

Online Appendix to Assortment Optimization Under History-Dependent Effects

EC.1. Meituan’s Grocery Flash Sale Channel

Figure EC.1 details Meituan’s flash sale channel. Figure EC.1a is the main page of the Meituan online store. The main page lists the entry of the grocery flash sale channel and highlights its remaining time in the current campaign. The flash sale channel runs each campaign on a daily basis. Hence, the remaining time in our example is no more than 24 hours. Figure EC.1b is the screenshot of the flash sale’s products. In each category, only a limited number of products are offered. Please note that the products are the same for all customers.

Meituan aggregates all orders during a flash sale campaign and forwards the order information to suppliers once the current campaign ends. Suppliers will then prepare the products and deliver them to the pick-up stations the next day. Based on our discussion with Meituan, they must determine the products for flash sales for a particular planning period ahead of time because Meituan needs lead time to negotiate contracts with suppliers. As shown in Figure EC.1c, the payment page details the delivery and pick-up information, such as the customer’s chosen pick-up station and the expected pick-up time. This time indicates when the product is expected to reach the pick-up station. If a customer opts for home delivery, the products will initially be sent to the pick-up station, after which the station’s staff will deliver them to the customer’s address. This case shows that inventory is not the key concern of the platform because all the products are delivered from suppliers to the pick-up station the next day.

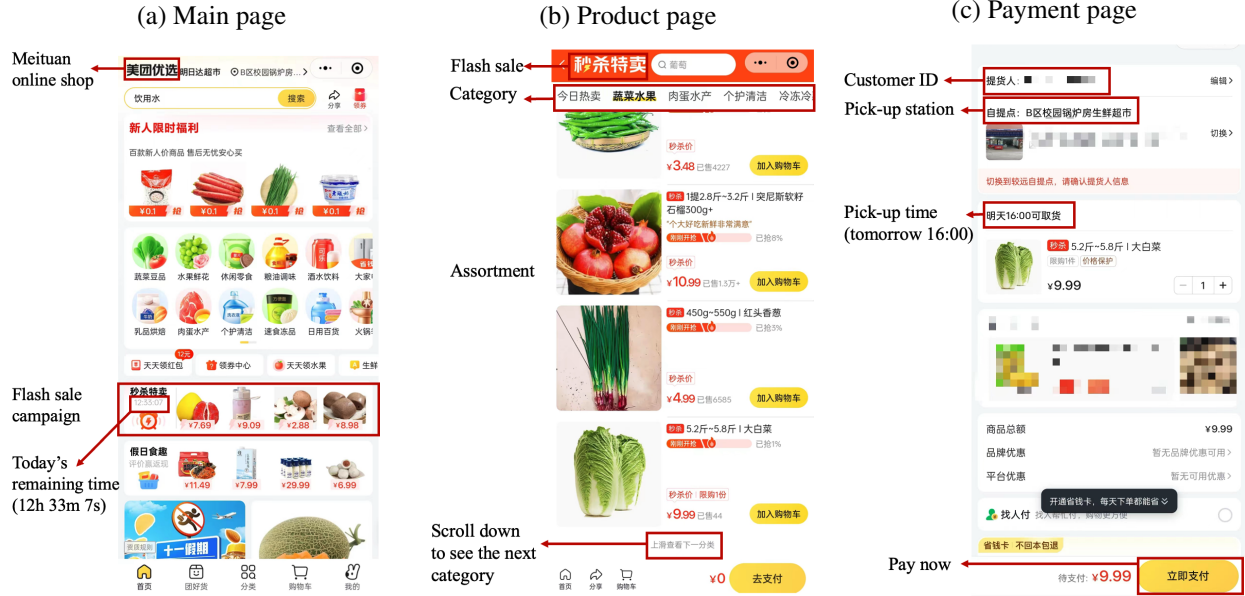
To the best of our knowledge, Meituan plans assortments for its flash sales channel independently. This makes our history-dependent assortment planning model well-suited to this relatively clean setting. If a platform also has a regular sales channel, the flash sale campaigns may include products that are simultaneously available outside of the promotion period, potentially influencing the demand of these regular channel products. This leaves an interesting problem of how to plan assortments for both channels under this additional intertemporal effect. In this setting, our model can handle assortment planning for a single channel by incorporating the terms of product availability in the other channel into the utility function.

EC.2. Proofs of Section 2

EC.2.1. Proof of Proposition 1

PROPOSITION 1. (HAP) *is NP-hard even when the planning horizon is two, the memory length is one, and history-dependent effects are negative.*

Figure EC.1 Screenshots of Meituan's grocery flash sale channel



Proof. To prove the NP-hardness of (HAP), we will reduce an NP-hard problem—the $3/4$ -Partition problem, to a special case of (HAP). The $3/4$ -Partition problem is defined as follows:

- **Input:** A set $\{c_1, c_2, \dots, c_N\}$ of non-negative integers.
- **Output:** True if and only if there exists a subset $S \subseteq [N]$ such that $\sum_{i \in S} c_i = (3/4) \sum_{i \in [N]} c_i$.

We will show the NP-hardness of the $3/4$ -Partition problem by showing that any instance of a *Partition* problem, a well-known NP-hard problem, can be reduced to an instance of the $3/4$ -Partition problem. The Partition problem is defined as follows:

- **Input:** A set $\{w_1, w_2, \dots, w_N\}$ of non-negative integers.
- **Output:** True if and only if there exists a subset $S \subseteq [N]$ such that $\sum_{i \in S} w_i = \sum_{i \in [N]/S} w_i$.

Notice that $\sum_{i \in S} w_i = \sum_{i \in [N]/S} w_i$ if and only if $\sum_{i \in S} w_i = (1/2) \sum_{i \in [N]} w_i$. Given an instance of the Partition problem, we solve it by solving a $3/4$ -Partition problem as follows. Define $c_i = w_i$ for $i \in [N]$ and $c_{N+1} = \sum_{i \in [N]} w_i$. Then, solve $3/4$ -Partition with input (c_1, \dots, c_{N+1}) . If the output is true and returns an index set S such that $\sum_{i \in S} c_i = \frac{3}{4} \sum_{i \in [N+1]} c_i$ then the output for Partition is true, that is, $\sum_{i \in S \setminus \{N+1\}} w_i = \sum_{i \in [N+1] \setminus S} w_i$ is a partition. The correctness follows from the fact that the set S contains $N + 1$. If the output is false, then the output for Partition is also false, since otherwise adding c_{N+1} into one of two sets yields a $3/4$ partition.

Next, we introduce a special case of (HAP). In such a special case, the planning horizon $T = 2$, the memory length $M = 1$, and all products have identical revenue, that is, $r_i = r > 0$ for $i \in [N]$. In

addition, we assume that the history-dependent effects are homogeneous, that is, for every $i \in [N]$, $\exp(\beta_i^1) = k$ for some $k \in (0, 1)$. Then, it turns out that **(HAP)** becomes

$$\max_{\mathbf{x}^1, \mathbf{x}^2 \in \{0,1\}^N} \frac{r}{2} \left\{ \frac{\sum_{i \in [N]} x_i^1 \exp(\beta_i^0)}{1 + \sum_{i \in [N]} x_i^1 \exp(\beta_i^0)} + \frac{\sum_{i \in [N]} x_i^2 \exp(\beta_i^0 + x_i^1 \beta_i^1)}{1 + \sum_{i \in [N]} x_i^2 \exp(\beta_i^0 + x_i^1 \beta_i^1)} \right\}. \quad (\text{SPECIAL})$$

We simplify **(SPECIAL)** by showing that the optimal assortment in the second period is $x_i^2 = 1$ for $i \in [N]$. Since all products have the same revenue, we can focus on maximizing the purchase probability. Given an assortment $\mathbf{x}^1 \in \{0, 1\}^N$, the purchase probability at the first period is a constant, and the purchase probability at the second period increases with $\sum_{i \in [N]} x_i^2 \exp(\beta_i^0 + x_i^1 \beta_i^1)$. Since $\exp(\beta_i^0 + x_i^1 \beta_i^1) > 0$ for each $i \in [N]$, then the optimal solution at the second period is $x_i^2 = 1$ for $i \in [N]$. Let $\nu_i = \exp(\beta_i^0)$ be the base attraction value and $V = \sum_{i \in [N]} \exp(\beta_i^0) = \sum_{i \in [N]} \nu_i$ denote the total base attraction value. Thus, solving **(SPECIAL)** equals to solving the following model:

$$\max_{\mathbf{x}^1 \in \{0,1\}^N} \frac{r}{2} \left\{ \frac{\sum_{i \in [N]} x_i^1 \nu_i}{1 + \sum_{i \in [N]} x_i^1 \nu_i} + \frac{V - \sum_{i \in [N]} x_i^1 \nu_i + k \sum_{i \in [N]} x_i^1 \nu_i}{1 + V - \sum_{i \in [N]} x_i^1 \nu_i + k \sum_{i \in [N]} x_i^1 \nu_i} \right\}. \quad (\text{EC.1})$$

Next, we show that the 3/4-Partition problem can be reduced to our special case **(SPECIAL)**. First, define $\nu_i = ac_i$ for $i \in [N]$, $k = 1/4$, and $C = \sum_{i \in [N]} c_i$, where

$$a = \frac{4(1 - \sqrt{1-k})}{C(3(1-k) + 3\sqrt{1-k} - 4)} = \frac{16 - 8\sqrt{3}}{C(6\sqrt{3} - 7)} > 0.$$

Hence, $V = \sum_{i \in [N]} \nu_i = aC$. Second, set the target average revenue as

$$\frac{r}{2} \left(\frac{3aC}{4 + 3aC} + \frac{7aC}{16 + 7aC} \right).$$

The Partition problem indeed has a solution if and only if there exists a $\mathbf{x}^1 \in \{0, 1\}^N$ such that

$$\begin{aligned} & \frac{r}{2} \left\{ \frac{\sum_{i \in [N]} x_i^1 \nu_i}{1 + \sum_{i \in [N]} x_i^1 \nu_i} + \frac{V - \sum_{i \in [N]} x_i^1 \nu_i + k \sum_{i \in [N]} x_i^1 \nu_i}{1 + V - \sum_{i \in [N]} x_i^1 \nu_i + k \sum_{i \in [N]} x_i^1 \nu_i} \right\} \\ &= \frac{r}{2} \left\{ \frac{a \sum_{i \in [N]} x_i^1 c_i}{1 + a \sum_{i \in [N]} x_i^1 c_i} + \frac{aC - (1-k)a \sum_{i \in [N]} x_i^1 c_i}{1 + aC - (1-k)a \sum_{i \in [N]} x_i^1 c_i} \right\} \\ &\leq \max_{y \in [0, C]} \frac{r}{2} \left\{ \frac{ay}{1 + ay} + \frac{aC - (1-k)ay}{1 + aC - (1-k)ay} \right\} = \frac{r}{2} \left(\frac{3aC}{4 + 3aC} + \frac{7aC}{16 + 7aC} \right). \end{aligned}$$

This holds because if we view the objective function in the last row as a one-dimensional function, denoted as

$$h(y) := \frac{r}{2} \left(\frac{ay}{1 + ay} + \frac{aC - (1-k)ay}{1 + aC - (1-k)ay} \right),$$

$h(y)$ is concave in y and achieves a unique maximum at $y = 3C/4$. \square

EC.3. Proofs in Section 3.1

EC.3.1. Proof of Proposition 2

To prove this proposition, we will invoke the following technical lemma.

