E-Companian to "Efficient Nested Estimation of CoVaR: A Decoupled Approach"

EC.1. Examples of Derivatives Prices With Closed-form Expressions

EC.1.1. Example 1: European Call Option Under Geometric Brownian Motion

The analytical pricing formula of European call option with an underlying asset whose price S_t follows a geometric Brownian motion, given by

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_t$$

is given by

$$C(S_t,t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

and $N(\cdot)$ is the standard normal cumulative distribution function.

EC.1.2. Example 2: Barrier Up-and-Out Call Option Under Geometric Brownian Motion

The closed-form pricing expression of barrier up-and-out call option, with an underlying asset whose price S_t follows a geometric Brownian motion, is given by

$$\begin{aligned} &C(S_t,t) = S_t\left[N\left(\delta_+\left(T-t,\frac{S_t}{K}\right)\right) - N\left(\delta_+\left(T-t,\frac{S_t}{B}\right)\right)\right] \\ &-e^{-r(T-t)}K\left[N\left(\delta_-\left(T-t,\frac{S_t}{K}\right)\right) - N\left(\delta_-\left(T-t,\frac{S_t}{B}\right)\right)\right] \\ &-B\left(\frac{S_t}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(T-t,\frac{B^2}{KS_t}\right)\right) - N\left(\delta_+\left(T-t,\frac{B}{S_t}\right)\right)\right] \\ &+e^{-r(T-t)}K\left(\frac{S_t}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T-t,\frac{B^2}{KS_t}\right)\right) - N\left(\delta_-\left(T-t,\frac{B}{S_t}\right)\right)\right], \\ &0 \le t < T, 0 < S_t < B, \end{aligned}$$

where

$$\delta_\pm(\tau,s) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln s + (r \pm \frac{1}{2}\sigma^2)\tau\right].$$

EC.1.3. Example 3: European Call Option Under Heston Model

Consider a European call option with an underlying asset whose price S_t follows the Heston model as follows,

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sqrt{v_t} S_t \mathrm{d}B_t^1 \\ \mathrm{d}v_t &= \kappa(\theta - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}B_t^2, \end{aligned}$$

where the correlation $corr(B_t^1, B_t^2) = \rho t$.

Then, the closed-form expression of the price of European call option is

$$C(S_t, v_t, t) = S_t P_1 - K e^{r(T-t)} P_2,$$

where

$$P_j(S_t, v_t, T; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} f_j(\ln S_t, v_t, T; \phi)}{i\phi}\right] d\phi, \quad j = 1, 2.$$

Specifically,

$$f_j(\ln S_t, v_t, t; \phi) = e^{C(T-t;\phi) + D(T-t;\phi)v_t + i\phi \ln S_t},$$

where

$$\begin{aligned} C(\tau;\phi) &= r\phi i\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d)\,\tau - 2\ln\left[\frac{1 - g e^{d\tau}}{1 - g}\right]\right\}, \\ D(\tau;\phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - g e^{d\tau}}\right] \end{aligned}$$

and

$$\begin{aligned} g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\ d &= \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2 (2u_j\phi i - \phi^2)}. \end{aligned}$$

Other parameters are defined as

$$u_1 = 1/2, \quad u_2 = -1/2, \quad a = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa + \lambda.$$

Note that the integrand $P_j(S_t, v_t, T; \ln K)$ is a smooth function (Heston 1993).

EC.2. Lemmas and Proofs

EC.2.1. Lemma 1 and Its Proof

Lemma 1. *Suppose Assumption 1 holds. In addition, for sufficiently large m and conditioned on fitted portfolio loss functions $\tilde{\mu}_m(\cdot)$, the joint density function $b_m(x,u)$ of $(\mu(Z), \eta_m(Z))$ and its partial derivatives $\partial_x b_m(x,u)$ and $\partial_x^2 b_m(x,u)$ exist for all x and u, and satisfies the following regularity conditions: there exist non-negative functions $\bar{b}_{m,i}(u), i = 0,1,2$ such that*

$$b_m(x,u) \le \bar{b}_{m,0}(u), \quad |\partial_x b_m(x,u)| \le \bar{b}_{m,1}(u), \quad \left|\partial_x^2 b_m(x,u)\right| \le \bar{b}_{m,2}(u) \tag{EC.1}$$

for all x and u, with

$$B_{r,i} \equiv \sup_m \int_{\mathbb{R}} |u|^r \bar{b}_{m,i}(u) < \infty \tag{EC.2}$$

for $r = 0,1,2,3$ and $i = 0,1,2$. Then for any sequence $\{x_m\}$ such that $x_m \to x$ as $m \to \infty$, we have $f_{\tilde{\mu}_m}(x_m) \to f_\mu(x)$ and $f'_{\tilde{\mu}_m}(x_m) \to f'_\mu(x)$ almost surely as $m \to \infty$.

Remark EC.1. *The regularity conditions specified in Lemma 1 and subsequent lemmas within this section align with those typically assumed in the nested simulation literature, e.g., Gordy and Juneja (2010). These conditions are expected to hold when the portfolio loss surfaces and their fitted approximations exhibit sufficient smoothness, aligning with the basic assumptions of our framework. Through our analysis, we assume that boundedness conditions are satisfied almost surely in line with the conventional practices in the literature. However, it is possible to modify these conditions to allow for stochastic boundedness, which would change our results from convergence almost surely to convergence in probability. Although this modification is feasible, it would significantly complicate the proofs without providing substantial additional insights or benefits. Therefore, we retain the original assumptions for clarity and conciseness in our analysis.*

Proof of Lemma 1. We represent $f_{\tilde{\mu}_m}$ and f_μ in terms of b_m. Notice that

$$\begin{aligned} F_{\tilde{\mu}_m}(x) &= \Pr{}_m\{\tilde{\mu}_m(Z) \le x\} = \Pr{}_m\{\mu(Z) + a_m \eta_m(Z) \le x\} = \int_{\mathbb{R}} \int_{-\infty}^{x - a_m u} b_m(s,u)\,ds\,du, \\ F_\mu(x) &= \Pr\{\mu(Z) \le x\} = \int_{\mathbb{R}} \int_{-\infty}^{x} b_m(s,u)\,ds\,du. \end{aligned}$$

Notice that for any $x' \neq x$, by (EC.1),

$$\left| \frac{\int_{-\infty}^{x' - a_m u} b_m(s,u)ds - \int_{-\infty}^{x - a_m u} b_m(s,u)ds}{x' - x} \right| = \left| \frac{\int_{x - a_m u}^{x' - a_m u} b_m(s,u)ds}{x' - x} \right| \le \bar{b}_{m,0}(u),$$

where $\bar{b}_{m,0}$ is integrable due to (EC.2). Therefore, by dominated convergence theorem, we can derive $f_{\tilde{\mu}_m}(x)$ by differentiating $F_{\tilde{\mu}_m}(x)$ and switch integration and differentiation, which yields

$$f_{\tilde{\mu}_m}(x) = \frac{d}{dx} \int_{\mathbb{R}} \int_{-\infty}^{x - a_m u} b_m(s,u)\,ds\,du = \int_{\mathbb{R}} \frac{d}{dx} \int_{-\infty}^{x - a_m u} b_m(s,u)\,ds\,du = \int_{\mathbb{R}} b_m(x - a_m u, u)\,du.$$

Similarly, we have

$$f_\mu(x) = \int_{\mathbb{R}} b_m(x,u)\,du.$$

As a result,

$$\begin{aligned} f_{\tilde{\mu}_m}(x_m) - f_\mu(x) &= \int_{\mathbb{R}} [b_m(x_m - a_m u, u) - b_m(x,u)]\,du \\ &= \int_{\mathbb{R}} \partial_x b_m(x^*, u) \cdot (x_m - a_m u - x)\,du \\ &= (x_m - x) \int_{\mathbb{R}} \partial_x b_m(x^*, u)\,du - a_m \int_{\mathbb{R}} u \cdot \partial_x b_m(x^*, u)\,du. \end{aligned}$$

where x^* lies between $x - a_m u$ and x. Therefore

$$\begin{aligned}|f_{\tilde{\mu}_m}(x_m) - f_\mu(x)| &\le |x_m - x| \cdot \int_{\mathbb{R}} |\partial_x b_m(x^*, u)|\, du + a_m \cdot \int_{\mathbb{R}} |u \cdot \partial_x b_m(x^*, u)|\, du \\ &\le |x_m - x| \cdot \int_{\mathbb{R}} \bar{b}_{m,1}(u) du + a_m \cdot \int_{\mathbb{R}} |u| \bar{b}_{m,1}(u) du \\ &\le |x_m - x| \cdot B_{0,1} + a_m \cdot B_{1,1}.\end{aligned}$$

As $m \to 0$, $x_m - x \to 0$ and $a_m \to 0$. Therefore, by (EC.2), we have $f_{\tilde{\mu}_m}(x_m) \to f_\mu(x)$.

