

E-companion for “Managing Newsvendors via Demand Shifting”

We provide six appendices in this e-companion. Section A describes the full model in which the platform specifies how to shift demand between any two sellers under any system state, and the mean-field inventory equilibrium induced by it. We establish that restricting attention to the base model where the platform only considers state-independent post-shift demand and Dirac inventory response is WLOG. Section B proves the results in Section A. Section C proves the optimal policy and equilibrium in Section 3. We prove the other results stated in Section 4–5 in Section D. Finally, we discuss four extensions in Section E and provide their proof in Section F.

A. Full Model

In this section, we formalize the complete model in which the platform commits to a fully specified, state-dependent demand shifting policy, and the sellers determine their inventory levels in response to both the platform’s policy and the actions of other sellers. We then state a generalized version of Theorem 1. For brevity, we do not restate the reduced-form model here (see Section 2), but instead introduce the additional components required for the complete model. The continuum of sellers’ profiles (d, q) induces a unit measure over $\mathbb{R}_+^2$. In particular, this is a product measure since the sellers’ inventory decisions are independent of the realized demand. Let μ and ν denote its two marginals, respectively. By the Glivenko–Cantelli theorem, μ coincides with the probability distribution of D . The system’s state can thus be solely represented by ν , the measure formed by the sellers’ inventory decisions. Let $\mathcal{M}$ denote the set of all unit (probability) measures defined on $\mathbb{R}_+$ and the natural σ -algebra. $\mathcal{M}$ is our state space.

The fully specified demand shifting policy A fully specified demand shifting policy ϕ_F is formally defined as a collection of mappings. Given the system’s state ν and any pair of sellers with initial demands $d_i(d_j)$ and inventory levels $q_i(q_j)$ respectively, the mapping $\phi_F^\nu : (d_i, q_i; d_j, q_j) \rightarrow \mathbb{R}_+$ represents the amount of demand originally intended for seller i that is reallocated to seller j . The amount of demand received by seller i under ϕ_F is

$$\phi^\nu(d_i, q_i) := d_i - \iint_{\mathbb{R}_+^2} \phi_F^\nu(d_i, q_i; d_j, q_j) dF(d_i) d\nu(q_j) + \iint_{\mathbb{R}_+^2} \phi_F^\nu(d_j, q_j; d_i, q_i) dF(d_i) d\nu(q_j). \quad (7)$$

In line with Definition 1, we introduce the following admissibility condition.

DEFINITION 3 (ADMISSIBLE POLICY (FULLY SPECIFIED)). ϕ_F is admissible if the induced post-shift demand ϕ (7) satisfies the following conditions. For any system's state ν , $\phi^\nu(\cdot, \cdot) \geq 0$, and

- (i) (FLOW BALANCE) For any $q \geq 0$, $\iint_{\mathbb{R}_+^2} \phi^\nu(d, q) dF(d) d\nu(q) = \mathbb{E}[D] = \int_{\mathbb{R}_+} ddF(d)$.
- (ii) (SHIFTING FEASIBILITY) For any $d, q \geq 0$, $\phi^\nu(d, q) > 0$ implies $\phi^\nu(d, q) \leq q$.

We denote by Φ_F the set of all admissible demand shifting policies. We further denote Φ the set of $\phi := \{\phi^\nu\}$ (see (7)), that is, the post-shift demand induced by an admissible policy $\phi_F \in \Phi_F$. In Definition 3, the admissibility of ϕ_F fully relies on conditions on ϕ . The next lemma formalizes that it suffices to specify the post-shift demand ϕ .

LEMMA 1. *It is without loss of optimality to consider the post-shift demands ϕ instead of the fully specified ϕ_F .*

The proof is provided in Section B.1, which is essentially done by construction. We therefore adopt this simplification in what follows, and (with slight abuse of notation) refer to ϕ as the demand shifting policy.

The seller's inventory problem and the mean-field game Analogous to the reduced-form model, given any system state ν and demand shifting policy ϕ , a representative seller solves the following profit-maximization problem:

$$\max_{q \geq 0} \{r_s \mathbb{E}_{D \sim F} [\phi^\nu(D, q) \wedge q] - c_s q\} = \max_{q \geq 0} \left\{ r_s \int_{\mathbb{R}_+} (\phi^\nu(d, q) \wedge q) dF(d) - c_s q \right\}, \quad (8)$$

Let $\mathcal{S}^{\phi, \nu}$ denote the set of optimal solutions to (8). Here the seller's inventory decision is state-dependent, leading to a more complicated system as compared to the reduced-form model. We adopt *mean-field equilibrium* as our solution concept for describing the sellers' emergent aggregate behavior in the system.

DEFINITION 4 (MEAN-FIELD EQUILIBRIUM). State ν is a mean-field equilibrium with respect to demand shifting policy $\phi \in \Phi$, if $\mathcal{S}^{\phi, \nu}$, the set of optimal solutions to the seller's profit optimization problem (8) contains the support of ν as a subset. Denote by $\mathcal{B}^\phi$ the set of all equilibria induced by ϕ .

In words, given the equilibrium state ν , the representative seller solves (8). In addition, there exists a resulting seller's (possibly randomized) optimal strategy that collectively induce ν as the system's aggregate state. Under the conditions of Definition 4, a straightforward optimal strategy of such for each seller is to directly implement ν as their strategy. This notion of equilibrium admits

the largest class of system's state consistent with individual optimality. More stringent equilibrium notions that account for the seller's behavioral limitations (e.g. lack of coordination) can also be considered, under which the set of equilibria is smaller and our theoretical results remain the same.

Denote by $v_s^{\phi, \nu} := \max_{q \geq 0} \{r_s \mathbb{E}[\phi^\nu(D, q) \wedge q] - c_s q\}$ the seller's optimal profit under policy ϕ and at state $\nu \in \mathcal{M}$, slightly generalizing the analogous notation introduced in Section 2.

The platform's demand shifting problem The platform's expected profit under a policy $\phi \in \Phi$ and a corresponding state $\nu \in \mathcal{M}$ is

$$v_p^{\phi, \nu} := r_p \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) \wedge q) dF(d) d\nu(q) - c_p \int_{\mathbb{R}_+} q d\nu(q) - \beta \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) - d)^+ dF(d) d\nu(q). \quad (9)$$

Comparing to (3), the platform's objective introduced in the reduced-form model, the only change in (9) is an additional integration with respect to the system's state ν . Analogous to (3), the platform solves

$$\sup_{\phi \in \Phi} \left\{ \sup_{\nu \in \mathcal{B}^\phi} v_p^{\phi, \nu} \right\} \quad s.t. \quad v_s^{\phi, \nu} \geq v_s^0. \quad (10)$$

In words, the platform, acting as a Stackelberg leader, selects a demand shifting policy ϕ together with the corresponding mean-field equilibrium ν to maximize her expected profit, subject to an individual rationality (IR) constraint ensuring that each seller is better off participating in the demand shifting program than opting out.

Connection to the reduced-form model The reduced-form model introduced in Section 2 can be viewed as a restricted version of the general model presented here. In particular, the state-independent policy class Π (see Definition 1) is a subset of Φ . Under $\pi \in \Pi$, the set of mean-field equilibria, as defined in Definition 4, reduces to the set of all measures supported on $\mathcal{S}^\pi$, the set of (state-independent) optimal solutions to the seller's problem. In the reduced-form model, we further simplify the problem by omitting all non-Dirac systems state. Thus, one may interpret the reduced-form model as one in which the platform can only choose state-independent policies and Dirac equilibria. However, this limitation does not in fact restrict the platform's ability. The optimal demand shifting policy turns out to be state-independent and retains the same defluctuating structure as characterized in Theorem 1, even under the general model. And it is without loss of optimality to only consider Dirac equilibria. We formalize the above in the next theorem, which is our main result in the full model. For any $a \in \mathbb{R}$, let $\delta(a)$ denote the Dirac measure at a .

THEOREM 3. *There exists a state-independent policy $\pi^* \in \Pi$ and $q^* \geq 0$, such that $(\pi^*, \delta(q^*))$ optimizes the platform's problem (10) in the full model. In particular, π^*, q^* coincide with the optimal policy and the seller's best-case (w.r.t. the platform) inventory response in the reduced-form model as appearing in Theorem 1 and Theorem 2 and specified in Definition 6–8 in Appendix C.1.*

B. Proof in Appendix A

There are two results in Section A, Lemma 1 and Theorem 3. We prove the former in Section B.1 and the latter in Section B.2.

B.1. Proof of Lemma 1

We provide the proof of Lemma 1 in this section. We first introduce the concept of shifting efficiency of a policy. For any seller, an efficient shifting policy either only shifts demand from other sellers into him, or only shifts that seller's demand to other sellers, but can not do both.

DEFINITION 5 (EFFICIENT POLICY). A policy ϕ_F is efficient if for any v and any $d, q \geq 0$,

$$\iint_{\mathbb{R}_+^2} \phi_F^v(d, q; d', q') dF(d) dv(q) \cdot \iint_{\mathbb{R}_+^2} \phi_F^v(d', q'; d, q) dF(d) dv(q) = 0$$

We proceed by proving a series of lemmas. First we show that any ϕ satisfying the conditions in Definition 3 can be realized by an efficient, admissible policy $\phi_F \in \Phi_F$.

LEMMA 2. *Any ϕ that satisfies the conditions of Definition 3 is the post-shift demand induced by some efficient admissible policy $\phi_F \in \Phi_F$.*

Proof of Lemma 2 We prove by construction. By the flow balance condition in Definition 3,

$$\iint_{\mathbb{R}_+^2} (\phi^v(d, q) - d)^+ dF(d) dv(q) = \iint_{\mathbb{R}_+^2} (d - \phi^v(d, q))^+ dF(d) dv(q).$$

Our construction is as follows. If $\iint_{\mathbb{R}_+^2} (\phi^v(d, q) - d)^+ dF(d) dv(q) = \iint_{\mathbb{R}_+^2} (d - \phi^v(d, q))^+ dF(d) dv(q) = 0$, we set $\phi_F^v(d, q; d', q') = 0$ for all $d, q; d', q'$. Otherwise, we let

$$\phi_F^v(d, q; d', q') = \frac{(d - \phi^v(d, q))^+ \cdot (\phi^v(d', q') - d')^+}{\iint_{\mathbb{R}_+^2} (\phi^v(d_0, q_0) - d_0)^+ dF(d_0) dv(q_0)}.$$

In other words, $\phi_F^v(d, q; d', q')$ is non-zero only when $\phi^v(d, q) < d$ and $\phi^v(d', q') > d'$. By definition, ϕ_F^v is non-negative and measurable (following from the measurability of ϕ^v .) We next show that ϕ_F indeed induces ϕ as its post-shift demand.

We first consider the case $\iint_{\mathbb{R}_+^2} (\phi^v(d, q) - d)^+ dF(d) d\nu(q) = \iint_{\mathbb{R}_+^2} (d - \phi^v(d, q))^+ dF(d) d\nu(q) = 0$. In this case, $\phi^v(d, q) - d \leq 0$ and simultaneously $d - \phi^v(d, q) \leq 0$, *a.e.* (w.r.t. F and ν). Hence $\phi^v(d, q) = d$, *a.e.* The ϕ_F^v -induced demand at any (d, q) equals $\phi^v(d, q)$, both equal to d . Next we consider the case $\iint_{\mathbb{R}_+^2} (\phi^v(d, q) - d)^+ dF(d) d\nu(q) = \iint_{\mathbb{R}_+^2} (d - \phi^v(d, q))^+ dF(d) d\nu(q) > 0$. For any d and q such that $\phi^v(d, q) < d$, we have

$$\begin{aligned} & \iint_{\mathbb{R}_+^2} \phi_F^v(d, q; d', q') dF(d) d\nu(q) \\ &= \frac{(d - \phi^v(d, q))^+}{\iint_{\mathbb{R}_+^2} (\phi^v(d_0, q_0) - d_0)^+ dF(d_0) d\nu(q_0)} \iint_{\mathbb{R}_+^2} (\phi^v(d', q') - d')^+ dF(d') d\nu(q'), \\ &= \frac{d - \phi^v(d, q)}{\iint_{\mathbb{R}_+^2} (\phi^v(d_0, q_0) - d_0)^+ dF(d_0) d\nu(q_0)} \iint_{\mathbb{R}_+^2} (\phi^v(d', q') - d')^+ dF(d') d\nu(q') \\ &= d - \phi^v(d, q), \end{aligned}$$

while $\iint_{\mathbb{R}_+^2} \phi_F^v(d', q'; d, q) dF(d') d\nu(q') = 0$. We thus have

$$\begin{aligned} & d - \iint_{\mathbb{R}_+^2} \phi_F^v(d, q; d', q') dF(d') d\nu(q') + \iint_{\mathbb{R}_+^2} \phi_F^v(d', q'; d, q) dF(d') d\nu(q') \\ &= d - d + \phi^v(d, q) = \phi^v(d, q). \end{aligned}$$

Now for (d, q) such that $\phi^v(d, q) > d$, similarly we can show that $\iint_{\mathbb{R}_+^2} \phi_F^v(d, q; d', q') dF(d') d\nu(q') = 0$ and $\iint_{\mathbb{R}_+^2} \phi_F^v(d', q'; d, q) dF(d') d\nu(q') = \phi^v(d, q) - d$, which implies that the post-shift demand of ϕ_F^v coincides with $\phi^v(d, q)$.

Finally, the efficiency of ϕ_F follows immediately from its construction and the definition of efficiency (Definition 5), thus completing the proof. $\square$

We next show that for efficient policies, both the seller's objective and the platform's objective can be represented as a function of ϕ .

LEMMA 3. *Suppose ϕ_F is efficient and induces post-shift demand ϕ . Then the seller's expected profit and the platform's expected profit under ϕ_F and a system's state ν can both be represented in terms of ϕ and ν .*

Proof The seller's objective is by definition a function of the post-shift demand, more specifically, the newsvendor-type objective, as given in (8). It remains to prove that the platform's objective can also be represented as a function of ϕ and ν . The expected revenue and the inventory cost are clearly both functions of ϕ and ν . It suffices to clarify that the shifting cost can also be represented

as a function of ϕ and ν under an efficient policy. By definition, at system's state ν , policy ϕ_F generates a shifting cost of

$$\beta \iint_{\mathbb{R}_+^2} \iint_{\mathbb{R}_+^2} \phi_F^\nu(d, q; d', q') dF(d) d\nu(q) dF(d') d\nu(q').$$

As ϕ_F is efficient, for a fixed pair of (d, q) , we have

$$\iint_{\mathbb{R}_+^2} \phi_F^\nu(d, q; d', q') dF(d') d\nu(q') = (d - \phi^\nu(d, q))^+.$$

Thus the shifting cost can be reformulated in terms of ϕ^ν as

$$\beta \iint_{\mathbb{R}_+^2} (d - \phi^\nu(d, q))^+ dF(d) d\nu(q) = \frac{\beta}{2} \iint_{\mathbb{R}_+^2} |\phi^\nu(d, q) - d| dF(d) d\nu(q),$$

where the equality holds due to flow balance. Combining the above, the platform's objective can also be represented as a function of ϕ and ν , as in (9), completing the proof. $\square$

Lemma 3 implies that at least for efficient policies, it suffices to specify the post-shift demand, as both the seller's objective and the platform's objective only depend on the post-shift demand. Lemma 2 then ensures the existence of an efficient policy that induces the specified post-shift demand. Next we prove that restricting to efficient policies is without loss of optimality.

LEMMA 4. *It suffices to consider efficient policies.*

Proof For any admissible policy ϕ_F that is not efficient, we construct an admissible efficient policy ϕ'_F that achieves larger platform profit. A policy ϕ_F is not efficient if there exists ν and d, q ,

$$\iint_{\mathbb{R}_+^2} \phi_F^\nu(d', q'; d, q) dF(d) d\nu(q) > 0, \quad \iint_{\mathbb{R}_+^2} \phi_F^\nu(d, q; d', q') dF(d) d\nu(q) > 0.$$

We denote the post-shift demand induced by ϕ_F by ϕ . We construct an efficient policy ϕ'_F as in the proof of Lemma 1, so that ϕ'_F also induces ϕ . We next prove that ϕ'_F is at least as good as ϕ_F in terms of the platform's profit. Indeed, by Lemma 3, the profit achieved by ϕ'_F , as an efficient policy, can be stated in terms of the induced post-shift demand ϕ . Then to show that ϕ'_F is a better policy than ϕ_F , it suffices to compare the substitution cost, in particular,

$$\iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) - d)^+ dF(d) d\nu(q) \leq \iint_{\mathbb{R}_+^2} \iint_{\mathbb{R}_+^2} \phi_F^\nu(d, q; d', q') dF(d) d\nu(q) dF(d') d\nu(q').$$

By definition of ϕ , we have

$$\iint_{\mathbb{R}_+^2} \phi_F^\nu(d', q'; d, q) dF(d) d\nu(q) = \phi^\nu(d, q) - d + \iint_{\mathbb{R}_+^2} \phi_F^\nu(d, q; d', q') dF(d') d\nu(q'),$$

from which it follows directly by the non-negativity of ϕ_F that

$$\iint_{\mathbb{R}_+^2} \phi_F^v(d', q'; d, q) dF(d) dv(q) \geq (\phi^v(d, q) - d)^+.$$

Integrating over d, q with respect to distribution F and system's state measure v on both sides yields the desired bound. $\square$

Combining the above lemmas, we conclude that it is indeed without loss of optimality to consider the reduced-form specification of post-shift demand, proving Lemma 1.

B.2. Proof of Theorem 3

We prove Theorem 3 in two key steps. First, we show the optimality of state-independent policies and Dirac mean-field equilibria in a relaxed version of problem (10) that does not require shifting feasibility. In particular, denote by $\bar{\Phi}$ a policy class that includes Φ , yet without requiring the shifting feasibility condition. Similarly, denote by $\bar{\Pi}$ a policy class that includes Π but without requiring shifting feasibility. Consider the following relaxed version of (10):

$$\sup_{\phi \in \bar{\Phi}} \left\{ \sup_{v \in \mathcal{B}^\phi} v_p^\phi \right\} \quad \text{s.t. } v_s^{\phi, v} \geq v_s^0. \quad (11)$$

we show that for any general policy $\phi \in \bar{\Phi}$ and any corresponding mean-field equilibrium v that solves (11), there exists a state-independent policy $\pi \in \bar{\Pi}$ and an equilibrium system state that is Dirac at a point $\bar{q}$, that also satisfies IR, and is better from both the seller's and the platform's perspective. This is stated and proved in Lemma 5 below.