LEMMA EC.1. *Let $f : \{0, 1\}^n \rightarrow \mathbb{R}$ and $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible affine transformation such that $L(\{0, 1\}^n) = \{0, 1\}^n$, and consider a composition $(f \circ L)(\mathbf{x}) := f(L(\mathbf{x}))$ for every $\mathbf{x} \in \{0, 1\}^n$. Then,*

$$\text{conc}(f)(\mathbf{x}) = \text{conc}(f \circ L)(L^{-1}(\mathbf{x})) \quad \text{for } \mathbf{x} \in [0, 1]^n.$$

Proof. Let $g(\mathbf{x}) := \text{conc}(f \circ L)(L^{-1}(\mathbf{x}))$ for $\mathbf{x} \in [0, 1]^n$. Clearly, $g(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \{0, 1\}^n$, and $g(\cdot)$ is a concave function. Thus, we show that $g(\mathbf{x}) \geq \text{conc}(f)(\mathbf{x})$ for every $\mathbf{x} \in [0, 1]^n$ since the concave envelope is the smallest concave function on $[0, 1]^n$ that overestimates $f(\cdot)$ on $\{0, 1\}^n$.

To show the opposite direction, we need the dual definition of the concave envelope (Rockafellar 1970), that is,

$$\text{conc}(f)(\mathbf{x}) = \max \left\{ \sum_{v \in V} f(v) \lambda_v \mid \mathbf{x} = \sum_{v \in V} v \lambda_v, \sum_{v \in V} \lambda_v = 1, \lambda_v \geq 0 \text{ for } v \in V \right\},$$

where $V := \{0, 1\}^n$. Now, consider a point $\bar{\mathbf{x}} \in [0, 1]^n$ and let $\bar{\mathbf{y}} = L^{-1}(\bar{\mathbf{x}})$. We will argue that $\text{conc}(f \circ L)(\bar{\mathbf{y}}) \leq \text{conc}(f)(\bar{\mathbf{x}})$. By the dual definition of concave envelope, there exists a convex multiplier $\{\bar{\lambda}_v\}_{v \in V}$ such that $\bar{\mathbf{y}} = \sum_{v \in V} v \bar{\lambda}_v$ and $\text{conc}(f \circ L)(\bar{\mathbf{y}}) = \sum_{v \in V} (f \circ L)(v) \bar{\lambda}_v$. It follows readily that $\bar{\mathbf{x}} = L(\bar{\mathbf{y}}) = \sum_{v \in V} L(v) \bar{\lambda}_v$, and thus

$$\text{conc}(f)(\bar{\mathbf{x}}) \geq \sum_{v \in V} f(L(v)) \cdot \bar{\lambda}_v = \sum_{v \in V} (f \circ L)(v) \cdot \bar{\lambda}_v = \text{conc}(f \circ L)(\bar{\mathbf{y}}),$$

where the first inequality holds due to the dual definition of $\text{conc}(f)(\cdot)$ and $\bar{\mathbf{x}}$ can be expressed as $\sum_{v \in V} L(v) \bar{\lambda}_v$. \square

PROPOSITION 2 *Constraint (6) is equivalent to the following system of linear inequalities.*

$$y_i^t \leq \alpha_i(\mathbf{h}_{i,0}^\sigma)(\gamma_i^t - \tilde{z}_{i\sigma(1)}^t) + \sum_{k \in [M]} \alpha_i(\mathbf{h}_{i,k}^\sigma)(\tilde{z}_{i\sigma(k)}^t - \tilde{z}_{i\sigma(k+1)}^t) \quad \text{for } \sigma \in \Omega$$

where $\tilde{z}_{i\sigma(M+1)}^t = 0$, $\tilde{z}_{i\sigma(k)}^t = z_{i\sigma(k)}^t$ if $\sigma(k) \notin I_i$, and $\tilde{z}_{i\sigma(k)}^t = \gamma_i^t - z_{i\sigma(k)}^t$ if $\sigma(k) \in I_i$.

Proof. We start with the satiation case. Since β_i are non-positive, by Corollary 3.14 in Tawarmalani et al. (2013), $\alpha_i(\cdot)$ is supermodular over $(x_i^{t-1}, \dots, x_i^{t-M})$. Then, it follows from Theorem 3.3 in Tawarmalani et al. (2013) that the concave envelope of $\alpha_i(\cdot)$ is given as:

$$\text{conc}(\alpha_i)(x_i^{t-1}, \dots, x_i^{t-M}) = \min_{\sigma \in \Omega} \left\{ \alpha_i(\mathbf{w}_0^\sigma)(1 - x_i^{t-\sigma(1)}) + \sum_{k \in [M]} \alpha_i(\mathbf{w}_k^\sigma)(x_i^{t-\sigma(k)} - x_i^{t-\sigma(k+1)}) \right\}, \quad (\text{EC.2})$$

where $\mathbf{w}_0^\sigma = \mathbf{0}$, $\mathbf{w}_k^\sigma = \mathbf{w}_{k-1}^\sigma + \mathbf{e}_{\sigma(k)}$ for $k \in [M]$, and $x_i^{t-\sigma(M+1)} = 0$. It follows readily that its perspective function can be represented as follows under the presence of constraint (4):

$$\text{pers}(\text{conc}(\alpha_i))(\gamma_i^t, \mathbf{z}_i^t) = \min_{\sigma \in \Omega} \left\{ \alpha_i(\mathbf{w}_0^\sigma)(\gamma_i^t - z_{i\sigma(1)}^t) + \sum_{k \in [M]} \alpha_i(\mathbf{w}_k^\sigma)(z_{i\sigma(k)}^t - z_{i\sigma(k+1)}^t) \right\},$$

where $z_{i\sigma(M+1)}^t = 0$

For the general case of mixed satiation and addiction effects, we transform the variable x_i^{t-m} to $1 - x_i^{t-m}$ if $\beta_i^m > 0$, and obtain a transformed attraction value function, defined as follows:

$$\begin{aligned} \bar{\alpha}_i(w^1, \dots, w^M) &:= \exp\left(\beta_i^0 + \sum_{k \in I_i} \beta_i^k(1 - w^k) + \sum_{k \in [M] \setminus I_i} \beta_i^k w^k\right) \\ &= \exp\left(\beta_i^0 + \sum_{k \in I_i} \beta_i^k + \sum_{k \in [M]} (-|\beta_i^k|)w^k\right). \end{aligned}$$

Now, the coefficients $-|\beta_i^k|$ are non-positive, and by Corollary 3.14 in [Tawarmalani et al. \(2013\)](#), the transformed function $\bar{\alpha}_i(\cdot)$ is supermodular. Thus, its concave envelope can be described using [\(EC.2\)](#).

Next, we utilize $\text{conc}(\bar{\alpha}_i)(\cdot)$ to characterize the concave envelope of the original attraction value function $\alpha_i(\cdot)$. Define $\tilde{x}_i^{t-m} = x_i^{t-m}$ if $m \in [M] \setminus I_i$, $\tilde{x}_i^{t-m} = 1 - x_i^{t-m}$ if $m \in I_i$, and $\tilde{x}_i^{t-\sigma(M+1)} = 0$. We then build the following connections:

$$\begin{aligned} \text{conc}(\alpha_i)(x_i^{t-1}, \dots, x_i^{t-M}) &= \text{conc}(\bar{\alpha}_i)(\tilde{x}_i^{t-1}, \dots, \tilde{x}_i^{t-M}) \\ &= \min_{\sigma \in \Omega} \left\{ \bar{\alpha}_i(\mathbf{w}_0^\sigma)(1 - \tilde{x}_i^{t-\sigma(1)}) + \sum_{k \in [M]} \bar{\alpha}_i(\mathbf{w}_k^\sigma)(\tilde{x}_i^{t-\sigma(k)} - \tilde{x}_i^{t-\sigma(k+1)}) \right\} \\ &= \min_{\sigma \in \Omega} \left\{ \alpha_i(\mathbf{h}_{i,0}^\sigma)(1 - \tilde{x}_i^{t-\sigma(1)}) + \sum_{k \in [M]} \alpha_i(\mathbf{h}_{i,k}^\sigma)(\tilde{x}_i^{t-\sigma(k)} - \tilde{x}_i^{t-\sigma(k+1)}) \right\} \end{aligned}$$

The first equality follows from Lemma [EC.1](#), the second equality holds by [\(EC.2\)](#), and the last equality holds since it follows from the definition of \mathbf{h} in [\(SWITCHNESTED\)](#) that $\bar{\alpha}_i(\mathbf{w}_k^\sigma) = \alpha_i(\mathbf{h}_{i,k}^\sigma)$.

Finally, we scale the concave envelope to obtain

$$\text{pers}(\text{conc}(\alpha_i))(\gamma_i^t, \mathbf{z}_i^t) = \min_{\sigma \in \Omega} \left\{ \alpha_i(\mathbf{h}_{i,0}^\sigma)(\gamma_i^t - \tilde{z}_{i\sigma(1)}^t) + \sum_{k \in [M]} \alpha_i(\mathbf{h}_{i,k}^\sigma)(\tilde{z}_{i\sigma(k)}^t - \tilde{z}_{i\sigma(k+1)}^t) \right\}.$$

This completes the proof. □

EC.3.2. Proof of Theorem 1

THEOREM 1 (**CONIC**) is a mixed-integer exponential cone formulation of problem (**HAP**).