Further notice that for any $x' \neq x$, by Equation (EC.1),

$$\left| \frac{b_m(x' - a_m u, u) - b_m(x - a_m u, u)}{x' - x} \right| = \left| \frac{\partial_x b_m(x'', u)(x' - x)}{x' - x} \right| \le \bar{b}_{m,1}(u),$$

where x'' lies between x and x', $\bar{b}_{m,1}$ is integrable due to Equation (EC.2). Therefore, by the dominated convergence theorem, we can derive $f'_{\tilde{\mu}_m}(x)$ by differentiating $f_{\tilde{\mu}_m}(x) = \int_{\mathbb{R}} b_m(x - a_m u, u) du$ and switch integration and differentiation, which yields

$$f'_{\tilde{\mu}_m}(x) = \int_{\mathbb{R}} b_m(x - a_m u, u) du = \int_{\mathbb{R}} \partial_x b_m(x - a_m u, u) du.$$

Similarly, we have

$$f'_\mu(x) = \int_{\mathbb{R}} \partial_x b_m(x, u) du.$$

As a result,

$$\begin{aligned}f'_{\tilde{\mu}_m}(x_m) - f'_\mu(x) &= \int_{\mathbb{R}} [\partial_x b_m(x_m - a_m u, u) - \partial_x b_m(x, u)] du \\ &= \int_{\mathbb{R}} \partial_x^2 b_m(x^{**}, u) \cdot (x_m - a_m u - x) du \\ &= (x_m - x) \int_{\mathbb{R}} \partial_x^2 b_m(x^{**}, u) du - a_m \int_{\mathbb{R}} u \cdot \partial_x^2 b_m(x^{**}, u) du.\end{aligned}$$

where x^{**} lies between $x - a_m u$ and x. Therefore

$$\begin{aligned}|f'_{\tilde{\mu}_m}(x_m) - f'_\mu(x)| &\le |x_m - x| \cdot \int_{\mathbb{R}} \left|\partial_x^2 b_m(x^{**}, u)\right| du + a_m \cdot \int_{\mathbb{R}} \left|u \cdot \partial_x^2 b_m(x^{**}, u)\right| du \\ &\le |x_m - x| \cdot \int_{\mathbb{R}} \bar{b}_{m,2}(u) du + a_m \cdot \int_{\mathbb{R}} |u| \bar{b}_{m,2}(u) du \\ &\le |x_m - x| \cdot B_{0,2} + a_m \cdot B_{1,2}.\end{aligned}$$

As $m \to 0$, $x_m - x \to 0$ and $a_m \to 0$. Therefore, by Equation (EC.2), we have $f'_{\tilde{\mu}_m}(x_m) \to f'_\mu(x)$. $\square$

EC.2.2. Lemma 2 and Its Proof

Lemma EC.2.1. *Let $\Phi_m(x;w)$ and $\phi_m(x;w)$ be the cumulative distribution function and density of $\mu(Z)+w\eta_m(Z)$ for any $w\in[0,a_m]$. Assume Assumption 1 and regularity conditions of Lemma 1 hold. In addition, assume $f_\mu(q)>0$. Then*

(1) $\phi_m(x;w)$ is continuous with respect to w and x.

(2) $\partial_w\Phi_m(x;w)$ exists and is continuous with respect to w and x.

(3) For any $w\in(0,a_m)$, $\mathrm{E}_m[\eta_m(Z)|\mu(Z)+w\eta_m(Z)=x]$ is continuous with respect to x at $x=q$ for sufficiently large m.

Proof of Lemma EC.2.1. We first represent $\Phi_m(x;w)$ and $\phi_m(x,w)$ in terms of b_m by

$$\Phi_m(x;w)=\Pr_m\{\mu(Z)+w\eta_m(Z)\le x\}=\int_{\mathbb{R}}\int_{-\infty}^{x-wu} b_m(s,u)dudv,$$
$$\phi_m(x;w)=\partial_x\Phi_m(x;w)=\int_{\mathbb{R}} b_m(x-wu,u)du,$$

where the formula for ϕ_m is derived by switching integration and differentiation under the regularity conditions on b_m and $\partial_x b_m$ in the similarly way as in the proof of Lemma 1. Then apply mean value theorem, and use (EC.1) and (EC.2), we have

$$\begin{aligned}
|\phi_m(x;w')-\phi_m(x;w)| &= \left|\int_{\mathbb{R}} b_m(x-w'u,u)-b_m(x-wu,u)du\right|\\
&= \left|\int_{\mathbb{R}} \partial_x b_m(s^{**},u)(w'-w)(-u)du\right|\\
&\le |w'-w|\int_{\mathbb{R}}|u|\bar{b}_{m,1}(u)du\\
&\le |w'-w|\cdot B_{1,1}.\\
|\phi_m(x';w)-\phi_m(x;w)| &= \left|\int_{\mathbb{R}} b_m(x'-wu,u)-b_m(x-wu,u)du\right|\\
&= \left|\int_{\mathbb{R}} \partial_x b_m(s^{***},u)(x'-x)du\right|\\
&\le |x'-x|\int_{\mathbb{R}}\bar{b}_{m,1}(u)du\\
&\le |x'-x|\cdot B_{0,1},
\end{aligned}$$

where s^{**} lies between $x-wu$ and $x-w'u$ and s^{***} lies between $x-wu$ and $x'-wu$. This implies that $\phi_m(x;w)$ is continuous with respect to both w and x.

Next, notice that

$$\left|\frac{\int_{-\infty}^{x-w'u} b_m(s,u)ds-\int_{-\infty}^{x-wu} b_m(s,u)ds}{w'-w}\right| = \left|\frac{\int_{x-w'u}^{x-wu} b_m(s,u)ds}{w'-w}\right| \le |u|\bar{b}_{m,0}(u),$$

Therefore, by dominated convergence theorem and (EC.2), we can derive $\partial_w \Phi_m(x;w)$ by differentiating $\Phi_m(x;w)$ and switch integration and differentiation, which yields

$$\partial_w \Phi_m(x;w) = \partial_w \int_{\mathbb{R}} \int_{-\infty}^{x-wu} b_m(s,u)dudv = \int_{\mathbb{R}} \partial_w \int_{-\infty}^{x-wu} b_m(s,u)dudv = -\int_{\mathbb{R}} u \cdot b_m(x-wu,u)du.$$

Similarly, apply mean value theorem, and use (EC.1) and (EC.2), we know $\partial_w \Phi_m(x;w)$ is continuous with respect to w and x under the regularity conditions on $u^2 \partial_x b_m$ and $u \partial_x b_m$, respectively.

Last, we have

$$\mathrm{E}_m[\eta_m(Z)|\mu(Z) + w\eta_m(Z) = x] = \frac{\int_{\mathbb{R}} u \cdot b_m(x-wu,u)du}{\phi_m(x,w)}.$$

We have shown that both the denominator and the numerator are continuous with respect to x. In addition, $\phi_m(x,w)$ is continuous with respect to w and $\phi_m(q,0) \equiv f_\mu(q) > 0$, and hence $\phi_m(x,w) > 0$ for $x = q$ and $w \in (0, a_m)$ for sufficiently large m. Therefore the conditional expectation is continuous at with respect to x at $x = q$ for sufficiently large m. $\square$

Lemma 2. *Under Assumption 1 and regularity conditions of Lemma 1 and EC.2.1, we have*

$$\mathcal{F}_q[\tilde{\mu}_m] - \mathcal{F}_q[\mu] = \mathcal{O}_{\mathrm{P}}(a_m).$$

Proof of Lemma 2. We first show that $\mathcal{F}_q[\mu + w\eta_m]$ is differentiable with respect to $w \in (0, a_m)$, and

$$\frac{d}{dw}\mathcal{F}_q[\mu + w\eta_m] = -\phi_m(q;w)\,\mathrm{E}_m[\eta_m(Z)|\mu(Z) + w\eta_m(Z) = q]. \tag{EC.3}$$

This result can be proved by applying Theorem 1 of Hong (2009), which states that for real number q and bi-variate function $h: \Omega \times (0, a_m) \to \mathbb{R}$, we have

$$p_q'(w) = -\phi(q;w)\,\mathrm{E}[\partial_w h(Z,w)|h(Z,w) = q], \tag{EC.4}$$

for any $w \in (0, a_m)$, where $p_q(w) = \Pr\{h(Z,w) \le q\}$, if the following conditions are satisfied:

(a) The path-wise derivative $\partial_w h(Z,w)$ exists w.p.1 for any $w \in (0, a_m)$, and there exists a function $k(Z)$ with $\mathrm{E}[k(Z)] < \infty$, such that $|h(Z,w_1) - h(Z,w_2)| \le k(Z)|w_2 - w_1|$ for all w_1, w_2.