We then proceed the proof of Theorem 3 by showing that an optimal state-independent policy and Dirac inventory response that solves (11) is given by Theorem 1–2 and specified in Definition 6–8 in Appendix C.1, which satisfies shifting feasibility, hence is also an optimal solution to the original problem (10).

LEMMA 5. *Consider any policy $\phi \in \bar{\Phi}$ and a mean-field equilibrium v under ϕ . Suppose $v_s^{\phi, v} \geq v_s^0$. Then there must exist a state-independent policy $\pi \in \bar{\Pi}$ and a Dirac mean-field equilibrium at $\bar{q}$, under which v_s^π and v_p^π are the corresponding expected profit of a seller and the platform, and that they satisfy $v_s^\pi \geq v_s^{\phi, v}$ and $v_p^\pi \geq v_p^{\phi, v}$.*

Proof Let $\bar{q}$ denote the mean of measure v . Namely,

$$\bar{q} := \int_{\mathbb{R}_+} q dv(q)$$

Define a state-independent policy

$$\pi(d, q) = \begin{cases} \int_{\mathbb{R}_+} \phi^\nu(d, q') d\nu(q') & \text{if } q = \bar{q}, \\ d & \text{otherwise.} \end{cases}$$

We now verify that $\pi \in \bar{\Pi}$, $\bar{q}$ is a seller's optimal solution under π , and that $v_s^\pi \geq v_s^{\phi, \nu}$ and $v_p^\pi \geq v_p^{\phi, \nu}$.

First check the flow balance condition in Definition 1. Note that $\int_{\mathbb{R}_+} \pi(d, q) dF(d) = \mathbb{E}D$ if $q \neq \bar{q}$, and

$$\int_{\mathbb{R}_+} \pi(d, \bar{q}) dF(d) = \iint_{\mathbb{R}_+^2} \phi^\nu(d, q') dF(d) d\nu(q') = \mathbb{E}D,$$

where the first equality is by the definition of π and the second equality is by the admissibility of ϕ (Definition 3). Thus, π satisfies flow balance for any possible state measure, i.e., $\pi \in \bar{\Pi}$.

We now show that $v_s^\pi \geq v_s^{\phi, \nu}$ and that $\bar{q}$ is a seller's optimal inventory under π . By definition, $v_s^{\phi, \nu} = \max_{q \geq 0} \{r_s \mathbb{E}[\phi^\nu(D, q) \wedge q] - c_s q\}$, and the solution set to this problem must contain the support of ν since the latter is a mean-field equilibrium by Definition 4. Therefore, for any q in the support of ν , $r_s \mathbb{E}[\phi^\nu(D, q) \wedge q] - c_s q = v_s^{\phi, \nu}$, hence

$$\int_{\mathbb{R}_+} (r_s \mathbb{E}[\phi^\nu(D, q) \wedge q] - c_s q) d\nu(q) = v_s^{\phi, \nu}.$$

We show that $\bar{q}$ is an optimal solution to the seller's problem under π , i.e.,

$$v_s^\pi = \max_{q \geq 0} \{r_s \mathbb{E}[\pi(D, q) \wedge q] - c_s q\} = r_s \mathbb{E}[\pi(D, \bar{q}) \wedge \bar{q}] - c_s \bar{q}.$$

To this end, we compute

$$\begin{aligned} r_s \mathbb{E}[\pi(D, \bar{q}) \wedge \bar{q}] - c_s \bar{q} &= r_s \int_{\mathbb{R}_+} (\pi(d, \bar{q}) \wedge \bar{q}) dF(d) - c_s \int_{\mathbb{R}_+} q d\nu(q) \\ &= r_s \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \phi^\nu(d, q) d\nu(q) \right) \wedge \left(\int_{\mathbb{R}_+} q d\nu(q) \right) dF(d) - c_s \int_{\mathbb{R}_+} q d\nu(q) \\ &\geq r_s \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) \wedge q) dF(d) d\nu(q) - c_s \int_{\mathbb{R}_+} q d\nu(q) \\ &= v_s^{\phi, \nu}, \end{aligned}$$

where the inequality follows from the concavity of $\wedge$ and Jensen's inequality. While for $q \neq \bar{q}$, by definition of π , we have

$$r_s \mathbb{E}[\pi(D, q) \wedge q] - c_s q = r_s \mathbb{E}[D \wedge q] - c_s q \leq v_s^0.$$

Recall that $v_s^{\phi, \nu} \geq v_s^0$ by assumption. Combining the above bounds, $r_s \mathbb{E}[\pi(D, \bar{q}) \wedge \bar{q}] - c_s \bar{q} \geq v_s^0 \geq r_s \mathbb{E}[\pi(D, q) \wedge q] - c_s q$ for all $q \neq \bar{q}$. This implies that $\bar{q}$ is an optimal solution to the seller's problem under π , and that $v_s^\pi \geq v_s^{\phi, \nu}$.

It only remains to show $v_p^\pi \geq v_p^{\phi, \nu}$. Recall that $\bar{q}$ is an optimal inventory response under π . Hence by definition of the platform's objective as defined in (2), $v_p^\pi \geq r_p \int_{\mathbb{R}_+} (\pi(d, \bar{q}) \wedge \bar{q}) dF(d) - c_p \bar{q} - \beta \int_{\mathbb{R}_+} (\pi(d, \bar{q}) - d)^+ dF(d)$. Therefore, it suffices to show that

$$\begin{aligned} r_p \int_{\mathbb{R}_+} (\pi(d, \bar{q}) \wedge \bar{q}) dF(d) - c_p \bar{q} - \beta \int_{\mathbb{R}_+} (\pi(d, \bar{q}) - d)^+ dF(d) &\geq v_p^{\phi, \nu} \\ &= r_p \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) \wedge \bar{q}) dF(d) d\nu(q) - c_p \int_{\mathbb{R}_+} q d\nu(q) - \beta \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) - d)^+ dF(d) d\nu(q). \end{aligned}$$

By a similar argument above, one can verify that

$$r_p \int_{\mathbb{R}_+} (\pi(d, \bar{q}) \wedge \bar{q}) dF(d) - c_p \bar{q} \geq r_p \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) \wedge \bar{q}) dF(d) d\nu(q) - c_p \int_{\mathbb{R}_+} q d\nu(q).$$

Therefore it suffices to show that $\beta \int_{\mathbb{R}_+} (\pi(d, \bar{q}) - d)^+ dF(d) \leq \beta \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) - d)^+ dF(d) d\nu(q)$.

Note that

$$\begin{aligned} \int_{\mathbb{R}_+} (\pi(d, \bar{q}) - d)^+ dF(d) &= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \phi^\nu(d, q) d\nu(q) - d \right)^+ dF(d) \\ &\leq \iint_{\mathbb{R}_+^2} (\phi^\nu(d, q) - d)^+ d\nu(q) dF(d), \end{aligned}$$

where the inequality is again due to Jensen's inequality and the convexity of $(\phi^\nu(d, q) - d)^+$ in $\phi^\nu(d, q)$. We thus have the desired result. $\square$

With Lemma 5, we can proceed to prove Theorem 3.

Proof of Theorem 3 We prove a stronger version of the theorem, that the feasible pair (π^*, q^*) is an optimal solution even to the relaxed problem (11) where shifting feasibility is not enforced. By Lemma 5, for any pair of $\phi \in \bar{\Phi}$ and $\nu \in \mathcal{B}^\phi$ that the IR constraint holds ($v_s^{\phi, \nu} \geq v_s^0$), there exists $\pi \in \bar{\Pi}$ and $\bar{q} \in \mathcal{S}^\pi$, such that the equilibrium $(\pi, \bar{q})$ is both better and feasible. That is, $v_p^\pi \geq v_p^{\phi, \nu}$, i.e., the platform objective value increases, and that $v_s^\pi \geq v_s^{\phi, \nu} \geq v_s^0$, i.e., IR holds. The relaxed platform's problem (11) can thus be reduced to a problem that optimizes over state-independent policies and Dirac equilibria ((12) in Appendix C.2). The rest of the proof largely coincides with that of Theorem 1–2 in Appendix C. In particular, there we show (π^*, q^*) is optimal to (12). Thus, (π^*, q^*) must also be optimal to (11). Combining the above completes the proof. $\square$

C. Optimal Policy and Proof in Section 3

In this section, we provide a detailed characterization of π^* in Theorem 1 and Theorem 2 in Section 3, and prove the theorems. Recall that $q^s := F^{-1}\left(1 - \frac{c_s}{r_s}\right)$, $q^p := F^{-1}\left(1 - \frac{c_p}{r_p}\right)$. We further introduce the following notation.

$$q^{\text{QD}} := F^{-1}\left(1 - \frac{c_p}{\beta}\right), \quad q^{\text{ED}} := F^{-1}\left(\frac{r_p - c_p}{\beta}\right).$$

Note that $q^{\text{QD}} < q^{\text{ED}}$ since $\beta < r_p$.

C.1. Optimal Defluctuating Policy

Now we specify policy π^* in Theorem 1.

DEFINITION 6 (INTERVAL-DEFLUCTUATING FUNCTIONS). Given a fixed random variable X with continuous and strictly increasing *c.d.f.* defined on $\mathbb{R}_+$, and $0 \leq a < b \leq +\infty$, we say a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ *defluctuates* the interval (a, b) with respect to X , if for each $c \in (a, \mathbb{E}[X|a < X < b])$,

$$g(x, c) = \begin{cases} x & x \notin (a, b_c), \\ c & x \in (a, b_c), \end{cases}$$

where $b_c \in (c, b)$ is the unique solution to the equation $\mathbb{E}[X|a < X < b_c] = c$. And for each $c \in [\mathbb{E}[X|a < X < b], b)$,

$$g(x, c) = \begin{cases} x & x \notin (a_c, b), \\ c & x \in (a_c, b), \end{cases}$$

where $a_c \in (a, c)$ is the unique solution to the equation $\mathbb{E}[X|a_c < X < b] = c$. For $c \notin (a, b)$, $g(x, c) = x$.

DEFINITION 7 (θ -DEFLUCTUATING POLICY). A demand shifting policy $\pi \in \Pi$ is called θ -defluctuating for a fixed $\theta > 0$, if π defluctuates $(0, L^{-1}(\theta))$ in the case $\theta < \mathbb{E}[D]$, and π defluctuates $(U^{-1}(\theta), \infty)$ in the case $\theta \geq \mathbb{E}[D]$, both with respect to $D \sim F$, where we recall $L(q) = \mathbb{E}(D|D \leq q)$ and $U(q) = \mathbb{E}(D|D > q)$.

We note that the θ -defluctuating policy of Definition 7 is a special case of the general defluctuating policy given in Definition 2. We state and prove the admissibility of defluctuating policies.

LEMMA 6. *A defluctuating policy is admissible.*

Proof Denote by π the defluctuating policy. It suffices to prove that π is a measurable, non-negative-valued function satisfying the conditions in Definition 1. By definition of a defluctuating policy, for any d, q , $\pi(d, q)$ equals either d or q , hence is non-negative. π is by definition clearly measurable. Next, we verify it satisfies the flow balance condition. By definition of the defluctuating policy, for any $q \geq 0$, there exists an interval $I(q)$, such that $\pi(d, q) = d$ for $d \in \mathbb{R}_+ \setminus I(q)$ and $\pi(d, q) = q$ for $d \in I(q)$. Therefore,

$$\begin{aligned} \mathbb{E}[\pi(D, q)] &= \int_{\mathbb{R}_+} \pi(d, q) dF(d), \\ &= \int_{I(q)} \pi(d, q) dF(d) + \int_{\mathbb{R}_+ \setminus I(q)} \pi(d, q) dF(d), \\ &= \int_{I(q)} q dF(d) + \int_{\mathbb{R}_+ \setminus I(q)} d dF(d), \\ &= \int_{I(q)} d dF(d) + \int_{\mathbb{R}_+ \setminus I(q)} d dF(d), \\ &= \int_{\mathbb{R}_+} d dF(d) = \mathbb{E}D, \end{aligned}$$

where the second to the last equality holds due to the property of a θ -defluctuating policy, particularly, $\mathbb{E}[D|D \in I(q)] = q$.

Finally, we verify that defluctuating policies satisfy the shifting feasibility condition, namely $\pi(d, q) > d$ implies $\pi(d, q) \leq q$. Indeed, recall that for any d, q , $\pi(d, q)$ is equal to either q or d . Therefore, when $\pi(d, q) > d$ it has to be equal to q . $\square$

Recall

$$q^* := \begin{cases} \mathbb{E}D \vee (q^{\text{QD}} \wedge U(q^s)) & \text{if } \mathbb{E}D \leq q^p, \\ \mathbb{E}D \wedge (q^{\text{ED}} \vee L(q^s)) & \text{if } \mathbb{E}D > q^p. \end{cases}$$

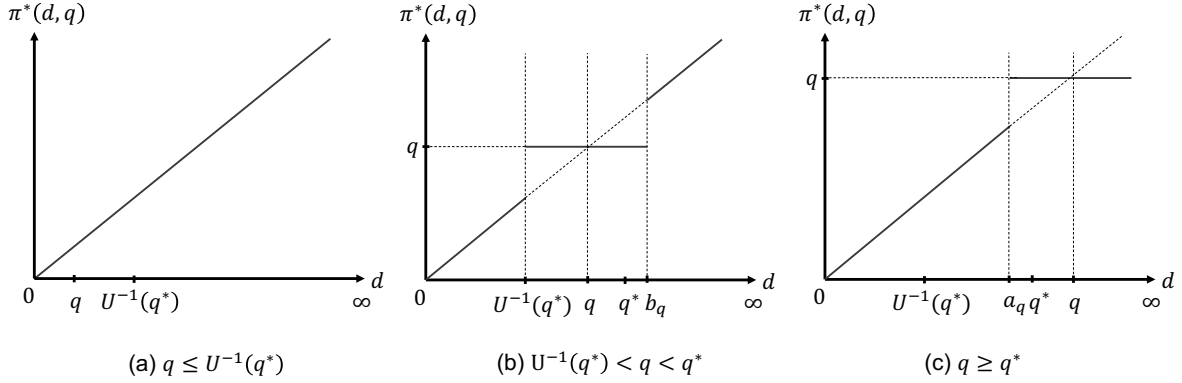
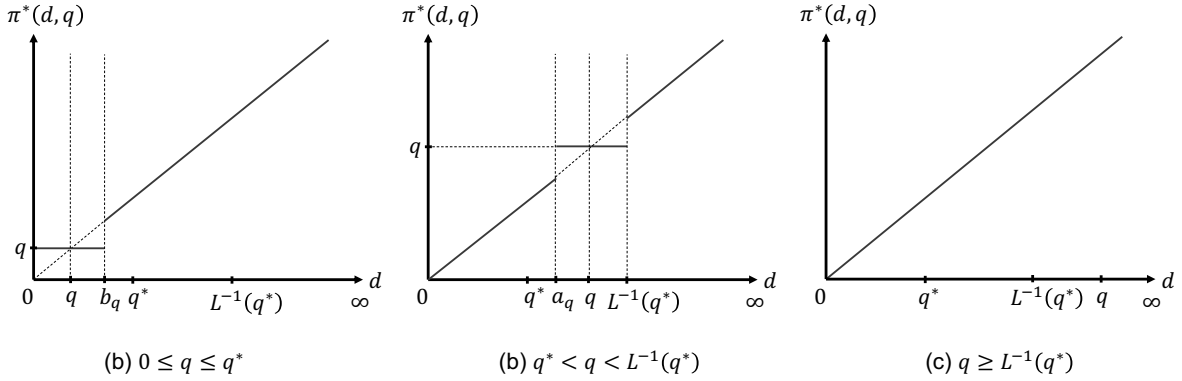
Following the terminology introduced in Definition 6–7, the complete specification of the optimal policy π^* has the following simple form. We also graphically illustrate π^* for the off-path in Figure 6–7, which correspond to the case of $\mathbb{E}D \leq q^p$ and $\mathbb{E}D > q^p$, respectively.

DEFINITION 8. π^* is q^* -defluctuating.

C.2. Proof of Theorem 1 and Theorem 2

Recall the platform's problem in (3). We have introduced notation v_p^* to denote its optimal value:

$$v_p^* := \sup_{\pi \in \Pi} v_p^\pi \quad \text{s.t. } v_s^\pi \geq v_s^0,$$

Figure 6 $\pi^*(d, q)$ versus d under the platform-optimal policy π^* in Definition 8. Suppose $\mathbb{E}D \leq q^P$.**Figure 7** $\pi^*(d, q)$ versus d under the platform-optimal policy π^* in Definition 8. Suppose $\mathbb{E}D > q^P$.

where we recall v_p^π is defined in (2) as

$$v_p^\pi = \max_{q \in \mathcal{S}^\pi} \left\{ r_p \int_{\mathbb{R}_+} (\pi(d, q) \wedge q) dF(d) - c_p q - \beta \int_{\mathbb{R}_+} (\pi(d, q) - d)^+ dF(d) \right\}$$

To facilitate the proof of the theorems, let's introduce some additional notation. Denote by $v_p^{\pi, q} := r_p \int_{\mathbb{R}_+} (\pi(d, q) \wedge q) dF(d) - c_p q - \beta \int_{\mathbb{R}_+} (\pi(d, q) - d)^+ dF(d)$ for any π and q . Recall that in Definition 1, we require the policy π to satisfy both the flow balance condition and the shifting feasibility condition to be admissible. It turns out that shifting feasibility is imposed out of practical/operational concerns and does not affect the optimal policy's structure. To reflect this point in the proof, we consider a relaxed platform's problem that does not require shifting admissibility. Recall that we have introduced $\bar{\Pi}$ to denote such a relaxed policy class. Let

$$\bar{v}_p^* := \sup_{\pi \in \bar{\Pi}} v_p^\pi \quad \text{s.t. } v_s^\pi \geq v_s^0. \quad (12)$$

Our proof proceeds by first showing that the pair (π^*, q^*) is a feasible solution to the original problem (3). Then we show that (π^*, q^*) gives an optimal policy and the corresponding best-case

seller's response to the relaxed problem (12). We then conclude that (π^*, q^*) is optimal to the original problem (3).

LEMMA 7. $\pi^* \in \Pi$. Under policy π^* , q^* is the best-case (w.r.t. the platform) seller's inventory response. Furthermore, $v_s^{\pi^*} = v_s^{\pi^*, q^*} \geq v_s^0$.

LEMMA 8. Policy π^* is optimal to problem (12), i.e., $\bar{v}_p^* = v_p^{\pi^*}$.

We defer the proof of the above lemmas to Section C.3. With the two lemmas, we complete the proof of Theorem 1 and Theorem 2.