Proof. By lifting, we equivalently decompose (**CHOICE**) into the choice constraints (2):

$$\gamma_i^t = \rho^t x_i^t \quad (2a)$$

$$z_{im}^t = \gamma_i^t x_i^{t-m} \quad \text{for } m \in [M] \quad (2b)$$

$$y_i^t = \text{pers}(\alpha_i)(\gamma_i^t, \mathbf{z}_i^t) \quad (2c).$$

Thus, proving the equivalence of (**CONIC**) and (**HAP**) suffices to show that constraints (3) (4) (5) and (7) construct an equivalent representation of the choice constraints (2). Clearly, since \mathbf{x} is binary, (3) (resp. (4)) models constraint (2a) (resp. (2b)). Next, we show constraints (5) and (7) provide an exact representation of (2c). By $\mathbf{x} \in \{0, 1\}^{N \times T}$ and constraints (3) and (4), constraint (5) equals to

$$y_i^t \geq \rho^t x_i^t \exp(\beta_i^0 + \sum_{m \in [M]} \beta_i^m x_i^{t-m}). \quad (\text{EC.3})$$

On the other hand, the constraint (7) is the concave envelope of $\alpha_i(\cdot)$ scaled by γ_i^t . That is, $y_i^t \leq \gamma_i^t \cdot \text{conc}(\alpha_i)(x_i^{t-1}, \dots, x_i^{t-M})$. In our case, the concave envelope α_i is the tightest concave extension α_i (see [Tawarmalani and Sahinidis 2002](#), Theorem 6). Thus, $\text{conc}(\alpha_i)(x_i^{t-1}, \dots, x_i^{t-M}) = \alpha_i(x_i^{t-1}, \dots, x_i^{t-M})$ for $x_i^{t-m} \in \{0, 1\}$ and $m \in [M]$. Therefore, constraint (7) equals to $y_i^t \leq \gamma_i^t \cdot \alpha_i(x_i^{t-1}, \dots, x_i^{t-M})$. Combined with constraint (3), constraint (7) equals to

$$y_i^t \leq \rho^t x_i^t \exp(\beta_i^0 + \sum_{m \in [M]} \beta_i^m x_i^{t-m}). \quad (\text{EC.4})$$

Constraints (EC.3) and (EC.4) suggest that constraints (5) and (7) are exact representation of (2c).

Recall that exponential cones are three-dimensional convex cones:

$$\mathcal{K}_{\text{exp}} = \{(x_1, x_2, x_3) \mid x_1 \geq x_2 \exp(x_3/x_2), x_2 > 0\} \cup \{(x_1, 0, x_3) \mid x_1 \geq 0, x_3 \leq 0\}.$$

By introducing new variable $w_i^t = \beta_i^0 \gamma_i^t + \sum_{m \in [M]} \beta_i^m z_{im}^t$, (5) can be represented by exponential cones:

$$(y_i^t, \gamma_i^t, w_i^t) \in \mathcal{K}_{\text{exp}}.$$

Therefore, (**CONIC**) is an equivalent mixed-integer exponential cone programming of (**HAP**).

EC.3.3. Proof of Theorem 2

THEOREM 2 *Assume the absence of cross-product and cross-period constraints. Then, the sequential-revenue-ordered policy solves (HAP) if the history-dependent effects are non-negative, that is, $\beta \geq 0$.*

Before proving the result, we present some preliminary results that will be used in our proof. Given a vector of attraction values $\nu \in \mathbb{R}_+^N$, consider a single-period assortment optimization problem under the MNL model given as follows:

$$\max_S \left\{ \text{MNL}(S, \nu) := \sum_{i \in S} r_i \nu_i / (1 + \sum_{i \in S} \nu_i) \mid S \subseteq [N] \right\}. \quad (\text{EC.5})$$

We assume that products are indexed such that $r_1 \geq \dots \geq r_N$. For $\kappa \in [N]$, we will use $[\kappa]$ to denote a revenue-ordered (RO) assortment. Talluri and Van Ryzin (2004) show the optimal revenue is achieved by a RO assortment $[\kappa]$ for some cut-off product $\kappa \in [N]$. Without loss of generality, we assume that there exists a unique optimal solution; otherwise, we will use the optimal RO assortment with the largest number of products.

LEMMA EC.2. *Let $\nu \in \mathbb{R}_+^N$ and assume that $r_1 \geq \dots \geq r_N$. Then, we obtain the following results:*

1. *The RO assortment $[\kappa]$ is optimal if and only if $r_\kappa \geq \text{MNL}([\kappa], \nu)$ and $r_{\kappa+1} < \text{MNL}([\kappa], \nu)$.*
2. *If the RO assortment $[\kappa]$ is optimal then, for every μ such that $\mu \geq \nu$, $\text{MNL}([\kappa], \mu) \geq \text{MNL}([\kappa], \nu)$.*

Proof of Lemma EC.2. Part 1 follows from Talluri and Van Ryzin (2004). To prove Part 2, we consider μ such that $\mu_j \geq \nu_j$ for $j \in [\kappa]$ but $\mu_i = \nu_i$ for all $i \notin [\kappa]$, and observe that

$$\text{MNL}([\kappa], \mu) - \text{MNL}([\kappa], \nu) = \begin{cases} \frac{\mu_j - \nu_j}{1 + \sum_{i \in [\kappa]} \mu_i} (r_j - \text{MNL}([\kappa], \nu)) & \text{if } j \in [\kappa] \\ 0 & \text{otherwise.} \end{cases}$$

Due to Part 1, we obtain that $r_j \geq \text{MNL}([\kappa], \nu)$ for $j \in [\kappa]$. This completes the proof. \square

The next result establishes that assortments generated by our policy are nested.

LEMMA EC.3. *Let $[\kappa^1], \dots, [\kappa^T]$ be a sequence of RO assortments generated by the sequential-revenue-ordered policy. Then, $\kappa^1 \geq \kappa^2 \geq \dots \geq \kappa^T$.*

Proof of Lemma EC.3. Let $\nu = (\nu^1, \dots, \nu^T)$ be corresponding attraction values, that is,

$$\nu_i^t := \alpha_i (\mathbb{1}(i \in [\kappa^{t-1}]), \dots, \mathbb{1}(i \in [\kappa^{t-M}])) \quad \text{for } t \in [T] \text{ and } i \in [N].$$

We prove that $N \geq \kappa^1 \geq \dots \geq \kappa^t \geq 1$ holds for every $t \in [T]$. Clearly, it holds for $\tau = 1$. Assume that it holds for some $1 \leq \tau < T$. Now, we observe that

$$r_{\kappa^{\tau+1}} < \text{MNL}([\kappa^\tau], \boldsymbol{\nu}^\tau) \leq \text{MNL}([\kappa^\tau], \boldsymbol{\nu}^{\tau+1}) \leq \text{MNL}([\kappa^{\tau+1}], \boldsymbol{\nu}^{\tau+1}),$$

where the first inequality follows from Part 1 of Lemma EC.2, the second inequality holds due to Part 2 of Lemma EC.2 since $\beta \geq 0$ and $\kappa^1 \geq \dots \geq \kappa^\tau$ imply that $\boldsymbol{\nu}^{\tau+1} \geq \boldsymbol{\nu}^\tau$, and the last inequality follows from the optimality of the RO assortment $[\kappa^{\tau+1}]$. Thus, $r_{\kappa^{\tau+1}} < \text{MNL}([\kappa^{\tau+1}], \boldsymbol{\nu}^{\tau+1})$. Therefore, by Part 1 of Lemma EC.2, we can conclude that $\kappa^{\tau+1} \leq \kappa^\tau$. \square

Now, we are ready to prove Theorem 2.

Proof. Let Z_* and Z_{RO} be the revenue generated from solving (HAP) and from using the sequential revenue-ordered policy. Clearly, our policy generates a feasible plan and thus $Z_{\text{RO}} \leq Z_*$. To show the reverse, we consider the revenue generated from solving the following optimization problem,

$$Z_{\text{U}} := \frac{1}{T} \sum_{t \in [T]} \max_{S^t} \left\{ \text{MNL}(S^t, \boldsymbol{\nu}_{\text{U}}^t) \mid S^t \subseteq [N] \right\},$$

where $\nu_{\text{U},i}^t$ is the attraction value of product i in period t obtained by assuming that the product i is offered in all past periods, that is,

$$\nu_{\text{U},i}^t := \alpha_i (\mathbb{1}(t-1 > 0), \dots, \mathbb{1}(t-M > 0)).$$

Moreover, for each $t \in [T]$, let $[\kappa_{\text{U}}^t]$ denote the RO assortment that maximizes $\text{MNL}(\cdot, \boldsymbol{\nu}_{\text{U}}^t)$. To complete the proof, we will show that $Z_* \leq Z_{\text{U}} \leq Z_{\text{RO}}$.

Now, we show that $Z_* \leq Z_{\text{U}}$. Let S_*^1, \dots, S_*^T be an optimal solution of (HAP). Let $\boldsymbol{\nu}_* = (\boldsymbol{\nu}_*^1, \dots, \boldsymbol{\nu}_*^T)$ denote the corresponding attraction values, that is, for $t \in [T]$ and $i \in [N]$,

$$\nu_{*,i}^t := \alpha_i (\mathbb{1}(i \in S_*^{t-1}), \dots, \mathbb{1}(i \in S_*^{t-M})),$$

where $S^s := \emptyset$ for $s \leq 0$. Then,

$$Z_* = \frac{1}{T} \sum_{t \in [T]} \text{MNL}(S_*^t, \boldsymbol{\nu}_*^t).$$

Now, for $t \in [T]$, let $\kappa_*^t \in [N]$ such that the RO assortment $[\kappa_*^t]$ maximizes $\text{MNL}(\cdot, \boldsymbol{\nu}_*^t)$. Then, $Z_* \leq Z_{\text{U}}$ follows by observing that

$$\text{MNL}(S_*^t, \boldsymbol{\nu}_*^t) \leq \text{MNL}([\kappa_*^t], \boldsymbol{\nu}_*^t) \leq \text{MNL}([\kappa_*^t], \boldsymbol{\nu}_{\text{U}}^t) \leq \text{MNL}([\kappa_{\text{U}}^t], \boldsymbol{\nu}_{\text{U}}^t),$$

where the first (resp. last) inequality holds since the right-hand-side is the optimal revenue that can be achieved under the attraction value ν_*^t (resp. ν_U^t), and the second inequality follows from Part 2 of Lemma EC.2 since $\beta \geq 0$ implies $\nu_U^t \geq \nu_*^t$.