(b) Let $\Phi(t;w)$ and $\phi(t;w)$ denote the cumulative distribution function and density of $h(Z,w)$. For any $w \in (0, a_m)$, $h(Z,w)$ has a continuous density $\phi(t;w)$ in a neighborhood of $t = q$, and $\partial_w \Phi(t;w)$ exists and is continuous with respect to both w and t at $t = q$.

(c) For any $w \in (0, a_m)$, $\mathrm{E}[\partial_w h(Z,w)|h(Z,w) = t]$ is continuous at $t = q$.

In order to prove (EC.3), we set $h(z,w) = \mu(z) + w\eta_m(z)$, and verify the validity of the above conditions. On the one hand, $\partial_w h(Z,w) = \eta_m(Z)$, and $|h(Z,w_1) - h(Z,w_2)| = |\eta_m(Z)(w_2 - w_1)|$. Therefore condition (a) holds with $k(Z) = \eta_m(Z)$. On the other hand, condition (b) and (c) also hold due to Lemma EC.2.1.

Therefore, we can apply (EC.4) to $p_q(w) = \Pr_m\{h(Z,w) \le q\} = \Pr_m\{\mu(Z) + w\eta_m(Z) \le q\} = \mathcal{F}_q[\mu + w\eta_m]$, which gives (EC.3).

Now notice that

$$\begin{aligned}
\mathcal{F}_q[\tilde{\mu}_m] - \mathcal{F}_q[\mu] &= \mathcal{F}_q[\mu + w\eta_m]\big|_{w=a_m} - \mathcal{F}_q[\mu + w\eta_m]\big|_{w=0} \\
&= \int_0^{a_m} \frac{d}{dw}\mathcal{F}_q[\mu + w\eta_m]dw \\
&= \int_0^{a_m} -\phi_{\tilde{\mu}_m}(q;w)\,\mathrm{E}[\eta_m(Z)|\mu(Z) + w\eta_m(Z) = q]dw,
\end{aligned}$$

and hence

$$|\mathcal{F}_q[\tilde{\mu}_m] - \mathcal{F}_q[\mu]| \le \int_0^{a_m} \phi_{\tilde{\mu}_m}(q;w)dw \cdot \|\eta_m\|_\infty.$$

Since

$$\begin{aligned}
\int_0^{a_m} \phi_{\tilde{\mu}_m}(q;w)dw &= \int_0^{a_m}\int_{\mathbb{R}} b_m(q - wu, u)dudw \\
&= \int_0^{a_m}\int_{\mathbb{R}} [b_m(q,u) - \partial_x b_m(q^*,u)\cdot wu]\,dudw \\
&\le \int_0^{a_m}\int_{\mathbb{R}} [b_m(q,u) + |\partial_x b_m(q^*,u)|\cdot w|u|]\,dudw \\
&= \int_0^{a_m}\int_{\mathbb{R}} b_m(q,u)dudw + \int_0^{a_m}\int_{\mathbb{R}} |\partial_x b_m(q^*,u)|\cdot w|u|dudw \\
&\le a_m\cdot f_\mu(q) + \frac{a_m^2}{2}\cdot B_{1,1} \\
&= \mathcal{O}(a_m),
\end{aligned}$$

where q^* is between q and $q - wu$. Therefore $\mathcal{F}_q[\tilde{\mu}_m] - \mathcal{F}_q[\mu] = \mathcal{O}(a_m)\cdot\|\eta_m\|_\infty = \mathcal{O}(\|\tilde{\mu}_m - \mu\|_\infty) = \mathcal{O}_{\mathrm{P}}(a_m)$. $\square$

EC.2.3. Lemma 3 and Its Proof

Lemma 3. *Suppose Assumption 1 holds. For sufficiently large m, $h_m(x,u,y,v)$, the joint density function of $(\mu(Z), \eta_m(Z), \pi(Z), \gamma_m(Z))$, and its derivative $\partial_x h_m(x,u,y,v)$, $\partial_y h_m(x,u,y,v)$ exist for all x,u,y and v. In addition, let $h_m^Y(x,u,y,v) = \int_{-\infty}^{y} h_m(x,u,\check{y},v)d\check{y}$. Its derivatives $\partial_x h_m^Y(x,u,y,v)$ and $\partial_x^2 h_m^Y(x,u,y,v)$ exist for all x,u,y and v. There exist non-negative functions $\bar{h}_{m,i}(u,v)$, $i = 0,1,2$, and $\bar{h}_{m,i}^Y(u,v)$, $i = 0,1,2$, such that*

$$\begin{aligned}
&h_m(x,u,y,v) \le \bar{h}_{m,0}(u,v), \quad |\partial_x h_m(x,u,y,v)| \le \bar{h}_{m,1}(u,v), \quad |\partial_y h_m(x,u,y,v)| \le \bar{h}_{m,2}(u,v), \\
&h_m^Y(x,u,y,v) \le \bar{h}_{m,0}^Y(u,v), \quad |\partial_x h_m^Y(x,u,y,v)| \le \bar{h}_{m,1}^Y(u,v), \quad |\partial_x^2 h_m^Y(x,u,y,v)| \le \bar{h}_{m,2}^Y(u,v)
\end{aligned}$$

for all x, u, y and v, with

$$H_i \equiv \sup_m \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|u|+|v|)^r \bar{h}_{m,i}(u,v)dudv < \infty,$$
$$H_i^Y \equiv \sup_m \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|u|+|v|)^r \bar{h}_{m,i}^Y(u,v)dudv < \infty$$

for $i = 0,1,2$ and $r = 1,2$. Then if $(x_m, y_m) \to (x,y)$ and $f_\mu(x) \neq 0$, we have

$$\partial_x G_{\tilde{\mu}_m, \tilde{\pi}_m}(x_m, y_m) \to \partial_x G_{\mu,\pi}(x,y), \quad \partial_y G_{\tilde{\mu}_m, \tilde{\pi}_m}(x_m, y_m) \to \partial_y G_{\mu,\pi}(x,y).$$

Proof of Lemma 3. First of all, by noticing that $b_m(x,u) = \int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x,u,y,v)dydv$, it is easy to check that the regularity conditions of Lemma 1 hold by setting

$$\bar{b}_{m,i}(u) = \int_{\mathbb{R}} \bar{h}_{m,i}(u,v)dv$$

for $i = 0,1$. Therefore both the conclusions of Lemma 1 and 2 hold.

Next, we consider the partial derivatives with respect to y. We start by representing $G_{\mu,\pi}$ and $G_{\tilde{\mu}_m,\tilde{\pi}_m}$ in terms of h_m by

$$\begin{aligned}
G_{\mu,\pi}(x,y) &= \Pr\{\pi(Z) \le y | \mu(Z) = x\} \\
&= \frac{\int_{\mathbb{R}} \int_{-\infty}^{y} \int_{\mathbb{R}} h_m(x,u,\check{y},v)dud\check{y}dv}{\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x,u,\check{y},v)dud\check{y}dv} \\
&= \frac{\int_{\mathbb{R}} \int_{-\infty}^{y} \int_{\mathbb{R}} h_m(x,u,\check{y},v)dud\check{y}dv}{f_\mu(x)}; \\
G_{\tilde{\mu}_m,\tilde{\pi}_m}(x,y) &= \Pr_m\{\pi(Z) + a_m\gamma_m(Z) \le y | \mu(Z) + a_m\eta_m(Z) = x\} \\
&= \frac{\int_{\mathbb{R}} \int_{-\infty}^{y} \int_{\mathbb{R}} h_m(x - a_m u, u, \check{y} - a_m v, v)dud\check{y}dv}{\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x - a_m u, u, \check{y} - a_m v, v)dud\check{y}dv} \\
&= \frac{\int_{\mathbb{R}} \int_{-\infty}^{y} \int_{\mathbb{R}} h_m(x - a_m u, u, \check{y} - a_m v, v)dud\check{y}dv}{f_{\tilde{\mu}_m}(x)}.
\end{aligned}$$

Since the integrations are interchangeable due to Fubini's theorem, we have

$$\partial_y G_{\mu,\pi}(x,y) = \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x,u,y,v)dudv}{f_\mu(x)}; \tag{EC.5}$$

$$\partial_y G_{\tilde{\mu}_m,\tilde{\pi}_m}(x_m,y_m) = \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x_m - a_m u, u, y_m - a_m v, v)dudv}{f_{\tilde{\mu}_m}(x_m)}. \tag{EC.6}$$

