frac{(1-w\Lambda)(1-w+2z)t\Lambda z^*}{2(0.5-\Lambda y^*)^2}\geq 0.$$

The above equations are decreasing functions of z^{*2}, so in both cases, we can have

$$t(1-2y^*)-\frac{2c}{(1-2\Lambda y^*)^2}\geq 0 \tag{EC.41}$$

Therefore, we can have $t(1-2y^*)-\frac{2c}{1-2\Lambda y^*}>t(1-2y^*)-\frac{2c}{(1-2\Lambda y^*)^2}\geq 0$, since we have $t\left(\frac{1}{2}-x_{AR}\right)-\frac{c}{1-2\Lambda x_{AR}}=0$. So $y^*<x_{AR}$ as $t\left(\frac{1}{2}-x\right)-\frac{c}{1-2\Lambda x}$ is a strictly decreasing function of x, thus $2y=x_A^*+x_B^*<2x_{AR}$.

Next we show that $x_A^*+x_B^*<2x_s$. At optimality, everyone must has non-negative utility (which will be proved in the proof of Proposition 8). Therefore,

$$v-tx_A^*-\frac{c}{0.5-\Lambda x_A^*}-\frac{c}{1-\Lambda(x_A^*+x_B^*+x_R^*)}+\frac{c}{1-\Lambda(x_A^*+x_B^*)}\geq 0.$$

That means $v-tx_A^*-\frac{c}{0.5-\Lambda x_A^*}\geq\frac{c}{1-\Lambda(x_A^*+x_B^*+x_R^*)}-\frac{c}{1-\Lambda(x_A^*+x_B^*)}>0$. Similarly, we can prove $x_A^*<x_s$.

(ii) We have proved that when $\Lambda\leq 1-\sqrt{\frac{c}{v-t/2}}$, $q_A^{eq}+q_B^{eq}+q_R^{eq}=x_A^*+x_B^*+x_R^*=1$; When $1-\frac{c}{v-\frac{t}{2}}\geq\Lambda>1-\sqrt{\frac{c}{v-\frac{t}{2}}}$, the equilibrium has full coverage. In this case, the social optimum has less throughput. When $\Lambda>1-\frac{c}{v-\frac{t}{2}}$, the equilibrium has throughput greater or equal to $1-\frac{c}{v-\frac{t}{2}}$, so next, we will show that social optimum has less throughput than $1-\frac{c}{v-\frac{t}{2}}$ when $v\geq t$, which completes the proof.

Since (EC.40) is a strictly decreasing function of w, the optimal solution must have $v-\frac{t}{2}-\frac{c}{(1-\Lambda w)^2}+\frac{(1+2y+2z-2w)t\Lambda z}{1-2\Lambda y}=0$. At the same time, we have $0.5-\Lambda y>\Lambda z$ and $1+2y+2z-2w=1-2(y-z)-2x_R\leq 1$, so we have $\frac{(1+2y+2z-2w)t\Lambda z}{1-2\Lambda y}\leq\frac{t}{2}$. Therefore, $v-\frac{c}{(1-\Lambda w)^2}\geq 0\Rightarrow\Lambda w\leq 1-\sqrt{\frac{c}{v}}\leq 1-\frac{c}{v-\frac{t}{2}}$. The last inequality holds because $v-\frac{t}{2}\geq 2c$ and $v\geq t$. ■

Proof of Proposition 8 Suppose x_A^*, x_B^*, x_R^* are the optimal solution to Problem (5). Without loss of generality, assume customers with $x \le x_A^*$ join queue A only; customers with $x \ge 1 - x_B^*$ join queue B only; customers with $1 - x_B^* - x_R^* \le x < 1 - x_B^*$ are assigned to multi-listing in the social optimum (as shown in the Figure 5). Next, we show that the social optimum could be achieved by the pricing scheme $P_A = u_A(x_A^*)$, $P_B = u_B(1-x_B^*) - u_R(1-x_B^*) + u_R(1 - x_B^* - x_R^*)$, $P_R = \max\{u_R(1 - x_B^* - x_R^*), 0\}$. We first show that each customer derives non-negative utility in the optimal solution. If there are customers with negative utility, removing such customers from the system reduces the waiting times for other players, resulting in higher utility for the entire system as W_A, W_B, W_R decrease when λ_A, λ_B or λ_R decrease. We have proved in Theorem 5 that $x_A^* > 0$, and $x_B^* \le x_A^*$, thus $u_B(1 - x_B^*) \ge u_A(x_A^*) \ge 0$. Next we show that $u_B(1 - x_B^*) > u_R(1 - x_B^*)$.

$$u_B(1 - x_B^*) > u_R(1 - x_B^*) \Leftrightarrow t(1 - 2x_B^*) > \frac{c}{0.5 - \Lambda x_B^*} \tag{EC.42}$$

(EC.42) can be proved from (EC.41). When $x_R^* > 0$, to ensure that $u_R(1 - x_B^* - x_R^*) \ge 0$, we have $P_B > 0$. If $x_R^* = 0$, $P_B = u_B(1 - x_B^*) - u_R(1 - x_B^*) + u_R(1 - x_B^* - x_R^*) = u_B(1 - x_B^*) \ge 0$. So P_B is non-negative in both cases.

Next, we will show that under such a price scheme, the equilibrium of the multi-listing system is the same as the social optimum. If customer x_0 joins the queue A only, then all customers with type $x \le x_0$ will also join queue A only because they will have higher utility than customer x_0 for joining the queue A only. So there must exist a threshold $x_A \le \frac{1}{2}$ such that all the people with $x \le x_A$ will join queue A only. Similarly, there must exist a $x_B \le \frac{1}{2}$ such that all the people with $x \ge 1 - x_B$ will join queue B only. All customers in between will be multi-listed or balk.

At the same time, $(x_A, x_B, x_R) = (x_A^*, x_B^*, x_R^*)$ is an equilibrium under price scheme $P_A = u_A(x_A^*)$, $P_B = u_B(1-x_B^*) - u_R(1 - x_B^*) + u_R(1 - x_B^* - x_R^*)$, $P_R = \max\{u_R(1 - x_B^* - x_R^*), 0\}$. Customers $x \le x_A^*$ join queue A only and customers with $x \ge 1 - x_B^*$ will join queue B only, customers with $1 - x_B^* - x_R^* \le x \le 1 - x_B^*$ will be multi-listed and the rest of customers will balk. Under this price scheme, customers with $x = x_A^*$ are indifferent between joining queue A and balking; customers with $x = 1 - x_B^*$ are indifferent between joining queue B and multi-listing. No one will make a different choice from the one in the social optimum.

It's easy to see that $P_B > P_R$ as $P_B = P_R + u_B(1 - x_B^*) - u_R(1 - x_B^*) > P_R$. We next prove that when $\Lambda \le 1 - \sqrt{\frac{c}{v - \frac{t}{2}}}$, $P_A > P_R$. In this case, social optimum has full coverage, therefore, $x_A^* = x_B^*$, so

$$P_A - P_R = u_A(x_A^*) - u_R(1 - x_B^* - x_R^*) = t(\frac{1}{2} - x_A^*) - \frac{c}{\mu - \Lambda x_A^*} + \frac{c}{2\mu - \Lambda(x_A^* + x_B^*)} > t(\frac{1}{2} - x_A^*) - \frac{c}{\mu - \Lambda x_A^*} \ge 0,$$

the inequality can be derived from (EC.41). So we can have $P_A > P_R$ in this case. ■

EC.5. Social Optimum and Revenue Maximization under CUE

For the CUE mode, we build on the results of Visschers et al. (2012) and derive in Lemma EC.3 the expressions of the expected response time for choosing option $i \in \{A, B, R\}$, $W_i(\lambda_A, \lambda_B, \lambda_R)$ and the probability that a multi-listing customer ends up having her request completed by Server A, $p_A(\lambda_A, \lambda_B, \lambda_R)$, as a function of $(\lambda_A, \lambda_B, \lambda_R)$. To obtain closed-form expressions, Visschers et al. (2012) assume that when a multi-listing customer arrives at an empty system, she is assigned to server A with probability $\frac{\lambda_R + \lambda_B}{2\lambda_R + \lambda_A + \lambda_B}$ and server B with probability $\frac{\lambda_R + \lambda_A}{2\lambda_R + \lambda_A + \lambda_B}$, i.e., she is less likely to be assigned to a server facing more single-listing demand. We follow this assumption. Note that in a symmetric case with $\lambda_A = \lambda_B$, she is assigned to each server with equal probability.

LEMMA EC.3. *In the CUE mode of the multi-listing system, given $(\lambda_A, \lambda_B, \lambda_R)$, the expected response time for being assigned to queue A is*

$$W_A(\lambda_A, \lambda_B, \lambda_R) = \frac{(\lambda_R+\lambda_A)(\lambda_R+\lambda_A+\lambda_B)((2\mu-\lambda_A-\lambda_B)^2\mu+\lambda_R(\lambda_A^2-2\lambda_A\mu+\lambda_B\mu))}{(2\mu-\lambda_R-\lambda_A-\lambda_B)(\mu-\lambda_A)\mu(\lambda_R^2(2\mu-\lambda_A-\lambda_B)+(\lambda_A+\lambda_B)(2\mu-\lambda_A-\lambda_B)\mu+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2))};$$

the expected response time for being assigned to queue B is

$$W_B(\lambda_A, \lambda_B, \lambda_R) = \frac{(\lambda_R+\lambda_B)(\lambda_R+\lambda_B+\lambda_A)((2\mu-\lambda_B-\lambda_A)^2\mu+\lambda_R(\lambda_B^2-2\lambda_B\mu+\lambda_A\mu))}{(2\mu-\lambda_R-\lambda_B-\lambda_A)(\mu-\lambda_B)\mu(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+(\lambda_B+\lambda_A)(2\mu-\lambda_B-\lambda_A)\mu+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2))};$$

the expected response time for multi-listing (option R) is

$$W_R(\lambda_A, \lambda_B, \lambda_R) = \frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(\lambda_R+\lambda_A)(2\mu-\lambda_B-\lambda_A)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)(2\mu-\lambda_R-\lambda_B-\lambda_A)};$$

the probability that a multi-listing customer ends up having her request completed by Server A is

$$p_A(\lambda_A, \lambda_B, \lambda_R) = \frac{(\lambda_B+\lambda_R)(\lambda_A-\mu)(\lambda_A-\lambda_B+\lambda_R+2\mu)}{\lambda_A^2(\lambda_R+\mu)+\lambda_B^2(\lambda_R+\mu)-2\lambda_R\mu(\lambda_R+2\mu)+\lambda_A(\lambda_R^2+\lambda_R\mu+2(\lambda_B-\mu)\mu)+\lambda_B(\lambda_R^2+\lambda_R\mu-2\mu^2)}.$$

Proof of Lemma EC.3 There are three types of customers $C = \{R, A, B\}$ with arrival rates λ_R, λ_A and λ_B, and two servers $C(m_1) = \{R, B\}, C(m_2) = \{R, A\}$, with service rates $\mu_1 = \mu_2 = \mu$.