Last, we show that $Z_U = Z_{RO}$. Let $[\kappa_{RO}^1], \dots, [\kappa_{RO}^T]$ be the sequence of RO assortments given by our policy, and let ν_{RO}^t be the corresponding attraction value, that is,

$$\nu_{RO,i}^t := \alpha_i(\mathbb{1}(i \in [\kappa_{RO}^{t-1}]), \dots, \mathbb{1}(i \in [\kappa_{RO}^{t-M}])),$$

where $[\kappa_{RO}^s] := \emptyset$ for $s \leq 0$. In general, $\beta \geq 0$ implies that $\nu_{RO}^t \leq \nu_U^t$. However, due to the nested structure on $\kappa_{RO}^1, \dots, \kappa_{RO}^T$ in Lemma EC.3, for each $t \in [T]$, we obtain that $\nu_{RO,k}^t = \nu_{U,k}^t$ for $k \in [\kappa_{RO}^t]$ and, thus

$$\text{MNL}([k], \nu_{RO}^t) = \text{MNL}([k], \nu_U^t) \quad \text{for } k \in [\kappa_{RO}^t]. \quad (\text{EC.6})$$

Moreover, since $[\kappa_{RO}^t]$ maximizes $\text{MNL}(\cdot, \nu_{RO}^t)$, Part 1 in Lemma EC.2 shows that

$$r_{\kappa_{RO}^t} \geq \text{MNL}([\kappa_{RO}^t], \nu_{RO}^t) > r_{\kappa_{RO}^{t+1}}. \quad (\text{EC.7})$$

By (EC.6) and (EC.7), we obtain $r_{\kappa_{RO}^t} \geq \text{MNL}([\kappa_{RO}^t], \nu_U^t) > r_{\kappa_{RO}^{t+1}}$. By invoking Part 1 in Lemma EC.2, we obtain that $\kappa_U^t = \kappa_{RO}^t$. Therefore, we can conclude that

$$\text{MNL}([\kappa_{RO}^t], \nu_{RO}^t) = \text{MNL}([\kappa_{RO}^t], \nu_U^t) = \text{MNL}([\kappa_U^t], \nu_U^t),$$

showing that $Z_U = Z_{RO}$. □

EC.3.4. Proof of Proposition 3

PROPOSITION 3 *Assume the absence of cross-product and cross-period constraints. Then, there exists an optimal assortment $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^T)$ of (HAP) such that $\{i \in [N] \mid \sum_{t \in [T]} x_i^t \geq 1\}$ is revenue-ordered.*

Proof. Let $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^T)$ be an optimal solution, and let $\nu^t = (\nu_i^t)_{i \in [N]}$ be the attraction value vector in period t , that is, $\nu_i^t = \alpha_i(x_i^{t-1}, \dots, x_i^{t-M})$. For each $t \in [T]$, let S^t denote $\{i \in [N] \mid x_i^t = 1\}$, and define $S := \cup_{t \in [T]} S^t$. Suppose that S is not revenue-ordered. We assume that products are indexed such that $r_1 \geq \dots \geq r_N$, then there exists a pair of products, i and j with $i < j$, such that $i \notin S$ but $j \in S$. We consider two possible cases on the revenue of the i^{th} product.

Suppose that there exists $\tau \in [T]$ such that $\text{MNL}(S^\tau, \boldsymbol{\nu}^\tau) \leq r_i$, where $\text{MNL}(\cdot, \cdot)$ is defined as in (EC.5). Let $\text{Rev}_{\text{HAP}}(Z^1, \dots, Z^T)$ be the objective value of (HAP) if Z^t is the assortment for the t^{th} period. It follows readily that

$$\begin{aligned} \text{Rev}_{\text{HAP}}(S^1, \dots, S^{\tau-1}, S^\tau \cup \{i\}, S^{\tau+1}, \dots, S^T) &= \frac{1}{T} \left(\sum_{t \in [T] \setminus \{\tau\}} \text{MNL}(S^t, \boldsymbol{\nu}^t) + \text{MNL}(S^\tau \cup \{i\}, \boldsymbol{\nu}^\tau) \right) \\ &\geq \frac{1}{T} \sum_{t \in [T]} \text{MNL}(S^t, \boldsymbol{\nu}^t) = \text{Rev}_{\text{HAP}}(S^1, \dots, S^T), \end{aligned}$$

where the first equality holds since $i \notin S^t$ for all $t \in [T]$. The inequality holds because $\text{MNL}(S^\tau \cup \{i\}, \boldsymbol{\nu}^\tau) = \lambda r_i + (1 - \lambda) \text{MNL}(S^\tau, \boldsymbol{\nu}^\tau) \geq \text{MNL}(S^\tau, \boldsymbol{\nu}^\tau)$, where $\lambda = \nu_i^\tau / (1 + \sum_{k \in S^\tau} \nu_k^\tau + \nu_i^\tau)$. If $\text{MNL}(S^\tau, \boldsymbol{\nu}^\tau) = r_i$, then we obtain a new optimal solution with the same revenue as that of S^1, \dots, S^T . If $\text{MNL}(S^\tau, \boldsymbol{\nu}^\tau) < r_i$, the condition is not possible because the above inequality contradicts the optimality of S^1, \dots, S^T .

Now, assume that $\text{MNL}(S^t, \boldsymbol{\nu}^t) > r_i$ for all $t \in [T]$. However, this is not possible as we argue next. Let τ be the last period such that $j \in S^\tau$. Then, we have $\text{MNL}(S^t, \boldsymbol{\nu}^t) > r_j$ for all $t \in [T]$. We obtain a contradiction to the optimality of S^1, \dots, S^T as follows.

$$\begin{aligned} \text{Rev}_{\text{HAP}}(S^1, \dots, S^{\tau-1}, S^\tau \setminus \{j\}, S^{\tau+1}, \dots, S^T) &= \frac{1}{T} \left(\sum_{t \in [T] \setminus \{\tau\}} \text{MNL}(S^t, \boldsymbol{\nu}^t) + \text{MNL}(S^\tau \setminus \{j\}, \boldsymbol{\nu}^\tau) \right) \\ &> \frac{1}{T} \sum_{t \in [T]} \text{MNL}(S^t, \boldsymbol{\nu}^t) = \text{Rev}_{\text{HAP}}(S^1, \dots, S^T), \end{aligned}$$

where the first equality holds due to $j \notin S^t$ for $t \geq \tau + 1$. The inequality holds because $\text{MNL}(S^\tau, \boldsymbol{\nu}^\tau) = \lambda r_j + (1 - \lambda) \text{MNL}(S^\tau \setminus \{j\}, \boldsymbol{\nu}^\tau) < \lambda \text{MNL}(S^\tau, \boldsymbol{\nu}^\tau) + (1 - \lambda) \text{MNL}(S^\tau \setminus \{j\}, \boldsymbol{\nu}^\tau)$, where $\lambda = \nu_j^\tau / (1 + \sum_{k \in S^\tau} \nu_k^\tau)$. The above inequality contradicts the optimality of S^1, \dots, S^T . \square

EC.4. Proofs of Section 4

We derive a graph representation of (HAP) for proving cyclic policies. With each instance of (HAP), we associate a directed graph $D = (V, A)$ with nodes V and arcs A , referred to as *assortment graph*, as follows. A node in the assortment graph is a tuple of historical assortments, $(\boldsymbol{x}^{t-1}, \dots, \boldsymbol{x}^{t-M})$. That is, a node $v = (v^1, \dots, v^M)$ in V has M elements, and each one is an assortment. The first element denotes the latest offered assortment, and the last one is the earliest one in memory. The number of nodes is 2^{NM} . We say a node $\mu = (\mu^1, \dots, \mu^M)$ is a predecessor of $v = (v^1, \dots, v^M)$ if $\mu^k = v^{k+1}$ for $k = 1, 2, \dots, M - 1$. Then, an ordered pair (μ, v) is an arc going

from μ to v if μ is a predecessor of v . Therefore, the arc set in the assortment graph is given as follows:

$$A := \{(\mu, v) \mid \text{if } \mu \text{ is predecessor of } v \text{ and } \mu, v \in V\}.$$

The number of arcs is 2^{NM+N} . The arc (μ, v) records the latest assortment v^1 and drops the earliest one μ^M , which defines the updating process of history assortments within memory. Then, we can define an arc's weight as

$$w(\mu, v) = \sum_{i \in [N]} r_i \pi_i(v^1, \mu^1, \dots, \mu^M)$$

where $\pi_i(\cdot)$ is the purchase probability of i given current assortment v^1 and historical assortments (μ^1, \dots, μ^M) defined in Section 2.1. Here, we omit the superscript of period t for simplicity.

EC.4.1. Proof of Theorem 3

THEOREM 3 *Given an instance of (INFTY), there exists a positive integer L^* such that L^* -cyclic policy is optimal to (INFTY).*

Proof. We show that finding an optimal assortment planning for (INFTY) equals finding the maximum mean cycle in the assortment graph, which is the cycle with the largest mean weight.

First, we show an assortment planning of (INFTY) maps to a path in the assortment graph. Suppose $\mathbf{x}^1, \mathbf{x}^2, \dots$ is an assortment planning of (INFTY), it defines a path in the assortment graph, that is, $v_0, v(1), v(2), \dots$, where $v_0 = (\mathbf{0}, \dots, \mathbf{0})$ and

$$v(t)^k = \begin{cases} \mathbf{x}^{t-k+1} & \text{if } t - k \geq 0 \\ \mathbf{0} & \text{otherwise,} \end{cases} \quad \text{for } k \in [M] \text{ and } t \in \mathbb{Z}_+.$$

It suggests $v(t)$'s first element records the assortment decision in period t , and the following $M - 1$ elements record the historical assortments before \mathbf{x}^t . Then, the mean weight of the path equals the average revenue of the assortment planning. That is,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t \in [T]} w(v(t-1), v(t)) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t \in [T]} \sum_{i \in [N]} r_i \pi_i(\mathbf{x}^t, \mathbf{x}^{t-1}, \dots, \mathbf{x}^{t-M}).$$

Next, we show that to solve (INFTY), it suffices to find the maximum mean cycle denoted as C^* . Since any assortment planning is a path in the assortment graph, solving (INFTY) equals finding an infinite-length path with the largest mean weight and the starting node v_0 . We call such a path an optimal path. Because the assortment graph has finite nodes and arcs, an optimal path will inevitably repeat cycles in the assortment graph. Given an infinite-length path, we can repeatedly

replace each cycle in the path with the maximum mean cycle. Finally, we obtain a path that repeats the maximum mean cycle forever and has a larger mean weight than the initial one. We do not need to consider the path's weight from the starting node v_0 to the maximum mean cycle because it will vanish when taking the average over an infinite horizon T .

Next, we show how to recover an assortment plan from the maximum mean cycle. Suppose the maximum mean cycle is $C^* = v(1), v(2), \dots, v(L^* - 1), v(0)$ where L^* is the length of the maximum mean cycle. We can construct a cyclic policy as follows.

$$\mathbf{x}^t = v^1(t \bmod L^*) \text{ for } t \in \mathbb{N}_+.$$

It suggests \mathbf{x}^t is the first element of $v(t \bmod L^*)$. Therefore, we prove that an optimal solution of (INFTY) is a cyclic policy. The average revenue generated by the cyclic policy is identical to the mean weight of the maximum mean cycle C^* , and the length of the cyclic policy is L^* . If we know L^* , we can compute the cyclic assortment planning via (CYCLE-CONIC). \square

EC.4.2. Proof of Theorem 4

THEOREM 4 *Assume that $M < N$. An $(M + 1)$ -cyclic policy is optimal for problem (INFTY) if the cross-period constraint is non-overlapping.*

Proof. We prove this result by showing any L -cyclic policy with length $L > M + 1$ will be dominated by the $(M + 1)$ -cyclic policy in terms of the expected revenue. Recall that we do not consider cyclic policies with length $L < M + 1$ because they contain empty sets.