Notice that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x_m - a_m u, u, y_m - a_m v, v)dudv - \int_{\mathbb{R}} \int_{\mathbb{R}} h_m(x,u,y,v)dudv \right| \\
&= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\partial_x h_m(x^*, u, y^*, v)(x_m - a_m u - x) + \partial_y h_m(x^*, u, y^*, v)(y_m - a_m v - y)\right] dudv \right| \\
&\le |x_m - x| \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_x h_m(x^*, u, y^*, v)|dudv + |y_m - y| \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_y h_m(x^*, u, y^*, v)|dudv
\end{aligned}$$

$$
\begin{aligned}
&+a_m\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|u|\cdot|\partial_x h_m(x^*,u,y^*,v)|dudv+\int_{\mathbb{R}}\int_{\mathbb{R}}|v|\cdot|\partial_y h_m(x^*,u,y^*,v)|dudv\right]\\
&\le |x_m-x|\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{h}_{m,1}(u,v)dudv+|y_m-y|\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{h}_{m,2}(u,v)dudv\\
&+a_m\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|u|\cdot\bar{h}_{m,1}(u,v)dudv+\int_{\mathbb{R}}\int_{\mathbb{R}}|v|\cdot\bar{h}_{m,}(u,v)dudv\right]\\
&\le |x_m-x|\cdot H_1+|y_m-y|\cdot H_2+a_m\cdot(H_1+H_2).
\end{aligned}
$$

As x_m-x, y_m-y and a_m converge to 0 when $m\to\infty$, the numerator of (EC.6) converges to that of (EC.5). In addition, by Lemma 1, we know that the denominator of (EC.6) converges to that of (EC.5). As a result, given that $f_\mu(x)\neq 0$, we have $\partial_y G_{\tilde{\mu}_m,\tilde{\pi}_m}(x_m,y_m)\to\partial_y G_{\mu,\pi}(x,y)$.

Last, we consider the partial derivatives with respect to x. We have

$$
\begin{aligned}
\partial_x G_{\mu,\pi}(x,y) &= \frac{1}{f_\mu(x)}\int_{\mathbb{R}}\int_{-\infty}^{y}\int_{\mathbb{R}}\partial_x h_m(x,u,\check{y},v)dud\check{y}dv\\
&\quad -\frac{f'_\mu(x)}{[f_\mu(x)]^2}\int_{\mathbb{R}}\int_{-\infty}^{y}\int_{\mathbb{R}}h_m(x,u,\check{y},v)dud\check{y}dv\\
&\equiv \frac{1}{f_\mu(x)}I_{m,1}(x,y)-\frac{f'_\mu(x)}{[f_\mu(x)]^2}I_{m,2}(x,y);\\
\partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(x,y) &= \frac{1}{f_{\tilde{\mu}_m}(x)}\int_{\mathbb{R}}\int_{-\infty}^{y}\int_{\mathbb{R}}\partial_x h_m(x-a_mu,u,\check{y}-a_mv,v)dud\check{y}dv\\
&\quad -\frac{f'_{\tilde{\mu}_m}(x)}{[f_{\tilde{\mu}_m}(x)]^2}\int_{\mathbb{R}}\int_{-\infty}^{y}\int_{\mathbb{R}}h_m(x-a_mu,u,\check{y}-a_mv,v)dud\check{y}dv\\
&\equiv \frac{1}{f_{\tilde{\mu}_m}(x)}\tilde{I}_{m,1}(x,y)-\frac{f'_{\tilde{\mu}_m}(x)}{[f_{\tilde{\mu}_m}(x)]^2}\tilde{I}_{m,2}(x,y).
\end{aligned}
$$

By Lemma 1, we know that $f_{\tilde{\mu}_m}(x_m)\to f_\mu(x)$ and $f'_{\tilde{\mu}_m}(x_m)\to f_\mu(x)$. Then it suffices to show that $\tilde{I}_{m,i}(x_m,y_m)\to I_{m,i}(x,y)$ for $i=1,2$. Actually,

$$
\begin{aligned}
&|\tilde{I}_{m,1}(x_m,y_m)-I_{m,1}(x,y)|\\
&=\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\int_{-\infty}^{y_m}\partial_x h_m(x_m-a_mu,u,\check{y}-a_mv,v)d\check{y}-\int_{-\infty}^{y}\partial_x h_m(x,u,\check{y},v)d\check{y}\right]dudv\right|\\
&=\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\int_{-\infty}^{y_m-a_mv}\partial_x h_m(x_m-a_mu,u,\check{y},v)d\check{y}-\int_{-\infty}^{y}\partial_x h_m(x,u,\check{y},v)d\check{y}\right]dudv\right|\\
&=\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\partial_x h_m^Y(x_m-a_mu,u,y_m-a_mv,v)-\partial_x h_m^Y(x,u,y,v)\right]dudv\right|\\
&=\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\left[(x_m-a_mu-x)\partial_x^2 h_m^Y(x^{**},u,y^{**},v)+(y_m-a_mv-y)\partial_x h_m(x^{**},u,y^{**},v)\right]dudv\right|\\
&\le |x_m-x|\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{h}_{m,2}^Y(u,v)dudv+|y_m-y|\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{h}_{m,1}(u,v)dudv\\
&\quad +a_m\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|u|\bar{h}_{m,2}^Y(u,v)dudv+\int_{\mathbb{R}}\int_{\mathbb{R}}|v|\bar{h}_{m,1}(u,v)dudv\right]\\
&\le |x_m-x|\cdot H_2^Y+|y_m-y|\cdot H_1+a_m\cdot(H_2^Y+H_1),
\end{aligned}
$$

and

$$\begin{aligned}
&|\tilde{I}_{m,2}(x_m,y_m)-I_{m,2}(x,y)| \\
&=\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\int_{-\infty}^{y_m} h_m(x_m-a_m u,u,\check{y}-a_m v,v)d\check{y}-\int_{-\infty}^{y} h_m(x,u,\check{y},v)d\check{y}\right]dudv\right| \\
&=\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\left[(x_m-a_m u-x)\partial_x h_m^Y(x^{***},u,y^{***},v)+(y_m-a_m v-y)h_m(x^{***},u,y^{***},v)\right]dudv\right| \\
&\le |x_m-x|\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{h}_{m,1}^Y(u,v)dudv+|y_m-y|\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{h}_{m,0}(u,v)dudv \\
&\qquad +a_m\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|u|\bar{h}_{m,1}^Y(u,v)dudv+\int_{\mathbb{R}}\int_{\mathbb{R}}|v|\bar{h}_{m,0}(u,v)dudv\right] \\
&\le |x_m-x|\cdot H_1^Y+|y_m-y|\cdot H_0+a_m\cdot(H_1^Y+H_0).
\end{aligned}$$

As x_m-x, y_m-y and a_m converge to 0 when $m\to\infty$, we know that $\tilde{I}_{m,i}(x_m,y_m)\to I_{m,i}(x,y)$ for $i=1,2$. Given that $f_\mu(x)\neq 0$, we have $\partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(x_m,y_m)\to\partial_x G_{\mu,\pi}(x,y)$. □

EC.2.4. Lemma 4 and Its Proof

Lemma EC.2.2. *Denote $\Psi_m(x,y;w,s)$ and $\psi_m(x,y;w,s)$ be the joint cumulative distribution and density function of $(\mu(Z)+w\eta_m(Z),\pi(Z)+s\gamma_m(Z))$ for $w,s\in[0,a_m]$, respectively. Recall that $\Phi_m(x;w)$ and $\phi_m(x;w)$ are the marginal cumulative distribution function and density of $\mu(Z)+w\eta_m(Z)$, respectively. Assume Assumption 1 and regularity conditions of Lemma 3 hold. In addition, let $h_m^X(x,u,y,v)=\int_{-\infty}^{x}h_m(\check{x},u,y,v)d\check{x}$ and its derivatives $\partial_y h_m^X(x,u,y,v)$ exist for all x,u,y and v. There exist non-negative functions $\bar{h}_{m,i}^X(u,v)$ $i=0,1$ such that*

$$h_m^X(x,u,y,v)\le\bar{h}_{m,0}^X(u,v),\quad |\partial_y h_m^X(x,u,y,v)|\le\bar{h}_{m,1}^X(u,v),$$

for all x,u,y and v, with

$$H_i^X\equiv\sup_m\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|u|+|v|)\bar{h}_{m,i}^X(u,v)dudv<\infty,$$

for $i=0,1,2$. Then we have

(a) For all $w,s\in(0,a_m)$, $\partial_w\Psi_m(x,y;w,s)$, $\partial_x^2\Psi_m(x,y;w,s)$, and $\partial_w\partial_x\Psi_m(x,y;w,s)$ exist and are continuous with respect to $w\in(0,a_m)$, x in a neighbourhood of q, and y in a neighbourhood of θ. Moreover, $\partial_w\Phi_m(x;w),\partial_x\phi_m(x;w),\partial_w\phi_m(x;w)$ exist and are continuous with respect to both $w\in(0,a_m)$ and x in a neighbourhood of q. For all $w\in(0,a_m)$, $\phi_m(q,w)>0$.