Proof of Theorem 1 By Lemma 7, $\pi^* \in \Pi$ is admissible and the IR condition is satisfied under π^* . Therefore, $v_p^{\pi^*} \leq v_p^*$. On the other hand, $v_p^* \leq \bar{v}_p^*$. By Lemma 8, we have $v_p^{\pi^*} = \bar{v}_p^*$, implying the inequality $v_p^{\pi^*} \leq v_p^* \leq \bar{v}_p^*$ must be tight. Combining with the fact that π^* is admissible, it must be optimal to problem (3). $\square$

Proof of Theorem 2 By Lemma 7, q^* is the best-case (w.r.t. the platform) seller's response under π^* . From the proof of Lemma 7 (see Section C.3), the seller's problem is equivalently reduced to a standard newsvendor problem with decision-independent demand distributions, from which the concavity of the objective function follow immediately. Also, by plugging in the expression of q^* , we derive the desired expression of the post-shift demand. Combining the above, we complete the proof of the theorem. $\square$

C.3. Proof of auxiliary lemmas

We now state and prove a number of results, and provide the proof of Lemma 7 and Lemma 8.

LEMMA 9. Recall the seller's reservation profit $v_s^0 = \max_{q \geq 0} \{r_s \mathbb{E}[D \wedge q] - c_s q\}$ and q^s is the corresponding (unique) optimal inventory level. We have $(r_s - c_s)L(q^s) = v_s^0 = r_s \mathbb{E}D - c_s U(q^s)$.

Proof We have

$$\begin{aligned}
 v_s^0 &= \max_{q \geq 0} \{r_s \mathbb{E}[D \wedge q] - c_s q\} \\
 &= r_s \mathbb{E}[D \wedge q^s] - c_s q^s \\
 &= r_s \left(\int_0^{q^s} d dF(d) + q^s \mathbb{P}(D \geq q^s) \right) - c_s q^s \\
 &= r_s \int_0^{q^s} d dF(d) + r_s q^s \frac{c_s}{r_s} - c_s q^s = r_s \int_0^{q^s} d dF(d).
 \end{aligned}$$

By definition, $\int_0^{q^s} ddF(d) = \mathbb{P}(D \leq q^s) \mathbb{E}[D|D \leq q^s] = \left(1 - \frac{c_s}{r_s}\right) L(q^s)$. Therefore, $v_s^0 = (r_s - c_s)L(q^s)$. On the other hand, $\int_0^{q^s} ddF(d) = \mathbb{E}D - \int_{q^s}^{\infty} ddF(d) = \mathbb{E}D - \frac{c_s}{r_s}U(q^s)$. Therefore, $v_s^0 = r_s \mathbb{E}D - c_s U(q^s)$, completing the proof. $\square$

Observe that the two equivalent expressions of v_s^0 from Lemma 9 have an interesting implication. In particular, the seller's expected profit in the newsvendor problem is the same under three possible random demand scenarios. The first demand scenario is simply the original demand $D \sim F$. The second demand scenario is the post-shift demand under the platform's substitution policy

$$\pi(D) = \begin{cases} L(q^s) & \text{if } D \leq q^s \\ D & \text{if } D > q^s. \end{cases}$$

In the third case, the demand becomes the post-shift demand under the platform's substitution policy

$$\pi(D) = \begin{cases} D & \text{if } D < q^s \\ U(q^s) & \text{if } D \geq q^s. \end{cases}$$

LEMMA 10. *Consider any $\pi \in \bar{\Pi}$ such that the IR condition is satisfied, i.e. $v_s^\pi \geq v_s^0$. Then it must be that $\mathcal{S}^\pi \subseteq [L(q^s), U(q^s)]$.*

Proof Fix a policy π . Consider the seller's problem in (1) under π :

$$\max_{q \geq 0} \{r_s \mathbb{E}_D [\pi(D, q) \wedge q] - c_s q\}.$$

We now argue that any $q \in [0, L(q^s)) \cup (U(q^s), \infty)$ must be suboptimal. Denote

$$v_s^\pi(q) := r_s \mathbb{E}_D [\pi(D, q) \wedge q] - c_s q. \quad (13)$$

$[\pi(D, q) \wedge q] \leq q$ implies that

$$v_s^\pi(q) \leq (r_s - c_s)q, \quad (14)$$

and $[\pi(D, q) \wedge q] \leq \pi(D, q)$ implies that

$$v_s^\pi(q) \leq r_s \mathbb{E}D - c_s q, \quad (15)$$

where we also used the flow balance condition that is required for $\bar{\Pi}$. By Lemma 9 and (14), for any $q < L(q^s)$, we have $v_s^\pi(q) \leq (r_s - c_s)q < (r_s - c_s)L(q^s) = v_s^0$. By Lemma 9 and (15), for any $q > U(q^s)$, we have $v_s^\pi(q) \leq r_s \mathbb{E}D - c_s q < r_s \mathbb{E}D - c_s U(q^s) = v_s^0$. Hence if any $q \in [0, L(q^s)) \cap (U(q^s), \infty)$ is optimal to the seller's problem, it must be that $v_s^\pi = v_s^\pi(q) < v_s^0$, violating the IR condition. Therefore, the desired statement is true and we complete the proof. $\square$

Lemma 10 implies that $\mathcal{S}^\pi \subseteq [L(q^s), U(q^s)]$. In other words, problem (12) can be equivalently stated as

$$\bar{v}_p^* := \sup_{\pi \in \bar{\Pi}} \bar{v}_p^\pi \quad s.t. \ v_s^\pi \geq v_s^0,$$

where $\bar{v}_p^\pi := \max_{q \in \mathcal{S}^\pi \cap [L(q^s), U(q^s)]} v_p^{\pi, q}$, and

$$v_p^{\pi, q} = r_p \int_{\mathbb{R}_+} (\pi(d, q) \wedge q) dF(d) - c_p q - \beta \int_{\mathbb{R}_+} (\pi(d, q) - d)^+ dF(d).$$

We now consider an upper bound of $\bar{v}_p^*$ where we further relax the IR condition as well as the requirement of $q \in \mathcal{S}^\pi$:

$$\bar{\bar{v}}_p^* := \sup_{\pi \in \bar{\Pi}, q \in [L(q^s), U(q^s)]} v_p^{\pi, q}, \quad (16)$$

We make two observations regarding the structure of the optimal solution to problem (16), summarized as the next two lemmas.

LEMMA 11. *To solve problem (16), it is WLOG to only consider (π, q) pairs that satisfy*

1. *if $q \leq \mathbb{E}D$, then $\pi(d, q) \geq q$ for all $d \geq 0$;*
2. *if $q > \mathbb{E}D$, then $\pi(d, q) \leq q$ for all $d \geq 0$.*

Proof We prove by showing that any pair (π, q) that violates the above conditions can be “upgraded” to $(\bar{\pi}, q)$ such that (1) the conditions in the statement of the lemma hold, and (2) the objective value (w.r.t. problem (16)) achieved by $(\bar{\pi}, q)$ is greater than or equal to that achieved by (π, q) .

Suppose (π, q) is such that $q \leq \mathbb{E}D$ yet $\pi(d, q) < q$ for some $d \geq 0$. Consider a modified policy $\bar{\pi}$. For any $d \geq 0$ such that $\pi(d, q) < q$, let $\bar{\pi}(d, q) = q$. For any $d \geq 0$ such that $\pi(d, q) \geq q$, let $q \leq \bar{\pi}(d, q) \leq \pi(d, q)$ (in an arbitrary way) so that the flow balance condition $\mathbb{E}\bar{\pi}(D) = \mathbb{E}D$ is satisfied. This is achievable since $q \leq \mathbb{E}D$. The so constructed $\bar{\pi}$ belongs to $\bar{\Pi}$. We now prove it performs no worse than π .

First, calculate the new substitution cost

$$\begin{aligned} \beta \mathbb{E}(\bar{\pi}(D, q) - D)^+ &= \beta \mathbb{E}(\bar{\pi}(D, q) - \pi(D, q) + \pi(D, q) - D)^+ \\ &\leq \beta \mathbb{E}(\bar{\pi}(D, q) - \pi(D, q))^+ + \beta \mathbb{E}(\pi(D, q) - D)^+ \\ &= \beta \mathbb{E}(q - \pi(D, q))^+ + \beta \mathbb{E}(\pi(D, q) - D)^+, \end{aligned}$$

where the last equality follows from the construction of $\bar{\pi}$. The objective value of problem (16) under $(\bar{\pi}, q)$ thus becomes

$$\begin{aligned} v_p^{\bar{\pi}, q} &= r_p \mathbb{E}(\bar{\pi}(D, q) \wedge q) - c_p q - \beta \mathbb{E}(\bar{\pi}(D, q) - D)^+ \\ &= r_p q - c_p q - \beta \mathbb{E}(\bar{\pi}(D, q) - D)^+ \\ &\geq r_p q - c_p q - \beta \mathbb{E}(q - \pi(D, q))^+ - \beta \mathbb{E}(\pi(D, q) - D)^+. \end{aligned}$$

On the other hand, the objective value of problem (16) under (π, q) is

$$\begin{aligned} v_p^{\pi, q} &= r_p \mathbb{E}(\pi(D, q) \wedge q) - c_p q - \beta \mathbb{E}(\pi(D, q) - D)^+ \\ &= r_p \mathbb{E}(q - (q - \pi(D, q))^+) - c_p q - \beta \mathbb{E}(\pi(D, q) - D)^+ \\ &= r_p q - c_p q - \beta \mathbb{E}(\pi(D, q) - D)^+ - r_p \mathbb{E}(q - \pi(D, q))^+. \end{aligned}$$

Comparing $v_p^{\bar{\pi}, q}$ with $v_p^{\pi, q}$ above, we get

$$v_p^{\bar{\pi}, q} - v_p^{\pi, q} \geq (r_p - \beta) \mathbb{E}(q - \pi(D, q))^+ \geq 0.$$

Next we consider the second case. Suppose (π, q) is such that $q > \mathbb{E}D$ yet $\pi(d, q) > q$ for some $d \geq 0$. Consider the following new modified version of π , again called $\bar{\pi}$. In this case, for any d such that $\pi(d, q) > q$, let $\bar{\pi}(d, q) = q$. For other d such that $\pi(d) \leq q$, we let $\pi(d, q) \leq \bar{\pi}(d) \leq q$ (in an arbitrary way) so that the flow balance condition $\mathbb{E}\bar{\pi}(D) = \mathbb{E}D$ is satisfied. This can be achieved since $q > \mathbb{E}D$. Similar to the previous case, we show that this new policy $\bar{\pi}$ performs no worse than π . The new substitution cost under $\bar{\pi}$ becomes

$$\begin{aligned} \beta \mathbb{E}(\bar{\pi}(D, q) - D)^+ &= \beta \mathbb{E}(D - \bar{\pi}(D, q))^+ \\ &= \beta \mathbb{E}(D - \pi(D, q) + \pi(D, q) - \bar{\pi}(D, q))^+ \\ &\leq \beta \mathbb{E}(D - \pi(D, q))^+ + \beta \mathbb{E}(\pi(D) - \bar{\pi}(D, q))^+ \\ &= \beta \mathbb{E}(D - \pi(D, q))^+ + \beta \mathbb{E}(\pi(D, q) - q)^+, \end{aligned}$$

where the first equality follows from the fact that $\mathbb{E}\bar{\pi}(D) = \mathbb{E}D$, and the last equality follows from the construction of $\bar{\pi}$. The objective value of problem (16) under $(\bar{\pi}, q)$ thus becomes

$$\begin{aligned} v_p^{\bar{\pi}, q} &= r_p \mathbb{E}(\bar{\pi}(D, q) \wedge q) - c_p q - \beta \mathbb{E}(\bar{\pi}(D, q) - D)^+ \\ &= r_p \mathbb{E}\bar{\pi}(D, q) - c_p q - \beta \mathbb{E}(\bar{\pi}(D, q) - D)^+ \end{aligned}$$

$$\geq r_p \mathbb{E}D - c_p q - \beta \mathbb{E}(D - \pi(D, q))^+ - \beta \mathbb{E}(\pi(D, q) - q)^+.$$

On the other hand,

$$\begin{aligned} v_p^{\pi, q} &= r_p \mathbb{E}(\pi(D, q) \wedge q) - c_p q - \beta \mathbb{E}(\pi(D, q) - D)^+ \\ &= r_p \mathbb{E}(\pi(D, q) - (\pi(D, q) - q)^+) - c_p q - \beta \mathbb{E}(D - \pi(D, q))^+ \\ &= r_p \mathbb{E}D - c_p q - \beta \mathbb{E}(D - \pi(D, q))^+ - r_p \mathbb{E}(\pi(D, q) - q)^+. \end{aligned}$$

Comparing the above two equations, we get

$$v_p^{\bar{\pi}, q} - v_p^{\pi, q} \geq (r_p - \beta) \mathbb{E}(\pi(D, q) - q)^+ \geq 0.$$

□

Lemma 11 states that in a centralized platform, excess demand and leftover inventory should not coexist on the platform even when the demand is requesting a different item initially, since if so, the platform could always earn an extra profit of $r_p - \beta > 0$ per unit by fulfilling this excess demand (which generates a revenue of r_p) using a leftover inventory (with substitution cost of β).

LEMMA 12. *To solve problem (16), it is WLOG to only consider (π, q) pairs that satisfy*

1. *if $q \leq \mathbb{E}D$, then $\mathbb{E}(\pi(D, q) - D)^+ = \mathbb{E}(q - D)^+$;*
2. *if $q > \mathbb{E}D$, then $\mathbb{E}(\pi(D, q) - D)^+ = \mathbb{E}(D - q)^+$.*

Proof The proof logic follows that of Lemma 11. Suppose $q \leq \mathbb{E}D$. By Lemma 11, we know that $\pi(D) \geq q$ for all D , hence the objective value in problem (16) becomes

$$\begin{aligned} v_p^{\pi, q} &= r_p q - c_p q - \beta \mathbb{E}(\pi(D, q) - D)^+ \\ &\leq r_p q - c_p q - \beta \mathbb{E}(q - D)^+. \end{aligned} \tag{17}$$

We show that π can be constructed to achieve the upper bound in (17). In particular, consider

$$\pi(d, q) = \begin{cases} q & \text{if } d \leq L^{-1}(q) \\ d & \text{otherwise} \end{cases}$$

One can easily verify that this π satisfies the flow balance constraint $\mathbb{E}\pi(D) = \mathbb{E}D$, and that it satisfies the condition in Lemma 11. Moreover, the inequality in (17) is tight under this π .

Now suppose that $q > \mathbb{E}D$. Lemma 11 implies that $\pi(d, q) \leq q$ for all $d \geq 0$, hence the objective value in problem (16) becomes

$$\begin{aligned} v_p^{\pi, q} &= r_p \mathbb{E} \pi(D, q) - c_p q - \beta \mathbb{E} (\pi(D, q) - D)^+ \\ &= r_p \mathbb{E} D - c_p q - \beta \mathbb{E} (D - \pi(D, q))^+ \\ &\leq r_p \mathbb{E} D - c_p q - \beta \mathbb{E} (D - q)^+, \end{aligned} \quad (18)$$

where the second equality follows from the fact that $\mathbb{E} \pi(D, q) = \mathbb{E} D$. Again, we show that π can be constructed to achieve the upper bound in (18). In particular, consider

$$\pi(d, q) = \begin{cases} d & \text{if } d \leq U^{-1}(q) \\ q & \text{otherwise} \end{cases}$$

One can, again, easily verify that this π satisfies the flow balance constraint $\mathbb{E} \pi(D) = \mathbb{E} D$, and that it satisfies the condition in Lemma 11. Moreover, the inequality in (18) is tight under this π .

The above argument shows that the restriction in the statement of Lemma 12 is WLOG, completing the proof. $\square$

Building on Lemma 11 and Lemma 12, the next lemma establishes that q^* and π^* defined in Theorem 1 is optimal to problem (16).

LEMMA 13. (π^*, q^*) in Theorem 1 is optimal to problem (16), i.e., $\bar{v}_p^* = v_p^{\pi^*, q^*}$.

Proof of Lemma 13 Recall problem (16):

$$\sup_{\pi \in \bar{\Pi}, q \in [L(q^s), U(q^s)]} r_p \mathbb{E} D (\pi(D, q) \wedge q) - c_p q - \beta \mathbb{E} (\pi(D, q) - D)^+,$$

We want to show that π^* and q^* (as defined in Theorem 1) are a pair of optimal solutions to this centralized problem.

From Lemma 11–12, problem (16) reduces to

$$\max_{q \in [L(q^s), U(q^s)]} \begin{cases} r_p \mathbb{E} D - c_p q - \beta \mathbb{E} (D - q)^+ & \text{if } q > \mathbb{E} D, \\ (r_p - c_p) q - \beta \mathbb{E} (q - D)^+ & \text{if } q \leq \mathbb{E} D. \end{cases} \quad (19)$$

It can be verified that the objective function in (19) is concave on intervals $[0, \mathbb{E} D]$ and $(\mathbb{E} D, \infty)$, respectively. At $q = \mathbb{E} D$, the objective function is continuous. Furthermore, the left derivative is greater than or equal to the right derivative at $q = \mathbb{E} D$. Combining these fact, the objective function

in (19) is concave on $[L(q^s), U(q^s)]$. More specifically, $r_p \mathbb{E}D - c_p q - \beta \mathbb{E}(D - q)^+$ is a concave function in q , and its unique unconstrained maximizer is obtained at $q = F^{-1}(1 - \frac{c_p}{\beta}) = q^{\text{QD}}$. Similarly, $(r_p - c_p)q - \beta \mathbb{E}(q - D)^+$ is also a concave function in q , with its unique unconstrained maximizer obtained at $q = F^{-1}(\frac{r_p - c_p}{\beta}) = q^{\text{ED}}$. Observe that $q^{\text{QD}} < q^p < q^{\text{ED}}$ and that $L(q^s) < \mathbb{E}D < U(q^s)$. We consider two regimes. In the first regime, $\mathbb{E}D \leq q^p$. Then $\mathbb{E}D < q^{\text{ED}}$. By concavity, the function $(r_p - c_p)q - \beta \mathbb{E}(q - D)^+$ is monotone in $[0, \mathbb{E}D]$, and its maximum value is achieved at $q = \mathbb{E}D$. Consequently, the maximum of the objective function of (19) is achieved in $[\mathbb{E}D, \infty)$. There are three cases. If $q^{\text{QD}} > U(q^s)$, the optimal value is achieved at $U(q^s)$; else if $q^{\text{QD}} \in [L(q^s), U(q^s)]$ and the optimal value is achieved at q^{QD} ; otherwise $q^{\text{QD}} < L(q^s)$, which further implies $q^{\text{QD}} < \mathbb{E}D$, and the optimal value is achieved at $\mathbb{E}D$. Combining the above, the optimal value is achieved at $\mathbb{E}D \vee (q^{\text{QD}} \wedge U(q^s))$. Similar arguments apply to the other regime and in summary we conclude that the solution to (19) is

$$q^* = \begin{cases} \mathbb{E}D \vee (q^{\text{QD}} \wedge U(q^s)) & \text{if } q^p \geq \mathbb{E}D, \\ \mathbb{E}D \wedge (q^{\text{ED}} \vee L(q^s)) & \text{if } q^p < \mathbb{E}D. \end{cases}$$

The corresponding optimal policy π^* also follows from the policy construction in the proof of Lemma 12.

