E-Companion for "Beyond $\mathcal{O}(\sqrt{T})$ Regret: Decoupling Learning and Decision-making in Online Linear Programming"

Appendix A: Proof of Results in Section 3

A.1. Auxiliary results

Lemma EC 1 (Hoeffding's inequality) *Let* $X_1, \dots, X_n$ *be independent random variables such that* $0 \le X_i \le u$ *almost surely. Then for all* $\zeta \ge 0$,

$$\mathbb{P}\left\{\tfrac{1}{n}\textstyle\sum_{i=1}^n X_i - \mathbb{E}[\tfrac{1}{n}\sum_{i=1}^n X_i] \ge \zeta\right\} \le \exp\{-2nu^{-2}\zeta^2\}.$$

Lemma EC 2 *Consider standard form LP* $\min_{\mathbf{A}\mathbf{x}=\mathbf{b},\mathbf{x}\ge\mathbf{0}}\langle \mathbf{c}, \mathbf{x}\rangle$ *and suppose both primal and dual problems are non-degenerate. Then the primal LP solution* $\mathbf{x}^\star$ *is unique and there exists* $\mu > 0$ *such that*

$$\langle \mathbf{c}, \mathbf{x}\rangle - \langle \mathbf{c}, \mathbf{x}^\star\rangle \ge \mu \|\mathbf{x} - \mathbf{x}^\star\|$$

for all primal feasible $\mathbf{x} \in \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$.

Proof. Denote $\|\mathbf{x}\|_{-\infty} = \min_j x_j$. Since both primal and dual problems are non-degenerate, $\mathbf{x}^\star$ is unique (Bertsimas and Tsitsiklis 1997). Denote (B, N) to be the partition of basic and non-basic variables, and we have $\mathbf{x}^\star = (\mathbf{x}_B^\star, \mathbf{x}_N^\star)$, where $\mathbf{x}_B^\star > \mathbf{0}$ and $\mathbf{x}_N^\star = \mathbf{0}$. Similarly, denote $\mathbf{s}$ to be the dual slack for $\mathbf{x}$, we can partition $\mathbf{s}^\star = (\mathbf{s}_B^\star, \mathbf{s}_N^\star)$ where $\mathbf{s}_B^\star = \mathbf{0}$ and $\mathbf{s}_N^\star > 0$. We have $\mathbf{A}_B\mathbf{x}_B^\star = \mathbf{b}$ by primal feasibility of $\mathbf{x}^\star$. With dual feasibility, $\mathbf{c}_N = \mathbf{A}_N^\top \mathbf{y}^\star + \mathbf{s}_N^\star, \mathbf{c}_B = \mathbf{A}_B^\top \mathbf{y}^\star$ for some $\mathbf{y}^\star$. Next, consider any feasible LP solution $\mathbf{x}$, and we can write

$$\mathbf{A}\mathbf{x} = \mathbf{A}_B\mathbf{x}_B + \mathbf{A}_N\mathbf{x}_N = \mathbf{b} = \mathbf{A}_B\mathbf{x}_B^\star.$$

Since $\mathbf{A}_B$ is non-degenerate, taking inverse on both sides gives $\mathbf{x}_B^\star = \mathbf{x}_B + \mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N$ and we deduce that

$$\begin{aligned}
\langle \mathbf{c}, \mathbf{x}\rangle - \langle \mathbf{c}, \mathbf{x}^\star\rangle &= \langle \mathbf{c}_B, \mathbf{x}_B\rangle + \langle \mathbf{c}_N, \mathbf{x}_N\rangle - \langle \mathbf{c}_B, \mathbf{x}_B^\star\rangle && \text{(EC.1)}\\
&= \langle \mathbf{c}_B, \mathbf{x}_B\rangle + \langle \mathbf{c}_N, \mathbf{x}_N\rangle - \langle \mathbf{c}_B, \mathbf{x}_B + \mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N\rangle && \text{(EC.2)}\\
&= \langle \mathbf{c}_N - \mathbf{A}_N^\top \mathbf{A}_B^{-\top}\mathbf{c}_B, \mathbf{x}_N\rangle &&\\
&= \langle \mathbf{A}_N^\top \mathbf{y}^\star + \mathbf{s}_N^\star - \mathbf{A}_N^\top \mathbf{A}_B^{-\top}\mathbf{A}_B^\top \mathbf{y}^\star, \mathbf{x}_N\rangle && \text{(EC.3)}\\
&= \langle \mathbf{s}_N^\star, \mathbf{x}_N\rangle \ge \|\mathbf{s}_N^\star\|_{-\infty}\|\mathbf{x}_N\|, && \text{(EC.4)}
\end{aligned}$$