(b) For all $w,s\in(0,a_m)$, $\partial_x\Psi_m(x,y;w,s)$, $\partial_s\Psi_m(x,y;w,s)$, and $\partial_s\partial_x\Psi_m(x,y;w,s)$ exist and are continuous with respect to $s\in(0,a_m)$, x in a neighbourhood of q, and y in a neighbourhood of θ.

Proof of Lemma EC.2.2. First of all, by definition, we have

$$\begin{aligned}
\Psi_m(x,y;w,s) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\infty}^{x-wu}\int_{-\infty}^{y-sv} h_m(\check{x},u,\check{y},v)d\check{y}d\check{x}dudv,\\
\psi_m(x,y;w,s) &= \int_{\mathbb{R}}\int_{\mathbb{R}} h_m(x-wu,u,y-sv,v)dudv,\\
\Phi_m(x;w) &= \lim_{y\to\infty}\Psi_m(x,y;w,s) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\infty}^{x} h_m(\check{x}-wu,u,y,v)d\check{x}dudvdy,\\
\phi_m(x;w) &= \int_{\mathbb{R}}\psi_m(x,y;w,s)dy = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} h_m(x-wu,u,y,v)dudvdy.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\partial_w\Psi_m(x,y;w,s) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\infty}^{y-sv} u\cdot h_m(x-wu,u,\check{y},v)d\check{y}dudv\\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} u\cdot h_m^Y(x-wu,u,y-sv,v)dudv,\\
\partial_x\Psi_m(x,y;w,s) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\infty}^{y-sv} h_m(x-wu,u,\check{y},v)d\check{y}dudv\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} h_m^Y(x-wu,u,y-sv,v)dudv,\\
\partial_x^2\Psi_m(x,y;w,s) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x h_m^Y(x-wu,u,y-sv,v)dudv,\\
\partial_w\partial_x\Psi_m(x,y;w,s) &= -\int_{\mathbb{R}}\int_{\mathbb{R}} u\cdot\partial_x h_m^Y(x-wu,u,y-sv,v)dudv,\\
\partial_s\Psi_m(x,y;w,s) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\infty}^{x-wu} v\cdot h_m(\check{x},u,y-sv,v)d\check{x}dudv\\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} v\cdot h_m^X(x-wu,u,y-sv,v)dudv,\\
\partial_s\partial_x\Psi_m(x,y;w,s) &= -\int_{\mathbb{R}}\int_{\mathbb{R}} v\cdot h_m(x-wu,u,y-sv,v)dudv.\\
\partial_w\Phi_m(x;w) &= -\int_{\mathbb{R}}\int_{\mathbb{R}} u\cdot h_m^Y(x-wu,u,\infty,v)dudv,\\
\partial_x\phi_m(x;w) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x h_m^Y(x-wu,u,\infty,v)dudv,\\
\partial_w\phi_m(x;w) &= -\int_{\mathbb{R}}\int_{\mathbb{R}} u\cdot\partial_x h_m^Y(x-wu,u,\infty,v)dudv.
\end{aligned}$$

Then the continuity of these functions follows from the regularity conditions in Lemma 3 and Lemma EC.2.2. □

Lemma EC.2.3 (Fu et al. 2023)**.** *Suppose $X(w)$ and $Y(s)$ are two random variables with parameters $w\in\Theta$, $s\in\Delta$, where Θ and Δ are open subset of $\mathbb{R}$. Let $f_{X,Y}(x,y;w,s)$ and $F_{X,Y}(x,y;w,s)$ be the joint probability density function and cumulative distribution function of $(X(w),Y(s))$, respectively, and $f_X(x;w)$ and $F_X(x;w)$ be the marginal probability density function and cumulative distribution function of $X(w)$, respectively. If the following conditions are satisfied:*

(a) *The pathwise derivative $\partial_w X(w), \partial_s Y(s)$ exist w.p.1 for any $w \in \Theta$ and $s \in \Delta$, and there exist random variables K and T with $\mathrm{E}[K], \mathrm{E}[T] < \infty$, which are not dependent on w and s, such that $|X(w_1) - X(w_2)| \le K|w_1 - w_2|$ and $|Y(s_1) - Y(s_2)| \le T|s_1 - s_2|$ hold for all $w_1, w_2 \in \Theta, s_1, s_2 \in \Delta$.*

(b) *For all $w \in \Theta, s \in \Delta, \partial_w F_{X,Y}(x,y;w,s)$, $\mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le y\}} \partial_w X(w)\right], \partial_x^2 F_{X,Y}(x,y;w,s)$, and $\partial_w \partial_x F_{X,Y}(x,y;w,s)$ exist and are continuous with respect to $w \in \Theta$, x in a neighbourhood of a, and y in a neighbourhood of b. Moreover, $\partial_w F_X(x;w), \partial_x f_X(x;w), \partial_w f_X(x;w)$, and $\mathrm{E}\left[I_{\{X(w)\le x\}} \partial_w X(w)\right]$ exist and are continuous with respect to both $w \in \Theta$ and x in a neighbourhood of a. For all $w \in \Theta$, $f_X(a,w) > 0$.*

(c) *For all $w \in \Theta, s \in \Delta, \partial_x F_{X,Y}(x,y;w,s)$, $\mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le y\}} \partial_s Y(s)\right]$, $\partial_s F_{X,Y}(x,y;w,s)$, and $\partial_s \partial_x F_{X,Y}(x,y;w,s)$ exist and are continuous with respect to $s \in \Delta$, x in a neighbourhood of a, and y in a neighbourhood of b.*

Then for $p_{a,b}(w,s) = \Pr\{Y(s) \le b | X(w) = a\}$, we have

$$\partial_w p_{a,b}(w,s) = \frac{\left\{-\partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le b\}} \partial_w X(w)\right] + p_{a,b}(w,s) \partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}} \partial_w X(w)\right]\right\}\big|_{x=a}}{f_X(a,w)}$$

$$\partial_s p_{a,b}(w,s) = \frac{-\partial_x \partial_y \mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le y\}} \partial_s Y(s)\right]\big|_{x=a,y=b}}{f_X(a,w)}$$

Lemma 4. *Under Assumptions 1 and regularity conditions of Lemma 3 and Lemma EC.2.2, we have*

$$\mathcal{G}_{q,\theta}[\tilde{\mu}_m, \tilde{\pi}_m] - \mathcal{G}_{q,\theta}[\mu, \pi] = \mathcal{O}_{\mathrm{P}}(a_m).$$

Proof of Lemma 4. Define $p_{q,\theta}(w,s) = \Pr\{\pi(Z) + s\gamma_m(Z) \le \theta | \mu(Z) + w\eta_m(Z) = q\}$, then

$$\mathcal{G}_{q,\theta}[\tilde{\mu}_m, \tilde{\pi}_m] = p_{q,\theta}(a_m, a_m), \quad \mathcal{G}_{q,\theta}[\mu, \pi] = p_{q,\theta}(0,0).$$

Under regularity conditions of Lemma 3, it is straightforward to check the conditions in Lemma EC.2.3 hold with $X(w) = \mu(Z) + w\eta_m(Z)$, $Y(s) = \pi(Z) + s\gamma_m(Z)$, $a = q$ and $b = \theta$. Consequently, we can apply the conclusions of Lemma EC.2.3 to derive partial derivatives of $p_{q,\theta}(w,s)$. In particular, with $\partial_w X(w) = \eta_m(Z)$, $\partial_s Y(s) = \gamma_m(Z)$, we have

$$\partial_w p_{q,\theta}(w,s) = \frac{\left\{-\partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le \theta\}} \eta_m(Z)\right] + p_{q,\theta}(w,s) \partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}} \eta_m(Z)\right]\right\}\big|_{x=q}}{\phi_m(q;w)}, \tag{EC.7}$$

$$\partial_s p_{q,\theta}(w,s) = \frac{-\partial_x \partial_y \mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le y\}} \gamma_m(Z)\right]\big|_{x=q,y=\theta}}{\phi_m(q;w)}. \tag{EC.8}$$