Given an L -cyclic policy satisfying the non-overlapping condition, $\mathbf{x}^1, \dots, \mathbf{x}^L$, the purchase probability of product i in period t is

$$\pi_i^t(\mathbf{x}^t, \mathbf{x}^{t-1}, \dots, \mathbf{x}^{t-M}) = \frac{x_i^t \exp(\beta_i^0)}{1 + \sum_{i \in [N]} x_i^t \exp(\beta_i^0)},$$

because $x_i^t + x_i^{t-1} + \dots + x_i^{t-M} \leq 1$ for $i \in [N]$ and $t \in [L]$. Hence, we can simplify the average revenue of such a cyclic policy as follows

$$\text{Rev}_L = \frac{1}{L} \sum_{t \in [L]} R(\mathbf{x}^t) = \frac{1}{L} \sum_{t \in [L]} \sum_{i \in [N]} r_i x_i^t \exp(\beta_i^0) / (1 + \sum_{i \in [N]} x_i^t \exp(\beta_i^0)).$$

For any L -cyclic policy with length $L > M + 1$, we show its average revenue is a convex combination of cyclic policies with length $M + 1$. We rewrite Rev_L as

$$\text{Rev}_L = \frac{(M + 1)R(\mathbf{x}^1) + \dots + (M + 1)R(\mathbf{x}^L)}{(M + 1)L}$$

$$\begin{aligned}
&= \frac{(R(\mathbf{x}^1) + R(\mathbf{x}^2) + \dots + R(\mathbf{x}^{M+1})) + \dots + (R(\mathbf{x}^L) + R(\mathbf{x}^1) + \dots + R(\mathbf{x}^M))}{(M+1)L} \\
&= \frac{1}{L} \frac{R(\mathbf{x}^1) + R(\mathbf{x}^2) + \dots + R(\mathbf{x}^{M+1})}{M+1} + \dots + \frac{1}{L} \frac{R(\mathbf{x}^L) + R(\mathbf{x}^1) + \dots + R(\mathbf{x}^M)}{M+1} \\
&\leq \text{Rev}_{M+1}^*.
\end{aligned}$$

The first equation holds by multiplying both the denominator and numerator by $M+1$. The second equation holds by re-arranging orders of $R(\mathbf{x}^1), \dots, R(\mathbf{x}^L)$. The third equation shows that we construct L cyclic policies with length $M+1$ by abstracting $M+1$ adjacent assortments in the cycle $\mathbf{x}^1, \dots, \mathbf{x}^L$. Each cyclic policy is a feasible solution of (CYCLE-CONIC) with length $M+1$ since they are non-overlapping. Then, we have the last inequality because Rev_{M+1}^* is generated by the optimal solution of (CYCLE-CONIC) with length $M+1$. Therefore, the $(M+1)$ -cyclic policy has the largest revenue under the non-overlapping condition.

EC.4.3. Proof of Theorem 5

THEOREM 5 $Z_{\text{BOUND-FREE}} \leq Z_{\text{CYCLE-CONIC}}$ if the cross-period constraint is non-overlapping.

Proof. In the following, we will show that $Z_{\text{BOUND-FREE}} \leq Z_{\text{McCORMICK}} \leq Z_{\text{CYCLE-CONIC}}$, where $Z_{\text{McCORMICK}}$ is the optimal objective value of the continuous relaxation of the following integer programming:

$$\max \quad \frac{1}{M+1} \sum_{t \in [M+1]} \sum_{i \in [N]} r_i u_i \gamma_i^t$$

$$\text{s. t.} \quad x_i^t + \sum_{m \in [M]} x_i^{\tau(m|t)} \leq 1 \quad \text{for } i \in [N] \quad (\text{EC.8a})$$

$$\rho^t + \sum_{i \in [N]} u_i \gamma_i^t = 1 \quad \text{for } t \in [M+1] \quad (\text{EC.8b})$$

$$\gamma_i^t \leq \rho_U^t x_i^t \quad \text{for } i \in [N] \text{ and } t \in [M+1] \quad (\text{EC.8c})$$

$$\gamma_i^t \leq \rho^t + \rho_L^t x_i^t - \rho_L^t \quad \text{for } i \in [N] \text{ and } t \in [M+1] \quad (\text{EC.8d})$$

$$\gamma_i^t \geq \rho^t + \rho_U^t x_i^t - \rho_U^t \quad \text{for } i \in [N] \text{ and } t \in [M+1] \quad (\text{EC.8e})$$

$$\gamma_i^t \geq \rho_L^t x_i^t \quad \text{for } i \in [N] \text{ and } t \in [M+1] \quad (\text{EC.8f})$$

$$\mathbf{x}^t \in \mathcal{X} \cap \{0, 1\}^N \quad \text{for } t \in [M+1], \quad (\text{EC.8g})$$

where constraints (EC.8c)-(EC.8f) are exactly (3), obtained by using the McCormick envelopes to linearize $\gamma_i^t = \rho^t x_i^t$.

First, we prove $Z_{\text{BOUND-FREE}} \leq Z_{\text{MCCORMICK}}$. Since the bound-free formulation (12) and formulation (EC.8) have identical objectives, it suffices to show that any feasible solution of the continuous relaxation of (12) is also feasible for that of (EC.8). More specifically, we need to show that constraints (EC.8c)-(EC.8f) are implied by (12b)-(12e) of the bound-free formulation. Constraints (12c), $\Gamma_{ij}^t \leq \gamma_i^t$ for $i \neq j$ from (12d), and $\Gamma_{ii}^t = \gamma_i^t$ from (12e) imply that

$$x_i^t \leq \left(1 + \sum_{j \in N} u_j\right) \gamma_i^t,$$

which is (EC.8f). Constraints (12b) (12c), $\Gamma_{ij}^t \leq \gamma_j^t$ for $i \neq j$ from (12d), and $\Gamma_{ii}^t = \gamma_i^t$ from (12e) imply that

$$x_i^t \leq \gamma_i^t + \sum_{j \in [N]} u_j \gamma_j^t = \gamma_i^t + 1 - \rho^t,$$

which is (EC.8e). Also, constraint (12c), $\Gamma_{ij}^t \geq 0$ for $i \neq j$ and $\Gamma_{ii}^t = \gamma_i^t$ imply

$$x_i^t \geq \gamma_i^t,$$

which is (EC.8c). Constraint (12b) (12c), $\Gamma_{ij}^t \geq \gamma_i^t + \gamma_j^t - \rho^t$ for $i \neq j$, $\Gamma_{ii}^t = \gamma_i^t$, and $\gamma_i^t \leq \rho^t$ imply

$$\begin{aligned} x_i^t &\geq \gamma_i^t + u_i \gamma_i^t + \sum_{j \neq i} u_j (\gamma_i^t + \gamma_j^t - \rho^t) = (1 + \sum_{j \in [N]} u_j) (\gamma_i^t - \rho^t) + (1 + u_i^t) \rho^t + \sum_{j \neq i} u_j \gamma_j^t \\ &= (1 + \sum_{j \in [N]} u_j) (\gamma_i^t - \rho^t) + u_i^t \rho^t + 1 - u_i^t \gamma_i^t \geq (1 + \sum_{j \in [N]} u_j) (\gamma_i^t - \rho^t) + 1, \end{aligned}$$

which is (EC.8d).

Next, we prove $Z_{\text{MCCORMICK}} \leq Z_{\text{CYCLE-CONIC}}$. Note that in this case, (CYCLE-CONIC) becomes

$$\begin{aligned} \max \quad & \frac{1}{M+1} \sum_{i \in [N]} \sum_{t \in [M+1]} r_i y_i^t \\ \text{s. t.} \quad & x_i^t + \sum_{m \in [M]} x_i^{\tau(m|t)} \leq 1 && \text{for } i \in [N] \\ & \rho^t + \sum_{i \in [N]} y_i^t = 1 && \text{for } t \in [M+1] \\ & \max\{0, \rho_U^t x_i^{\tau(m|t)} + \gamma_i^t - \rho_U^t\} \leq z_{im}^t \leq \min\{\gamma_i^t, \rho_U^t x_i^{\tau(m|t)}\} && \text{for } t \in [M+1], m \in [M] \\ & && \text{(EC.9a)} \\ & (3), (5), (10b) && \text{for } t \in [M+1], i \in [N] \\ & \mathbf{x}^t \in \mathcal{X} \cap \{0, 1\}^N && \text{for } t \in [M+1], \end{aligned}$$

where (3) and (EC.9a) are McCormick envelopes linearizing $\gamma_i^t = \rho^t x_i^t$ and $z_{im}^t = \gamma_i^t x_i^{\tau(m|t)}$, respectively. (5) and (10b) are convex and concave extensions of y_i^t , respectively. Letting $(\mathbf{x}, \boldsymbol{\rho}, \boldsymbol{\gamma})$ be a feasible solution of the continuous relaxation of (EC.8), and define $y_i^t = \gamma_i^t u_i$ and $z_{im}^t = 0$ for all $t \in [M + 1]$ and $i \in [N]$. To complete the proof, we only need to show that $(\mathbf{x}, \boldsymbol{\rho}, \boldsymbol{\gamma}, \mathbf{y}, \mathbf{z})$ is feasible to (CYCLE-CONIC). Clearly, constraints (3), (5) and (10b) are satisfied. Next, we verify constraint (EC.9a) is satisfied. Clearly, $\min\{\gamma_i^t, \rho_U^t x_i^{\tau(m|t)}\} \geq 0 = z_{im}^t$. On the other hand,

$$\begin{aligned} \rho_U^t x_i^{\tau(m|t)} + \gamma_i^t - \rho_U^t &\leq \rho_U^t x_i^{\tau(m|t)} + x_i^t \rho_U^t - \rho_U^t \\ &= \rho_U^t (x_i^{\tau(m|t)} + x_i^t - 1) \\ &\leq 0 = z_{im}^t, \end{aligned}$$

where the first inequality holds because of (EC.8c) and the second inequality holds because of (EC.8a). \square

EC.4.4. Example of the Unstable Maximum Mean Cycle Length

The following example shows that a slight fluctuation in one parameter could result in a different maximum mean cycle length if there are no non-overlapping constraints.

EXAMPLE EC.1. Consider a case where a firm sells three products to customers with a memory length of one. There are no cross-period or cross-product constraints. The outside option has utility 0. Table EC.1 provides a detailed parameter setup. The two-cyclic policy has larger revenue

Product	Price	Base utility	One-period history-dependent effect
1	6	1.3	-0.5
2	16	-0.6	-0.8
3	17	0.2	-0.9

than the three-cyclic policy. However, once the base utility of product 3 increases from 0.2 to 0.3, the average revenue of the two-cyclic policy is smaller than that of the three-cyclic policy. \square

EC.5. Additional Material to Section 6

EC.5.1. Multilinear Extension Based Formulation

In this subsection, we introduce the multilinear extension-based formulation of (HAP). For the nonlinear structure (CHOICE), we first use the multilinear extension to present the attraction value

function $\alpha_i(\cdot)$ (O'Donnell 2014), and then recursively apply McCormick envelopes to derive a linear representation. It is well-understood that both the quality and the size of the resulting linear representations depend on the recursive sequence, and finding an optimal recursive sequence amounts to solving a difficult combinatorial optimization problem (Speakman and Lee 2017, Khajavirad 2023). Speakman and Lee (2017) show that the relaxation quality depends on the sequence of recursion, but we do not exploit the best sequence in this paper. Even though the number of introduced variables and constraints increases exponentially with the memory length, this formulation is solvable under a small M .