First, we need to calculate the activation rate for all the cases, to satisfy the assignment condition, we have the activation rate as follows:

$$\lambda_{m_1}(\{m_2\}) = \lambda_R + \lambda_B,$$
$$\lambda_{m_2}(\{m_1\}) = \lambda_R + \lambda_A,$$
$$\lambda_{m_1}(\{\emptyset\}) = (\lambda_R+\lambda_B+\lambda_A)\frac{\lambda_R+\lambda_B}{2\lambda_R+\lambda_B+\lambda_A},$$
$$\lambda_{m_2}(\{\emptyset\}) = (\lambda_R+\lambda_B+\lambda_A)\frac{\lambda_R+\lambda_A}{2\lambda_R+\lambda_B+\lambda_A}.$$

Therefore we have the steady-state probability from Equation (32) in Visschers et al. (2012):

$$\pi(\cdot, m_1) = \frac{(\lambda_R+\lambda_B+\lambda_A)\frac{\lambda_R+\lambda_B}{2\lambda_R+\lambda_B+\lambda_A}}{\mu}\frac{1}{1-\frac{\lambda_B}{\mu}}\pi(0) = \frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)}{(\mu-\lambda_B)(2\lambda_R+\lambda_B+\lambda_A)}\pi(0),$$

$$\pi(\cdot, m_2) = \frac{(\lambda_R+\lambda_B+\lambda_A)\frac{\lambda_R+\lambda_A}{2\lambda_R+\lambda_B+\lambda_A}}{\mu}\frac{1}{1-\frac{\lambda_A}{\mu}}\pi(0) = \frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_A)}{(\mu-\lambda_A)(2\lambda_R+\lambda_B+\lambda_A)}\pi(0),$$

$$\pi(\cdot, m_2, m_1) = \frac{\left((\lambda_R+\lambda_B+\lambda_A)\frac{\lambda_R+\lambda_B}{2\lambda_R+\lambda_B+\lambda_A}\right)(\lambda_R+\lambda_A)}{\mu\cdot 2\mu}\frac{1}{1-\frac{\lambda_B}{\mu}}\frac{1}{1-\frac{\lambda_R+\lambda_B+\lambda_A}{2\mu}}\pi(0)$$
$$= \frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(\lambda_R+\lambda_A)}{(\mu-\lambda_B)(2\mu-\lambda_R-\lambda_B-\lambda_A)(2\lambda_R+\lambda_B+\lambda_A)}\pi(0),$$

$$\pi(\cdot, m_1, m_2) = \frac{\left((\lambda_R+\lambda_B+\lambda_A)\frac{\lambda_R+\lambda_A}{2\lambda_R+\lambda_B+\lambda_A}\right)(\lambda_R+\lambda_B)}{\mu\cdot 2\mu}\frac{1}{1-\frac{\lambda_A}{\mu}}\frac{1}{1-\frac{\lambda_R+\lambda_B+\lambda_A}{2\mu}}\pi(0)$$
$$= \frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(\lambda_R+\lambda_A)}{(\mu-\lambda_A)(2\mu-\lambda_R-\lambda_B-\lambda_A)(2\lambda_R+\lambda_B+\lambda_A)}\pi(0).$$

so we have:

$$\pi(0)=\frac{(2\lambda_R+\lambda_B+\lambda_A)(2\mu-\lambda_R-\lambda_B-\lambda_A)(\mu-\lambda_B)(\mu-\lambda_A)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)},$$

$$\pi(\cdot,m_1)=\frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(2\mu-\lambda_R-\lambda_B-\lambda_A)(\mu-\lambda_A)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)},$$

$$\pi(\cdot,m_2)=\frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_A)(2\mu-\lambda_R-\lambda_B-\lambda_A)(\mu-\lambda_B)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)},$$

$$\pi(\cdot,m_1,m_2)=\frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(\lambda_R+\lambda_A)(\mu-\lambda_B)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)},$$
$$\pi(\cdot,m_2,m_1)=\frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(\lambda_R+\lambda_A)(\mu-\lambda_A)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)}.$$

According to equation (33) in Visschers et al. (2012), taking the derivative with respect to s and substituting $s=0$ gives the expected waiting time for type R and B, A:

$$\begin{aligned}E(W_R)&=(\pi(\cdot,m_1,m_2)+\pi(\cdot,m_2,m_1))\frac{1}{2\mu-\lambda_R-\lambda_B-\lambda_A}\\&=\frac{(\lambda_R+\lambda_B+\lambda_A)(\lambda_R+\lambda_B)(\lambda_R+\lambda_A)(2\mu-\lambda_B-\lambda_A)}{\mu\left(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+\mu(2\mu-\lambda_B-\lambda_A)(\lambda_B+\lambda_A)+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2)\right)(2\mu-\lambda_R-\lambda_B-\lambda_A)},\\E(W_B)&=\pi(\cdot,m_1)\frac{1}{\mu-\lambda_B}+\pi(\cdot,m_2,m_1)\left(\frac{1}{\mu-\lambda_B}+\frac{1}{2\mu-\lambda_R-\lambda_B-\lambda_A}\right)+\pi(\cdot,m_1,m_2)\frac{1}{2\mu-\lambda_R-\lambda_B-\lambda_A}\\&=\end{aligned}$$

$$\frac{(\lambda_R+\lambda_B)(\lambda_R+\lambda_B+\lambda_A)((2\mu-\lambda_B-\lambda_A)^2\mu+\lambda_R(\lambda_B^2-2\lambda_B\mu+\lambda_A\mu))}{(2\mu-\lambda_R-\lambda_B-\lambda_A)(\mu-\lambda_B)\mu(\lambda_R^2(2\mu-\lambda_B-\lambda_A)+(\lambda_B+\lambda_A)(2\mu-\lambda_B-\lambda_A)\mu+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2))},$$

similarly $E(W_A)=$

$$\frac{(\lambda_R+\lambda_A)(\lambda_R+\lambda_A+\lambda_B)((2\mu-\lambda_A-\lambda_B)^2\mu+\lambda_R(\lambda_A^2-2\lambda_A\mu+\lambda_B\mu))}{(2\mu-\lambda_R-\lambda_A-\lambda_B)(\mu-\lambda_A)\mu(\lambda_R^2(2\mu-\lambda_A-\lambda_B)+(\lambda_A+\lambda_B)(2\mu-\lambda_A-\lambda_B)\mu+\lambda_R(4\mu^2-\mu\lambda_B-\mu\lambda_A-\lambda_B^2-\lambda_A^2))}.$$

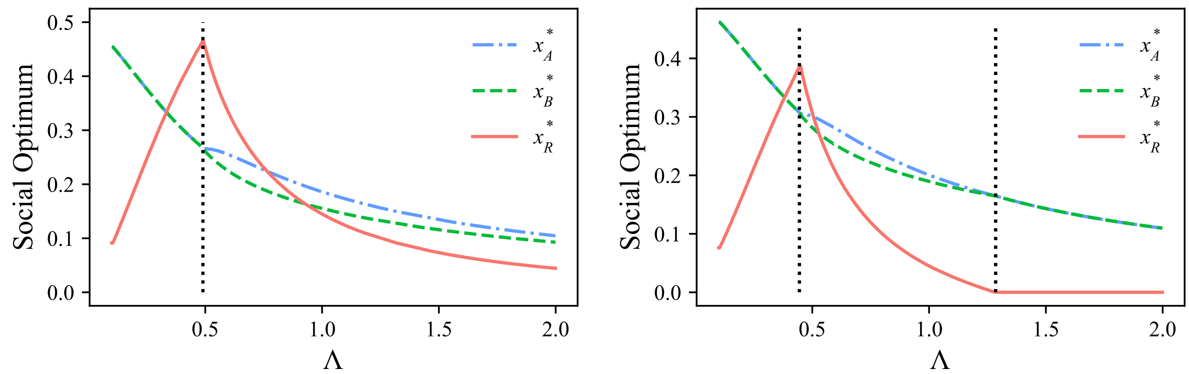
The total expected response time $W_R=E_R+\frac{1}{\mu}$, $W_B=E_B+\frac{1}{\mu}$ and $W_A=E_A+\frac{1}{\mu}$. And for the multi-listing customers, the service probability is

$$\begin{aligned}p_A=&\frac{\mu(\pi(\cdot,m_2)+\pi(\cdot,m_1,m_2)+\pi(\cdot,m_2,m_1))-\lambda_A}{\mu(\pi(\cdot,m_2)+\pi(\cdot,m_1,m_2)+\pi(\cdot,m_2,m_1))-\lambda_A+\mu(\pi(\cdot,m_1)+\pi(\cdot,m_1,m_2)+\pi(\cdot,m_2,m_1))-\lambda_B}\\=&\frac{(\lambda_B+\lambda_R)(\lambda_A-\mu)(\lambda_A-\lambda_B+\lambda_R+2\mu)}{\lambda_A^2(\lambda_R+\mu)+\lambda_B^2(\lambda_R+\mu)-2\lambda_R\mu(\lambda_R+2\mu)+\lambda_A(\lambda_R^2+\lambda_R\mu+2(\lambda_B-\mu)\mu)+\lambda_B(\lambda_R^2+\lambda_R\mu-2\mu^2)}.\end{aligned}$$

■

The expressions in Lemma EC.3 are generalizations of the expressions provided in Table 1 by allowing for $\lambda_A\neq\lambda_B$. As we can see, asymmetry ($\lambda_A\neq\lambda_B$) significantly complicates the expressions, precludes analytical traceability, and necessitates a numerical investigation. Our goal is to show that many key insights from the CUS case extend to the CUE case.