where (EC.1) uses $\mathbf{x}_N^\star = \mathbf{0}$, (EC.2) plugs in $\mathbf{x}_B^\star = \mathbf{x}_B + \mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N$, (EC.3) plugs in $\mathbf{c}_N = \mathbf{A}_N^\top \mathbf{y}^\star + \mathbf{s}_N^\star$ and $\mathbf{c}_B = \mathbf{A}_B^\top \mathbf{y}^\star$, (EC.4) uses the fact that $\mathbf{s}_N^\star > \mathbf{0}$ and $\langle \mathbf{s}_N^\star, \mathbf{x}_N\rangle \geq \|\mathbf{s}_N\|_{-\infty}\|\mathbf{x}_N\|_1 \geq \|\mathbf{s}_N\|_{-\infty}\|\mathbf{x}_N\|$. Re-arranging the terms,

$$\|\mathbf{x}_N\| \leq \|\mathbf{s}_N^\star\|_{-\infty}^{-1}(\langle \mathbf{c}, \mathbf{x}\rangle - \langle \mathbf{c}, \mathbf{x}^\star\rangle). \tag{EC.5}$$

On the other hand, we have

$$\begin{aligned}
\|\mathbf{x} - \mathbf{x}^\star\|^2 &= \|\mathbf{x}_B - \mathbf{x}_B^\star\|^2 + \|\mathbf{x}_N - \mathbf{x}_N^\star\|^2 \\
&= \|\mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N\|^2 + \|\mathbf{x}_N\|^2 && \text{(EC.6)} \\
&= \langle \mathbf{x}_N, (\mathbf{A}_N^\top \mathbf{A}_B^{-\top}\mathbf{A}_B^{-1}\mathbf{A}_N + \mathbf{I})\mathbf{x}_N\rangle \\
&\leq \big(\tfrac{\|\mathbf{A}_N\|^2}{\sigma_{\min}(\mathbf{A}_B)^2} + 1\big)\|\mathbf{x}_N\|^2 \\
&\leq \big(\tfrac{\|\mathbf{A}_N\|^2}{\sigma_{\min}(\mathbf{A}_B)^2} + 1\big)\big(\tfrac{\langle \mathbf{c},\mathbf{x}\rangle - \langle \mathbf{c},\mathbf{x}^\star\rangle}{\|\mathbf{s}_N^\star\|_{-\infty}}\big)^2, && \text{(EC.7)}
\end{aligned}$$

where (EC.6) again plugs in $\mathbf{x}_B^\star = \mathbf{x}_B + \mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N$ and $\mathbf{x}_N^\star = \mathbf{0}$; (EC.7) uses the relation (EC.5) and $\sigma_{\min}(A)$ denotes the minimum singular value of matrix A. Taking square root on both sides gives

$$\|\mathbf{x} - \mathbf{x}^\star\| \leq (\tfrac{\|\mathbf{A}_N\|^2}{\sigma_{\min}(\mathbf{A}_B)^2} + 1)^{1/2}\tfrac{1}{\|\mathbf{s}_N^\star\|_{-\infty}}[\langle \mathbf{c}, \mathbf{x}\rangle - \langle \mathbf{c}, \mathbf{x}^\star\rangle] \leq \tfrac{\|\mathbf{A}_N\| + \sigma_{\min}(\mathbf{A}_B)}{\sigma_{\min}(\mathbf{A}_B)\|\mathbf{s}_N^\star\|_{-\infty}}[\langle \mathbf{c}, \mathbf{x}\rangle - \langle \mathbf{c}, \mathbf{x}^\star\rangle]$$

and another rearrangement of the inequality completes the proof. □

Lemma EC 3 (Learning algorithm for Hölder growth (Xu et al. 2017)) *Consider stochastic optimization problem* $\min_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{y}) := \mathbb{E}_\xi[f(\mathbf{y},\xi)]$ *with optimal set* $\mathcal{Y}^\star$ *and suppose the following conditions hold:*

1. *there exists some* $\mathbf{y}^1 \in \mathcal{Y}$ *such that* $f(\mathbf{y}^1) - f(\mathbf{y}^\star) \leq \varepsilon_0$*,*
2. $\mathcal{Y}^\star$ *is a nonempty compact set,*
3. *there exists some constant* G *such that* $\|f'(\mathbf{y},\xi)\| \leq G$ *for all* ξ*,*
4. *there exists some constant* $\lambda > 0$ *and* $\theta \in (0,1]$ *such that for all* $\mathbf{y} \in \mathcal{Y}$

$$f(\mathbf{y}) - f(\mathbf{y}^\star) \geq \lambda \cdot \operatorname{dist}(\mathbf{y}, \mathcal{Y}^\star)^{1/\theta}.$$