We represent the attraction value function via a sum of multiple multilinear functions. Given memory length $M \in \mathbb{Z}_+$, $\{S_k \mid S_k \subseteq [M], k \in [2^M]\}$ is the collection of all subsets of $[M]$. For every $S_k, k \in [2^M]$, let χ^{S_k} be its indicator vector, that is, the m^{th} coordinate of χ^{S_k} is 1 if and only if $m \in S_k$. We can rewrite the attraction value function as

$$\alpha_i(x_i^{t-1}, \dots, x_i^{t-M}) = \sum_{k \in [2^M]} \left(\prod_{m \in S_k} x_i^{t-m} \right) \cdot a_{ik} \quad \text{for } i \in [N],$$

where $a_{ik} = \alpha_i(\chi^{S_k}) - \sum_{j: S_j \subset S_k} a_{ij}$.

Then, introduce θ_{ik}^t to present the multilinear term, that is,

$$\theta_{ik}^t = x_i^t \cdot \left(\prod_{m \in S_k} x_i^{t-m} \right) = \prod_{m \in S_k \cup \{0\}} x_i^{t-m} \quad \text{for } i \in [N], t \in [T], k \in [2^M].$$

We recursively use McCormick envelopes to linearize θ_{ik}^t . The idea is to sequentially introduce artificial variables and reduce the number of variables in the multilinear term. First, select any two components $p, q \in S_k \cup \{0\}$. Then, introduce a variable $h_i^t(p, q)$ that corresponds to the bilinear product $x_i^{t-p} x_i^{t-q}$ and the multilinear term changes to

$$\theta_{ik}^t = h_i^t(p, q) \cdot \prod_{m \in S_k \cup \{0\} \setminus \{p, q\}} x_i^{t-m}.$$

We can use McCormick envelopes to linearize $h_i^t(p, q) = x_i^{t-p} x_i^{t-q}$,

$$\max\{0, x_i^{t-p} + x_i^{t-q} - 1\} \leq h_i^t(p, q) \leq \min\{x_i^{t-p}, x_i^{t-q}\}.$$

This procedure can be recursively applied to the remaining parts until θ_{ik}^t is completely linearized. We use a notation $\mathcal{H}(i, t, k)$ to present the corresponding McCormick envelopes.

Note that $\theta_{i,k}^t \in \{0, 1\}$, we also apply the McCormick envelopes to linearize $\rho^t \theta_{i,k}^t$ denoted by $w_{i,k}^t$. That is,

$$w_{i,k}^t \leq \min\{\rho_L^t \theta_{i,k}^t + \rho^t - \rho_L^t, \rho_U^t \theta_{i,k}^t\} \quad (\text{EC.10a})$$

$$w_{i,k}^t \geq \max\{\rho_L^t \theta_{i,k}^t, \rho_U^t \theta_{i,k}^t + \rho^t - \rho_U^t\}. \quad (\text{EC.10b})$$

Finally, we obtain a mixed-integer linear formulation of **(HAP)** as follows.

$$\begin{aligned} \max \quad & \frac{1}{T} \sum_{t \in [T]} \sum_{i \in [N]} r_i y_i^t \\ \text{s. t.} \quad & \mathbf{x}^t \in \{0, 1\}^N \cap \mathcal{X} \text{ and } \mathbf{x} \in \mathcal{P} && \text{for } t \in [T] && (\text{MULTILINEAR}) \\ & \rho^t + \sum_{i \in [N]} y_i^t = 1 && \text{for } t \in [T] \\ & y_i^t = \sum_{k \in [2^M]} w_{i,k}^t a_{ik} && \text{for } t \in [T] \text{ and } i \in [N] \\ & \theta_{i,k}^t \in \mathcal{H}(i, t, k) && \text{for } t \in [T], i \in [N] \text{ and } k \in [2^M] \\ (\text{EC.10}) \quad & && \text{for } t \in [T], i \in [N] \text{ and } k \in [2^M]. \end{aligned}$$

EC.5.2. A Projected Cutting-Plane Implementation for Bound-Free Formulation (12)

The bound-free formulation (12) has $\mathcal{O}(N^2)$ variables due to the presence of Γ variables. We design a cutting-plane procedure to implement (12) for computation efficiency. Recall that (12) is reformulation of **((M+1)-CYCLIC)**. In our implementation, we first reformulate **((M+1)-CYCLIC)** as a base model. Next, we use a projected separation oracle to cut off infeasible solutions for K rounds.

We linearize **((M+1)-CYCLIC)** to a base model as follows. First, following the idea in Section 3.1, we introduce variable ρ^t to denote $1/(1 + \sum_{j \in [N]} u_j x_j^t)$ with constraint $\rho^t + \sum_{j \in [N]} u_j x_j^t \rho^t = 1$. Next, let γ_j^t denote the bilinear product $x_j^t \rho^t$ and apply McCormick envelopes to linearize the bilinear term. We also notice that, for $t \in [M+1]$,

$$\rho^t (1 + \sum_{j \in [N]} x_j^t u_j) \geq 1,$$

which is a valid convex constraint and is representable via the second-order cone. Let $w^t = 1 + \sum_{j \in [N]} x_j^t u_j$ for $t \in [M + 1]$ and add the above constraint into our formulation. Finally, we obtain a mixed-integer second-order conic formulation of **((M+1)-CYCLIC)** as follows.

$$\begin{aligned}
& \max \quad \frac{1}{M+1} \sum_{t \in [M+1]} \sum_{i \in [N]} r_i u_i \gamma_i^t \\
& \text{s. t.} \quad x_i^t + \sum_{m \in [M]} x_i^{\tau(m|t)} \leq 1 && \text{for } i \in [N] \\
& \quad \rho^t + \sum_{i \in [N]} u_i \gamma_i^t = 1 && \text{for } t \in [M+1] \\
& \quad \gamma_i^t \leq \min\{\rho_L^t x_i^t + \rho^t - \rho_L^t, \rho_U^t x_i^t\} && \text{for } i \in [N], t \in [M+1] \quad \text{(BASE)} \\
& \quad \gamma_i^t \geq \max\{\rho_L^t x_i^t, \rho_U^t x_i^t + \rho^t - \rho_U^t\} && \text{for } i \in [N], t \in [M+1] \\
& \quad \rho^t + w^t \geq \|\rho^t - w^t, 2\| && \text{for } t \in [M+1] \\
& \quad w^t = 1 + \sum_{j \in [N]} x_j^t u_j && \text{for } t \in [M+1] \\
& \quad \mathbf{x}^t \in \{0, 1\}^N && \text{for } t \in [M+1],
\end{aligned}$$

where ρ_L^t and ρ_U^t are lower and upper bound of ρ^t for $t \in [M + 1]$, respectively. Here, although we need the bound of ρ^t when implementing (12), we do not require it to be very tight, and we use it only for getting an equivalent formulation of **((M+1)-CYCLIC)**.

We tighten the base formulation through a projected separation oracle (Algorithm 2). Given a solution $(\hat{\mathbf{x}}, \hat{\gamma}, \hat{\rho})$, the separation oracle checks whether the projection of (12c) in the space of the base formulation is violated for each $i \in [N]$ and $t \in [M + 1]$. More specifically, we replace Γ_{ij}^t in (12c) with the bounds in (12d) and (12e), then we have

$$\gamma_i^t (1 + u_i) + \sum_{j \in [N], j \neq i} u_j \min\{0, \gamma_i^t + \gamma_j^t - \rho^t\} \leq x_i^t \leq \gamma_i^t + \sum_{j \in [N]} u_j \min\{\gamma_i^t, \gamma_j^t\}. \quad \text{(EC.11)}$$

If the current solution violates (EC.11), we add the corresponding constraint to the base model. We detail each step in Algorithm 2.

Running the separation oracle once is a round of cut generation. We add new cuts for K rounds and the final formulation as BF- K , where BF denotes ‘‘bound-free’’. In each round $k \in [K]$, the cutting-plane algorithm solves the continuous relaxation of current formulation BF- $(k - 1)$ and obtain an optimal solution $(\hat{\mathbf{x}}, \hat{\gamma}, \hat{\rho})$. Then, the algorithm calls the separation oracle (Algorithm 2) to generate new cuts. The algorithm finally solves BF- K with binary constraints and obtains an optimal integer solution. We present details of this implementation process in Algorithm 3.

Algorithm 2 Projection-based separation oracle for (BASE)

```

1: Input:  $(\hat{x}, \hat{\gamma}, \hat{\rho})$  and base utility  $u$ 
2: Cuts  $\leftarrow \emptyset$ ,  $A \leftarrow \emptyset$ ,  $B \leftarrow \emptyset$ , and  $C \leftarrow \emptyset$ 
3: for  $i \in [N]$  and  $t \in [M + 1]$  do
4:   for  $j \in [N]$  do
5:     if  $\hat{\gamma}_i^t + \hat{\gamma}_j^t \geq \hat{\rho}_t$  and  $i \neq j$  then
6:        $A \leftarrow A \cup \{j\}$ 
7:     end if
8:     if  $\hat{\gamma}_j^t \geq \hat{\gamma}_i^t$  then
9:        $B \leftarrow B \cup \{j\}$ 
10:    else
11:       $C \leftarrow C \cup \{j\}$ 
12:    end if
13:  end for
14:  lower-bound  $\leftarrow (1 + u_i) \cdot \hat{\gamma}_i^t + \sum_{j \in A} u_j (\hat{\gamma}_i^t + \hat{\gamma}_j^t - \hat{\rho}_t)$ 
15:  upper-bound  $\leftarrow \hat{\gamma}_i^t + \sum_{j \in B} u_j \hat{\gamma}_i^t + \sum_{j \in C} u_j \hat{\gamma}_j^t$ 
16:  if  $\hat{x}_i^t \leq$  lower-bound then
17:    Cuts  $\leftarrow x_i^t \geq (1 + u_i) \gamma_i^t + \sum_{j \in A} u_j (\gamma_i^t + \gamma_j^t - \rho_t)$ 
18:  else if  $\hat{x}_i^t \geq$  upper-bound then
19:    Cuts  $\leftarrow x_i^t \leq \gamma_i^t + \sum_{j \in B} u_j \gamma_i^t + \sum_{j \in C} u_j \gamma_j^t$ 
20:  end if
21: end for
22: Output: Cuts