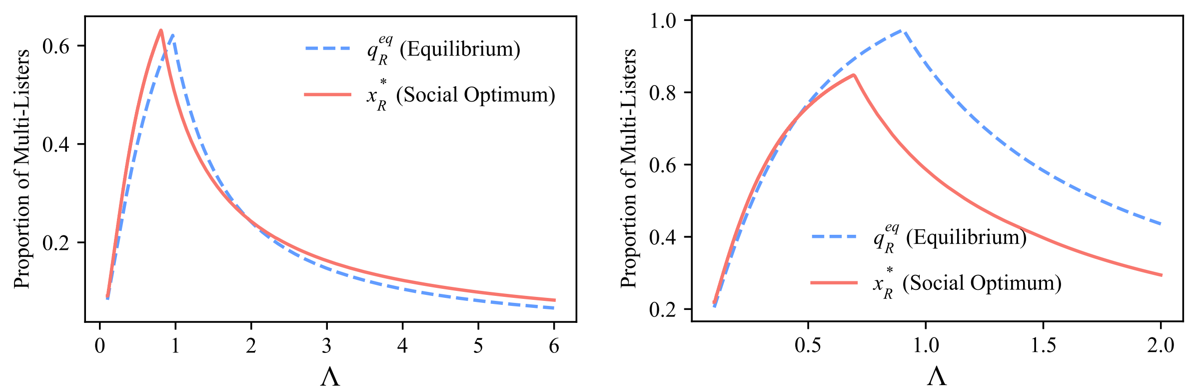
Figure EC.2 Socially Optimal Routing Under CUE



(a) $t = 0.5$ (b) $t = 0.6$

Note. $v = 0.7$, $c = 0.1$.

Figure EC.3 Social Optimum vs. Equilibrium Under CUE: Proportion of Multi-Listing Customers



(a) $c = 0.1$ (b) $c = 0.25$

Note. $v = 3$, $t = 0.5$.

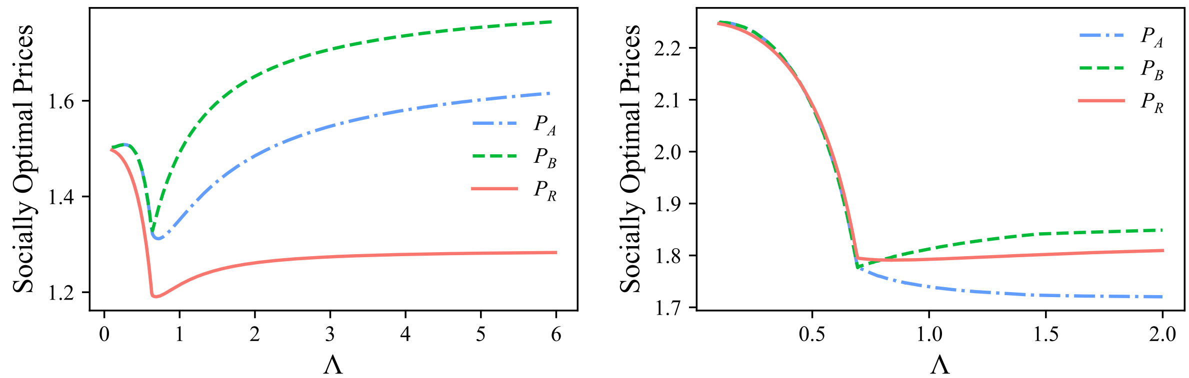
EC.5.1. Social Optimum

First, Figure EC.2 confirms that the structural insights from Proposition 7 and Figure 6 into the socially optimal routing policy carry over from the CUS case to the CUE case. Importantly, in the CUE mode, we also observe asymmetry in the socially optimal routing when unit misfit cost t is low and demand size Λ is large or when t is high and Λ is intermediate.

Second, Figure EC.3 presents the comparison between the social optimum and the equilibrium in the CUE mode. It echoes the message from Theorem 5 and Figure 7 that customers under-choose multi-listing in equilibrium relative to what they should in the social optimum when the demand size Λ is small but it could be the reverse case for other demand sizes.

Third, Figure EC.4 presents the socially optimal prices in the CUE mode. Similar to the CUS case (Proposition 8 and Figure EC.4), we observe that the socially optimal prices in the CUE mode are also nonnegative and that when

Figure EC.4 Socially Optimal Prices Under CUE



(a) $t = 2$ (b) $t = 0.5$

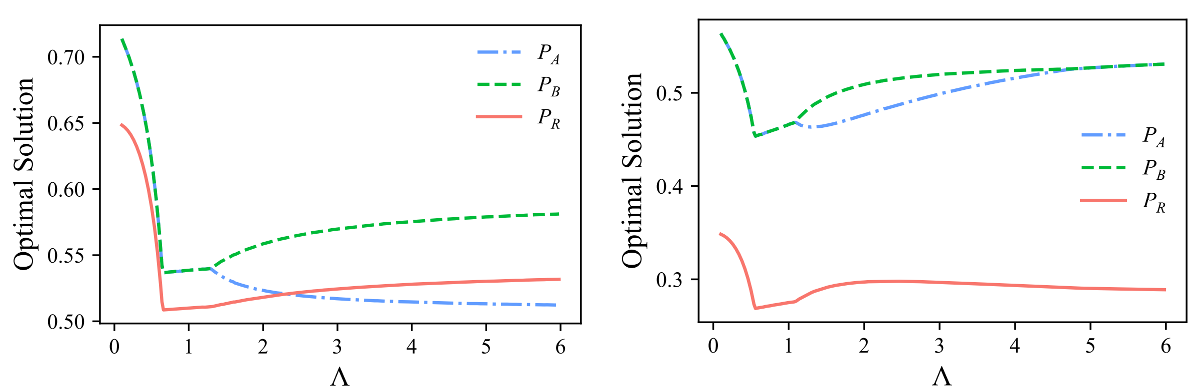
Note. $v = 3$, $c = 0.25$.

the demand size is small, the socially optimal price for multi-listing is lower than that for single-listing with either server. However, just like the CUS case, we observe that in the CUE mode, this relationship may change when the demand size is large. In particular, Figure EC.4-(b) shows the possibility of the socially optimal multi-listing price being slightly higher than both single-listing prices in the CUE mode. Finally, Figure EC.4 shows that the optimal pricing is asymmetric for sufficiently large demand size.

EC.5.2. Revenue Maximization

For the CUE model, characterization of the revenue-maximizing equilibrium is analytically intractable. However, we numerically find that the revenue-maximizing prices can still be asymmetric, as illustrated in Figure EC.5. We observe similar structural properties to those for the CUS model summarized in Proposition 9 and illustrated in Figure 9.

Figure EC.5 Revenue-Maximizing Prices Under CUE



(a) $t = 0.3$ (b) $t = 0.9$

Note. $v = 1, c = 0.1$.

EC.6. Proof of Proposition 9 in Section 7

As shown in the proof of Proposition 8, given prices (P_A, P_B, P_R) with $P_A \leq P_B$ (without loss of generality), there exist (x_A, x_B, x_R) with $x_B \leq x_A \leq 1/2$ such that customers with $x \in [0, x_A]$ choose to single-list with server A; customers with $x \in [1-x_B, x_B]$ choose to single-list with server B; and customers with $x \in [1-x_B-x_R, 1-x_B)$ choose multi-listing and the remaining customers balk. Moreover, customer $1-x_B$ must be indifferent between multi-listing and single-listing with server B, which gives $u_B(1-x_B) - P_B = u_R(1-x_B) - P_R$. Under revenue-maximizing prices, we must have $u_A(x_A) - P_A = 0$ and $u_R(1-x_B-x_R) - P_R = 0$. Therefore, the revenue-maximizing prices satisfy

$$P_A = u_A(x_A), \quad P_B = u_B(1-x_B) - u_R(1-x_B) + u_R(1-x_B-x_R), \quad P_R = u_R(1-x_B-x_R). \tag{EC.43}$$

Note that $P_A = P_B \Leftrightarrow x_A = x_B$. Hence, we reformulate the revenue-maximization problem as an optimization problem over (x_A, x_B, x_R) and examine when $x_A = x_B$ (which implies $P_A = P_B$) and when $x_A \neq x_B$ (which implies $P_A \neq P_B$). Substituting (EC.43) into the revenue-maximization problem under CUS gives

$$\begin{aligned}
\max_{x_A,x_B,x_R} R(x_A,x_B,x_R) = \quad & \Lambda x_A \left[v - tx_A - \tfrac{c}{0.5-\Lambda x_A} - \tfrac{c}{1-\Lambda(x_A+x_B+x_R)} + \tfrac{c}{1-\Lambda(x_A+x_B)}\right] + \\
& \Lambda x_B \left[v - tx_B - \tfrac{c}{0.5-\Lambda x_B} - \tfrac{c}{1-\Lambda(x_A+x_B+x_R)} + \tfrac{c}{1-\Lambda(x_A+x_B)} - \tfrac{\Lambda(x_A-x_B)}{1-\Lambda x_A-\Lambda x_B} t x_R\right] + \\
& \Lambda x_R \left[v - \tfrac{t}{2} + t\tfrac{\Lambda(x_A-x_B)(1/2-x_B-x_R)}{1-\Lambda(x_A+x_B)} - \tfrac{c}{1-\Lambda(x_A+x_B+x_R)}\right] \\
s.t. \quad & 0 \leq x_B \leq x_A \leq 0.5 \\
& 0 \leq x_R \leq 1 - x_A - x_B \\
& \Lambda x_A < 0.5 \\
& \Lambda(x_A + x_B + x_R) < 1.
\end{aligned} \tag{EC.44}$$

After variable substitution $x_A = y+z$, $x_B = y-z$ and $x_A + x_B + x_R = w$ in (EC.44), the problem becomes

$$\begin{aligned}
\max_{y,z,w} R(y,z,w) = \; & wv - 2t(y^2+z^2) - \tfrac{cw}{1-\Lambda w} - \tfrac{cy(1-2\Lambda y)+2c\Lambda z^2}{(0.5-\Lambda y)^2-\Lambda^2 z^2} + \tfrac{cy}{0.5-\Lambda y} - \tfrac{t}{2}(w-2y) + \tfrac{2t\Lambda z(w-2y)(0.5-w+2z)}{1-2\Lambda y} \\
s.t. \quad & 0 \leq y \leq 0.5, \\
& \Lambda(y+z) < 0.5, \\
& 0 \leq z \leq y \leq \tfrac{w}{2} \leq \tfrac{1}{2}.
\end{aligned} \tag{EC.45}$$

So we have

$$\frac{\partial R(y,z,w)}{\partial y} = t(1-4y) + \frac{2c}{(1-2\Lambda y)^2} - c\frac{(0.5-\Lambda y)^2 + \Lambda^2 z^2}{\left((0.5-\Lambda y)^2 - \Lambda^2 z^2\right)^2} - \frac{(1-w\Lambda)(0.5-w+2z)t\Lambda z}{(0.5-\Lambda y)^2}. \tag{EC.46}$$

At the same time, when $y = 0$, we must have $z = 0$ and therefore $\frac{\partial S(y,z,w)}{\partial y}|_{y=0} = t - 2c > 0$ because Assumption 1 (ii), so we must have $y > 0$ in optimal solution, i.e., $2y = x_A^* + x_B^* > 0$, some customers are single-listed.