□

We now provide the proof of Lemma 7.

Proof of Lemma 7 We first consider the regime $\mathbb{E}D \leq q^p$. Recall that the seller's inventory response is $\mathcal{S}^{\pi^*}$, the set of optimal solutions to

$$\max_{q \geq 0} \{r_s \mathbb{E}_D[\pi^*(D, q) \wedge q] - c_s q\}. \quad (20)$$

π^* is q^* -defluctuating. In the regime $\mathbb{E}D \leq q^p$, we have $q^* = \mathbb{E}D \vee (q^{\text{QD}} \wedge U(q^s)) \geq \mathbb{E}D$. Thus π^* defluctuates the interval $(U^{-1}(q^*), \infty)$. We begin by proving that under π^* ,

$$\mathbb{E}[\pi^*(D, q) \wedge q] = \mathbb{E}[\pi^*(D, q^*) \wedge q]. \quad (21)$$

In words, even if π^* is a demand shifting policy that depends on the seller's inventory decision, from the seller's perspective the resulting demand appears as if it were independent of that decision. Consequently, the seller's problem is as if it were a standard newsvendor problem with the demand $\pi^*(D, q^*)$. By Definition 7, for $q \geq q^*$,

$$\pi^*(D, q) = \begin{cases} D & \text{if } D \leq U^{-1}(q), \\ q & \text{if } D > U^{-1}(q). \end{cases}$$

For $q \in (U^{-1}(q^*), q^*)$ and c_q specified uniquely by $\mathbb{E}[D|U^{-1}(q^*) < D < c_q] = q$,

$$\pi^*(D, q) = \begin{cases} D & \text{if } D \leq U^{-1}(q^*), \\ q & \text{if } U^{-1}(q^*) < D < c_q, \\ D & \text{if } D \geq c_q. \end{cases}$$

For $0 \leq q \leq U^{-1}(q^*)$, the policy does not shift demand and $\pi^*(D, q) = D$. Combining the above, we have that for $q \geq q^*$

$$\begin{aligned} \mathbb{E}[\pi^*(D, q) \wedge q] &= \int_0^{U^{-1}(q)} d dF(d) + \int_{U^{-1}(q)}^{\infty} q dF(d) \\ &= \int_0^{U^{-1}(q)} d dF(d) + \int_{U^{-1}(q)}^{\infty} d dF(d) = \mathbb{E}D, \end{aligned}$$

where we used the definition of U . For $U^{-1}(q^*) < q < q^*$,

$$\begin{aligned} \mathbb{E}[\pi^*(D, q) \wedge q] &= \int_0^{U^{-1}(q^*)} d dF(d) + \int_{U^{-1}(q^*)}^{c_q} q dF(d) + \int_{c_q}^{\infty} d \wedge q dF(d) \\ &= \int_0^{U^{-1}(q^*)} d dF(d) + \int_{U^{-1}(q^*)}^{\infty} q dF(d), \end{aligned}$$

where we used the fact that $c_q > q$. For $0 \leq q < U^{-1}(q^*)$

$$\mathbb{E}[\pi^*(D, q) \wedge q] = \mathbb{E}[D \wedge q] = \int_0^q d dF(d) + \int_q^{\infty} q dF(d).$$

On the other hand, if we fix q^* (recalling $\mathbb{E}D \leq q^p$), then

$$\pi^*(D, q^*) = \begin{cases} D & \text{if } D \leq U^{-1}(q^*) \\ q^* & \text{if } D > U^{-1}(q^*). \end{cases}$$

For $q \geq q^*$,

$$\begin{aligned} \mathbb{E}[\pi^*(D, q^*) \wedge q] &= \int_0^{U^{-1}(q^*)} (d \wedge q) dF(d) + \int_{U^{-1}(q^*)}^{\infty} (q \wedge q^*) dF(d) \\ &= \int_0^{U^{-1}(q^*)} d dF(d) + \int_{U^{-1}(q^*)}^{\infty} q^* dF(d) \\ &= \int_0^{U^{-1}(q^*)} d dF(d) + \int_{U^{-1}(q^*)}^{\infty} d dF(d) = \mathbb{E}D, \end{aligned}$$

where we used $q \geq q^* > U^{-1}(q^*)$ and the definition of U . For $U^{-1}(q^*) < q < q^*$,

$$\begin{aligned}\mathbb{E}[\pi^*(D, q^*) \wedge q] &= \int_0^{U^{-1}(q^*)} (d \wedge q) dF(d) + \int_{U^{-1}(q^*)}^{\infty} (q \wedge q^*) dF(d) \\ &= \int_0^{U^{-1}(q^*)} d dF(d) + \int_{U^{-1}(q^*)}^{\infty} q dF(d),\end{aligned}$$

and for $0 \leq q < U^{-1}(q^*)$,

$$\begin{aligned}\mathbb{E}[\pi^*(D, q^*) \wedge q] &= \int_0^{U^{-1}(q^*)} (d \wedge q) dF(d) + \int_{U^{-1}(q^*)}^{\infty} (q \wedge q^*) dF(d) \\ &= \int_0^q d dF(d) + \int_q^{U^{-1}(q^*)} q dF(d) + \int_{U^{-1}(q^*)}^{\infty} q dF(d) \\ &= \int_0^q d dF(d) + \int_q^{\infty} q dF(d).\end{aligned}$$

Comparing the above terms yields (21).

With the above understanding, we now prove that q^* is an optimal seller's inventory decision. Indeed, we have

$$\begin{aligned}\mathbb{P}(\pi^*(D, q^*) \geq q^*) &= \mathbb{P}(D > U^{-1}(q^*)), \\ &= \mathbb{P}(D > U^{-1}(\mathbb{E}D \vee (q^{\text{QD}} \wedge U(q^s))), \\ &= \min \left\{ \mathbb{P}(D > U^{-1}(\mathbb{E}D)), \mathbb{P}(D > U^{-1}(q^{\text{QD}} \wedge U(q^s))) \right\} \\ &\geq \min \{ \mathbb{P}(D > 0), \mathbb{P}(D > q^s) \} = \min \{ 1, \frac{c_s}{r_s} \} = \frac{c_s}{r_s},\end{aligned}$$

where we use $U^{-1}(\mathbb{E}D) = 0$ and $q^{\text{QD}} \wedge U(q^s) \leq U(q^s)$. Notice that by construction $\pi^*(D, q^*) \leq q^*$ *a.s.*. Combining with the bound $\mathbb{P}(\pi^*(D, q^*) \geq q^*) \geq \frac{c_s}{r_s}$, we conclude that q^* is an optimal solution to the newsvendor problem

$$\max_{q \geq 0} \{ r_s \mathbb{E}[\pi^*(D, q^*) \wedge q] - c_s q \}.$$

By (21), we further conclude that q^* is an optimal solution to (20). That is, $q^* \in \mathcal{S}^{\pi^*}$ is a seller's inventory response to π^* .

Finally, we prove that q^* is the best-case inventory response w.r.t. the platform's objective. Recall that the platform's objective is equal to

$$v_p^{\pi^*, q} = r_p \int_{\mathbb{R}_+} (\pi^*(d, q) \wedge q) dF(d) - c_p q - \beta \int_{\mathbb{R}_+} (\pi^*(d, q) - d)^+ dF(d).$$

From the previous argument, $\mathcal{S}^{\pi^*}$ coincides with the $(1 - \frac{c_s}{r_s})$ -quantile of $\pi^*(D, q^*)$, which is uniquely q^* when $\mathbb{P}(\pi^*(D, q^*) = q^*) > \frac{c_s}{r_s}$, or equivalently, $q^{\text{QD}} < U(q^s)$. Consequently, $\mathcal{S}^{\pi^*}$ is a singleton and q^* is automatically the best-case response w.r.t. the platform. Otherwise, when $q^{\text{QD}} \geq U(q^s)$, the seller's problem has multiple inventory responses $[U^{-1}(q^*), q^*]$. We show that in this case q^* is the platform optimal seller response.

By construction, for any $q \in [U^{-1}(q^*), q^*]$, $\pi^*(d, q) = q$ on the interval $(U^{-1}(q^*), b_{q^*})$, where b_{q^*} is such that the flow balance condition is satisfied. Note that this is a partial defluctuation. By the policy construction in Lemma 11–12, to the platform's interest, a complete defluctuation (on the interval $(U^{-1}(q), \infty)$) would be more favorable since it would completely eliminate lost sale or left-over inventory. Mathematically, under such a complete defluctuation at q , the platform's profit is captured by the objective in (19), serving as an upper bound on $v_p^{\pi^*, q}$ for all $q \in [U^{-1}(q^*), q^*]$. Particularly, the objective in (19) is equal to $v_p^{\pi^*, q^*}$ at $q = q^*$. From the analysis of the optimal solution to (19) in Lemma 13, we conclude that the objective function of (19) is monotone increasing in q for $q \leq q^*$ (in the regime $\mathbb{E}D \leq q^p$), as (i) $\mathbb{E}D < U(q^s) \leq q^{\text{QD}} < q^{\text{ED}}$ implies that for $q < \mathbb{E}D$, the objective function is monotone increasing, and (ii) the objective function of (19) is concave on $[\mathbb{E}D, \infty)$, and q^* achieves the maximal objective value in the interval $[L(q^s), U(q^s)]$, where $L(q^s) < \mathbb{E}D < U(q^s)$. This further implies that p^* achieves the maximal value with respect to function $v_p^{\pi^*, q}$ among all seller responses in $\mathcal{S}^{\pi^*}$. Combining the above, we conclude that q^* is indeed the best seller's response for the platform under π^* .

In the regime $\mathbb{E}D > q^p$, we can similarly show that the seller's problem can be reduced to a standard newsvendor problem, with q^* being the best seller's response for the platform. We omit the similar proof details. The fact that the IR condition holds, namely, $v_s^{\pi^*, q^*} \geq v_s^0$ can be verified directly from the above reasoning in both regimes, which we also omit. $\square$

With the above results, we are ready to finish the proof of Lemma 8.

Proof of Lemma 8 On one hand, by Lemma 13, $\bar{v}_p^* = v_p^{\pi^*, q^*}$. On the other hand, by Lemma 7, q^* is the best-case (w.r.t. the platform) seller's response to policy $\pi^* \in \Pi \subseteq \bar{\Pi}$, under which the IR condition is satisfied. This implies that $v_p^{\pi^*, q^*} = \max_{q \in \mathcal{S}^{\pi^*}} v_p^{\pi^*, q} = v_p^{\pi^*} \leq \bar{v}_p^*$. Combining the above, we conclude that $\bar{v}_p^* \leq \bar{v}_p^*$. Since $\bar{v}_p^*$ is an upper bound on $\bar{v}_p^*$, the above inequalities all hold in equality, in particular, $\bar{v}_p^* = v_p^{\pi^*}$, completing the proof. $\square$

D. Proof of Other Results

We prove Proposition 1–3 in this appendix.

Proof of Proposition 1 The proof is nearly identical to the proof of Lemma 13. Thus we omit.

□

Proof of Proposition 2 Note that under π^* , the seller's expected profit is

$$v_s^* = \begin{cases} (r_s - c_s)q^* & \text{if } q^* \leq \mathbb{E}D \\ r_s \mathbb{E}D - c_s q^* & \text{if } q^* \geq \mathbb{E}D. \end{cases}$$

Then by Lemma 9, $v_s^* > v_s^0$ if and only if $q^* \in (L(q^s), U(q^s))$. Similar to Corollary 1, the equivalent condition on β can be deduced accordingly. □

Proof of Proposition 3 We want to compare $\mathbb{E}(\pi^*(D, q^*) \wedge q^*)$ with $\mathbb{E}(D \wedge q^s)$. First observe from Theorem 1 that if $\mathbb{E}D \leq q^p$, then $\mathbb{E}(\pi^*(D, q^*) \wedge q^*) = \mathbb{E}D \geq \mathbb{E}(D \wedge q^s)$.

Now consider the case $\mathbb{E}D > q^p$. In this case, $\mathbb{E}(\pi^*(D, q^*) \wedge q^*) = q^* = \mathbb{E}D \wedge \left(F^{-1}\left(\frac{r_p - c_p}{\beta}\right) \vee L(q^s)\right)$. On the other hand,

$$\begin{aligned} \mathbb{E}(D \wedge q^s) &= \int_0^{q^s} D dF(D) + q^s(1 - F(q^s)) \\ &\geq \int_0^{q^s} D dF(D) + \mathbb{E}(D|D \leq q^s)(1 - F(q^s)) \\ &= L(q^s). \end{aligned}$$

Therefore, $\mathbb{E}(\pi^*(D, q^*) \wedge q^*) < \mathbb{E}(D \wedge q^s)$ if and only if $F^{-1}\left(\frac{r_p - c_p}{\beta}\right) < \mathbb{E}(D \wedge q^s)$. This is equivalent to

$$\frac{r_p - c_p}{\beta} < F(\mathbb{E}(D \wedge q^s)),$$

i.e.,

$$\beta > \frac{F(q^p)}{F(\mathbb{E}(D \wedge q^s))} \cdot r_p.$$

□

E. Extensions

In this section, we provide four extensions of the base model to illustrate the robustness of the main results. In Section E.1, we relax the assumption that the unit shifting cost is a constant

to allow for nonlinear total shifting cost. In Section E.2, we consider heterogeneous sellers with different cost primitives, demand distributions and shifting costs. In Section E.3, we study the case of shared shifting cost, which can be applied to a decentralized multi-location newsvendor problem with possible transshipment arrangements. Finally in Section E.4, we consider proactive demand shifting, as opposed to the reactive demand shifting as a function of seller inventory in the base model. In all extensions, our main structural insight regarding the optimality of a defluctuating policy is preserved.

E.1. Nonlinear shifting cost

Assume the platform's total cost of shifting is a general function of the total amount of demand shifted, i.e., $B\left(\frac{1}{2} \int_{\mathbb{R}_+} |\pi(d, q) - d| dF(d)\right)$ when the sellers' inventory is set at q . For example, when the unit shifting cost is a constant β as in the base model, we have

$$B\left(\frac{1}{2} \int_{\mathbb{R}_+} |\pi(d, q) - d| dF(d)\right) = \frac{\beta}{2} \int_{\mathbb{R}_+} |\pi(d, q) - d| dF(d).$$

We impose the following mild assumption on the new shifting cost structure.

ASSUMPTION 1. *The platform's total shifting cost $B(\cdot)$ is a convex nondecreasing differentiable function. Moreover, the marginal shifting cost $B'(\cdot)$ is always smaller than r_p .*

The nondecreasingness requirement is straightforward. The convexity assumption, on the other hand, reflects the diseconomies of scale in the price discounts for shifting. For example, in a constant elasticity demand model or a linear demand model, the average discount, i.e., the average shifting cost, increases in the quantity shifted. Finally, we consider imperfectly substitutable products, for which demand shifting is always profitable, and this is captured by $B'(\cdot) \leq r_p$. Under this general form of shifting cost, we modify the platform's problem in (3) to be

$$\sup_{\pi^B \in \Pi} \max_{q^B \in \mathcal{S}^{\pi^B}} \left\{ r_p \int_{\mathbb{R}_+} (\pi^B(d, q) \wedge q) dF(d) - c_p q - B\left(\frac{1}{2} \int_{\mathbb{R}_+} |\pi^B(d, q) - d| dF(d)\right) \right\}, \quad \text{s.t. } v_s^\pi \geq v_s^0. \quad (22)$$

The next theorem characterizes a platform's optimal shifting policy and its corresponding inventory best response. As in the base model, the shifting policy employs a defluctuating structure.

THEOREM 4. *Consider the platform's problem in (22). Then there exists a defluctuating policy (see Definition 2) π^B that is optimal. Let*

$$q^B := \begin{cases} \mathbb{E}D \vee \left(\tilde{q}_1 \wedge U\left(F^{-1}\left(1 - \frac{c_s}{r_s}\right)\right) \right) & \text{if } \mathbb{E}D \leq F^{-1}\left(1 - \frac{c_p}{r_p}\right), \\ \mathbb{E}D \wedge \left(\tilde{q}_2 \vee L\left(F^{-1}\left(1 - \frac{c_s}{r_s}\right)\right) \right) & \text{if } \mathbb{E}D > F^{-1}\left(1 - \frac{c_p}{r_p}\right), \end{cases}$$

where $\tilde{q}_1$ and $\tilde{q}_2$ solve

$$\tilde{q}_1 = F^{-1} \left(1 - \frac{c_p}{B'(\mathbb{E}(D - \tilde{q}_1)^+)} \right), \quad \tilde{q}_2 = F^{-1} \left(\frac{r_p - c_p}{B'(\mathbb{E}(\tilde{q}_2 - D)^+)} \right),$$

respectively. Under π^B , the seller's problem is concave and q^B is the unique optimal inventory response. Moreover, at equilibrium, a seller's post-shift demand is

$$\pi^B(d, q^B) = \begin{cases} d & \text{if } d < U^{-1}(q^B), \\ q^B & \text{otherwise.} \end{cases}, \quad \text{if } \mathbb{E}D \leq F^{-1} \left(1 - \frac{c_p}{r_p} \right);$$

$$\pi^B(d, q^B) = \begin{cases} q^B & \text{if } d \leq L^{-1}(q^B), \\ d & \text{otherwise.} \end{cases}, \quad \text{if } \mathbb{E}D > F^{-1} \left(1 - \frac{c_p}{r_p} \right).$$

The intuition behind the optimal policy under nonlinear shifting cost is similar to that in the base model. In particular, we first argue that the platform should always (at equilibrium) not have simultaneous left-over inventory and lost sales. Then, depending on whether there is only left-over inventory or only lost sales, the platform should determine the optimal level of inventory in equilibrium based on her tradeoff between the benefit of shifting and the cost of it. When there is only left-over inventory in the system (say, the inventory level is $q > \mathbb{E}D$), the marginal benefit of more shifting is to reduce inventory by one unit, hence a cost savings of c_p . At the same time, given the right tail-defluctuating structure of the post-shift demand in this case, the marginal cost of more shifting is $B'(\mathbb{E}(D - q)^+) \cdot \left(-\frac{d\mathbb{E}(D - q)^+}{dq} \right) = (1 - F(q))B'(\mathbb{E}(D - q)^+)$. The optimal equilibrium q should be chosen that equates the marginal cost and benefit, i.e., at $q = F^{-1} \left(1 - \frac{c_p}{B'(\mathbb{E}(D - q)^+)} \right)$. Similarly, when there are only lost sales in the system, (say, the inventory level is $q < \mathbb{E}D$), the marginal benefit of more shifting is to increase inventory by one unit, hence reducing lost sale by one unit which results in a profit increase of $r_p - c_p$. On the other hand, given the left tail-defluctuating structure of the post-shift demand in this case, the cost of more shifting (to increase one unit of inventory) is $B'(\mathbb{E}(q - D)^+) \cdot \left(\frac{d\mathbb{E}(q - D)^+}{dq} \right) = F(q)B'(\mathbb{E}(q - D)^+)$. The optimal equilibrium q should hence be chosen to equate the marginal cost and benefit, i.e., at $q = F^{-1} \left(\frac{r_p - c_p}{B'(\mathbb{E}(q - D)^+)} \right)$.