Under regularity conditions of Lemma 3, the partial derivatives and expectations are interchangeable, and thus

$$\begin{aligned}\partial_x \mathrm{E}\left[I_{\{X(w)\le x\}} I_{\{Y(s)\le y\}} \eta_m(Z)\right] &= \partial_x \int_{-\infty}^{x} \mathrm{E}\left[I_{\{Y(s)\le y\}} \eta_m(Z) | X(w) = t\right] \phi_m(t;w) dt \\ &= \mathrm{E}\left[I_{\{Y(s)\le y\}} \eta_m(Z) | X(w) = x\right] \phi_m(x;w),\end{aligned}$$

and similarly

$$\partial_x \mathrm{E}\left[I_{\{X(w)\le x\}}\eta_m(Z)\right] = \mathrm{E}\left[\eta_m(Z)|X(w)=x\right]\phi_m(x;w),$$
$$\partial_x \mathrm{E}\left[I_{\{X(w)\le x\}}I_{\{Y(s)\le y\}}\gamma_m(Z)\right] = \mathrm{E}\left[I_{\{Y(s)\le y\}}\gamma_m(Z)|X(w)=x\right]\phi_m(x;w).$$

Furthermore, under regularity conditions of Lemma 3, the partial derivatives ∂_x and ∂_y are also interchangeable, and thus

$$\begin{aligned}\partial_x\partial_y \mathrm{E}\left[I_{\{X(w)\le x\}}I_{\{Y(s)\le y\}}\gamma_m(Z)\right] &= \partial_y\left\{\mathrm{E}\left[I_{\{Y(s)\le y\}}\gamma_m(Z)|X(w)=x\right]\phi_m(x;w)\right\}\\ &= \phi_m(x;w)\partial_y\int_{-\infty}^{y}\mathrm{E}\left[\gamma_m(Z)|X(w)=x,Y(s)=\check{y}\right]f_{Y|X}(\check{y}|x)d\check{y}\\ &= \mathrm{E}\left[\gamma_m(Z)|X(w)=x,Y(s)=y\right]\psi_m(x,y;w,s),\end{aligned}$$

which indicates

$$\left|\partial_x\partial_y \mathrm{E}\left[I_{\{X(w)\le x\}}I_{\{Y(s)\le y\}}\gamma_m(Z)\right]\right| \le \|\gamma_m\|_\infty \psi_m(x,y;w,s) = \mathcal{O}_{\mathrm{P}}(1). \tag{EC.9}$$

We also have

$$\begin{aligned}\partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}}\eta_m(Z)\right] &= \partial_x\left\{\mathrm{E}\left[\eta_m(Z)|X(w)=x\right]\phi_m(x;w)\right\}\\ &= \partial_x\int_{\mathbb{R}} u b_m(x-wu,u)du\\ &= \int_{\mathbb{R}} u\partial_x b_m(x-wu,u)du.\end{aligned}$$

Due to the regularity conditions, we have

$$\left|\partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}}\eta_m(Z)\right]\right| \le \int_{\mathbb{R}}|u|\bar{b}_{m,1}(u)du \le B_{1,1}. \tag{EC.10}$$

Similarly,

$$\begin{aligned}\partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}}I_{\{Y(s)\le y\}}\eta_m(Z)\right] &= \partial_x\left\{\mathrm{E}\left[I_{\{Y(s)\le \theta\}}\eta_m(Z)|X(w)=x\right]\phi_m(x;w)\right\}\\ &= \partial_x\int_{-\infty}^{y-sv}\int_{\mathbb{R}}\int_{\mathbb{R}} u h_m(x-wu,u,\check{y},v)dudvd\check{y}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} u\partial_x h_m^Y(x-wu,u,y-sv,v)dudv.\end{aligned}$$

Due to the regularity conditions, we have

$$\left|\partial_x^2 \mathrm{E}\left[I_{\{X(w)\le x\}}I_{\{Y(s)\le \theta\}}\eta_m(Z)\right]\right| \le \int_{\mathbb{R}}\int_{\mathbb{R}}|u|\bar{h}_{m,1}(u,v)dudv \le H_1^Y. \tag{EC.11}$$

Substitute (EC.9), (EC.10) and (EC.11) into (EC.7) and (EC.8), we known both $\partial_w p_{q,\theta}(w,s)$ and $\partial_w p_{q,\theta}(w,s)$ are of order $\mathcal{O}_{\mathrm{P}}(1)$ as $m\to\infty$. Now define $g(t)=\mathcal{G}_{q,\theta}[\mu+t\eta_m,\pi+t\gamma_m]$, $t\in[0,a_m]$. Notice that $g(t)=p_{q,\theta}(t,t)$. Therefore

$$g'(s) = \partial_w p_{a,b}(s,s) + \partial_s p_{a,b}(s,s) = \mathcal{O}_{\mathrm{P}}(1).$$

Therefore

$$\mathcal{G}_{q,\theta}[\tilde{\mu}_m, \tilde{\pi}_m] - \mathcal{G}_{q,\theta}[\mu, \pi] = g(a_m) - g(0) = \int_0^{a_m} g'(s)ds = \mathcal{O}_{\mathrm{P}}(a_m).$$

□

EC.2.5. Lemma EC.2.4 and Its Proof

Lemma EC.2.4. *(a) Under the regularity conditions of Lemma 1, we have*

$$F_{\tilde{\mu}_m}(\tilde{q}_m) = F_{\tilde{\mu}_m}(q) + f_{\tilde{\mu}_m}(q)(\tilde{q}_m - q) + o(\tilde{q}_m - q).$$

(b) Under the regularity conditions of Lemma 3, we have

$$G_{\tilde{\mu}_m,\tilde{\pi}_m}(\tilde{q}_m, \tilde{\theta}_m) - G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta) = \partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \theta)(\tilde{q}_m - q) + \partial_y G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta)(\tilde{\theta}_m - \theta) + o(\tilde{\theta}_m - \theta),$$

as $m \to \infty$, where q^* is some number between $\tilde{q}_m$ and q.

Proof of Lemma EC.2.4. (a) By the mean value theorem, we have

$$F_{\tilde{\mu}_m}(\tilde{q}_m) = F_{\tilde{\mu}_m}(q) + f_{\tilde{\mu}_m}(q)(\tilde{q}_m - q) + \frac{1}{2} f'_{\tilde{\mu}_m}(q^*)(\tilde{q}_m - q)^2$$

for some q^* between q and $\tilde{q}_m$. Following the proof of Lemma 1, we can derive $f'_{\tilde{\mu}_m}(x)$ by switching integration and differentiation, which yields

$$f'_{\tilde{\mu}_m}(x) = \frac{d}{dx} \int_{\mathbb{R}} b_m(x - a_m u, u)du = \int_{\mathbb{R}} \partial_x b_m(x - a_m u, u)du.$$

Hence, under the regularity conditions of Lemma 1, $|f'_{\tilde{\mu}_m}(x)| \le \int_{\mathbb{R}} |\partial_x b_m(x - a_m u, u)|du \le \int_{\mathbb{R}} \bar{b}_{m,1}(u)du \le B_{0,1}$. As a result,

$$F_{\tilde{\mu}_m}(\tilde{q}_m) = F_{\tilde{\mu}_m}(q) + f_{\tilde{\mu}_m}(q)(\tilde{q}_m - q) + o(\tilde{q}_m - q).$$

(b) By the mean value theorem, we have

$$\begin{aligned} G_{\tilde{\mu}_m,\tilde{\pi}_m}(\tilde{q}_m, \tilde{\theta}_m) - G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \tilde{\theta}_m) &= \partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \tilde{\theta}_m)(\tilde{q}_m - q), \\ G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \tilde{\theta}_m) - G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta) &= \partial_y G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta)(\tilde{\theta}_m - \theta) + \frac{1}{2}\partial_{yy} G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta^*)(\tilde{\theta}_m - \theta)^2, \\ \partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \tilde{\theta}_m) &= \partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \theta) + \partial_{xy} G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \theta^{**})(\tilde{\theta}_m - \theta), \end{aligned}$$

where θ^*, θ^{**} are between $\tilde{\theta}_m$ and θ, and q^* is between $\tilde{q}_m$ and q. Therefore we have

$$G_{\tilde{\mu}_m,\tilde{\pi}_m}(\tilde{q}_m, \tilde{\theta}_m) - G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta) = \partial_x G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \theta)(\tilde{q}_m - q) + \partial_y G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta)(\tilde{\theta}_m - \theta) + R_m.$$

where $R_m = \partial_{xy} G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*, \theta^{**})(\tilde{q}_m - q)(\tilde{\theta}_m - \theta) + \frac{1}{2}\partial_{yy} G_{\tilde{\mu}_m,\tilde{\pi}_m}(q, \theta^*)(\tilde{\theta}_m - \theta)^2$. Therefore, it suffice to prove that $R_m = o(\tilde{\theta}_m - \theta)$ as $m \to \infty$.