We first prove that when $x_R^* > 0$ and $\Lambda > 2$, we will have $x_A^* > x_B^*$. After variable substitution $x_A = y+z$, $x_B = y-z$ in (EC.44), the social optimal problem becomes

$$\begin{aligned}
\max_{y,z,x_R} R(y,z,x_R) = \; & (2y+x_R)v - 2t(y^2+z^2) - \tfrac{c(2y+x_R)}{1-\Lambda(2y+x_R)} - \tfrac{cy(1-2\Lambda y)+2c\Lambda z^2}{(0.5-\Lambda y)^2-\Lambda^2 z^2} + \tfrac{cy}{0.5-\Lambda y} - \tfrac{t}{2}x_R + \tfrac{2t\Lambda z x_R(0.5-2y-x_R+2z)}{1-2\Lambda y} \\
s.t. \quad & 0 \leq y \leq 0.5, \\
& \Lambda(y+z) < 0.5, \\
& 0 \leq z \leq y, \\
& 0 \leq x_R \leq 1-2y.
\end{aligned} \tag{EC.47}$$

From (EC.47), we have

$$\frac{\partial R(y,z,x_R)}{\partial z} = -4tz - \frac{2c(0.5-\Lambda y)\Lambda z}{\left((0.5-\Lambda y)^2-\Lambda^2 z^2\right)^2} + \frac{4t\Lambda z x_R}{0.5-\Lambda y} + \frac{2x_R(0.5-2y-x_R)\Lambda t}{1-2\Lambda y}.$$

When $\Lambda > 2$, we must have $2y + x_R < 0.5$, so and if $x_R^* > 0$ and suppose y^* is the optimal solution, we must have $\frac{\partial R(y^*,z,x_R^*)}{\partial z}|_{z=0} = \frac{2x_R^*(0.5-2y^*-x_R^*)\Lambda t}{1-\Lambda y^*} > 0$, therefore, we must have $z^* > 0$ in the optimal solution, that is $x_A^* > x_B^*$.

We then prove that when $x_A^* + x_B^* + x_R^* = 1$, we must have $x_A^* = x_B^*$. When $x_A^* + x_B^* + x_R^* = 1$, after substituting $x_A = y + z$ and $x_B = y - z$ and $x_R = 1 - 2y$ in (EC.47), we have

$$\begin{aligned}
\max_{y,z} S(y,z) = \; & -2t(y^2+z^2) - \tfrac{c(0.5-\Lambda y)}{\Lambda\left((0.5-\Lambda y)^2-\Lambda^2 z^2\right)} + \tfrac{cy}{0.5-\Lambda y} + ty + \tfrac{\Lambda t z(1-2y)(2z-0.5)}{0.5-\Lambda y} \\
s.t. \quad & 0 \le y \le 0.5, \\
& \Lambda(y+z) < 0.5, \\
& 0 \le z \le y.
\end{aligned}$$

So we have

$$\begin{aligned}
\frac{\partial S(y,z)}{\partial z} &= -4tz - \frac{2c(0.5-\Lambda y)\Lambda z}{\left((0.5-\Lambda y)^2-\Lambda^2 z^2\right)^2} + \frac{(1-2y)(8z-1)\Lambda t}{1-2\Lambda y} \\
&= \frac{t}{1-2\Lambda y}\left(\Lambda(1-2y)(8z-1) - 4z(1-2\Lambda y)\right) - \frac{2c(0.5-\Lambda y)\Lambda z}{\left((0.5-\Lambda y)^2-\Lambda^2 z^2\right)^2} \\
&= \frac{t}{1-2\Lambda y}\left(-4z(1-\Lambda) - (1-4z)\Lambda(1-2y)\right) - \frac{2c(0.5-\Lambda y)\Lambda z}{\left((0.5-\Lambda y)^2-\Lambda^2 z^2\right)^2} < 0,
\end{aligned}$$

which means optimal $z^* = 0$, i.e., optimal solution must have $x_A^* = x_B^*$.

We next prove that when $\Lambda \le 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$, the revenue optimum is full coverage without balking, while $\Lambda \ge 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$, the social optimum has balking. From problem (EC.45), we have

$$\frac{\partial S(y,z,w)}{\partial w} = v - \frac{t}{2} - \frac{c}{(1-\Lambda w)^2} + \frac{(1+4y+4z-4w)t\Lambda z}{1-2\Lambda y} \tag{EC.48}$$

It is a strictly decreasing function of w. If $w = 1$, we have $z = 0$, then $\frac{\partial S(y,z,w)}{\partial w} = v - \frac{t}{2} - \frac{c}{(1-\Lambda)^2}$, to have $w = 1$, we must have $\frac{\partial S(y,z,w)}{\partial w} = v - \frac{t}{2} - \frac{c}{(1-\Lambda)^2} \ge 0$, that is $\Lambda \le 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$.

So when $\Lambda \le 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$, the revenue optimum has no balking, when $\Lambda > 1 - \sqrt{\frac{c}{v-\frac{t}{2}}}$, the revenue optimum has balking.

Finally, we prove that when $t \le v$ and $\Lambda > 2$, we must have $x_R^* > 0$. From (EC.37), we have

$$\frac{\partial S(y,z,x_R)}{\partial x_R}|_{x_R=0} = v - \frac{t}{2} - \frac{c}{(1-2\Lambda y)^2} + \frac{(0.5-2y+2z)\Lambda t z}{1-2\Lambda y} > v - \frac{t}{2} - \frac{c}{(1-2\Lambda y)^2} > \frac{t}{2} - 2ty - \frac{c}{(1-2\Lambda y)^2} \ge 0,$$

when $\Lambda > 2$, we must have $2y < 0.5$, thus the first inequality holds, the last inequality will be proved in the following (EC.49), and $v - \frac{t}{2} > \frac{t}{2} - 2ty$ because $t \le v$ and $y > 0$.

(EC.46) is a decreasing function of z. Therefore, when $z = 0$, we must have its value non-negative, which means

$$t(1-4y) - \frac{2c}{(1-2\Lambda y)^2} \ge 0. \tag{EC.49}$$

■

EC.7. Proofs and More Results for the n-Server System in Section 8

EC.7.1. Proof of Proposition 10

Before presenting the proof, we first provide the expressions of $(x_m^n, \Lambda_{n_1}, \Lambda_{n_2}, \kappa_n, x_{R0}^n)$:

For even n:

$$x_m^n := \begin{cases} x_{AR}^{ne}, & \text{if } t \le \frac{4(n-1)}{n}v, \text{ or } \Lambda < \Lambda_{n_2} \text{ and } t > \frac{4(n-1)}{n}v, \\ x_s^n, & \text{if } \Lambda \ge \Lambda_{n_2} \text{ and } t > \frac{4(n-1)}{n}v. \end{cases}$$

For odd n:

$$x_m^n := \begin{cases} \frac{1}{2n}, & \text{if } t \le \frac{4n}{n+1}v \text{ and } \Lambda < 1 - \frac{c}{v - \frac{t}{4}(1+\frac{1}{n^2})}, \\ x_{AR}^{no}, & \text{if } t \le \frac{4n}{n+1}v \text{ and } \Lambda > 1 - \frac{c}{v - \frac{t}{4}(1+\frac{1}{n^2})}, \\ x_s^n, & \text{if } t > \frac{4n}{n+1}v. \end{cases}$$

$$\Lambda_{n_1} := 1 - \frac{c}{v - t/4}, \quad \Lambda_{n_2} := \frac{2t\left(1 - \frac{c}{v-t/4}\right)}{tn^2 - 4vn(n-1)}, \quad \kappa_n := \frac{1 - 2\Lambda x_m^n - \frac{c}{v-t/4}}{\Lambda(1 - 2x_m^n)},$$

$$x_{AR}^{ne} := \frac{2t + \Lambda tn - \sqrt{(2+\Lambda n)^2 t^2 - 8\Lambda nt(t - 4c(n-1))}}{8\Lambda nt},$$

$$x_{AR}^{no} := \frac{2t + \Lambda t + \Lambda tn - \sqrt{(2+\Lambda+\Lambda n)^2 t^2 - 8\Lambda t((n+1)t - 4cn^2)}}{8\Lambda nt},$$

$$x_s^n := \frac{t + 2\Lambda nv - \sqrt{(t + 2\Lambda nv)^2 - 8\Lambda nt(v - nc)}}{4\Lambda nt},$$

$$x_{R0}^n := \frac{2t + \Lambda t + 4\Lambda n^2(v - t/4) - \sqrt{(2t + \Lambda t + 4\Lambda n^2(v-t/4))^2 - 8\Lambda t(4n^2(v-c) - (n^2-1)t)}}{8\Lambda nt}.$$

Next, we proceed with the proof. When $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$, the expected utility of multi-listing is

$$u_R(x) = v - \frac{t}{n}\sum_{i=1}^{n} d_i(x) - \frac{c}{n\mu - n\lambda - \lambda_R}.$$

$$\sum_{i=1}^{n} d_i(x) = \begin{cases} \frac{n}{4}, & \text{if } n \text{ is even,} \\ \frac{n}{4} - \frac{1}{4n} + \min\{x - \frac{\lfloor nx \rfloor}{n}, \frac{\lfloor nx \rfloor}{n} + \frac{1}{n} - x\}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, for $x \in (0, 1/(2n))$,

$$u_R(x) = v - \frac{t}{n}\sum_{i=1}^{n} d_i(x) - \frac{c}{n\mu - n\lambda - \lambda_R} = \begin{cases} v - \frac{t}{4} - \frac{c}{n\mu - n\lambda - \lambda_R}, & \text{if n is even,} \\ v - \frac{t}{n}x - \frac{t}{4}(1 - \frac{1}{n^2}) - \frac{c}{n\mu - n\lambda - \lambda_R}, & \text{if n is odd.} \end{cases}$$

When $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$, $W_1 = W_2 = \dots = W_n$. Thus, if a customer chooses to single-list, it is optimal for her to single-list with the nearest server (which gives the highest expected utility). Hence, for $x \in (0, 1/(2n))$, the expected utility of single-listing (with the nearest server, server 1) is

$$u_S(x) = \max_{i \in \{1,\dots,n\}} u_i(x) = v - tx - \frac{c}{n\mu - n\lambda - \lambda_R} - \frac{c(n-1)}{n\mu - n\lambda}.$$

We next characterize customer strategy for $x \in (0, 1/(2n))$. The remaining x follows from symmetry.