Above is a heuristic argument for the platform's desired inventory level in the centralized control problem. When the sellers make their own inventory decisions, the platform needs to respect the sellers' inventory incentives. This equilibrium constraint is reflected by the $U \left(F^{-1} \left(1 - \frac{c_s}{r_s} \right) \right)$ upper bound and $L \left(F^{-1} \left(1 - \frac{c_s}{r_s} \right) \right)$ lower bound in the sellers' inventory response. The full proof of Theorem 4 is provided in Section F.1.

E.2. Heterogeneous sellers

Consider a system with K different types of sellers. A type- i seller is described by $(F_i, r_s^i, c_s^i, r_p^i, c_p^i)$, which includes the seller's original demand D_i independently following distribution F_i , his unit sales revenue r_s^i , unit inventory cost c_s^i , as well as the platform's unit sales revenue r_p^i and unit inventory cost c_p^i of the seller's product. Let $\lambda_i \in (0, 1)$ be the proportion of type- i sellers in the system, with $\lambda_1 + \dots + \lambda_K = 1$. In addition, assume that the platform's unit shifting cost between any pair of sellers is described by a matrix B , where its entry β_{ij} denotes the cost of shifting an original demand for type- i seller to a type- j seller. Also assume that the sellers' original demands are independent. We impose the following assumptions on the shifting cost matrix B .

ASSUMPTION 2. $0 \leq \beta_{ij} < r_p^j, \forall i, j = 1, \dots, K$.

ASSUMPTION 3. $\beta_{ij} + \beta_{jk} \geq \beta_{ik}, \forall i, j, k = 1, \dots, K$

Assumption 2 implies that it is always profitable to shift demand at equilibrium whenever there coexists any unsatisfied demand and leftover inventory (see Lemma 17 below). Assumption 3 requires a triangle inequality satisfied by the shifting cost of any three types, which says that it is not cheaper to shift a demand from i to j then to k compared with shifting it directly from i to k . This assumption is natural and crucial in establishing an efficiency property of the optimal policy (see Lemma 15).

We characterize each third-party seller by the profile $\mathbf{z} = (z_d, z_q, z_t) \in \mathbb{R}_+^2 \times \mathcal{K}$, where the first two arguments, as in the base model, capture the seller's initial demand and inventory decision, and the third argument captures the seller's type. Let $\bar{\nu}$ be the measure induced by sellers' profiles. As in the base model, the measure formed by the initial demand is exogenous, and the system's state can be solely determined by the inventory levels. We denote ν_i the normalized conditional measure over the inventory levels of type- i sellers (see Appendix F.2 for a formal and more detailed specification). With a slight abuse of notation, we denote ν the joint measures $\{\nu_1, \dots, \nu_K\}$. As in Section A, ν represents the system's state.

Similar to Section A (and slightly abusing the notation), the platform's shifting policy ϕ_F is formally defined as a collection of mappings. In particular, $\phi_F^\nu : \mathbb{R}_+^2 \times \mathcal{K} \times \mathbb{R}_+^2 \times \mathcal{K} \rightarrow \mathbb{R}_+$ is a measurable function that specifies the amount of demand substitution between any pair of sellers. More specifically, for any $\mathbf{z}, \mathbf{z}' \in \mathbb{R}_+^2 \times \mathcal{K}$ and system state ν , $\phi_F^\nu(\mathbf{z}, \mathbf{z}')$ denotes the amount of demand substituted from a seller of profile $\mathbf{z}$ to a seller of profile $\mathbf{z}'$. If no such substitution occurs, we let

$\phi_F^v(\mathbf{z}, \mathbf{z}') = 0$. A policy ϕ_F induces a post-shift demand for each seller of profile $\mathbf{z}$ under any system's state ν , denoted by ϕ , defined as follows

$$\phi^v(\mathbf{z}) = z_d - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^v(\mathbf{z}, \mathbf{y}) d\bar{\nu}(\mathbf{y}) + \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^v(\mathbf{y}, \mathbf{z}) d\bar{\nu}(\mathbf{y}). \quad (23)$$

For ϕ_F to be admissible, we impose a number of conditions on the post-shift demand it induces, analogous to that of Definition 3 in the base model.

DEFINITION 9 (ADMISSIBLE POLICY WITH HETEROGENEOUS SELLERS). A measurable, non-negative function $\phi_F : \mathbb{R}_+^2 \times \mathcal{K} \times \mathbb{R}_+^2 \times \mathcal{K} \rightarrow \mathbb{R}_+$ is an admissible policy if ϕ , the post-shift demand it induces, satisfies the following conditions.

1. (FLOW BALANCE) For any ν , $\int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi^v(\mathbf{z}) d\bar{\nu}(\mathbf{z}) = \sum_{i=1}^K \lambda_i \mathbb{E}[D_i]$.
2. (SHIFTING FEASIBILITY) For any ν and any $\mathbf{z} \in \mathbb{R}_+^2 \times \mathcal{K}$, if $\phi^v(\mathbf{z}) \geq z_d$, then $\phi^v(\mathbf{z}) \leq z_q$.

We again denote by Φ_F the set of admissible policies and Φ the set of post-shift demand induced by $\phi_F \in \Phi_F$.

LEMMA 14. *It is WLOG to adopt the post-shift demand as our policy representation.*

The lemma confirms that similar to the base model in Appendix A, we can use the policy reduction by specifying its induced post-shift demand in the heterogeneous setting. The proof is also similar, which we defer to Appendix F.2. We generalize the seller and the platform's problems in (8) and (10) to the heterogeneous setting, respectively, as follows. For each $i \in 1 \dots, K$, we introduce the type- i seller's problem.

$$\max_{q \geq 0} \{r_s^i \mathbb{E}[\phi^v((D_i, q, i)) \wedge q] - c_s^i q\} = \max_{q \geq 0} \left\{ r_s^i \int_{\mathbb{R}_+} (\phi^v((d, q, i)) \wedge q) dF_i(d) - c_s^i q \right\}. \quad (24)$$

We denote the above optimal value by $v_s^{\phi, \nu, i}$. We also denote by $v_s^{0, i} := \max_{q \geq 0} \{r_s^i \mathbb{E}[D_i \wedge q] - c_s^i q\}$ the reservation profit for $i = 1, \dots, K$. Under ϕ , the sellers' joint inventory decision-making problem constitutes a nonatomic game, and ν in (24) should be a mean-field equilibrium of the system in the sense of Definition 4 in Section A. Precisely, denote $\mathcal{S}^{\phi, \nu} \subseteq \mathbb{R}_+^K$ the set of optimal solutions to (24) for $i = 1, \dots, K$. Then for ν to be a mean field equilibrium, $\mathcal{S}^{\phi, \nu}$ must contain the support of ν . We again denote by $\mathcal{B}^\phi$ the set of equilibria induced by ϕ .

Extending (9), the platform's objective is

$$v_p^{\phi, \nu} := \int_{\mathbb{R}_+^2 \times \mathcal{K}} r_p^{z_t} (\phi^v(\mathbf{z}) \wedge z_q) d\bar{\nu}(\mathbf{z}) - \int_{\mathbb{R}_+^2 \times \mathcal{K}} c_p^{z_t} z_q d\bar{\nu}(\mathbf{z}) - C(\phi, \nu), \quad (25)$$

where $C(\phi, \nu)$ represents the shifting cost and emerges from the solution to a matching problem, and its explicit expression is given in Section F.2 and in particular Lemma 16. The platform's problem is

$$\sup_{\phi \in \Phi} \left\{ \sup_{\nu \in \mathcal{B}\phi} v_p^{\phi, \nu} \right\} \quad s.t. \quad v_s^{\phi, \nu, i} \geq v_s^{0, i}, \quad i = 1, \dots, K. \quad (26)$$

For simplicity, we only focus on Dirac equilibrium responses (for each type) in the heterogeneous setting. The next theorem establishes the defluctuating structure of an optimal shifting policy at the equilibrium, which justifies the robustness of the structural insight derived in our base model..

THEOREM 5. *Consider the platform's problem in (26). Then, there exists an optimal shifting policy ϕ_F^* (with induced post-shift demand ϕ^*) that has the following defluctuating structure at its corresponding equilibrium inventory responses $\vec{q}^* = (q_1^*, \dots, q_K^*)$.*

$$\phi^{*, \delta(\vec{q}^*)}((d_i, q_i^*, i)) = \begin{cases} q_i^* & \text{if } d_i \leq y_i, \\ d_i & \text{otherwise,} \end{cases} \quad \text{for some } \vec{y} < \vec{q}^*, \text{ if } \sum_{i=1}^K \lambda_i q_i^* \leq \sum_{i=1}^K \lambda_i \mathbb{E} D_i;$$

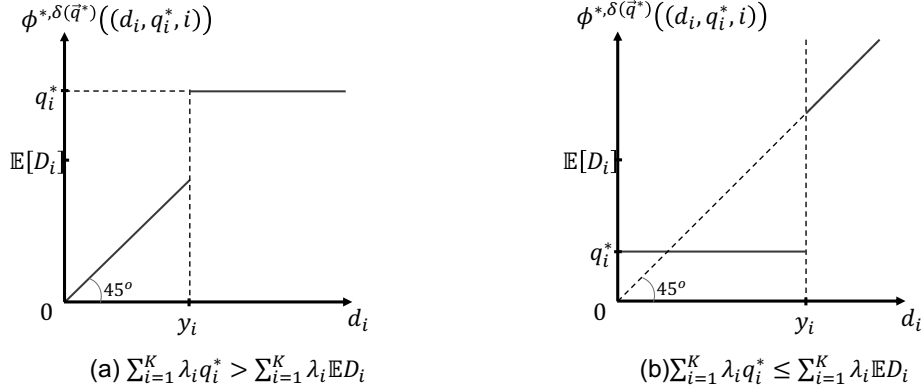
$$\phi^{*, \delta(\vec{q}^*)}((d_i, q_i^*, i)) = \begin{cases} d_i & \text{if } d_i < y_i, \\ q_i^* & \text{otherwise.} \end{cases} \quad \text{for some } \vec{y} \leq \vec{q}^*, \text{ if } \sum_{i=1}^K \lambda_i q_i^* > \sum_{i=1}^K \lambda_i \mathbb{E} D_i,$$

where we slightly abuse notation $\delta(\vec{q}^*)$ to represent the system's state.

Several observations are immediate from Theorem 5. First of all, as in the base model, the optimality of a defluctuating shifting policy is preserved in the case of heterogeneous sellers. More specifically, at equilibrium, if the system's total inventory [demand] exceeds total demand [inventory], then the platform should defluctuate the right [left] tail of each seller's original demand distribution, such that the original tail is replaced by a new point mass at the seller's equilibrium inventory response level. This is illustrated in Figure 8. Note that the induced new post-shift demand may not be mean-preserving for each seller as in the base model, since heterogeneity might drive the platform to favor certain types of sellers to sell more, and she may do so by shifting demand from sellers of other types.

As in the base model, the defluctuating structure characterized in Theorem 5 can greatly simplify the platform's demand design problem, from an infinite-dimensional one to a finite-dimensional one. We do note that, however, in order to fully characterize the optimal policy, one still needs to determine the value of $\vec{q}^*$ and $\vec{y}$. This requires transforming the equilibrium condition into constraints on achievable values of $\vec{q}^*$ and $\vec{y}$. We leave this for future study.

The full proof of Theorem 5 is in Section F.2.

Figure 8 $\phi^{*,\delta(\bar{q}^*)}((d_i, q_i^*, i))$ versus d_i under the platform-optimal policy ϕ^* in Theorem 5.

E.3. Shared shifting cost

Assume the shifting cost β is not entirely paid by the platform, but shared by the platform and the seller. In particular, let $\alpha \in [0, 1]$ be the proportion of shifting cost paid by the platform. That is, for each unit of demand shifted, the platform pays $\alpha\beta$, and the seller who receives¹ this demand pays $(1 - \alpha)\beta$. We assume $\alpha\beta < r_p$ and $(1 - \alpha)\beta < r_s$, so that it is profitable for the platform to shift demand and for the seller to satisfy a shifted demand.

This setting can be interpreted as decentralized multi-location newsvendors with possible transshipment arrangement. There is an online retailer (i.e., the platform) owning multiple stores selling a homogeneous product. Each store is operated by a manager (seller) who decentrally decides the order quantity of his product. Demand comes online which requests a product from the closest physical store. The central retailer who controls the fulfillment can decide to ship the product from the requested store A or from another store B to the customer. Whenever she uses another location B to satisfy a demand arising in location A, the local store in location B receives the sales revenue but needs to incur an additional transshipment cost of β to satisfy this demand. This corresponds to the special case of $\alpha = 0$ described in the previous paragraph.

In the general model of shared shifting cost with a cost sharing parameter α , similar to (3), the platform's problem is

$$\begin{aligned} & \sup_{\pi^t \in \Pi} \max_{q^t \geq 0} \left\{ r_p \int_{\mathbb{R}_+} (\pi^t(d, q^t) \wedge q^t) dF(d) - c_p q^t - \alpha\beta \int_{\mathbb{R}_+^2} (\pi^t(d, q^t) - d)^+ dF(d) \right\} \\ & \text{s.t. } q^t \in \arg \max_{q \geq 0} \left\{ r_s \mathbb{E}_D [\pi^t(d, q) \wedge q] - c_s q - (1 - \alpha)\beta \int_{\mathbb{R}_+} (\pi^t(d, q) - d)^+ dF(d) \right\} \end{aligned} \quad (27)$$

$$r_s \mathbb{E}_D [\pi^t(d, q^t) \wedge q^t] - c_s q^t - (1 - \alpha) \beta \int_{\mathbb{R}_+} (\pi^t(d, q^t) - d)^+ dF(d) \geq v_s^0. \quad (28)$$

The first constraint is the equilibrium requirement, i.e., q^t is the seller's best inventory response induced by policy π^t . The second constraint is individual rationality. That is, the seller's expected profit under policy π must be no less than that absent demand shifting.

The next theorem characterizes a platform's optimal demand shifting policy and its corresponding inventory best response. Again, as in the base model, the demand shifting policy employs a defluctuating structure.

THEOREM 6. *Consider the platform's problem in (27). Then there exists a defluctuating policy (see Definition 2) π^t that is optimal. Let*

$$q^t := \begin{cases} \left(\left(\mathbb{E}D \vee F^{-1} \left(1 - \frac{c_p}{\alpha\beta} \right) \right) \wedge \tilde{q}_1^t \right) \vee \tilde{q}_2^t & \text{if } \mathbb{E}D \leq F^{-1} \left(1 - \frac{c_p}{r_p} \right), \\ \left(\left(\mathbb{E}D \wedge F^{-1} \left(\frac{r_p - c_p}{\alpha\beta} \right) \right) \wedge \tilde{q}_1^t \right) \vee \tilde{q}_2^t & \text{if } \mathbb{E}D > F^{-1} \left(1 - \frac{c_p}{r_p} \right), \end{cases}$$

where $\tilde{q}_1^t$ and $\tilde{q}_2^t$ solve

$$\tilde{q}_1^t = U(q^s) - \frac{(1 - \alpha)\beta \mathbb{E}(D - \tilde{q}_1^t)^+}{c_s}, \quad \tilde{q}_2^t = L(q^s) + \frac{(1 - \alpha)\beta \mathbb{E}(\tilde{q}_2^t - D)^+}{r_s - c_s},$$

respectively. Under π^t , q^t is the optimal inventory response for the seller. Moreover, at equilibrium, a seller's post-shift demand is

$$\pi^t(d, q^t) = \begin{cases} d & \text{if } d < U^{-1}(q^t), \\ q^t & \text{otherwise.} \end{cases}, \quad \text{if } \mathbb{E}D \leq q^t;$$

$$\pi^t(d, q^t) = \begin{cases} q^t & \text{if } d \leq L^{-1}(q^t), \\ d & \text{otherwise.} \end{cases}, \quad \text{if } \mathbb{E}D > q^t.$$

The insight behind the optimal policy structure and the corresponding inventory response in Theorem 6 are similar to those in the base model. In particular, for any desired equilibrium inventory level $\tilde{q}^t$, it is WLOG for the platform to choose a defluctuating policy with defluctuating window $[0, L^{-1}(\tilde{q}^t)]$ (if $\tilde{q}^t < \mathbb{E}D$) or $(U^{-1}(\tilde{q}^t), \infty)$ (if $\tilde{q}^t \geq \mathbb{E}D$), since it benefits both the platform and the seller (conditional on $\tilde{q}^t$ being fixed). This reduces the platform's problem to a constraint problem of choosing $\tilde{q}^t$, as (19) in the base model, where β is replaced by $\alpha\beta$ and the interval constraint $q \in$

$[L(q^s), U(q^s)]$ is replaced by $\tilde{q}^t \in [\tilde{q}_2^t, \tilde{q}_1^t]$. The former adjustment is a straightforward consequence of the platform's partial bearing of the demand shifting cost. The latter adjustment is due to the seller's IR constraint affected by him paying partially for the demand shifting cost. Notice that when $\alpha = 1$, i.e., $(1 - \alpha)\beta = 0$, the platform pays for all the demand shifting cost, then $[\tilde{q}_2^t, \tilde{q}_1^t]$ becomes $[L(q^s), U(q^s)]$, and π^t and q^t reduce to π^* and q^* in the base model (see Theorem 1–2). As $(1 - \alpha)\beta$ increases, the seller pays for more of the shifting cost, so the benefit of demand shifting reduces for the seller, not enough to compensate his loss for deviating too far away from his original inventory incentive q^s . As a result, the IR constraint requires the equilibrium inventory to be closer to q^s , which results in the interval constraint $[\tilde{q}_2^t, \tilde{q}_1^t]$ to shrink towards the singleton q^s . The extreme point is achieved when $(1 - \alpha)\beta = r_s$.