Following equation (EC.6) in the proof of Lemma 3,

$$\partial_y G_{\tilde{\mu}_m,\tilde{\pi}_m}(x,y) = \frac{\int_{\mathbb{R}}\int_{\mathbb{R}} h_m(x-a_m u, u, y-a_m v, v)dudv}{f_{\tilde{\mu}_m}(x)}.$$

By further taking derivative with respect to y, we have

$$\partial_y^2 G_{\tilde{\mu}_m,\tilde{\pi}_m}(x,y) = \frac{\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_y h_m(x-a_m u, u, y-a_m v, v)dudv}{f_{\tilde{\mu}_m}(x)},$$

and hence

$$\left|\partial_y^2 G_{\tilde{\mu}_m,\tilde{\pi}_m}(q,\theta^*)\right| \le \frac{\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{h}_{m,2}(u,v)dudv}{f_{\tilde{\mu}_m}(q)} \le \frac{H_2}{f_{\tilde{\mu}_m}(q)} \to \frac{H_2}{f_{\mu}(q)}.$$

Similarly, by further taking derivative with respect to x, we have

$$\begin{aligned}\partial_{xy}^2 G_{\tilde{\mu}_m,\tilde{\pi}_m}(x,y) = &\frac{1}{f_{\tilde{\mu}_m}(x)}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_x h_m(x-a_m u, u, y-a_m v, v)dudv \\ &- \frac{f'_{\tilde{\mu}_m}(x)}{[f_{\tilde{\mu}_m}(x)]^2}\int_{\mathbb{R}}\int_{\mathbb{R}} h_m(x-a_m u, u, y-a_m v, v)dudv,\end{aligned}$$

and hence

$$\begin{aligned}\left|\partial_{xy}^2 G_{\tilde{\mu}_m,\tilde{\pi}_m}(q^*,\theta^{**})\right| &\le \frac{1}{f_{\tilde{\mu}_m}(q^*)}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{h}_{m,1}(u,v)dudv + \frac{|f'_{\tilde{\mu}_m}(q^*)|}{[f_{\tilde{\mu}_m}(q^*)]^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{h}_{m,0}(u,v)dudv \\ &\le \frac{1}{f_{\tilde{\mu}_m}(q^*)}H_1 + \frac{|f'_{\tilde{\mu}_m}(q^*)|}{[f_{\tilde{\mu}_m}(q^*)]^2}H_0 \\ &\to \frac{1}{f_{\mu}(q)}H_1 + \frac{|f'_{\mu}(q)|}{[f_{\mu}(q)]^2}H_0,\end{aligned}$$

where for the last limit we have used Lemma 1 and Proposition 2. Combine the above results with the definition of R_m, we can conclude that $R_m = o(\tilde{\theta}_m - \theta)$. □

EC.3. Other Proofs

Proof Sketch of Proposition 1. Given that the complete proof of this proposition is both tedious and standard, adhering closely to established methods in the literature, we provide only a sketch of the proof to highlight the key steps and underlying logic. Let $\xi_j(Z)$ and $\zeta_j(Z)$ denote the portfolio pricing error for the $j-$th inner-level observation of $\mu(z)$ and $\pi(z)$ respectively, and let $\bar{\xi}^l(Z) = \frac{1}{l}\sum_{j=1}^l \xi_j(Z)$ and $\bar{\zeta}^l(Z) = \frac{1}{l}\sum_{j=1}^l \zeta_j(Z)$ be the zero-mean average pricing errors for the portfolios. Following the analytical scheme of Gordy and Juneja (2010), we take the surrogates $\hat{X}(Z) = \mu(Z) + \bar{\xi}^l(Z)$ and $\hat{Y}(Z) = \pi(Z) + \bar{\zeta}^l(Z)$ and analyze them instead of analyzing $\bar{X}$ and $\bar{Y}$ directly. Let $\hat{\xi}^l = \sqrt{l}\bar{\xi}^l$ and $\hat{\zeta}^l = \sqrt{l}\bar{\zeta}^l$ so that they have nontrivial limit distribution as $l \to \infty$. Let $G_{\mu,\pi}(x,y) = \Pr\{\pi(Z) \le y | \mu(Z) = x\}$ and $G_{\hat{X},\hat{Y}}(x,y) = \Pr\{\hat{Y}(Z) \le y | \hat{X}(Z) = x\}$. Define the joint density of $(\mu, \pi, \hat{\xi}_l, \hat{\zeta}_l)$ as $\phi(s,t,u,v)$. With the regularity conditions on the joint density and its partial derivatives similar to Assumption 1 in Gordy and Juneja (2010), we have

$$G_{\hat{X},\hat{Y}}(x,y) - G_{\mu,\pi}(x,y) = \mathcal{O}(1/l). \tag{EC.12}$$

Let $\hat{\theta}_{k,h,l}$ be the CoVaR estimator of standard nested estimator and θ be the (α,β)-CoVaR of $(\mu(Z),\pi(Z))$, which satisfies $\mathrm{P}\{\pi(Z)\leq\theta|\mu(Z)=q_\alpha\}=G_{\mu,\pi}(q_\alpha,\theta)=\beta$, where q_α is the α-VaR of $\mu(Z)$. To derive the mean squared error (MSE) $\mathrm{E}[(\hat{\theta}_{k,h,l}-\theta)^2]$, we introduce another two auxiliary variables. We define $\hat{\theta}_{h,l}$ such that $\mathrm{P}\{\hat{Y}(Z)\leq\hat{\theta}_{h,l}|\hat{X}(Z)=\bar{X}_{\lceil h\alpha\rceil}\}=G_{\hat{X},\hat{Y}}(\bar{X}_{\lceil h\alpha\rceil},\hat{\theta}_{h,l})=\beta$, where $\bar{X}_{\lceil h\alpha\rceil}$ is the $\lceil h\alpha\rceil$-th order statistics of $\bar{X}$. Besides, $\hat{\theta}_l$ is defined as the (α,β)-CoVaR of $(\hat{X}(Z),\hat{Y}(Z))$, which satisfies $\mathrm{P}\{\hat{Y}(Z)\leq\hat{\theta}_l|\hat{X}(Z)=\hat{q}\}=G_{\hat{X},\hat{Y}}(\hat{q},\hat{\theta}_l)=\beta$, where $\hat{q}$ is the α-VaR of $\hat{X}(Z)$.

The MSE, $\mathrm{E}[(\hat{\theta}_{k,h,l}-\theta)^2]$, can be decomposed into

$$\mathrm{E}[(\hat{\theta}_{k,h,l}-\theta)^2]=\underbrace{\mathrm{VaR}(\hat{\theta}_{k,h,l})}_{variance}+\underbrace{(\mathrm{E}[\hat{\theta}_{k,h,l}]-\theta)^2}_{bias^2}.$$

It's obvious that the order of variance term is $\mathcal{O}(1/k)$ since it involves estimating the quantile with k i.i.d observations. Then, we analyze the bias term by further decompose the bias into:

$$\mathrm{E}[\hat{\theta}_{k,h,l}]-\theta=\underbrace{\mathrm{E}[\mathrm{E}[\hat{\theta}_{k,h,l}|\bar{X}_{\lceil h\alpha\rceil}]-\hat{\theta}_{h,l}]}_{I}+\underbrace{\mathrm{E}[\hat{\theta}_{h,l}-\hat{\theta}_l]}_{II}+\underbrace{\hat{\theta}_l-\theta}_{III},$$

and analyze them term by term.

For term I, $\mathrm{E}[\hat{\theta}_{k,h,l}|\bar{X}_{\lceil h\alpha\rceil}]-\hat{\theta}_{h,l}$ is intrinsically the difference between the $\lceil k\beta\rceil$-th order statistics and the β-th quantile of $\hat{Y}|\hat{X}=\bar{X}_{\lceil h\alpha\rceil}$. Under the regularity conditions on conditional density of $\hat{Y}|\hat{X}$ and the marginal density of $\hat{X}$, we can obtain that $\mathrm{E}[\mathrm{E}[\hat{\theta}_{k,h,l}|\bar{X}_{\lceil h\alpha\rceil}]-\hat{\theta}_{h,l}]]=\mathcal{O}(1/k)$ according to Equation (4.6.3) of David and Nagaraja (2004).