If n is even, $u_S(x) = v - tx - \frac{c}{1 - n\lambda - \lambda_R} - \frac{(n-1)c}{1 - n\lambda}$ $u_R(x) = v - \frac{t}{4} - \frac{c}{1 - n\lambda - \lambda_R}$. Let x_{AR}^n denote the intersection point of utility curves $u_S(\cdot)$ and $u_R(\cdot)$, so that $u_S(x_{AR}^n) = u_R(x_{AR}^n)$. Let x_s^n denote the intersection point of curve $u_S(\cdot)$ and the horizontal line $y = 0$, so that $u_S(x_s^n) = 0$. It is clear that customers over $[0, x_m^n]$ will choose single-listing, where $x_m^n = \min\{x_{AR}^n, x_s^n\}$. Thus, $\lambda = 2\Lambda x_m^n$. Substituting the expressions into $u_S(x_{AR}^n) = u_R(x_{AR}^n)$ gives:

$$t\left(\frac{1}{4} - x_{AR}^n\right) - \frac{c(n-1)}{1 - 2n\Lambda x_{AR}^n} = 0 \tag{EC.50}$$

With conditions in Assumption 2 (ii) and (iii), (EC.50) has a unique solution

$$\frac{2t + \Lambda tn - \sqrt{(2+\Lambda n)^2 t^2 - 8\Lambda nt(t - 4c(n-1))}}{8\Lambda nt} \in [0, \frac{1}{2n}]. \tag{EC.51}$$

When $\Lambda \le 1 - \frac{c}{v - t/4}$, we have $u_R(x_{AR}^n) = v - \frac{t}{4} - \frac{c}{1-\Lambda} \ge 0$, for all $x \in [0, \frac{1}{2n}]$. That implies curve $u_S(\cdot)$ intersects $u_R(\cdot)$ first and thus $x_m^n = x_{AR}^n \le x_s^n$. And customers over $(x_m^n, \frac{1}{2n} - x_m^n)$ must all choose R.

When $\Lambda > 1 - \frac{c}{v - t/4}$, not all customers in $(x_m^n, \frac{1}{2n} - x_m^n)$ choose R. Then they will balk with a certain probability or all balk. In either case, we have $u_S(x_m^n) = 0$ and $u_R(x_m^n) \le 0$, which leads to the following conditions.

$$u_S(x_m^n) = v - tx_m^n - \frac{c}{1 - 2n\Lambda x_m^n - \lambda_R} - \frac{(n-1)c}{1 - 2\Lambda n x_m^n} = 0, \tag{EC.52}$$

$$u_R(x_m^n) = v - \frac{t}{4} - \frac{c}{1 - 2n\Lambda x_m^n - \lambda_R} \le 0. \tag{EC.53}$$

If all customers over $(x_m^n, \frac{1}{2n} - x_m^n)$ balk, then $\lambda_R = 0$. Then (EC.52) and (EC.53) lead to the conditions $\Lambda \ge \Lambda_{n_2} := \frac{2t(1 - c/(v - t/4))}{tn^2 - 4vn(n-1)}$ and $t > \frac{4(n-1)}{n} v$, in this case, $\lambda_R = 0$, $x_{AR}^n = x_s^n = \frac{t/2 + \Lambda nv - \sqrt{(t/2 + \Lambda nv)^2 - 2\Lambda nt(v - cn)}}{2\Lambda nt}$ is the unique solution in $(0, \frac{1}{2n})$ to $v - tx - \frac{nc}{1 - 2\Lambda nx} = 0$. When $t \le \frac{4(n-1)}{n} v$ or $t > \frac{4(n-1)}{n} v$ and $\Lambda \in (\Lambda_{n_1}, \Lambda_{n_2})$, (EC.53) holds with equality for some $\lambda_R > 0$, so we have customers over $(x_m^n, \frac{1}{2n} - x_m^n)$ choosing R with probability $\frac{\lambda_R}{\Lambda - 2n\Lambda x_m^n} = \frac{1 - 2n\Lambda x_m^n - c/(v - t/4)}{\Lambda(1 - 2nx_m^n)}$, where $x_m^n = x_{AR}^n := \frac{2t + \Lambda tn - \sqrt{(2+\Lambda n)^2 t^2 - 8\Lambda nt(t - 4c(n-1))}}{8\Lambda nt}$ is the unique solution in $(0, \frac{1}{2n})$ to (EC.50).

If n is odd, $u_S(x) = v - tx - \frac{c}{1 - 2\Lambda x_m^n - \lambda_R} - \frac{(n-1)c}{1 - 2n\Lambda x_m^n}$ $u_R(x) = v - \frac{tx}{n} - \frac{t}{4}(1 - \frac{1}{n^2}) - \frac{c}{1 - 2\Lambda n x_m^n - \lambda_R}$. Customers with $x \in [0, x_m^n]$ choose single-listing. For customers over interval $(x_m^n, \frac{1}{2n} - x_m^n)$, there exists x_R such that customers choose option R if and only if $x \in [x_m^n, x_R]$. Let x_{AR}^n denote the intersection point of utility curves $u_S(\cdot)$ and $u_R(\cdot)$, so that $u_S(x_{AR}^n) = u_R(x_{AR}^n)$. Let x_{R0}^n denote the intersection point of curve $u_R(\cdot)$ and the horizontal line $y = 0$, so that $u_R(x_{R0}^n) = 0$. Let x_s^n denote the intersection point of curve $u_S(\cdot)$ and the horizontal line $y = 0$, so that $u_S(x_s^n) = 0$. By plugging in the expression, we get the following equation,

$$\frac{t}{n}\left(\frac{n+1}{4n} - x_{AR}^n\right) - \frac{c}{1 - 2n\Lambda x_{AR}^n} = 0 \tag{EC.54}$$

$$v - \frac{t}{n} x_{R0}^n - \frac{t}{4}(1 - \frac{1}{n^2}) - \frac{c}{1 - 2n\Lambda x_{R0}^n} = 0 \tag{EC.55}$$

$$v - tx_s^n - \frac{cn}{1 - 2n\Lambda x_s^n} = 0 \tag{EC.56}$$

By comparing (EC.55) and (EC.56), we deduce that if $t \le \frac{4n}{n+1} v$, $x_{R0}^n \ge x_s^n$, $x_{R0}^n \ge x_{AR}^n$; while if $t > \frac{4n}{n+1} v$, $x_{R0}^n < x_s^n$, $x_{R0}^n < x_{AR}^n$. Since $\frac{4cn^2}{n+1} < t < \frac{4cn^2}{n-1}$, (EC.54) has a unique solution

$$x_{AR}^n = \frac{2t + \Lambda t + \Lambda tn - \sqrt{(2 + \Lambda + \Lambda n)^2 t^2 - 8\Lambda t((n+1)t - 4cn^2)}}{8\Lambda nt} \in [0, \frac{1}{2n}]. \tag{EC.57}$$

When $\Lambda > 1 - \frac{c}{v - \frac{t}{4}(1 + \frac{1}{n^2})}$, (EC.55) has a unique solution

$$x_{R0}^n = \frac{2t + \Lambda t + 4\Lambda n^2(v - t/4) - \sqrt{(2t + \Lambda t + 4\Lambda n^2(v - t/4))^2 - 8\Lambda t(4n^2(v - c) - (n^2 - 1)t)}}{8\Lambda nt} \in [0, \frac{1}{2n}]. \tag{EC.58}$$

When $\Lambda > 1 - \frac{cn}{v-\frac{t}{2n}}$, (EC.56) has a unique solution

$$x_s^n = \frac{t + 2\Lambda nv - \sqrt{(t+2\Lambda nv)^2 - 8\Lambda nt(v-nc)}}{4\Lambda nt} \in [0, \frac{1}{2n}]. \tag{EC.59}$$

So if $t \le \frac{4n}{n+1}v$, when $\Lambda \le 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}$, we have $u_R(\frac{1}{2n}) = v - \frac{t}{4}(1+\frac{1}{n^2}) - \frac{c}{1-\Lambda} \ge 0$. That implies full coverage: customers over $[0, x_{AR}^n]$ choose single-listing and all customers over $(x_{AR}^n, \frac{1}{2n} - x_{AR}^n)$ must all choose R. When $\Lambda > 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}$, customers over $[0, x_{AR}^n]$ choose single-listing, and customers choose option R if and only if $x \in [x_{AR}^n, x_{R0}^n]$.

So if $t > \frac{4n}{n+1}v$, $x_{R0}^n < x_s^n$, $x_{R0}^n < x_{AR}^n$, combined with condition $t < \frac{4cn^2}{n-1}$, we can have $cn > \frac{(n-1)v}{n+1}$, thus $t > \frac{4n}{n+1}v > 2n(v-nc)$, which lead to $1 - \frac{cn}{v-\frac{t}{2n}} < 0$, so we must have $u_S(\frac{1}{2n}) = v - \frac{t}{2n} - \frac{cn}{1-\Lambda} < 0$. That implies customers over $[0, x_s^n]$ choose single-listing. Since $x_{R0}^n < x_s^n$ when $t > \frac{4n}{n+1}v$, customers between $(x_s^n, 1/(2n)]$ have $u_R(x) < 0$, thus all balk. ■

EC.7.2. Additional Analytical Results

COROLLARY EC.4 (**Throughput and Social Welfare Under Multi-listing**). *In the CUS setting, the throughput of a multi-listing system with $n > 2$ servers at the symmetric equilibrium is given as follows.*

- ***Even** n:*

$$Th_m^n = \begin{cases} \Lambda, & \text{if } \Lambda < \Lambda_{n_1}, \\ 1 - \frac{c}{v-t/4}, & \text{if } \Lambda \ge \Lambda_{n_1} \text{ and } t \le \frac{4(n-1)}{n}v, \text{ or } \Lambda_{n_1}^c \le \Lambda < \Lambda_{n_2} \text{ and } t > \frac{4(n-1)}{n}v, \\ 2n\Lambda x_s^n, & \text{if } \Lambda \ge \Lambda_{n_2} \text{ and } t > \frac{4(n-1)}{n}v. \end{cases}$$

- ***Odd** n:*

$$Th_m^n = \begin{cases} \Lambda, & \text{if } t \le \frac{4n}{n+1}v, \text{ and } \Lambda < 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, \\ 2n\Lambda x_{R0}^n, & \text{if } t \le \frac{4n}{n+1}v \text{ and } \Lambda > 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, \\ 2n\Lambda x_s^n, & \text{if } t > \frac{4n}{n+1}v. \end{cases}$$

Social welfare at the symmetric equilibrium is given as follows.