The above extension assumes that how the shifting cost is shared among different parties is fixed. If, instead, how the shifting cost is shared among the sellers must be negotiated among them, we discuss below that our model and results can still be extended. In particular, suppose the store managers see their own demand, each determines his own inventory, and can cooperate via inventory pooling (using one store's inventory to satisfy another store's demand), but need to jointly agree on how to allocate the excess revenue/cost (sales revenue, inventory cost, transshipment cost) from pooling to each store. This becomes similar to the cooperative game studied in Anupindi et al. (2001) and Yang et al. (2021), and it is known that there exists a proportional allocation rule that achieves the first-best solution (see Anupindi et al. (2001)). Note that the first-best solution here corresponds to when all the inventory and transshipment decisions are centrally made to maximize the total expected profit from the stores. That is, solve

$$\sup_{q \geq 0, \pi^c \in \Pi^c} r_s \mathbb{E}[\pi^c(D) \wedge q] - c_s q - \frac{\beta}{2} \mathbb{E}|\pi^c(D) - D|,$$

where the optimal solution q^c corresponds to each store's inventory level, and $\pi^c(D)$ corresponds to the transshipment plan (in the form of post-transshipment demand), with $\frac{\beta}{2} \mathbb{E}|\pi^c(D) - D|$ being the total transshipment cost. The solution to this problem is the same as Proposition 1, with r_p, c_p replaced by r_s, c_s .

E.4. Proactive demand shifting

In the base model, we assumed that the platform can commit to a demand shifting policy that reacts to the seller's inventory level. In this section, we consider an alternative in which the platform

commits to a proactive demand shifting plan that is independent of the seller's inventory. In this case, the platform solves

$$\begin{aligned} & \sup_{\pi^a \in \Pi^a, q^a \geq 0} \left\{ r_p \mathbb{E}[\pi^a(D) \wedge q^a] - c_p q^a - \frac{\beta}{2} \mathbb{E}|\pi^a(D) - D| \right\} \\ & \text{s.t. } q^a \in \arg \max_{q \geq 0} \{ r_s \mathbb{E}[\pi^a(D) \wedge q] - c_s q \}. \end{aligned}$$

Note that the demand shifting policy π^a in the above formulation does not take q as an input, and only depends on the realization of demand D . The admissibility set Π^a is the set of all $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ measurable functions that satisfy flow balance $\mathbb{E}\pi^a(D) = \mathbb{E}D$. The constraint corresponds to the seller's self-interested inventory response. Note that we don't impose shifting feasibility and IR in this problem.

Solving this problem, despite the absence of shifting feasibility and IR, we obtain the same optimal policy and inventory response as in the base model (focusing on the equilibrium only) in Section 3.

THEOREM 7. *The platform's optimal proactive demand shift policy is*

$$\pi^a(d) = \begin{cases} d & \text{if } d < U^{-1}(q^a) \\ q^a & \text{otherwise} \end{cases}$$

if $\mathbb{E}D \leq q^a$, and

$$\pi^a(d) = \begin{cases} q^a & \text{if } d \leq L^{-1}(q^a) \\ d & \text{otherwise} \end{cases}$$

if $\mathbb{E}D > q^a$, where q^a is the corresponding inventory response defined the same as q^* in Theorem 2.

Note that we obtain the same result as in the base model, hence IR holds anyway under this optimal policy and inventory response even though we do not impose it a priori. This implies that the interest alignment between the platform and the sellers under demand shifting is quite strong. When we restrict the platform's policy design from reactive to proactive, we no longer need IR (as in the pairwise state-dependent policy design) or shifting feasibility (as in the state-independent post-shift policy design) to further constrain the platform. It turns out that flow balance and the seller being self-interested (i.e., solving a newsvendor problem) suffice to guarantee him at least his reservation profit.

F. Proof of Results in Section E

In this appendix, we prove the results stated in the four extensions in Section E.

F.1. Proof of results in Section E.1

Proof of Theorem 4 The proof is analogous to the proof of Theorem 1–2 in Section C. The only part that potentially needs adjustments is the proof of Lemma 7–8, which states that the proposed policy and inventory response are optimal to a relaxed platform’s problem that ignores the shifting feasibility constraint. Their proof (see Section C.3) contains five key lemmas, Lemma 9–13. Lemma 9–10 state that the desired symmetric inventory response S^* at equilibrium must be in the interval $[L(q^s), U(q^s)]$, whose proof does not depend on the shifting cost structure, hence remains valid. Lemma 11–13 prove that the proposed policy is optimal to a centralized platform control problem (controlling both shifting and inventory) with a constraint that inventory must be in the interval $[L(q^s), U(q^s)]$. Specifically, Lemma 11 shows that at equilibrium, there should not coexist excess demand and leftover inventory in the system. One can see from the proof of Lemma 11 that this remains valid as long as the marginal shifting cost is always smaller than the platform’s sales revenue r_p (which is true by Assumption 1). Then we show the optimality of the shifting policy’s defluctuating structure in Lemma 12. One can also easily verify that the proof of Lemma 12 extends with the new shifting cost structure. It then only remains to find the optimal value of constrained inventory level $q^B \in [L(q^s), U(q^s)]$ in Lemma 13, utilizing the optimality of the defluctuating structure in the previous step. This is straightforward algebra since the problem now reduces to a constraint concave maximization problem (concavity follows by Assumption 1) with a convex feasible set.

□

F.2. Proof of results in Section E.2

Additional notation. Denote by $\bar{\nu}_i$ the induced normalized conditional measure over $\mathbb{R}_+^2$ for type $i \in \mathcal{K}$, namely, $\bar{\nu}_i(A) = \frac{1}{\lambda_i} \bar{\nu}(A \times \{i\})$ for any $A \in \mathbb{R}_+^2$. Similar to the base model, here $\bar{\nu}_i$ is a product measure, and we formally define ν_i as the marginal measure in $\bar{\nu}_i$ of the inventory levels. In the special case that $\nu_i = \delta(q_i)$ for some $q_i \geq 0, i = 1, \dots, K$, namely, the sellers of the same type set the same type-specific inventory level, we sometimes adopt the notation $\nu = \delta(\vec{q})$ in place of ν for simplicity.

We first prove Lemma 14. We break it down into a sequence of lemmas. Analogous to Definition 5 in the base model, we first define an efficiency notion of the shifting-policy, which states that the platform either shifts demand to a seller, or shifts his demand to other sellers, but cannot do both.

DEFINITION 10 (EFFICIENT POLICIES). A policy ϕ_F is efficient if for any $\nu, \mathbf{z}$,

$$\int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}') \cdot \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^\nu(\mathbf{z}', \mathbf{z}) d\bar{\nu}(\mathbf{z}') = 0$$

LEMMA 15. *Under Assumption 3, it is WLOG to only consider efficient policies.*

Proof For any admissible policy ϕ_F that is not efficient, we construct an admissible efficient policy ϕ'_F that achieves (possibly weakly) larger platform profit. A policy ϕ_F is not efficient if there exists ν and $\mathbf{z}$

$$\int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^\nu(\mathbf{z}', \mathbf{z}) d\bar{\nu}(\mathbf{z}') > 0, \quad \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}') > 0,$$

More specifically,

$$\sum_{i=1}^K \int_{\mathbb{R}_+^2} \phi_F^\nu(\mathbf{z}', \mathbf{z}) I(z'_t = i) d\bar{\nu}(\mathbf{z}') > 0, \quad \sum_{i=1}^K \int_{\mathbb{R}_+^2} \phi_F^\nu(\mathbf{z}, \mathbf{z}') I(z'_t = i) d\bar{\nu}(\mathbf{z}') > 0,$$

equivalently,

$$\sum_{i=1}^K \int_{\mathbb{R}_+^2} \lambda_i \phi_F^\nu(\mathbf{z}', \mathbf{z}) d\bar{\nu}_i(\mathbf{z}') > 0, \quad \sum_{i=1}^K \int_{\mathbb{R}_+^2} \lambda_i \phi_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}_i(\mathbf{z}') > 0,$$

which implies that there exists type $j, k \in \mathcal{K}$, such that

$$\int_{\mathbb{R}_+^2} \phi_F^\nu(\mathbf{z}', \mathbf{z}) d\bar{\nu}_j(\mathbf{z}') = \epsilon_j > 0, \quad \int_{\mathbb{R}_+^2} \phi_F^\nu(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_k(\mathbf{z}'') = \epsilon_k > 0.$$

WLOG we assume $\epsilon_j \geq \epsilon_k > 0$, the argument for the other direction is similar. We construct ϕ'_F according to the following:

$$\begin{aligned} \phi_F'^\nu(\mathbf{z}, \mathbf{z}'') &= 0, & \forall \mathbf{z}'' \text{ s.t. } z''_t &= k \\ \phi_F'^\nu(\mathbf{z}', \mathbf{z}) &= \left(1 - \frac{\epsilon_k}{\epsilon_j}\right) \phi_F^\nu(\mathbf{z}', \mathbf{z}), & \forall \mathbf{z}' \text{ s.t. } z'_t &= j \\ \phi_F'^\nu(\mathbf{z}', \mathbf{z}'') &= \frac{1}{\epsilon_j} \phi_F^\nu(\mathbf{z}', \mathbf{z}) \phi_F^\nu(\mathbf{z}, \mathbf{z}''), & \forall \mathbf{z}', \mathbf{z}'' \text{ s.t. } z'_t &= j, z''_t = k \\ \phi'_F &= \phi_F, & \text{otherwise} \end{aligned}$$

The so-defined ϕ'_F is obviously non-negative since $\epsilon_j \geq \epsilon_k$. To ensure admissibility of ϕ'_F , it suffices to show that ϕ'_F induces the same post-shift demand ϕ as ϕ_F . To this end, we need to check ϕ' at (1) $\mathbf{z}$, (2) $\mathbf{z}'$ for all $z'_t = j$ and (3) $\mathbf{z}''$ for all $z''_t = k$. For (1), we have

$$\phi'^V(\mathbf{z}) = \phi^V(\mathbf{z}) - \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_k(\mathbf{z}'') + \frac{\epsilon_k}{\epsilon_j} \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}', \mathbf{z}) d\bar{\nu}_j(\mathbf{z}') = \phi^V(\mathbf{z}) - \epsilon_k + \epsilon_k = \phi^V(\mathbf{z}).$$

Here in the first equality we account for the only changes of ϕ'_F , namely, the demand shifted to all $\mathbf{z}''$ of type k from $\mathbf{z}$ becomes 0, while the demand shifted into $\mathbf{z}$ from all $\mathbf{z}'$ of type j is reduced in proportion by $\frac{\epsilon_k}{\epsilon_j}$, and the second equality follows from definition. For (2), we have

$$\begin{aligned} \phi'^V(\mathbf{z}') &= \phi^V(\mathbf{z}) - \frac{\epsilon_k}{\epsilon_j} \phi_F^V(\mathbf{z}, \mathbf{z}') + \frac{1}{\epsilon_j} \int_{(\mathbb{R}^+)^2} \phi_F^V(\mathbf{z}', \mathbf{z}) \phi_F^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_k(\mathbf{z}'') \\ &= \phi^V(\mathbf{z}) - \frac{\epsilon_k}{\epsilon_j} \phi_F^V(\mathbf{z}', \mathbf{z}) + \frac{\epsilon_k}{\epsilon_j} \phi_F^V(\mathbf{z}', \mathbf{z}) = \phi^V(\mathbf{z}), \end{aligned}$$

where in the first equality we account for the changes of ϕ'_F , namely, the demand shifted from $\mathbf{z}'$ to $\mathbf{z}$ is reduced in proportion by $\frac{\epsilon_k}{\epsilon_j}$, and the demand shifted from $\mathbf{z}'$ to any $\mathbf{z}''$ of type k becomes $\frac{1}{\epsilon_j} \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}', \mathbf{z}) \phi_F^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_k(\mathbf{z}'')$, and the second equality follows from definition. Finally, for (3), we have

$$\begin{aligned} \phi'^V(\mathbf{z}'') &= \phi^V(\mathbf{z}'') - \phi_F(\mathbf{z}, \mathbf{z}'') + \frac{1}{\epsilon_j} \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}', \mathbf{z}) \phi_F^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_j(\mathbf{z}') \\ &= \phi^V(\mathbf{z}'') - \phi_F(\mathbf{z}, \mathbf{z}'') + \phi_F(\mathbf{z}, \mathbf{z}'') = \phi^V(\mathbf{z}''), \end{aligned}$$

where in the first equality we account for the changes of post-shift demand of seller z'' under ϕ'_F , namely, the demand shifted from $\mathbf{z}$ to $\mathbf{z}''$ becomes 0, and the demand shifted from any $\mathbf{z}'$ of type j to $\mathbf{z}''$ becomes $\frac{1}{\epsilon_j} \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}', \mathbf{z}) \phi_F^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_j(\mathbf{z}')$, and the second equality follows from definition. Thus ϕ'_F remain admissible, and it induces the same post-shift demand ϕ as ϕ_F . In particular, we observe that under ϕ'_F we have

$$\int_{\mathbb{R}_+^2} \phi'^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_k(\mathbf{z}'') = 0.$$

Now since ϕ_F also induces ϕ , the first two terms in the platform's objective (25) remains the same as the sellers' inventory responses remain the same (which only depend on their post-shift demands). We now argue that the shifting cost becomes smaller under ϕ'_F . Indeed, the difference between the shifting cost under ϕ'_F and that under ϕ_F can be accounted for as follows

$$-\beta_{ik} \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}, \mathbf{z}'') d\bar{\nu}_k(\mathbf{z}'') - \beta_{ji} \frac{\epsilon_k}{\epsilon_j} \int_{\mathbb{R}_+^2} \phi_F^V(\mathbf{z}', \mathbf{z}) d\bar{\nu}_j(\mathbf{z}')$$

$$\begin{aligned}
& + \beta_{jk} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{1}{\epsilon_j} \phi_F^v(\mathbf{z}', \mathbf{z}) \phi_F^v(\mathbf{z}, \mathbf{z}'') d\bar{v}_j(\mathbf{z}') d\bar{v}_k(\mathbf{z}'') \\
& = -\beta_{ik} \epsilon_k - \beta_{ji} \frac{\epsilon_k}{\epsilon_j} + \beta_{jk} \frac{\epsilon_j \times \epsilon_k}{\epsilon_j} \\
& = \epsilon_k (-\beta_{ik} - \beta_{ji} + \beta_{jk}) \leq 0. \quad (\text{by Assumption 3})
\end{aligned}$$

The above procedure eliminates the inefficiency at $\mathbf{z}$ while keeping the post-shift demands unchanged for all sellers, yet reducing the shifting cost. It can be repeated until no such $\mathbf{z}$ exists, namely, the policy becomes efficient. We thus complete our proof. $\square$

LEMMA 16. *Suppose ϕ is a feasible post-shift demand. Then there exists an efficient admissible policy ϕ_F which induces ϕ , and the platform's expected profit under ϕ_F and system's state ν can be stated in terms of ϕ , in particular,*

$$v_p^{\phi_F, \nu} = v_p^{\phi, \nu} = \int_{\mathbb{R}_+^2 \times \mathcal{K}} r_p^{z_t}(\phi^v(\mathbf{z}) \wedge z_q) d\bar{\nu}(\mathbf{z}) - \int_{\mathbb{R}_+^2 \times \mathcal{K}} c_p^{z_t} z_q d\nu((z_q, z_t)) - C(\phi, \nu),$$

where

$$\begin{aligned}
C(\phi, \nu) &= \min_{x_{ij} \geq 0} \sum_{i=1}^K \sum_{j=1}^K \beta_{ij} x_{ij} \\
s.t. \quad & \sum_{j=1}^K x_{ji} = \lambda_i \int_{\mathbb{R}_+^2} (\phi^v((d_i, q_i, i)) - d_i)^+ d\bar{v}_i((d_i, q_i)), \quad \forall i \in \mathcal{K} \\
& \sum_{j=1}^K x_{ij} = \lambda_i \int_{\mathbb{R}_+^2} (d_i - \phi^v((d_i, q_i, i)))^+ d\bar{v}_i((d_i, q_i)), \quad \forall i \in \mathcal{K}.
\end{aligned} \tag{29}$$

No admissible policy ϕ'_F that induces ϕ can achieve strictly larger platform profit.

REMARK 1. When ν is a Dirac measure with respect to a vector $\vec{q}$, we adopt the notation $C(\phi, \vec{q})$ for simplicity.

Proof We first prove $v_p^{\phi, \nu}$ is an upper bound on the platform's profit under any efficient policy ϕ'_F that induces ϕ and at the system's state ν . Indeed, assume ϕ'_F is such an efficient policy. Since ϕ'_F induces ϕ , hence the first two terms in the platform's objective and the seller's responses are exactly as were given in the definition of $v_p^{\phi, \nu}$. It remains to show $C(\phi, \nu)$, as defined in (29), serves as a lower bound on the shifting cost incurred by ϕ'_F . By Definition 9, the post-shift demand of a seller with profile $\mathbf{z}$ is

$$\phi^v(\mathbf{z}) = z_d - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F'^v(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}') + \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F'^v(\mathbf{z}', \mathbf{z}) d\bar{\nu}(\mathbf{z}').$$

Since ϕ'_F is efficient, it must be that for any $\mathbf{z} \in \mathbb{R}_+^2 \times \mathcal{K}$,

$$(\phi^v(\mathbf{z}) - z_d)^+ = \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi'^v_F(\mathbf{z}', \mathbf{z}) d\bar{\nu}(\mathbf{z}')$$

and

$$(z_d - \phi^v(\mathbf{z}))^+ = \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi'^v_F(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}').$$

Thus

$$x_{ij} = \lambda_i \lambda_j \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \phi'^v_F((d_i, q_i, i), (d_j, q_j, j)) d\bar{\nu}_i((d_i, q_i)) d\bar{\nu}_j((d_j, q_j)), \quad \forall i, j \in \mathcal{K} \quad (30)$$

satisfy the constraints of (29) and is a feasible solution to (29). Observe that the shifting cost under ϕ'_F is $\sum_{i=1}^K \sum_{j=1}^K \beta_{ij} x_{ij}$. Thus $C(\phi, \nu)$ is a lower bound on the shifting cost incurred by any efficient admissible policy that induces ϕ .