For term II, we apply Taylor's expansion to $G_{\hat{X},\hat{Y}}(\hat{\theta}_{h,l},\bar{X}_{\lceil h\alpha\rceil})$ at $(\hat{\theta}_l,\hat{q})$, we obtain that $\hat{\theta}_{h,l}-\hat{\theta}_l=\mathcal{O}(\bar{X}_{\lceil h\alpha\rceil}-\hat{q})$ and thus $\mathrm{E}[\hat{\theta}_{h,l}-\hat{\theta}_l]=\mathcal{O}(\mathrm{E}[\bar{X}_{\lceil h\alpha\rceil}-\hat{q}])=\mathcal{O}(1/h)$ according to Lemma 2 of Hong (2009).

For term III, by applying Taylor's expansion to $G_{\hat{X},\hat{Y}}(\hat{\theta}_l,\hat{q})$ on (θ,q_α) and using the fact that $G_{\hat{X},\hat{Y}}(\hat{\theta}_l,\hat{q})=G_{\mu,\pi}(q_\alpha,\theta)=\beta$, we can derive that

$$\hat{\theta}_l-\theta=\mathcal{O}(\underbrace{G_{\hat{X},\hat{Y}}(q_\alpha,\theta)-G_{\mu,\pi}(q_\alpha,\theta)}_{(i)})+\mathcal{O}(\underbrace{\hat{q}-q}_{(ii)}),$$

where term (i) is of order $\mathcal{O}(1/l)$ by Equation (EC.12) and term (i) is of order $\mathcal{O}(1/l)$ by Proposition 2 of Gordy and Juneja (2010).

Therefore, the bias term is of order $\mathcal{O}(1/k+1/h+1/l)$. Overall, the order of MSE is $\mathcal{O}(1/k+1/h^2+1/l^2)$. Notice that the first two terms of this rate, $1/k$ and $1/h^2$, align with the results derived in Huang et al. (2024) under the assumption that the closed form expressions of $\mu(Z)$ and $\pi(Z)$ are available, while the third therm, $1/l^2$, arises from the inner-level estimation with l sample observations. Since $\Gamma=khl$, we have the best convergence rate of RMSE is $\mathcal{O}(\Gamma^{-1/4})$, with $l=c_1\Gamma^{1/4}$, $h=c_2\Gamma^{1/4}$ and $k=\frac{1}{c_1c_2}\Gamma^{1/2}$, for some constants $c_1,c_2>0$. $\square$

Proof of Proposition 5. We can rewrite $\mu(z)$ by

$$\begin{aligned}\mu(z) &= \mathrm{E}[\varphi(Z)|Z_0 = z]\\ &= \mathrm{E}[\varphi(\psi_T(S_1, S_2, \cdots, S_n))|\psi_0(S_0) = z]\\ &= \iint \cdots \int \varphi(\psi_T(s_1, s_2 \cdots, s_n)) p_{0,1}(\psi_0^{-1}(z), s_1) \prod_{k=2}^{n} p_{k-1,k}(s_{k-1}, s_k) \mathrm{d}s_1 \mathrm{d}s_2 \cdots \mathrm{d}s_n.\end{aligned}$$

Let $\nu(s) = \mu(\psi_0(s))$, then $\mu(z) = \nu(\psi_0^{-1}(z))$. Since ψ_0^{-1} is smooth, we only need to prove the smoothness of $\nu(\cdot)$. Actually, under condition (a) we can apply Fubini's theorem, and

$$\begin{aligned}\nu(s) &= \iint \cdots \int \varphi(\psi_T(s_1, s_2 \cdots, s_n)) p_{0,1}(s, s_1) \prod_{k=2}^{n} p_{k-1,k}(s_{k-1}, s_k) \mathrm{d}s_1 \mathrm{d}s_2 \cdots \mathrm{d}s_n\\ &= \int p_{0,1}(s, s_1) \left[\int \cdots \int \varphi(\psi_T(s_1, s_2 \cdots, s_n)) \prod_{k=2}^{n} p_{k-1,k}(s_{k-1}, s_k) \mathrm{d}s_2 \cdots \mathrm{d}s_n\right] \mathrm{d}s_1\\ &= \int p_{0,1}(s, s_1) \phi(s_1) \mathrm{d}s_1.\end{aligned}$$

For given s and $j \in \{1, \cdots, d\}$, define a mapping $\zeta_{s,j} : \mathbb{R} \to \mathbb{R}^q$, such that $\zeta_j(t)$ has the j-th entry equal to $t \in \mathbb{R}$ while all the other entries identical to the corresponding entries of s. Denote e_j to be a q-dimensional vector with the jth entry being 1 and all the other entries being 0, and $s(j)$ to be the j-th entry of s, then by applying Leibniz integral rule,

$$p_{0,1}(s, s_1) = p_{0,1}(\zeta_{s,j}(s(j)), s_1) = p_{0,1}(\zeta_{s,j}(t_0), s_1) + \int_{t_0}^{s(j)} \partial^{e_j} p_{0,1}(\zeta_{s,j}(t), s_1) dt.$$

Therefore

$$\begin{aligned}\nu(s) &= \int p_{0,1}(\zeta_{s,j}(t_0), s_1)\phi(s_1)\mathrm{d}s_1 + \int \int_{t_0}^{s(j)} \partial^{e_j} p_{0,1}(\zeta_{s,j}(t), s_1)\phi(s_1) dt \mathrm{d}s_1\\ &= \int p_{0,1}(\zeta_{s,j}(t_0), s_1)\phi(s_1)\mathrm{d}s_1 + \int_{t_0}^{s(j)} \int \partial^{e_j} p_{0,1}(\zeta_{s,j}(t), s_1)\phi(s_1)\mathrm{d}s_1 dt\end{aligned}$$

where the changing the order of integration is validate by Fubini's theorem under condition (d). This condition also indicates the inner integral is a continuous function of t, and hence $D^{e_j}\nu(s)$ exists and

$$D^{e_j}\nu(s) = \int \partial^{e_j} p_{0,1}(\zeta_{s,j}(s(j)), s_1)\phi(s_1)\mathrm{d}s_1 = \int \partial^{e_j} p_{0,1}(s, s_1)\phi(s_1)\mathrm{d}s_1.$$

Because $\partial^{e_j} p_{0,1}(s, s_1)$ is continuous and has integrable derivatives with respect to s due to condition (d), the partial derivative $\partial^{e_j}\nu(s)$ is also continuous for all $j = 1, \cdots, d$. This indicates $\nu(s)$ is differentiable. Similarly, we can establish the higher order differentiability of $\nu(s)$ through induction on the order. $\square$

EC.4. Numerical Experiments for Validating the Theoretical Convergence Rates of Decoupled Estimators

This section presents numerical results to validate the theoretical convergence rates derived in Section 4.2 for the decoupled estimators of CoVaR. The experimental design follows a similar setup as in Section 5, where we consider two portfolios consisting of stocks, up-and-out barrier call options, and geometric Asian options. However, we use a moderate-dimensional setting with $q = 10$ underlying stocks, resulting in a 30-dimensional problem ($q = 30$). This lower-dimensional setting is chosen because, for high-dimensional problems, observing a clear pattern in convergence rates typically requires very large sample sizes. In such cases, hyperparameter tuning for smoothing techniques such as kernel ridge regression and neural networks becomes computationally expensive and impractical to implement on a standard PC.

Specifically, we consider decoupled estimators with four smoothing techniques: linear regression (LR), kernel smoothing (KS), kernel ridge regression (KRR), and neural networks (NN). For each technique, we vary the computational budget Γ from 10^3 to 10^5 and fit a log–log regression of the relative RMSE (r-RMSE) on Γ. The results are presented in Figure EC.1, which reports the fitted slopes and R^2 values, with identical y-axis ranges across all subplots to facilitate visual comparison.

Consistent with our theoretical results, LR, KRR, and NN exhibit slopes close to $-1/2$, indicating that r-RMSE $\propto \Gamma^{-1/2}$. In contrast, KS converges more slowly. As discussed in Section 4.2, KS suffers from the curse of dimensionality, and this example demonstrates that this issue arises even in the moderate-dimensional setting. In comparison, KRR and NN overcome this limitation and exhibit superior rates of convergence. Furthermore, KRR and NN deliver the highest accuracy among all methods. For example, their relative RMSEs fall below 1.5% at $\Gamma = 5 \times 10^4$. By contrast, LR attains a similar convergence slope but incurs larger RMSE at at the same Γ due to the irreducible bias introduced by its basis functions. For NN, we observe a relatively smaller R^2 compared to LR or KRR; this is expected because the performance of NN is more sensitive to hyperparameter tuning.

Figure EC.1 Empirical Convergence Rates of Decoupled Estimators with Different Smoothing Techniques