- ***Even** n:*

$$SW_m^n = \begin{cases} \Lambda(v - \frac{t}{4} - \frac{c}{1-\Lambda} + nt(x_{AR}^{ne})^2), & \text{if } \Lambda < \Lambda_{n_1}, \\ n\Lambda t(x_{AR}^{ne})^2, & \text{if } \Lambda \ge \Lambda_{n_1} \text{ if } t \le \frac{4(n-1)}{n}v, \text{ or } \Lambda_{n_1} \le \Lambda < \Lambda_{n_2} \text{ if } t > \frac{4(n-1)}{n}v, \\ n\Lambda t(x_s^n)^2, & \text{if } \Lambda \ge \Lambda_{n_2} \text{ and } t > \frac{4(n-1)}{n}v. \end{cases}$$

- ***Odd** n:*

$$SW_m^n = \begin{cases} \Lambda\left(v - \frac{t}{4} - \frac{c}{1-\Lambda} + (n-1)t(x_{AR}^{no})^2\right), & \text{if } t \le \frac{4n}{n+1}v, \text{ and } \Lambda < 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, \\ \Lambda t(x_{R0}^n)^2 + (n-1)\Lambda t(x_{AR}^{no})^2, & \text{if } t \le \frac{4n}{n+1}v, \text{ and } \Lambda \ge 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, \\ n\Lambda t(x_s^n)^2. & \text{if } t > \frac{4n}{n+1}v. \end{cases}$$

PROPOSITION EC.1 (**Single-Listing Equilibrium**). *The single-listing system has a unique equilibrium at which customers with $x \in [0, x_s^n]$ choose queue A; those with $x \in [1/n - x_s^n, 1/n]$ choose queue B; and those with $x \in (x_s, 1/n - x_s)$ balk. In particular, $x_s = \frac{1}{2n}$ (that is, no customers balk) if $\Lambda \le \Lambda_s^n := 1 - \frac{cn}{v-\frac{t}{2n}}$.*

COROLLARY EC.5 **(Throughput and Social Welfare Under Single-listing).** *The equilibrium throughput of a n-server single-listing system is given by*

$$Th_s^n = \begin{cases} \Lambda, & \text{if } \Lambda \le \Lambda_s^n, \\ 2n\Lambda x_s^n, & \text{if } \Lambda > \Lambda_s^n. \end{cases}$$

Social welfare at the equilibrium is given by

$$SW_s^n = \begin{cases} \Lambda(v - \frac{t}{4n} - \frac{nc}{1-\Lambda}), & \text{if } \Lambda \le \Lambda_s^n, \\ n\Lambda t(x_s^n)^2, & \text{if } \Lambda > \Lambda_s^n. \end{cases}$$

PROPOSITION EC.2 **(Pooling Equilibrium).** *The n-server pooling system has a unique equilibrium:*

- ***when** n **is even:***
  (i) *all customers join the queue if* $\Lambda \le 1 - \frac{c}{v-t/4}$*;*
  (ii) *customers join the queue with probability* $\left(1 - \frac{c}{v-t/4}\right)/\Lambda$ *and balk with probability* $1 - \left(1 - \frac{c}{v-t/4}\right)/\Lambda$ *if* $\Lambda > 1 - \frac{c}{v-t/4}$*;*
- ***when** n **is odd:***
  (i) *all customers join the queue if* $\Lambda \le 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}$*;*
  (ii) *customers with* $x \in [0, x_{R0}^n] \cup [\frac{1}{n} - x_{R0}^n]$ *join the queue, while others balk.*

COROLLARY EC.6 **(Throughput and Social Welfare Under Pooling).**

(i) When n is even, the throughput of a n-server pooling system at the symmetric equilibrium is given by $Th_p^n = \min\{\Lambda, 1 - \frac{c}{v-t/4}\}$*. Social welfare in equilibrium is given by*

$$SW_p^n = \begin{cases} \Lambda\left(v - \frac{t}{4} - \frac{c}{1-\Lambda}\right), & \text{if } \Lambda \le 1 - \frac{c}{v-t/4}, \\ 0, & \text{if } \Lambda > 1 - \frac{c}{v-t/4}. \end{cases}$$

(ii) When n is odd

$$Th_p^n = \begin{cases} \Lambda, & \text{if } \Lambda \le 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, \\ 2\Lambda n x_{R0}^n, & \text{if } \Lambda > 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}. \end{cases}$$

Social welfare in equilibrium is given by

$$SW_p^n = \begin{cases} \Lambda\left(v - \frac{t}{4} - \frac{c}{1-\Lambda}\right), & \text{if } \Lambda \le 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, \\ t\Lambda(x_{R0}^n)^2, & \text{if } \Lambda > 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}. \end{cases}$$

The proofs of the results in this subsection are straightforward and thus omitted.

EC.7.3. Proofs of Theorems 6,7, and 8

Proof of Theorem 6 The result follows from the throughput expressions in Corollaries EC.4, EC.5, EC.6, and the observation that $2n\Lambda x_s^n > 1 - \frac{c}{v-t/4}$ if and only if $t > \frac{4(n-1)}{n}v$ and $\Lambda > \Lambda_{n_2}$ for even n and $x_s^n > x_{R0}^n$ if and only if $t > \frac{4n}{n+1}v$ for odd n. ■

Proof of Theorem 7 We prove this theorem by proving Theorem EC.1 for even n and Theorem EC.2 for odd n, stated as follows:

THEOREM EC.1 **(Social Welfare Comparison for Even** n**).** *In the CUS mode, for even n, multi-listing achieves higher social welfare than pooling* ($SW_m^n > SW_p^n$) *but achieves weakly lower social welfare than single-listing* ($SW_m^n \le SW_s^n$) *when* Λ *is high. See Table EC.5 for more specific social welfare comparison.*

Table EC.5 Social Welfare Comparison for Even n Under CUS

Condition	Comparison
$\Lambda \leq \Lambda_s^n, v \leq (n+2)c$ or $\Lambda \leq \Lambda_s^n, t \leq 4nc < \frac{4n}{n+2}v$ or $1-4nc/t < \Lambda \leq \Lambda_s^n, 4nc < t \leq \frac{4n}{n+2}v$	$SW_m^n > SW_p^n > SW_s^n$
$\Lambda < 1-4nc/t, 4nc < t \leq \frac{4n}{n+2}v$ or $\Lambda \leq \Lambda_s, t > \frac{4n}{n+2}v > 4nc$	$SW_m^n > SW_s^n > SW_p^n$
$\Lambda > \Lambda_1^c, t \leq \frac{4(n-1)}{n}v$ or $\Lambda_1^c < \Lambda < \Lambda_2^c, t > \frac{4(n-1)}{n}v$	$SW_s^n > SW_m^n > SW_p^n$
$\Lambda \geq \Lambda_2^c, t > \frac{4(n-1)}{n}v$	$SW_s^n = SW_m^n > SW_p^n$

THEOREM EC.2 **(Social Welfare Comparison for Odd** n**).** *In the CUS mode, for odd n, multi-listing always achieves higher social welfare than pooling ($SW_m^n > SW_p^n$) and higher social welfare than single-listing when $\Lambda \leq \Lambda_s^n$ ($SW_m^n > SW_s^n$). More specific social welfare comparison is summarized in Table EC.6.*

Table EC.6 Social Welfare Comparison for odd n

Condition	Comparison
$\Lambda \leq \Lambda_s^n, v \leq (n+2)c$ or $\Lambda \leq \Lambda_s^n, t \leq 4nc < \frac{4n}{n+2}v$ or $1-4nc/t < \Lambda \leq \Lambda_s^n, 4nc < t \leq \frac{4n}{n+2}v$	$SW_m^n > SW_p^n > SW_s^n$
$\Lambda < 1-4nc/t, 4nc < t \leq \frac{4n}{n+2}v$ or $\Lambda \leq \Lambda_s, \frac{4n}{n+1}v > t > \frac{4n}{n+2}v > 4nc$ or $\Lambda < 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}, t > \frac{4n}{n+1}v$	$SW_m^n > SW_s^n > SW_p^n$

Table EC.7 Social Welfare for Even n

<table>
<tr><td>Λ</td><td>$[0, \Lambda_s^n]$</td><td>$[\Lambda_s^n, \Lambda_{n_1}]$</td><td>$[\Lambda_{n_1}, \Lambda_{n_2}]$</td><td>$[\Lambda_{n_2}, \infty]$</td></tr>
<tr><td>Pooling</td><td colspan="2">$\Lambda(v - \frac{t}{4} - \frac{c}{1-\Lambda})$</td><td colspan="2">0</td></tr>
<tr><td>Single</td><td>$\Lambda(v - \frac{t}{4n} - \frac{nc}{1-\Lambda})$</td><td colspan="3">$n\Lambda t(x_s^n)^2$</td></tr>
<tr><td>Multi</td><td colspan="2">$\Lambda(v - \frac{t}{4} - \frac{c}{1-\Lambda} + nt(x_{AR}^{ne})^2)$</td><td>$\Lambda t(x_{AR}^{ne})^2$</td><td>$\Lambda t(x_{AR}^{ne})^2$if $t \leq \frac{4(n-1)}{n}v$; $\Lambda t(x_s^n)^2$ if $t > \frac{4(n-1)}{n}v$</td></tr>
</table>

Proof of Theorem EC.1 When $\Lambda < \Lambda_s^n$, it is straightforward to show that $SW_m^n(\Lambda)$ dominates both $SW_s^n(\Lambda)$ and $SW_p^n(\Lambda)$ by comparing those expressions in Corollary EC.4, Corollary EC.5 and Corollary EC.6, and noting

$$SW_m^n(\Lambda) - SW_s^n(\Lambda) = \Lambda\left(\frac{(n-1)c}{1-\Lambda} - t\left(\frac{n-1}{4n} - n(x_{AR}^{ne})^2\right)\right) \geq \Lambda\left(\frac{(n-1)c}{1-\Lambda} - t(\frac{1}{4} - x_{AR}^{ne})\right) = \Lambda(\frac{(n-1)c}{1-\Lambda} - \frac{(n-1)c}{1-2n\Lambda x_{AR}^{ne}}) > 0.$$

We next compare $SW_s^n(\Lambda)$ and $SW_p^n(\Lambda)$ when $\Lambda < \Lambda_s^n$.

Solving $SW_s^n(\Lambda) < SW_p^n(\Lambda)$ shows that it is equivalent to $\Lambda > 1 - 4nc/t$. (i) When $t < 4nc$, we have $1 - 4nc/t < 0$, so $SW_s^n < SW_p^n$. (ii) When $4nc < t < \frac{4n}{n+2}v$, $0 < 1 - 4nc/t < \Lambda_s^n$, so $SW_s^n > SW_p$ when $\Lambda < 1 - 4nc/t$, $SW_s^n < SW_p$ when $1 - 4nc/t < \Lambda < \Lambda_s^n$. (iii) When $t > \frac{4n}{n+2}v > 4nc$, $1 - 4nc/t > \Lambda_s^n$, so $SW_s^n > SW_p$ when $\Lambda < \Lambda_s^n$. (iiii) If $v \leq (n+2)c$, we must have $t < 4nc$, thus $1 - 4nc/t < 0$ and $SW_s^n < SW_p^n$.