```

EC.5.3. A Cutting-Plane Implementation for Large Memory Length

One of the main difficulties of solving (CONIC) is that for each $i \in [N]$ and $t \in [T]$, the number of linear inequalities used to describe the concave envelope of the attraction value function in (7) is $M!$, which grows factorially as M becomes large. In the following, we will implement constraints in (7) in the so-called lazy fashion. More specifically, we use the `Callback` routine of Gurobi to implement (7) as lazy constraints. Constraints in (7) are removed from (CONIC) and placed in a lazy constraint pool. Then, Gurobi solves the relaxed problem and checks whether inequalities in the pool are violated at each integer solution generated in the branch-and-bound tree. If a violated

Algorithm 3 A projected-cutting-plane implementation of bound-free formulation (12)

-
- 1: **Input:** Formulation (**BASE**) (denoted as BF-0) and a positive integer K
 - 2: $k \leftarrow 1$
 - 3: **while** $k \leq K$ **do**
 - 4: $(\hat{\mathbf{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}}) \leftarrow$ an optimal solution to the continuous relaxation of BF- $(k-1)$
 - 5: Cuts \leftarrow Call the separation oracle, Algorithm 2, to separate $(\hat{\mathbf{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}})$
 - 6: BF- $k \leftarrow$ Add Cuts into BF- $(k-1)$ formulation
 - 7: $k \leftarrow k+1$
 - 8: **end while**
 - 9: Solve BF- K with binary constraints of \mathbf{x}
 - 10: **Output:** An optimal solution of BF- K , $(\mathbf{x}^*, \boldsymbol{\gamma}^*, \boldsymbol{\rho}^*)$
-

inequality is found, it is added to the node, the integer solution is cut off, and the node is resolved. We elaborate on our implementation as follows.

Since Gurobi does not support the exponential function, we linearize the exponential conic constraint (5) using the subgradient inequalities of the continuous relaxation of $\alpha_i(\cdot)$ and treat the subgradient inequalities as a lazy constraint as well. Note that the continuous relaxation $\tilde{\alpha}_i(\cdot)$ is a closed convex function over $[0, 1]^M$. Thus, by Rockafellar (1970), for $(x_i^{t-1}, \dots, x_i^{t-M}) \in [0, 1]^M$

$$\tilde{\alpha}_i(x_i^{t-1}, \dots, x_i^{t-M}) = \max_{\mathbf{w}} \left\{ \tilde{\alpha}_i(\mathbf{w}) + \sum_{m \in [M]} (x_i^{t-m} - w^m) \beta_i^m \tilde{\alpha}_i(\mathbf{w}) \mid \mathbf{w} \in [0, 1]^M \right\}.$$

where $\beta_i^m \tilde{\alpha}_i(\mathbf{w})$ is the m^{th} partial derivate of $\nabla \tilde{\alpha}_i(\mathbf{w})$. Therefore, using the variable γ_i^t to scale the subgradient inequalities, the exponential conic constraint (5) is equivalent to

$$y_i^t \geq \gamma_i^t \alpha_i(\mathbf{w}) + \sum_{m \in [M]} (z_{im}^t - \gamma_i^t w^m) \beta_i^m \alpha_i(\mathbf{w}) \quad \text{for } \mathbf{w} \in [0, 1]^M. \quad (\text{EC.12})$$

To implement (7) and (EC.12) as lazy constraints in Gurobi, we use Algorithm 4 to solve the separation problem. Given a point (solution) $(\hat{\mathbf{y}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$ in the space of these two constraints, for each product $i \in [N]$ and period $t \in [T]$, we check whether constraints in (7) and (EC.12) is violated. For constraint (7), we obtain a permutation $\hat{\sigma}$ by sorting $\{g_{im}^t\}_{m \in [M]}$ in a non-increasing order, that is, $g_{i\hat{\sigma}(1)}^t \geq \dots \geq g_{i\hat{\sigma}(M)}^t$, where $g_{im}^t = \hat{z}_{im}^t$ if $m \notin I_i$ and $g_{im}^t = \hat{\gamma}_i^t - \hat{z}_{im}^t$ if $m \in I_i$. The constraint in (7) corresponding to the permutation $\hat{\sigma}$ is

$$y_i^t \leq \alpha_i(\mathbf{h}_{i0}^{\hat{\sigma}})(\gamma_i^t - \hat{z}_{i\hat{\sigma}(1)}^t) + \sum_{k \in [M]} \alpha_i(\mathbf{h}_{ik}^{\hat{\sigma}})(\hat{z}_{i\hat{\sigma}(k)}^t - \hat{z}_{i\hat{\sigma}(k+1)}^t). \quad (\text{EC.13})$$

The given point $(\hat{\mathbf{y}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$ satisfies (7) if and only if it satisfies the constraint (EC.13). For the convex part, we directly check the subgradient constraint obtained at historical assortments $(\hat{w}^1, \dots, \hat{w}^M) = (\hat{z}_{i1}^t / \hat{\gamma}_i^t, \dots, \hat{z}_{iM}^t / \hat{\gamma}_i^t)$:

$$y_i^t \geq \gamma_i^t \alpha_i(\hat{w}^1, \dots, \hat{w}^M) + \sum_{m \in [M]} (z_{im}^t - \gamma_i^t \hat{w}^m) \beta_i^m \alpha_i(\hat{w}^1, \dots, \hat{w}^M). \quad (\text{EC.14})$$

The given point $(\hat{\mathbf{y}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$ satisfies (EC.12) if and only if it satisfies the constraint (EC.14). We summarize the above procedure in Algorithm 4

The idea of separating constraint (7) is as follows. Since constraint (7) is the scaled Lovász extension of the attraction value function, the separation problem reduces to identifying the correct linear inequality from its piecewise-linear representation. Each linear piece corresponds to a specific ordering (permutation) of the coordinates, that is, the historical assortments $(x_i^{t-1}, \dots, x_i^{t-M})$. Given a candidate solution $\hat{\mathbf{x}}$, the permutation $\hat{\sigma}$ obtained by sorting the historical assortments in a non-increasing order identifies which simplex contains the current point and determines the linear interpolation that yields the Lovász extension. For example, when the memory length is two, in Figure 2a, nodes in the blue shaded triangle satisfy $x_i^{t-1} \geq x_i^{t-2}$, which corresponds to permutation (1, 2). This permutation maps to the nested vertices $(0, 0) - (1, 0) - (1, 1)$ that define the Lovász extension for these nodes. In the mixed case shown in Figure 2b, we sort the switched historical assortments, i.e., $1 - x_i^{t-1} \geq x_i^{t-2}$, again obtaining permutation (1, 2). Since the linear scaling (by γ_i^t) preserves the relative ordering, we can directly sort g_{im}^t , as done in our algorithm. We provide a formal proof of the separation procedure of both constraint (7) and (EC.12) in the following Lemma EC.4.

LEMMA EC.4. *Let $\mathcal{Y} = \{(\mathbf{y}, \boldsymbol{\gamma}, \mathbf{z}) \mid (7), (\text{EC.12})\}$. Algorithm 4 solves the separation problem of set \mathcal{Y} in polynomial time.*

Proof of Lemma EC.4: We need to show that if current solution $(\hat{\mathbf{y}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}}) \notin \mathcal{Y}$, then the returned cuts can separate current solution and \mathcal{Y} . First, it is easy to check that the returned cuts are valid inequalities for \mathcal{Y} . That is, each point $(\mathbf{y}, \boldsymbol{\gamma}, \mathbf{z}) \in \mathcal{Y}$ satisfies (EC.13) and (EC.14).

Next, we discuss two cases when the returned cuts could cut off the current solution. We argue that it is sufficient to consider positive $\hat{\gamma}_i^t$ because the integer solution must be $\hat{x}_i^t = \hat{\gamma}_i^t = z_{i1}^t = \dots = z_{iM}^t = \hat{y}_i^t = 0$ if $\hat{\gamma}_i^t = 0$, which is feasible. If $(\mathbf{y}, \boldsymbol{\gamma}, \mathbf{z})$ violates at least one constraint (7). That is,

$$\hat{y}_i^t > \hat{\gamma}_i^t \alpha_i(\hat{z}_{i1}^t / \hat{\gamma}_i^t, \dots, \hat{z}_{iM}^t / \hat{\gamma}_i^t)$$

$$\begin{aligned}
&= \min_{\sigma} \left\{ \alpha_i(\mathbf{h}_{i0}^{\sigma})(\hat{\gamma}_i^t - g_{i\sigma(1)}^t) + \sum_{k \in [M]} \alpha_i(\mathbf{h}_{ik}^{\sigma})(g_{i\sigma(k)}^t - g_{i\sigma(k+1)}^t) \right\} \\
&= \alpha_i(\mathbf{h}_{i0}^{\hat{\sigma}})(\hat{\gamma}_i^t - g_{i\hat{\sigma}(1)}^t) + \sum_{k \in [M]} \alpha_i(\mathbf{h}_{ik}^{\hat{\sigma}})(g_{i\hat{\sigma}(k)}^t - g_{i\hat{\sigma}(k+1)}^t),
\end{aligned}$$

where g_{im}^t is switched with \hat{z}_{im}^t defined in Algorithm 4. The first and second equalities hold by the definition of the Lovász function of $\alpha_i(\cdot)$. Clearly, such a solution can be cut off by (EC.13). If $(\hat{\mathbf{y}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$ violates at least one constraint of (EC.12). That is,

$$\hat{y}_i^t < \hat{\gamma}_i^t \alpha_i(\hat{z}_{i1}^t/\hat{\gamma}_i^t, \dots, \hat{z}_{iM}^t/\hat{\gamma}_i^t) = \hat{\gamma}_i^t \alpha_i(\hat{w}^1, \dots, \hat{w}^M) + \sum_{m \in [M]} (\hat{z}_{im}^t - \hat{\gamma}_i^t \hat{w}^m) \beta_i^m \alpha_i(\hat{w}^1, \dots, \hat{w}^M),$$

where the equality hold by the definition $\hat{w}^m = \hat{z}_{im}^t/\hat{\gamma}_i^t$. Such a solution can be cut off by (EC.14).