On the other hand, we can construct an efficient policy ϕ_F that induces ϕ as the post-shift demand and incurs $C(\phi, \nu)$ as its shifting cost. Since ϕ is feasible, it is induced by an admissible policy. By Lemma 15, there then exists an efficient policy ϕ'_F that induces ϕ as its post-shift demand (thus the seller's responses remain unchanged). By the argument above, there thus exists $\{x_{ij}\}_{1 \leq i, j \leq K}$ defined using ϕ'_F that is feasible to Problem (29). Therefore, the feasible region of the LP is non-empty. Let $\tilde{x}_{ij}$ be an optimal solution to (29). Define ϕ_F :

$$\phi_F^v(\mathbf{z}, \mathbf{z}') = \begin{cases} \frac{\tilde{x}_{z_t, z'_t} (z_d - \phi^v(\mathbf{z}))^+ (\phi^v(\mathbf{z}') - z'_d)^+}{\lambda_{z_t} \lambda_{z'_t} \int_{\mathbb{R}_+^2} (\phi^v((\mathbf{y}, z'_t)) - y_d)^+ d\bar{\nu}_{z'_t}(\mathbf{y}) \cdot \int_{\mathbb{R}_+^2} (y_d - \phi^v((\mathbf{y}, z_t)))^+ d\bar{\nu}_{z_t}(\mathbf{y})} & \text{if } \int_{\mathbb{R}_+^2} (\phi^v((\mathbf{y}, z'_t)) - y_d)^+ d\bar{\nu}_{z'_t}(\mathbf{y}) \cdot \int_{\mathbb{R}_+^2} (y_d - \phi^v((\mathbf{y}, z_t)))^+ d\bar{\nu}_{z_t}(\mathbf{y}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that the so-defined ϕ_F induced ϕ as its post-shift demand, and that

$$\lambda_i \lambda_j \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \tilde{\phi}_F^v((d_i, q_i, i), (d_j, q_j, j)) d\bar{\nu}_i((d_i, q_i)) d\bar{\nu}_j((d_j, q_j)) = \tilde{x}_{ij}.$$

In other words, the shifting cost under ϕ_F is $\sum_{i=1}^K \sum_{j=1}^K \beta_{ij} \tilde{x}_{ij}$, which is equal to $C(\phi, \nu)$ by optimality of $\tilde{x}_{ij}$. The lemma hence follows.

□

Proof of Lemma 14 The proof follows immediately from Lemma 15 and Lemma 16. □

We next set to prove Theorem 5. We first state and prove an analogy of Lemma 11, which establishes that at equilibrium, there should not remain both unsatisfied demand and leftover inventory at the same time.

LEMMA 17. Let $(\phi, \vec{q})$ be the pair of post-shift demand and the corresponding equilibrium inventory responses induced by an optimal shifting policy ϕ_F . Then $(\phi, \vec{q})$ satisfies

1. if $\sum_{i=1}^K \lambda_i q_i \leq \sum_{i=1}^K \lambda_i \mathbb{E} D_i$, then $\phi^{\delta(\vec{q})}((d_i, q_i, i)) \geq q_i$ for all i, d_i a.s.;
2. if $\sum_{i=1}^K \lambda_i q_i > \sum_{i=1}^K \lambda_i \mathbb{E} D_i$, then $\phi^{\delta(\vec{q})}((d_i, q_i, i)) \leq q_i$ for all i, d_i a.s.

Proof Suppose that $\sum_{i=1}^K \lambda_i q_i \leq \sum_{i=1}^K \lambda_i \mathbb{E} D_i$ and that $\phi^{\delta(\vec{q})}((d_i, q_i, i)) < q_i$ for some (d_i, q_i, i) . Let I be the set of such (d_i, q_i, i) . Let J denote all (d_j, q_j, j) such that $\phi^{\delta(\vec{q})}((d_j, q_j, j)) \geq q_j$. Denote by $\epsilon_J := \int_J (\phi^{\delta(\vec{q})}(\mathbf{z}') - q_{z'_t}) d\bar{\nu}(\mathbf{z}')$ and $\epsilon_I := \int_I (q_{z_t} - \phi^{\delta(\vec{q})}(\mathbf{z})) d\bar{\nu}(\mathbf{z})$. Observe that

$$\begin{aligned} \epsilon_J - \epsilon_I &= \int_J (\phi^{\delta(\vec{q})}(\mathbf{z}') - q_{z'_t}) d\bar{\nu}(\mathbf{z}') - \int_I (q_{z_t} - \phi^{\delta(\vec{q})}(\mathbf{z})) d\bar{\nu}(\mathbf{z}) \\ &= \int_{I \cup J} \phi^{\delta(\vec{q})}(\mathbf{z}) d\bar{\nu}(\mathbf{z}) - \int_{I \cup J} q_{z_t} d\bar{\nu}(\mathbf{z}) \\ &= \sum_{i=1}^K \lambda_i \mathbb{E} D_i - \sum_{i=1}^K \lambda_i q_i \geq 0, \end{aligned}$$

thus $\epsilon_J \geq \epsilon_I$. If $\epsilon_I > 0$, we consider a new admissible policy $\bar{\phi}_F$, defined as follows.

$$\bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}', \mathbf{z}) = \begin{cases} \phi_F^{\delta(\vec{q})}(\mathbf{z}', \mathbf{z}) + \frac{1}{\epsilon_J} (\phi^{\delta(\vec{q})}(\mathbf{z}') - q_{z'_t}) (q_{z_t} - \phi^{\delta(\vec{q})}(\mathbf{z})) & \text{if } \mathbf{z} \in I, \mathbf{z}' \in J \\ \phi_F^{\delta(\vec{q})}(\mathbf{z}', \mathbf{z}) & \text{otherwise.} \end{cases}$$

Let's first check admissibility. For any $\mathbf{z}' \in J$, the induced post-shift demand $\bar{\phi}^{\delta(\vec{q})}(\mathbf{z}') = \phi^{\delta(\vec{q})}(\mathbf{z}') - \int_I \frac{1}{\epsilon_J} (\phi^{\delta(\vec{q})}(\mathbf{z}') - q_{z'_t}) (q_{z_t} - \phi^{\delta(\vec{q})}(\mathbf{z})) d\bar{\nu}(\mathbf{z})$, which is equal to $\phi^{\delta(\vec{q})}(\mathbf{z}') - \frac{\epsilon_I}{\epsilon_J} (\phi^{\delta(\vec{q})}(\mathbf{z}') - q_{z'_t})$ by definition. Since $\epsilon_I \leq \epsilon_J$, the above is further lower bounded by $q_{z'_t}$. Hence $\bar{\phi}^{\delta(\vec{q})}(\mathbf{z}')$ is positive and remains larger than $q_{z'_t}$ for $\mathbf{z}' \in J$. The flow balance condition holds automatically. On the other hand, for any $\mathbf{z} \in I$, the post-shift demand under $\bar{\phi}_F$ becomes $\bar{\phi}^{\delta(\vec{q})}(\mathbf{z}) = \phi^{\delta(\vec{q})}(\mathbf{z}) + \int_J \frac{1}{\epsilon_J} (\phi^{\delta(\vec{q})}(\mathbf{z}') - q_{z'_t}) (q_{z_t} - \phi^{\delta(\vec{q})}(\mathbf{z})) d\bar{\nu}(\mathbf{z}')$, which is equal to q_{z_t} by definition. For all other v and $\mathbf{z}''$, $\bar{\phi}^v(\mathbf{z}'') = \phi^v(\mathbf{z}'')$ remains unchanged. The shifting feasibility of $\bar{\phi}_F$ thus follows from that of ϕ_F . Thus, $\bar{\phi}_F$ is admissible. Furthermore, under $\bar{\phi}_F$, the condition in the statement of the lemma is satisfied. It remains to prove that the IR condition in (25) holds. Observe that the sellers' inventory response under $\bar{\phi}_F$ remains to be $\vec{q}$. This is true since a type- i (for any i) seller's expected profit remains the same when his choice of inventory is different from q_i (since the policy is unadjusted for these sample paths), and non-decreases when his choice of inventory is q_i (this follows from the fact that whenever his post-shift demand decreases (i.e. when the seller is in J), it never becomes smaller than q_i , hence the demand still clears up the inventory and the profit does not change).

Combining the above, $\bar{\phi}_F \in \Phi_F$ and $\vec{q}$ remains the equilibrium inventory responses under $\bar{\phi}_F$ at which the IR condition holds.

We now show that $\bar{\phi}_F$ is a strictly better policy than ϕ_F as the former generates a larger platform's profit. To this end, let's first bound the new shifting cost.

$$\begin{aligned}
& \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&= \int_I \int_J \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') + \int_I \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus J} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&\quad + \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus I} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&= \int_I \int_J \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') + \int_I \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus J} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&\quad + \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus I} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&= \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&\quad + \int_I \int_J \beta_{\mathbf{z}_t, \mathbf{z}'_t} \left(\bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') - \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') \right) d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&< \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') + \int_I r_p^i \left(q_i - \phi^{\delta(\vec{q})}(d_i, q_i, i) \right) d\bar{\nu}(d_i, q_i, i),
\end{aligned}$$

where the second equality follows from the construction of $\bar{\phi}_F$, and the last inequality follows from Assumption 2 and the definition of $\bar{\phi}_F$. The objective value of problem (25) under $\bar{\phi}$ (with the same inventory response $\vec{q}$, system inventory measure $\nu = \delta(\vec{q})$) thus becomes

$$\begin{aligned}
v_p^{\bar{\phi}_F, \vec{q}} &= \int_{\mathbb{R}_+^2 \times \mathcal{K}} r_p^i (\bar{\phi}^\nu((d_i, q_i, i)) \wedge q_i) d\bar{\nu}((d_i, q_i, i)) - \sum_{i=1}^K \lambda_i c_p^i q_i \\
&\quad - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&= \int_{\mathbb{R}_+^2 \times \mathcal{K}} r_p^i q_i d\bar{\nu}((d_i, q_i, i)) - \sum_{i=1}^K \lambda_i c_p^i q_i - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \bar{\phi}_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&> \int_{\mathbb{R}_+^2 \times \mathcal{K}} r_p^i q_i d\bar{\nu}((d_i, q_i, i)) - \sum_{i=1}^K \lambda_i c_p^i q_i - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\
&\quad - \int_I r_p^i \left(q_i - \phi^{\delta(\vec{q})}((d_i, q_i, i)) \right) d\bar{\nu}((d_i, q_i, i)) \\
&= \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus I} r_p^i q_i d\bar{\nu}((d_i, q_i, i)) - \sum_{i=1}^K \lambda_i c_p^i q_i - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_t, \mathbf{z}'_t} \phi_F^{\delta(\vec{q})}(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}')
\end{aligned}$$

$$+ \int_I r_p^i \phi^{\delta(\vec{q})}((d_i, q_i, i)) d\bar{\nu}((d_i, q_i, i)).$$

On the other hand, the objective value of problem (25) under ϕ (with inventory response $\vec{q}$, system inventory measure $\nu = \delta(\vec{q})$) is

$$\begin{aligned} v_p^{\phi_F, \vec{q}} &= \int_{\mathbb{R}_+^2 \times \mathcal{K}} r_p^i (\phi^\nu((d_i, q_i, i)) \wedge q_i) d\bar{\nu}((d_i, q_i, i)) - \sum_{i=1}^K \lambda_i c_p^i q_i \\ &\quad - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_i, \mathbf{z}'_i} \phi_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\ &= \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus I} r_p^i (\phi^\nu((d_i, q_i, i)) \wedge q_i) d\bar{\nu}((d_i, q_i, i)) + \int_I r_p^i (\phi^\nu((d_i, q_i, i)) \wedge q_i) d\bar{\nu}((d_i, q_i, i)) \\ &\quad - \sum_{i=1}^K \lambda_i c_p^i q_i - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_i, \mathbf{z}'_i} \phi_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}') \\ &= \int_{(\mathbb{R}_+^2 \times \mathcal{K}) \setminus I} r_p^i q_i d\bar{\nu}((d_i, q_i, i)) + \int_I r_p^i \phi^\nu((d_i, q_i, i)) d\bar{\nu}((d_i, q_i, i)) - \sum_{i=1}^K \lambda_i c_p^i q_i \\ &\quad - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \beta_{\mathbf{z}_i, \mathbf{z}'_i} \phi_F^\nu(\mathbf{z}, \mathbf{z}') d\bar{\nu}(\mathbf{z}) d\bar{\nu}(\mathbf{z}'). \end{aligned}$$

Comparing $v_p^{\phi_F, \vec{q}}$ with the bounds derived for $v_p^{\tilde{\phi}_F, \vec{q}}$ above, we get the desired result $v_p^{\tilde{\phi}_F, \vec{q}} > v_p^{\phi_F, \vec{q}}$. Thus under the assumption $\epsilon_I > 0$, we can strictly improve the platform's profit with a modified policy, and the original policy ϕ_F can not be optimal. By contradiction $\epsilon_I > 0$ cannot happen. $\epsilon_I = 0$ implies the condition in the statement of the lemma, thus completing the proof.

The other case $\sum_{i=1}^K \lambda_i q_i > \sum_{i=1}^K \lambda_i \mathbb{E} D_i$ is similar hence we omit its proof for brevity. $\square$

Lemma 17 states that in a centralized platform, excess demand and leftover inventory should not coexist on the platform even when the demand is requesting a different item initially, since if so, the platform could always earn an extra profit of $r_p - \beta > 0$ per unit by fulfilling this excess demand (which generates a revenue of r_p) using a leftover inventory (with substitution cost of β).

Utilizing Lemma 15–17, we can further sharpen the structural properties of the optimal demand shifting policy.

LEMMA 18. *There exists an optimal shifting policy ϕ_F to problem (25), with the pair of induced post-shift demand and the corresponding equilibrium inventory responses $(\phi, \vec{q})$ that has the following defluctuating structure:*

1. if $\sum_{i=1}^K \lambda_i q_i \leq \sum_{i=1}^K \lambda_i \mathbb{E} D_i$, then $\phi^{\delta(\vec{q})}((d_i, q_i, i)) = q_i$ for all $d_i \leq y_i$ for some $y_i \geq q_i$, and $\phi^{\delta(\vec{q})}((d_i, q_i, i)) = d_i$ for all $d_i > y_i$, $\forall i \in \mathcal{K}$;

2. if $\sum_{i=1}^K \lambda_i q_i > \sum_{i=1}^K \lambda_i \mathbb{E} D_i$, then $\phi^{\delta(\vec{q})}((d_i, q_i, i)) = q_i$ for all $d_i \geq y_i$ for some $y_i \in [0, q_i]$, and $\phi^{\delta(\vec{q})}((d_i, q_i, i)) = d_i$ for all $d_i < y_i$, $\forall i \in \mathcal{K}$.

Proof In case 1, when $d_i \leq q_i$, $\phi^{\delta(\vec{q})}((d_i, q_i, i)) \geq q_i \geq d_i$ by Lemma 17, and further $\phi^{\delta(\vec{q})}((d_i, q_i, i)) \leq q_i$ by shifting feasibility, hence equality must prevail. When $d_i > q_i$, first $\phi^{\delta(\vec{q})}((d_i, q_i, i)) \geq q_i$ by Lemma 17. Then if $\phi^{\delta(\vec{q})}((d_i, q_i, i)) > d_i$, this violates shifting feasibility since $d_i > q_i$. Therefore it must be that $\phi^{\delta(\vec{q})}((d_i, q_i, i)) \in [q_i, d_i]$ when $d_i > q_i$. Consider an optimal policy ϕ_F^* , let $\vec{q}^*$ be the corresponding inventory response. We show that we can WLOG modify ϕ_F^* to $\tilde{\phi}_F^*$ with the structure given in the statement of the lemma, such that the inventory response remains to be $\vec{q}^*$ and the platform's total profit is unchanged. Let $y_i^* \geq q_i^*$ be such that

$$\int_{q_i^*}^{y_i} (d_i - q_i^*) dF_i(d_i) = \int_{q_i^*}^{\infty} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - q_i^*) dF_i(d_i). \quad (31)$$

Its existence is guaranteed since $\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \in [q_i^*, d_i]$ when $d_i > q_i^*$.

We first define an intermediate policy $\tilde{\phi}_F$ as follows

$$\tilde{\phi}_F^{\delta(\vec{q}^*)}(\mathbf{z}, \mathbf{z}') = \begin{cases} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}, \mathbf{z}') + \frac{(z'_d - \phi^{*,\delta(\vec{q}^*)}(\mathbf{z}'))(-q_{z'_t}^* + \phi^{*,\delta(\vec{q}^*)}(\mathbf{z}))}{\int_{y_i^*}^{\infty} (y_d - \phi^{*,\delta(\vec{q}^*)}((y_d, q_{z'_t}^*, z'_t))) dF_{z'_t}(y_d)} & \text{if } z_t = z'_t, z'_d > y_i^*, z_d \leq y_i^* \\ \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}, \mathbf{z}') & \text{otherwise.} \end{cases}$$

It can be verified that $\tilde{\phi}_F^{\delta(\vec{q}^*)}$ is admissible and it induces a post-shift demand

$$\tilde{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) = \begin{cases} q_i^* & \text{if } d_i \leq y_i^* \\ d_i & \text{if } d_i > y_i^*. \end{cases} \quad (32)$$

Now, we use a technique similar to the proof of Lemma 16 to construct an efficient admissible policy $\tilde{\phi}_F^*$ that induces $\tilde{\phi}^*$ as its post-shift demand. Under $\tilde{\phi}_F^*$, the seller's profit is the same as under ϕ^* for any $\vec{q}$, since $\tilde{\phi}^*$ only modifies the excess demand on the equilibrium path $\vec{q}^*$ which does not affect the seller's profit, and is identical to ϕ^* for any other $\vec{q}$, hence the inventory response is unchanged. On the other hand, the platform's profit also remains the same under $\tilde{\phi}^*$. To see this, we observe that since $\vec{q}^*$ remains the sellers' inventory responses and $\tilde{\phi}^*$ only modifies the excess demand, the first two terms in the platform's profit objective (25) remain unchanged. Moreover, the shifting cost also remains the same, namely, $C(\tilde{\phi}^*, \delta(\vec{q}^*)) = C(\phi^*, \delta(\vec{q}^*))$. This is because by (31)

and (32), the constraints on the RHS of (29) remains the same as those under policy ϕ^* . Therefore, Lemma 16 indicates that the modified policy $\tilde{\phi}_F^*$ is optimal as well.

Now suppose the optimal policy ϕ_F^* induces a post-shift demand ϕ^* and its corresponding inventory response $\vec{q}^*$ that are of case 2. Similarly, we show that we can modify ϕ_F^* to another policy $\hat{\phi}_F^*$ that induces no worse platform profit. By Lemma 17, $\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \leq q_i^*$ for all d_i . We show that optimality of ϕ^* must also imply the following WLOG.