When $\Lambda > \Lambda_{n_2}$ and $t > \frac{4(n-1)}{n}v$, we have $SW_m^n(\Lambda) = \Lambda t(x_s^n)^2$, in which case $SW_m^n(\Lambda) = SW_s^n(\Lambda)$.

When $\Lambda > \Lambda_1^c$ and $t \leq \frac{4(n-1)}{n}v$, or when $\Lambda_{n_1} < \Lambda \leq \Lambda_{n2}$ and $t > \frac{4(n-1)}{n}v$, $SW_m(\Lambda) = \Lambda t(x_{AR}^{ne})^2$. We next prove $x_{AR}^{ne} < x_s^n$. We can have

$$v - tx_{AR}^{ne} - \frac{nc}{1-2n\Lambda x_{AR}^{ne}} > v - tx_{AR}^{ne} - \frac{c}{1-2n\Lambda x_{AR}^{ne} - \lambda_R} - \frac{c(n-1)}{1-2\Lambda x_{AR}^{ne}} \geq 0.$$

In the same time, we have $v - tx_s^n - \frac{nc}{1-2n\Lambda x_s^n} = 0$. We then deduce that $x_{AR}^{ne} < x_s^n$ as $v - tx - \frac{nc}{1-2n\Lambda x}$ is strictly decreasing in x.As a result, $SW_m^n(\Lambda) = \Lambda t(x_{AR}^{ne})^2 < \Lambda t(x_s^n)^2 = SW_s^n(\Lambda)$. ■

Proof of Theorem EC.2 $SW_m^n(\Lambda) > SW_p^n(\Lambda)$ follows from the expressions in Corollaries EC.4 and EC.6; in particular, when $t > \frac{4n}{n+1}v$, we have shown in the proof of Proposition 10 that $x_{R0}^n < x_s^n$, which implies $SW_m^n(\Lambda) > SW_p^n(\Lambda)$. Next, we compare $SW_m^n(\Lambda)$ and $SW_s^n(\Lambda)$. When $\Lambda < \Lambda_s^n$, $SW_m^n(\Lambda) > SW_s^n(\Lambda)$ because

$$\begin{aligned} SW_m^n(\Lambda) - SW_s^n(\Lambda) &= (n-1)\Lambda\left(\frac{c}{1-\Lambda} - \frac{t}{n}(\frac{1}{4} - n(x_{AR}^{no})^2)\right) \\ &\geq (n-1)\Lambda\left(\frac{c}{1-\Lambda} - \frac{t}{n}(\frac{n+1}{4n} - x_{AR}^{no})\right) \\ &= (n-1)\Lambda(\frac{c}{1-\Lambda} - \frac{c}{1-2n\Lambda x_{AR}^{no}}) > 0. \end{aligned}$$

We next compare $SW_s^n(\Lambda)$ and $SW_p^n(\Lambda)$ when $\Lambda < \min\{1 - \frac{c}{v - \frac{t}{4}(1+\frac{1}{n^2})}, \Lambda_s^n\}$.

When $t < \frac{4n}{n+1}v$, $1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})} > \Lambda_s^n$. Solving $SW_s^n(\Lambda) < SW_p^n(\Lambda)$ shows that it is equivalent to $\Lambda > 1 - 4nc/t$. (i) When $t < 4nc$, we have $1 - 4nc/t < 0$, so $SW_s^n < SW_p^n$. (ii) When $4nc < t < \frac{4n}{n+2}v$, $0 < 1 - 4nc/t < \Lambda_s^n$, so $SW_s^n > SW_p$ when $\Lambda < 1 - 4nc/t$, $SW_s^n < SW_p$ when $1 - 4nc/t < \Lambda < \Lambda_s^n$. (iii) When $t > \frac{4n}{n+2}v > 4nc$, $1 - 4nc/t > \Lambda_s^n$, so $SW_s^n > SW_p$ when $\Lambda < \Lambda_s^n$. (iiii) If $v \leq (n+2)c$, we must have $t < 4nc$, thus $1 - 4nc/t < 0$ and $SW_s^n < SW_p^n$.

When $t > \frac{4n}{n+1}v$, $1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})} < \Lambda_s^n$, and $1 - \frac{4n}{t} > 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})}$, so we must have $SW_s^n > SW_p^n$ when $\Lambda < 1 - \frac{c}{v-\frac{t}{4}(1+\frac{1}{n^2})} < \Lambda_s^n$. ■

Proof of Theorem 8 The matching probability with server i is

$$p_i = \frac{\mu_i - \lambda_i}{\sum_{j=1}^n \mu_j - \sum_{j=1}^n \lambda_j} = \begin{cases} p_A = \frac{\frac{1}{n} - 2\Lambda x_A}{1 - n\Lambda(x_A + x_B)}, & \text{if } i \text{ is odd,} \\ p_B = \frac{\frac{1}{n} - 2\Lambda x_B}{1 - n\Lambda(x_A + x_B)}, & \text{if } i \text{ is even.} \end{cases}$$

If a customer x chooses multi-listing, her expected utility is

$$U_R(x) = \sum_{i=1}^n p_i[v - td_i(x)] - \frac{c}{1 - n\Lambda(x_A + x_B) - \lambda_R}.$$

Let $\tilde{x}$ be the arc distance between customer x and her closest odd-indexed server. If n is a multiple of 4, then $\sum_{i=1}^n p_i d_i(x)$ will be independent of x because:

$$\begin{aligned} \sum_{i=1}^n p_i d_i(x) = p_A\left(\tilde{x} + (\tilde{x} + \frac{2}{n}) + \cdots + (\tilde{x} + \frac{n/2-2}{n}) + (\frac{n/2}{n} - \tilde{x}) + (\frac{n/2-2}{n} - \tilde{x}) + \cdots + (\frac{2}{n} - \tilde{x})\right) \\ + p_B\left((\frac{1}{n} + \tilde{x}) + (\frac{3}{n} + \tilde{x}) + \cdots + (\frac{n/2-1}{n} + \tilde{x}) + (\frac{n/2-1}{n} - \tilde{x}) + (\frac{n/2-3}{n} - \tilde{x}) + \cdots + (\frac{1}{n} - \tilde{x})\right) = \frac{1}{4}. \end{aligned}$$

If n is not a multiple of 4 (but still an even number),

$$\begin{aligned} \sum_{i=1}^n p_i d_i(x) &= p_A\left(\tilde{x} + (\tilde{x} + \frac{2}{n}) + \cdots + (\tilde{x} + \frac{n/2-1}{n}) + (\frac{n/2-1}{n} - \tilde{x}) + (\frac{n/2-3}{n} - \tilde{x}) + \cdots + (\frac{2}{n} - \tilde{x})\right) \\ &\quad + p_B\left((\frac{1}{n} + \tilde{x}) + (\frac{3}{n} + \tilde{x}) + \cdots + (\frac{n/2-2}{n} + \tilde{x}) + (\frac{n/2}{n} - \tilde{x}) + (\frac{n/2-2}{n} - \tilde{x}) + \cdots + (\frac{1}{n} - \tilde{x})\right) \\ &= p_A\tilde{x} + p_B(\frac{1}{n} - \tilde{x}) + \frac{n^2-4}{4n^2} = \frac{1}{4} + \frac{\Lambda(x_A - x_B)(\frac{1}{n} - 2\tilde{x})}{1 - n\Lambda(x_A + x_B)}. \end{aligned}$$

The social welfare maximization problem is

$$\begin{aligned} \max_{x_A, x_B, x_R} \; & n\Lambda x_A\left[v - \frac{tx_A}{2} - cW_A(x_A, x_B, x_R)\right] + n\Lambda x_B\left[v - \frac{tx_B}{2} - cW_B(x_A, x_B, x_R)\right] \\ & + n\Lambda x_R\left[v - \left(\sum_{i=1}^n p_i d_i(\frac{1}{n} - x_B - \frac{x_R}{2})\right)t - cW_R(x_A, x_B, x_R)\right], \\ \text{s.t.} \quad & x_A + x_B + x_R \leq \frac{1}{n}, \; x_A \geq x_B \geq 0, x_R \geq 0. \end{aligned}$$

If n is a multiple of 4, the social welfare maximization problem is

$$\begin{aligned}\max_{x_A,x_B,x_R}\ & n\Lambda x_A\left[v-\tfrac{tx_A}{2}-\tfrac{cn}{1-2n\Lambda x_A}-\tfrac{c}{1-n\Lambda(x_A+x_B+x_R)}+\tfrac{c}{1-n\Lambda(x_A+x_B)}\right]+\\ & n\Lambda x_B\left[v-\tfrac{tx_B}{2}-\tfrac{cn}{1-2n\Lambda x_B}-\tfrac{c}{1-n\Lambda(x_A+x_B+x_R)}+\tfrac{c}{1-n\Lambda(x_A+x_B)}\right]+\\ & n\Lambda x_R\left[v-\tfrac{t}{4}-\tfrac{c}{1-n\Lambda(x_A+x_B+x_R)}\right]\\ \text{s.t.}\ & x_A+x_B+x_R\le\tfrac{1}{n},\ x_A\ge x_B\ge 0, x_R\ge 0.\end{aligned}$$

It is equivalent to

$$\begin{aligned}\max_{x_A,x_B,x_R}\ & x_A\left(v-\tfrac{tx_A}{2}-\tfrac{cn}{1-2n\Lambda x_A}\right)+x_B\left(v-\tfrac{tx_B}{2}-\tfrac{cn}{1-2n\Lambda x_B}\right)+x_R\left(v-\tfrac{t}{4}\right)\\ & -\tfrac{c(x_A+x_B+x_R)}{1-n\Lambda(x_A+x_B+x_R)}+\tfrac{c(x_A+x_B)}{1-n\Lambda(x_A+x_B)}\\ \text{s.t.}\ & x_A+x_B+x_R\le\tfrac{1}{n},\ x_A\ge x_B\ge 0, x_R\ge 0.\end{aligned}$$

The optimal solution must be symmetric with $x_A^*=x_B^*$.