□

The last element in our implementation is to select a starting formulation before calling Gurobi. We solve a collection of relaxed models to obtain continuous solutions and call the separation oracle to add new cuts. The relaxed models contain a subset of constraints (7) and (EC.12), defined as follows:

$$\begin{aligned}
&\max_{\mathbf{x}, \mathbf{y}, \rho, \boldsymbol{\gamma}, \mathbf{z}} \quad \frac{1}{T} \sum_{t \in [T]} \sum_{i \in [N]} r_i y_i^t \\
&\text{s. t.} \quad \mathbf{x}^t \in [0, 1]^N \cap \mathcal{X} \text{ and } \mathbf{x} \in \mathcal{P} \quad \text{for } t \in [T] \\
&\quad \rho^t + \sum_{i \in [N]} y_i^t = 1 \quad \text{for } t \in [T] \tag{RLX_k} \\
&\quad (3), (4) \quad \text{for } t \in [T] \text{ and } i \in [N] \\
&\quad (7) \quad \text{for } \sigma \in \Omega_{it}^k, t \in [T] \text{ and } i \in [N] \\
&\quad (\text{EC.12}) \quad \text{for } \tilde{\mathbf{w}} \in W_{it}^k, t \in [T] \text{ and } i \in [N],
\end{aligned}$$

where $\Omega_{it}^k \subset \Omega$ (resp. $W_{it}^k \subset [0, 1]^M$) is a subset of permutations (resp. nodes). Given a continuous solution, we call the separation oracle (Algorithm 4) to add new cuts and update RLX_k to $\text{RLX}_{(k+1)}$. We add K rounds of cuts and set the final RLX_K with binary constraints as a starting model for `Callback`. We present details of the implementation process in Algorithm 5.

EC.6. Supplementary Numerical Studies

EC.6.1. Performance under Cross-product and Cross-period Constraints

In this numerical study, we add cross-product and cross-period constraints to understand how such constraints impact the computational performance of our formulations. We consider the cardinality

Algorithm 4 Separation oracle for (7) and (EC.12)

-
- 1: Input: $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$
 - 2: Cuts $\leftarrow \emptyset$
 - 3: **for** $i \in [N]$ and $t \in [T]$ **do**
 - 4: Generate $\hat{\sigma}$ by sorting $\{g_{i\hat{\sigma}(m)}^t\}_{m \in [M]}$ such that $g_{i\hat{\sigma}(1)}^t \geq \dots \geq g_{i\hat{\sigma}(M)}^t$, where $g_{im}^t = \hat{z}_{im}^t$ if $m \notin I_i$ and $g_{im}^t = \hat{\gamma}_i^t - \hat{z}_{im}^t$ if $m \in I_i$.
 - 5: **if** (EC.13) is violated **then**
 - 6: Cuts \leftarrow (EC.13)
 - 7: **end if**
 - 8: Find the historical assortments $(\hat{w}^1, \dots, \hat{w}^M) = (\hat{z}_{i1}^t / \hat{\gamma}_i^t, \dots, \hat{z}_{iM}^t / \hat{\gamma}_i^t)$
 - 9: **if** (EC.14) is violated **then**
 - 10: Cuts \leftarrow (EC.14)
 - 11: **end if**
 - 12: **end for**
 - 13: Output: Cuts
-

Algorithm 5 A cutting-plane algorithm for (CONIC) with large memory length

-
- 1: Input: Parameters $\mathbf{r}, \{\beta_i^0\}_{i \in [N]}, \boldsymbol{\beta}, M, N, T, \{\Omega_{it}^1, W_{it}^1\}_{i \in [N], t \in [T]}$, and K
 - 2: $k \leftarrow 1$
 - 3: **while** $k \leq K$ **do**
 - 4: $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$ Solve (RLX $_k$)
 - 5: Cuts \leftarrow Call the separation oracle, Algorithm 4, to separate $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}, \hat{\mathbf{z}})$
 - 6: RLX $_{k+1} \leftarrow$ Add Cuts into RLX $_k$
 - 7: $k \leftarrow k + 1$
 - 8: **end while**
 - 9: Add binary constraints to the final RLX $_K$ and solve it through the Callback routine of Gurobi to call the separation oracle Algorithm 4
 - 10: Output: An optimal integer solution $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\rho}^*, \boldsymbol{\gamma}^*, \mathbf{z}^*)$
-

constraint for each period:

$$\sum_{i \in [N]} x_i^t \leq C \quad \text{for } t \in [T]. \quad (\text{EC.15})$$

That is, the number of offered products in each period is no more than C , a positive integer. For the cross-period constraint, we consider that the number of times provided for a product $i \in [N]$ shall not exceed K . That is,

$$\sum_{t \in [T]} x_i^t \leq K \quad \text{for } i \in [N]. \quad (\text{EC.16})$$

We focus on the satiation effects and generate parameters as follows. The revenue and base utility of a product are randomly generated from uniform distributions with ranges of $[1, 10]$ and $[-1, 1]$, respectively. The history-dependent effect β_i^1 is uniformly sampled from $[-1, 0]$ for each $i \in [N]$. We fix $N = 30, T = 5$ and $M = 1$. We generate 5 synthetic instances for each configuration of $C \in \{5, 10, 15, 30\}$ and $K \in \{2, 3, 4, 5\}$. We use the MILP model, Env, to solve each instance within 3600 seconds and with an optimality gap 0.5%.

Table EC.2 Computation time, end gap, and root gap of Env formulation under weak satiation effects with cardinality constraint (EC.15) and offering constraint (EC.16) ($N = 30, T = 5, M = 1$)

C	K	#sol	$T_{\text{opt}(s)}$	$G_{\text{end}(\%)}$	$G_{\text{root}(\%)}$	C	K	#sol	$T_{\text{opt}(s)}$	$G_{\text{end}(\%)}$	$G_{\text{root}(\%)}$
5	2	1	801.20	2.15	13.58	15	2	1	796.22	1.93	13.23
5	3	4	509.65	1.73	7.49	15	3	4	385.27	1.46	7.03
5	4	4	369.20	1.28	5.81	15	4	4	207.41	0.86	4.86
5	5	4	139.18	1.18	4.41	15	5	5	86.74	0	3.26
10	2	1	960.74	1.96	13.22	30	2	1	871.66	1.86	13.22
10	3	4	339.98	1.42	7.03	30	3	4	439.71	1.44	7.03
10	4	4	187.44	0.71	4.86	30	4	4	214.48	0.93	4.86
10	5	5	82.12	0	3.27	30	5	5	89.89	0	3.26

Table EC.2 summarizes the computation performance of Env under constraints of (EC.15) and (EC.16). The first two columns are the limitations of cardinality size and offering times, respectively. The third column #sol indicates the number of instances solved to optimality within 3600 seconds. T_{opt} is the average computation time of solved instances and G_{end} is the average end gap of unsolved instances. The last column G_{root} is the average root gap of all instances. Table EC.2 shows that our formulation is more sensitive to the cross-period offering constraint (EC.16) than the cross-product cardinality constraint (EC.15). For instance, under the same offering times K , the average computation time, end gap, and root gap are stable under different cardinality sizes C . However, under the same C , if we increase the offering times K from 2 to 4, the average computation time, end gap, and root gap shrink by more than one-half. It is because our formulation does not characterize the problem structure in the space of cross-period choice probabilities, (y_i^t, y_i^{t+1}) . Although not all instances are solved to optimality, the end gap is no more than 2.5%. It indicates the robustness and efficiency of our formulation.

EC.6.2. Performance of Instances with Large Memory Length

This numerical study demonstrates that our (CONIC) formulation, combined with the cutting-plane algorithm, can solve instances with moderately large memory length.

We fix the number of products $N = 20$ and planning horizon $T = M + 1$ for $M \in \{4, 5, 6\}$. We consider both addiction and satiation effects. Specifically, a product has an addiction (resp. satiation) effect with probability θ (resp. $1 - \theta$), where $\theta \in \{0, 0.1, 0.2\}$. For products with addiction (resp. satiation) effects, its $\{\beta_i^m\}_{m \in [M]}$ are uniformly sampled from $[0, 1]$ (resp. $[-1, 0]$). For each configuration of M, T , and θ , we randomly generate 10 instances. The revenue and the base utility of a product are from uniform distributions with ranges of $[1, 10]$ and $[-1, 1]$, respectively. To initialize the cutting-plane algorithm, we set Ω_{it}^1 by randomly generating 2, 10, and 20 different permutations over $[M]$ for memory lengths of 4, 5, and 6, respectively. We initialize W_{it}^1 by adding $\mathbf{w} \in \{0, 1\}^M$ satisfying $\|\mathbf{w}\|_2^2 \in \{0, 1, 2, M\}$. When computing the starting model for `Callback`, we do not fix K rounds to add cuts. Instead, we call the separation oracle until the reduction of the objective value of (RLX_k) is less than a small constant ϵ . We set $\epsilon = 10^{-8}$ in the numerical study.

Table EC.3 Average computation time of solved instances and average end gap of unsolved instances

M	T	θ	#sol	Time (s)	$G_{\text{end}}(\%)$
4	5	0	10	317.13	0
4	5	0.1	10	2.94	0
4	5	0.2	10	17.00	0
5	6	0	9	273.89	1.6
5	6	0.1	9	1632.32	3.19
5	6	0.2	10	170.72	0
6	7	0	6	3416.57	2.96
6	7	0.1	8	963.53	3.8
6	7	0.2	7	18.16	2.75

Table EC.3 summarizes the computation results, where we set a time limit of 7200 seconds or an optimality gap tolerance of 0.5%. The first three columns show the memory length, planning horizon, and the proportion of products with positive effects. The last three columns present the number of solved instances, the average time of solved instances (including set-up time of the starting model and `Callback` time), and the average end gap of unsolved instances. Table EC.3 shows that the cutting-plane algorithm can solve most instances to optimality within two hours, even when $M = 6$. For instance, when $M = 6$ and $\theta = 0.2$, 7 instances are solved to optimality using only 18.16 seconds on average. While some instances cannot achieve optimality within two hours, their end gaps are small and no more than 4%.

EC.6.3. Performance of $(M + 1)$ -cyclic Policy

In this numerical study, we evaluate the $(M + 1)$ -cyclic policy on synthetic data and compare its performance with Sequential – RO and Sequential – LOSPO. We fix the number of products $N = 30$ and vary the memory length $M \in \{1, 2\}$ and planning horizon $T \in \{10, 30, 50\}$. We generate product revenue, base utility, and history-dependent effects in the same way as the numerical study in Section 6.1 and only consider strong satiation effects. For each combination of M and T , we generate five instances.

Table EC.4 shows the average revenue gap of the $(M + 1)$ -cyclic policy and two heuristics against the best feasible solution obtained by solving the envelope-based formulation, ENV, in 7200 seconds. In general, the $(M + 1)$ -cyclic policy has a very small revenue loss, which is no more than 1% in most cases and about only 2% of revenue losses of the two heuristics. The revenue gap can even be negative, indicating that the cyclic policy can outperform the best feasible solution obtained within the time limit. Moreover, the revenue gap shrinks when T increases, which matches the asymptotical optimality of the $(M + 1)$ -cyclic policy.

Table EC.4 Revenue gap (%) of $(M + 1)$ -cyclic policy and heuristics
($N = 30$)

M	T	$(M + 1)$ -cyclic	Sequential – RO	Sequential – LOSPO
1	10	0.5	12.79	11.42
1	30	0.01	14.51	13.16
1	50	-0.08	14.86	13.51
2	10	1.03	27.63	26.12
2	30	-0.48	35.94	34.79
2	50	-0.68	37.74	36.67