CLAIM 1. $\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) = q_i^*$ if $d_i \geq q_i^*$, and $\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \in [d_i, q_i^*]$ if $d_i \in [0, q_i^*]$, $\forall i \in \mathcal{K}$.

Proof We first show that one of two cases must prevail WLOG. That is, either

$$\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) = 0$$

for all $i \in \mathcal{K}$ (in which case we are done combining with Lemma 17); or

$$\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) > 0$$

and

$$\int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) = 0$$

for all $i \in \mathcal{K}$.

Suppose neither is true. Then it must be that $\int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) > 0$ and $\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) > 0$ for some $i \in \mathcal{K}$. We show that we can construct a (perhaps weakly) better policy than ϕ^* which incurs a (perhaps weakly) smaller shifting cost, while the other terms (the seller's problem and inventory response, the first two terms of the platform's profit) remain unchanged. Define

$$\Delta_i := \min \left(\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i), \right. \\ \left. \int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) \right), \forall i \in \mathcal{K}.$$

Let's denote

$$\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) := \delta_{1,i} \\ \int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) := \delta_{2,i}.$$

We specify the construction in the case $\delta_{2,i} \leq \delta_{1,i}$, namely $\Delta_i = \delta_{2,i}$, and the other case is similar. Let's denote by ϕ_F^* the corresponding admissible policy that induces ϕ^* . We first construct an intermediate admissible policy $\hat{\phi}_F$ as follows

$$\begin{aligned} \hat{\phi}_F^{\delta(\vec{q}^*)}(\mathbf{z}; d_i, q_i^*, i) &= 0, \quad \forall d_i < \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)), \mathbf{z} \in \mathbb{R}_+^2 \times \mathcal{K} \\ \hat{\phi}_F^{\delta(\vec{q}^*)}(\mathbf{z}; d_i, q_i^*, i) &= \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}; d_i, q_i^*, i) \\ &\quad + \frac{1}{\delta_{1,i}} \int_0^{q_i^*} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}; z_d, q_i^*, i) I\left(z_d < \phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)\right) dF_i(z_d) \left(d_i \wedge q_i^* - \phi^{*,\delta(\vec{q}^*)}(d_i, q_i^*, i)\right), \\ &\quad \forall d_i \wedge q_i^* \geq \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)), \mathbf{z} \in \mathbb{R}_+^2 \times \mathcal{K} \\ \hat{\phi}_F &= \phi_F^*, \quad \text{otherwise.} \end{aligned}$$

Let $\hat{\phi}^*$ denote the induced post-shift demand. Let's examine such a construction. First of all, the policy is admissible. Indeed, all $\hat{\phi}_F^v(\mathbf{z}, \mathbf{z}')$ are non-negative. Next, we show it satisfies shifting feasibility. For any $\mathbf{z}$ such that $z_t \neq i$, the difference between $\hat{\phi}^{*,\vec{q}^*}(\mathbf{z})$ and $\phi^{*,\vec{q}^*}(\mathbf{z})$ can be accounted for as

$$\begin{aligned} \phi^{*,\vec{q}^*}(\mathbf{z}) - \hat{\phi}^{*,\vec{q}^*}(\mathbf{z}) &= \int_0^{q_i^*} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}; z_d, q_i^*, i) I\left(z_d < \phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)\right) dF_i(z_d) \\ &\quad - \frac{1}{\delta_{1,i}} \left(\int_0^{q_i^*} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}; z_d, q_i^*, i) I\left(z_d < \phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)\right) dF_i(z_d) \right) \\ &\quad \times \left(\int_0^\infty \left(d_i \wedge q_i^* - \phi^{*,\delta(\vec{q}^*)}(d_i, q_i^*, i)\right) I\left(d_i \wedge q_i^* \geq \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))\right) dF_i(z_d) \right). \end{aligned}$$

Observe that

$$\begin{aligned} &\int_0^\infty \left(d_i \wedge q_i^* - \phi^{*,\delta(\vec{q}^*)}(d_i, q_i^*, i)\right) I\left(d_i \wedge q_i^* \geq \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))\right) dF_i(z_d) \\ &= \int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^\infty (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) = \delta_{1,i}. \end{aligned}$$

We thus conclude that $\phi^{*,\vec{q}^*}(\mathbf{z}) = \hat{\phi}^{*,\vec{q}^*}(\mathbf{z})$. For any $\mathbf{z}$ such that $z_t = i$, on the one hand, the above argument still applies hence the amount of shifting-outs are unchanged under $\hat{\phi}_F$ comparing to ϕ_F^* . The difference between $\hat{\phi}_{\text{FULL}}^*(\mathbf{z})$ and $\phi_{\text{FULL}}^*(\mathbf{z})$ is determined by the change in the amount of shifting-ins. Consider two cases. (1), $z_d < \phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)$. By design the shifting-in under $\hat{\phi}_F$ is zero. Since ϕ_F^* is efficient, $z_d < \phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)$ implies that there is no shifting-out demand from $\mathbf{z}$ under ϕ_F^* . Combining the above, we conclude that if $z_d < \phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)$ then $\hat{\phi}^{*,\vec{q}^*}(\mathbf{z}) = z_d$. (2), $z_d \wedge q_i^* \geq$

$\phi^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i)$. In this case, by efficiency of ϕ_F^* , it must be that $\int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}; z_d, q_i^*, i) d\bar{\nu}(\mathbf{z}) = 0$.

Hence

$$\begin{aligned} & \hat{\phi}^{*,\vec{q}^*}(\mathbf{z}) \\ &= z_d - \int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^{*,\delta(\vec{q}^*)}(z_d, q_i^*, i; \mathbf{z}') d\bar{\nu}(\mathbf{z}') \\ & \quad + \frac{1}{\delta_{1,i}} \int_{\mathbb{R}_+^2 \times \mathcal{K}} \int_0^{q_i^*} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}'; z_d'', q_i^*, i) I\left(z_d'' < \phi^{*,\delta(\vec{q}^*)}(z_d'', q_i^*, i)\right) dF_i(z_d'') d\bar{\nu}(\mathbf{z}') \\ & \quad \left(z_d \wedge q_i^* - \phi^{*,\delta(\vec{q}^*)}(\mathbf{z})\right) \\ &= \phi^{*,\vec{q}^*}(\mathbf{z}) + \frac{\delta_{2,i}}{\delta_{1,i}} \left(z_d \wedge q_i^* - \phi^{*,\delta(\vec{q}^*)}(\mathbf{z})\right) \leq z_d \wedge q_i^*, \end{aligned}$$

where in the last equality, we use the fact that (by the efficiency of ϕ_F^*),

$$\int_{\mathbb{R}_+^2 \times \mathcal{K}} \phi_F^{*,\delta(\vec{q}^*)}(\mathbf{z}'; z_d'', q_i^*, i) d\bar{\nu}(\mathbf{z}') = \left(-z_d'' + \phi^{*,\delta(\vec{q}^*)}(z_d'', q_i^*, i)\right)^+$$

and that

$$\int_0^{q_i^*} \left(-z_d'' + \phi^{*,\delta(\vec{q}^*)}(z_d'', q_i^*, i)\right) I\left(z_d'' < \phi^{*,\delta(\vec{q}^*)}(z_d'', q_i^*, i)\right) dF_i(z_d'') = \delta_{2,i}$$

Combining the above, we have verified that $\hat{\phi}_F$ is an admissible policy which induces $\hat{\phi}^*$ with the desired property as given in the claim. Furthermore, we have

$$\int_0^{q_i^*} (\hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) = \int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) - \Delta_i$$

and

$$\begin{aligned} & \int_0^{q_i^*} (d_i - \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) \\ &= \int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) - \Delta_i. \end{aligned}$$

We remark that the so-constructed $\hat{\phi}_F$ is not necessarily efficient. However, by Lemma 15 and Lemma 16, there exists an efficient admissible policy $\hat{\phi}_F^*$ which induces $\hat{\phi}^*$ and the platform's profit under $\hat{\phi}_F^*$ achieves what is given in Lemma 16.

We now check the platform's profit under $\hat{\phi}_F^*$. Recall that $\hat{\phi}^*$ satisfies Lemma 17, which implies that the seller still satisfies all his demands. Furthermore,

$$\int_{\mathbb{R}_+} \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) dF_i(d_i) = \int_0^{q_i^*} \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) dF_i(d_i) + \int_{q_i^*}^{\infty} \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) dF_i(d_i)$$

$$\begin{aligned}
&= \int_0^{q_i^*} d_i dF_i(d_i) - \int_0^{q_i^*} \left(d_i - \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \right) dF_i(d_i) \\
&\quad + q_i^* \mathbb{P}(D_i \geq q_i^*) - \int_{q_i^*}^{\infty} \left(q_i^* - \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \right) dF_i(d_i) \\
&= \int_0^{q_i^*} d_i dF_i(d_i) - \int_0^{q_i^*} \left(d_i - \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \right)^+ dF_i(d_i) \\
&\quad + \int_0^{q_i^*} \left(\hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i \right)^+ dF_i(d_i) \\
&\quad + q_i^* \mathbb{P}(D_i \geq q_i^*) - \int_{q_i^*}^{\infty} \left(q_i^* - \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \right) dF_i(d_i).
\end{aligned}$$

By construction, all $\hat{\phi}$ can be replaced by ϕ in the above equation and does not change the value, namely, the above is equal to

$$\begin{aligned}
&= \int_0^{q_i^*} d_i dF_i(d_i) - \int_0^{q_i^*} \left(d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \right)^+ dF_i(d_i) \\
&\quad + \int_0^{q_i^*} \left(\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i \right)^+ dF_i(d_i) \\
&\quad + q_i^* \mathbb{P}(D_i \geq q_i^*) - \int_{q_i^*}^{\infty} \left(q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) \right) dF_i(d_i).
\end{aligned}$$

Now we reverse the previous analysis and the above is equal to $\int_{\mathbb{R}_+} \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) dF_i(d_i)$. Thus we reach the conclusion that $\int_{\mathbb{R}_+} \hat{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) dF_i(d_i) = \int_{\mathbb{R}_+} \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) dF_i(d_i)$. Combining with the fact that the seller satisfies all his demand under both $\hat{\phi}^*$ and ϕ^* due to Lemma 17, we thus conclude that the seller's profit remains unchanged at q_i^* under either ϕ^* or $\hat{\phi}^*$. Since $\vec{q}^*$ is the equilibrium response with respect to ϕ^* , and ϕ^* and $\hat{\phi}^*$ are the same for all other inventory levels $\vec{q} \neq \vec{q}^*$, the above implies that the seller's inventory response under $\hat{\phi}^*$ remains $\vec{q}^*$. Since both the total sales and the inventory levels remain unchanged, the first two terms in the platform's objective also remains the same.

We now examine the shifting cost. Our construction of $\hat{\phi}_F$ never changes the shifting cost, as it only changes how shifted demand is allocated among different sellers within the same type, but the total amount of shifted demand between any two types remain the same. By upgrading $\hat{\phi}_F$ to $\hat{\phi}_F^*$, we further reduce the shifting cost according to the proof of Lemma 15, and the shifting cost incurred by $\hat{\phi}_F^*$ is smaller than that of the original policy.

Therefore, $\hat{\phi}^*$ is (perhaps weakly) better than ϕ^* for the platform, and we can WLOG only consider policies that belong to one of the two cases stated at the beginning of this proof. That is, either $\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) = 0$ for

all $i \in \mathcal{K}$; or $\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) > 0$ and $\int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) = 0$ for all $i \in \mathcal{K}$.

If it is the former case, combining with Lemma 17 the claim follows. Therefore we only need to consider the latter case. In this case, $\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) > 0$ and $\int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) = 0$ for all $i \in \mathcal{K}$. We now show that we can construct a modified policy $\bar{\phi}^*$ from ϕ^* with a lower inventory response hence lower platform inventory cost at equilibrium, while the platform's sales revenue and shifting costs remain unchanged.

Let $\bar{q}_i^* \in [0, q_i^*]$ be such that

$$\int_{\bar{q}_i^*}^{\infty} (d_i - \bar{q}_i^*) dF_i(d_i) = \int_0^{\infty} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i).$$

Existence of $\bar{q}_i^*$ is guaranteed since

$$\int_0^{q_i^*} (d_i - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)))^+ dF_i(d_i) + \int_{q_i^*}^{\infty} (q_i^* - \phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i))) dF_i(d_i) > 0$$

and $\int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i)^+ dF_i(d_i) = 0$. Next we show that $\bar{q}_i^*$ is the new inventory response under the modified policy $\bar{\phi}^*$ (modified only when $v = \delta(\vec{q}^*)$ or $v = \delta(\vec{\bar{q}}^*)$):

$$\bar{\phi}^{*,v}(d_i, \bar{q}_i^*, i) = \begin{cases} d_i & \text{if } d_i \leq \bar{q}_i^* \\ \bar{q}_i^* & \text{if } d_i > \bar{q}_i^*. \end{cases}$$

It's easy to verify that $\int_{\mathbb{R}_+} \bar{\phi}^{*,v}(d_i, \bar{q}_i^*, i) = \int_{\mathbb{R}_+} \phi^{*,v}((d_i, q_i^*, i))$ (for $v = \delta(\vec{q}^*)$ or $v = \delta(\vec{\bar{q}}^*)$ only) from the construction of $\bar{q}_i^*$ and $\bar{\phi}^*$. Therefore, by deviating from q_i^* to $\bar{q}_i^*$, a type- i seller can get the same sales revenue, lower inventory cost, hence a higher profit. The other off-path profits remain unchanged, so the inventory response under $\bar{\phi}^*$ is $\vec{\bar{q}}^*$. Also, the shifting cost $C(\bar{\phi}^{*,\delta(\vec{\bar{q}}^*)})$ when the inventory response is $\vec{\bar{q}}^*$ is the same as the previous shifting cost under ϕ^* , since the RHS of the constraints are unaffected by the construction of $\bar{\phi}^*$. Finally, the inventory cost under $\bar{\phi}^*$ is smaller than under ϕ^* , since $\vec{\bar{q}}^* \leq \vec{q}^*$. This means the platform's overall profit under $\bar{\phi}^*$ is better than under ϕ^* . The claim follows. $\square$

Given Claim 1, similar to case 1, we can WLOG derive a modified policy $\tilde{\phi}_F^*$ from ϕ_F^* with the desired structure such that the inventory response remains to be $\vec{q}^*$ and the platform's total profit is unchanged. More specifically, let $\tilde{\phi}^*$ denote the induced post-shift demand. Then it satisfies

$$\tilde{\phi}^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) = \begin{cases} d_i & \text{if } d_i \leq y_i^* \\ q_i^* & \text{if } d_i > y_i^*, \end{cases}$$

where $y_i^* \in [0, q_i^*]$ is such that

$$\int_{y_i^*}^{q_i^*} (q_i^* - d_i) dF_i(d_i) = \int_0^{q_i^*} (\phi^{*,\delta(\vec{q}^*)}((d_i, q_i^*, i)) - d_i) dF_i(d_i).$$

Existence of such y_i^* is guaranteed by Claim 1. The argument is similar to case 1 and we omit for brevity. $\square$

Theorem 5 follows immediately.

F.3. Proof of results in Section E.3

Proof of Theorem 6 The proof is analogous to the proof of Theorem 1–2 in Section C with the following adjustments. First, the IR constraint translates into an interval constraint of the seller's inventory response, as stated in Lemma 10. With the shared transshipment cost, the interval becomes $[\tilde{q}_1^t \wedge \tilde{q}_2^t, \tilde{q}_1^t \vee \tilde{q}_2^t]$. The second adjustment to the proof of the base model is in the proof of Lemma 13, where (19) is replaced by

$$\max_{q \in [\tilde{q}_1^t \wedge \tilde{q}_2^t, \tilde{q}_1^t \vee \tilde{q}_2^t]} \begin{cases} r_p \mathbb{E}D - c_p q - \alpha \beta \mathbb{E}(D - q)^+ & \text{if } q > \mathbb{E}D, \\ (r_p - c_p)q - \alpha \beta \mathbb{E}(q - D)^+ & \text{if } q \leq \mathbb{E}D. \end{cases}$$

The objective function is still concave as (19). Solving this constraint optimization problem by taking derivatives yields the desired result. $\square$

F.4. Proof of results in Section E.4

Proof of Theorem 7 The proof is analogous to the proof of Theorem 1–2 with the following adjustments. Recall that in the proof of the base model, we reduce the platform's problem to one of choosing a desired equilibrium inventory level, where we incorporate the optimal shifting cost given that inventory level in the objective function. This problem is expressed in (19). In this extension, (19) is modified to be

$$\max_{q \geq 0} \begin{cases} r_p \mathbb{E}D - c_p q - \beta \int_{q^s}^q (q - D) dF(D) & \text{if } q > U(q^s), \\ r_p \mathbb{E}D - c_p q - \beta \mathbb{E}(D - q)^+ & \text{if } \mathbb{E}D < q \leq U(q^s), \\ (r_p - c_p)q - \beta \mathbb{E}(q - D)^+ & \text{if } L(q^s) < q \leq \mathbb{E}D, \\ (r_p - c_p)q - \beta \int_q^{q^s} (D - q) dF(D) & \text{if } q < L(q^s). \end{cases} \quad (33)$$

The new expressions for $q > U(q^s)$ and $q < L(q^s)$ are due to the requirement that the inventory level q must be an inventory best response under the proactive demand shifting policy. That is, the choice

of q must satisfy the seller's critical fractile in the post-shift demand. When $q \in [L(q^s), U(q^s)]$, and under the demand shifting policy given by Lemma 11–12, the quantile at q in the post-shift demand satisfies the seller's critical fractile. When $q \notin [L(q^s), U(q^s)]$, the demand shifting policy given by Lemma 11–12 does not satisfy the critical fractile constraint at q . Therefore the platform must shift more demand to make sure that the post-shift probability mass at q is large enough, in particular, equals $1 - \frac{c_s}{r_s}$ if $q < L(q^s)$ and equals $\frac{c_s}{r_s}$ if $q > U(q^s)$. The most cost-effective way to achieve this is to let $\pi(d) = q$ for all $d \in [0, q^s]$, $\pi(d) = d$ for $d > q^s$ if $q < L(q^s)$; and let $\pi(d) = q$ for all $d \geq q^s$, $\pi(d) = d$ for $d < q^s$ if $q > U(q^s)$. One can easily verify that the resulting platform profit is given by (33). The rest of the proof is similar to the base model, where we utilize the concavity structure of (33) and take its derivative to find the maximizer. $\square$

Endnotes

¹The result is not affected if the seller whose demand is shifted out from pays $(1 - \alpha)\beta$.