If n is not a multiple of 4 but still an even number, the social welfare maximization problem is

$$\begin{aligned}\max_{x_A,x_B,x_R}\ & n\Lambda x_A\left[v-\tfrac{tx_A}{2}-\tfrac{cn}{1-2n\Lambda x_A}-\tfrac{c}{1-n\Lambda(x_A+x_B+x_R)}+\tfrac{c}{1-n\Lambda(x_A+x_B)}\right]+\\ & n\Lambda x_B\left[v-\tfrac{tx_B}{2}-\tfrac{cn}{1-2n\Lambda x_B}-\tfrac{c}{1-n\Lambda(x_A+x_B+x_R)}+\tfrac{c}{1-n\Lambda(x_A+x_B)}\right]+\\ & n\Lambda x_R\left[v-\tfrac{t}{4}+t\tfrac{\Lambda(x_A-x_B)(1/n-2x_B-x_R)}{1-n\Lambda(x_A+x_B)}-\tfrac{c}{1-n\Lambda(x_A+x_B+x_R)}\right]\\ \text{s.t.}\ & x_A+x_B+x_R\le\tfrac{1}{n},\ x_A\ge x_B\ge 0, x_R\ge 0.\end{aligned}$$

After variable substitution $x_A=y+z$, $x_B=y-z$, the problem becomes

$$\begin{aligned}\max_{y,z,x_R}\ S_n(y,z,x_R)=\ & (2y+x_R)v-t(y^2+z^2)-\tfrac{c(2y+x_R)}{1-n\Lambda(2y+x_R)}-\tfrac{2cny(1-2n\Lambda y)+4n^2c\Lambda z^2}{(1-2n\Lambda y)^2-4n^2\Lambda^2z^2}+\tfrac{2cy}{1-2n\Lambda y}-\tfrac{t}{4}x_R+\tfrac{2x_Rt\Lambda z(1/n-2y-x_R+2z)}{1-2n\Lambda y}\\ \text{s.t.}\ & 2y+x_R\le\tfrac{1}{n},\ y\ge z\ge 0, x_R\ge 0.\end{aligned}$$

So we have

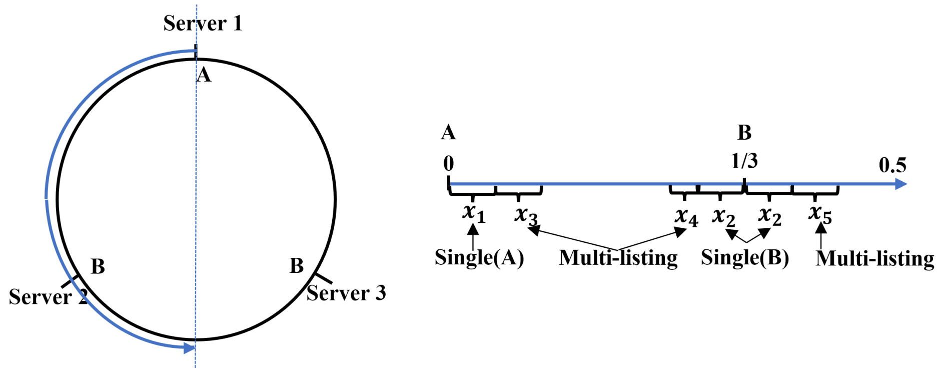
$$\frac{\partial S_n(y,z,x_R)}{\partial z}=-2tz-\frac{8n^2c(1-2n\Lambda y)\Lambda z}{\left((1-2n\Lambda y)^2-4n^2\Lambda^2z^2\right)^2}+\frac{8x_Rt\Lambda z}{1-2n\Lambda y}+\frac{2x_Rt\Lambda(1/n-2y-x_R)}{1-2n\Lambda y}.$$

So when $x_R^*>0$ and $x_A^*+x_B^*+x_R^*<\frac{1}{n}$, suppose y^* is the optimal solution, we must have $\frac{1}{n}-2y^*-x_R^*>0$, thus $\frac{\partial S_n(y^*,z,x_R^*)}{\partial z}|_{z=0}=\frac{2x_R^*t\Lambda(1/n-2y^*-x_R^*)}{1-2n\Lambda y^*}>0$, therefore, we must have $z^*>0$ in the optimal solution, that is $x_A^*>x_B^*$. ■

EC.7.4. Social Optimum for Odd n

Motivated by the equilibrium structure characterized in Proposition 10 for odd n, we propose the following routing policy. Starting from server 1, label servers alternately as A or B until server $(n+1)/2$ (i.e., server 1 as A, server 2 as B, server 3 as A, etc). Then label server $(n+3)/2$ with the same letter as server $(n+1)/2$, then label the rest of the servers alternately as A or B. For customers on each $A-B$ segment of the circle, let customers located within x_1 (x_2) units from server A (B) single-list with server A (B); assign customers between x_1 units and x_1+x_3 units away from server A and those between x_2 units and x_2+x_4 units away from server B to multi-listing and the rest of the customers

Figure EC.6 Proposed Routing Policy for 3 Servers



on the segment to balking. On the segment between servers $(n+1)/2$ and $(n+3)/2$, if it is a $A-A$ $(B-B)$ segment, then let customers within x_1 (x_2) units from either server single-list with that server; assign customers between x_1 (x_2) and x_1+x_6 (x_2+x_5) units away from either server to multi-listing and the rest to balking. See Figure EC.6 for an illustration of the proposed routing policy for $n=3$.

Next, we examine the case $n=3$ more closely. The matching probability with server i is

$$p_i = \frac{\mu_i - \lambda_i}{\sum_{j=1}^n \mu_j - \sum_{j=1}^n \lambda_j} = \begin{cases} p_A = \frac{\frac{1}{3}-2\Lambda x_A}{1-2\Lambda(x_A+2x_B)}, & \text{if } i \text{ is } 1, \\ p_B = \frac{\frac{1}{3}-2\Lambda x_B}{1-2\Lambda(x_A+2x_B)}, & \text{if } i \text{ is 2 or 3.} \end{cases}$$

If a customer $x \in [0,1)$ chooses multi-listing, her expected utility is

$$U_R(x) = v - t\sum_{i=1}^n p_i d_i(x) - \frac{c}{1-2\Lambda(x_A+2x_B)-\lambda_R},$$

where

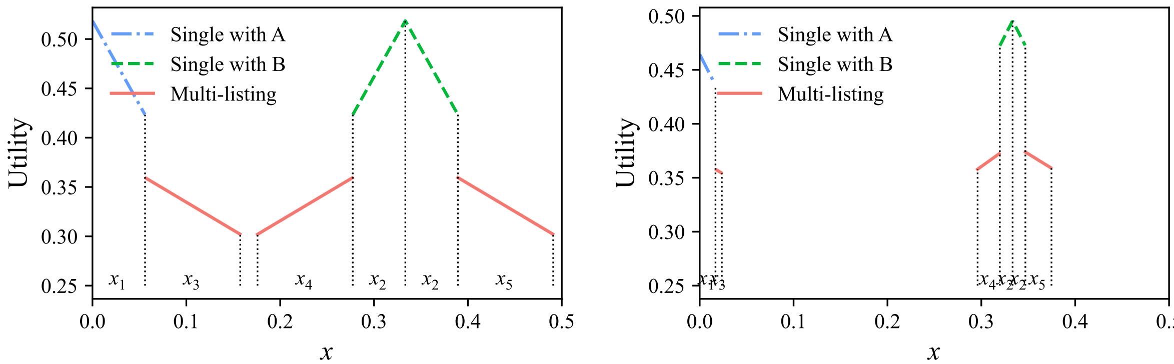
$$\sum_{i=1}^n p_i d_i(x) = \begin{cases} p_A x + \frac{1}{3}(1-p_A), & \text{if } 0 \le x \le \frac{1}{6}, \\ (2p_A-1)x + \frac{1}{2}(1-p_A), & \text{if } \frac{1}{6} \le x \le \frac{1}{3}, \\ p_A x + \frac{1}{6}(1-p_A), & \text{if } \frac{1}{3} \le x \le \frac{1}{2}, \\ -p_A x + \frac{1}{6}(1+5p_A), & \text{if } \frac{1}{2} \le x \le \frac{2}{3}, \\ (1-2p_A)x + \frac{1}{3}(3p_A-1), & \text{if } \frac{2}{3} \le x \le \frac{5}{6}, \\ -p_A x + \frac{1}{3}(1+2p_A), & \text{if } \frac{5}{6} \le x < 1, \end{cases}$$

Thus, the social-welfare maximization problem is

$$\begin{aligned} \max_{x_1,x_2,x_3,x_4,x_5} \quad & 2\Lambda x_1\left[v - \frac{tx_1}{2} - \frac{c}{1/3-2\Lambda x_1} - \frac{c}{1-2\Lambda(x_1+2x_2+x_3+x_4+x_5)} + \frac{c}{1-2\Lambda(x_1+2x_2)}\right] + \\ & 4\Lambda x_2\left[v - \frac{tx_2}{2} - \frac{c}{1/3-2\Lambda x_2} - \frac{c}{1-2\Lambda(x_1+2x_2+x_3+x_4+x_5)} + \frac{c}{1-2\Lambda(x_1+2x_2)}\right] + \\ & 2\Lambda x_3\left[v - t\left(\frac{1/3-2\Lambda x_1}{1-2\Lambda(x_1+2x_2)}(x_1+x_3/2) + \frac{1}{3}\left(1-\frac{1/3-2\Lambda x_1}{1-2\Lambda(x_1+2x_2)}\right)\right) - \frac{c}{1-2\Lambda(x_1+2x_2+x_3+x_4+x_5)}\right] + \\ & 2\Lambda x_4\left[v - t\left(\left(2\frac{1/3-2\Lambda x_1}{1-2\Lambda(x_1+2x_2)}-1\right)(1/3-x_2-x_4/2) + \frac{1}{2}\left(1-\frac{1/3-2\Lambda x_1}{1-2\Lambda(x_1+2x_2)}\right)\right) - \frac{c}{1-2\Lambda(x_1+2x_2+x_3+x_4+x_5)}\right] + \\ & 2\Lambda x_5\left[v - t\left(\frac{1/3-2\Lambda x_1}{1-2\Lambda(x_1+2x_2)}(1/3+x_2+x_5/2) + \frac{1}{6}\left(1-\frac{1/3-2\Lambda x_1}{1-2\Lambda(x_1+2x_2)}\right)\right) - \frac{c}{1-2\Lambda(x_1+2x_2+x_3+x_4+x_5)}\right] \\ \text{s.t.} \quad & x_1+x_3 \le \frac{1}{6},\ x_2+x_4 \le \frac{1}{6},\ x_2+x_5 \le \frac{1}{6},\ x_1,x_2,x_3,x_4,x_5 \ge 0. \end{aligned}$$

Our numerical study suggests that the optimal solution to the above problem can be symmetric (as illustrated in Figure EC.7-(a)) or asymmetric (as illustrated in Figure EC.7-(b)).

Figure EC.7 Social Optimum Within the Proposed Policy Class for $n = 3$



(a) Symmetric when $v = 1, c = 0.1, t = 1.7, \Lambda = 0.6$

(b) Asymmetric when $v = 1, c = 0.1, t = 1.7, \Lambda = 2.9$