*Then, there is a first-order method (**Algorithm 5**, Algorithm 1, 2, and 4 of Xu et al. (2017)) that outputs some* $\hat{\mathbf{y}}$ *such that* $f(\hat{\mathbf{y}}) - f(\mathbf{y}^\star) \leq \varepsilon$ *after*

$$T_\varepsilon \geq \{\max\{9, 1728\{\log(\tfrac{1}{\delta}) + \log\lceil \log_2(\tfrac{2\varepsilon_0}{\varepsilon})\rceil\}\}\tfrac{2^{2(1-\theta)}\lambda^{-2\theta}G^2}{\varepsilon^{2(1-\theta)}} + 1\}\lceil \log_2(\tfrac{2\varepsilon_0}{\varepsilon})\rceil$$

iterations with probability at least $1 - \delta$*.*

Lemma EC 4 (Last-iterate convergence of stochastic subgradient (Liu and Zhou 2024)) *Consider stochastic optimization problem* $\min_{\mathbf{y}\geq\mathbf{0}} f(\mathbf{y}) := \mathbb{E}_\xi[f(\mathbf{y},\xi)]$. *Suppose the following conditions hold:*

1. *There exist* $M \geq 0$ *such that*

$$f(\mathbf{x}) - f(\mathbf{y}) - \langle f'(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle \leq M\|\mathbf{x}-\mathbf{y}\|$$

   for all $\mathbf{x},\mathbf{y}$ *and* $f'(\mathbf{y}) \in \partial f(\mathbf{y})$,
2. *It is possible to compute* $\mathbf{g}_\mathbf{y}$ *such that* $\mathbb{E}[\mathbf{g}_\mathbf{y}] = f'(\mathbf{y})$,
3. $\mathbb{E}[\|\mathbf{g}_\mathbf{y} - f'(\mathbf{y})\|^2] \leq \sigma^2$.

Then, the last iterate of the projected subgradient method with stepsize α*:* $\mathbf{y}^{t+1} = [\mathbf{y}^t - \alpha\mathbf{g}^t]_+$ *satisfies*

$$\mathbb{E}[f(\mathbf{y}^{T+1}) - f(\mathbf{y})] \leq \frac{\|\mathbf{y}^1-\mathbf{y}\|^2}{T\alpha} + 2\alpha(M^2+\sigma^2)(1+\log T)$$

for all $\mathbf{y} \geq \mathbf{0}$.

Lemma EC. 4 is an application of Theorem C.1, equation (24) of Liu and Zhou (2024) with $L = 0, h(\mathbf{y}) = 0$ and $\psi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$.

A.2. Dual learning algorithm

We include two algorithms in Xu et al. (2017) that can exploit **A4** and achieve the sample complexity in **Lemma 1**. **Algorithm 5** is the baseline algorithm and **Algorithm 6** is its parameter-free variant that adapts to unknown λ. Note that the algorithm has an explicit projection routine onto $\mathcal{Y}' = \{\mathbf{y} \geq \mathbf{0} : \|\mathbf{y}\| \leq \frac{\bar{c}}{\underline{d}}\}$. According to Xu et al. (2017), given parameters $(\delta, \varepsilon, \varepsilon_0, \gamma, G)$, **Algorithm 5** is configured as follows:

$$K = \lceil \log_2(\tfrac{2\varepsilon_0}{\varepsilon})\rceil, \quad D_1 = \frac{2^{1-\gamma}\lambda^{-\theta}\varepsilon_0}{\varepsilon^{1-\gamma}}, \quad t = \max\{9, 1728\log(\tfrac{K}{\delta})\}\frac{G^2D_1^2}{\varepsilon_0^2}.$$

A.3. Verification of the examples

A.3.1. Continuous support The result is a direct application of Proposition 2 of Li and Ye (2022).

A.3.2. Finite support Denote $\{(\xi_k, \boldsymbol{\alpha}_k)\}_{k=1}^K$ to be the support of LP data associated with distribution $\mathbf{p} \in \mathbb{R}^K$. i.e., there are K types of customers and customers of type k arrive with probability p_k. We can write the distributional dual problem as

$$\min_{(\mathbf{y},\sigma)\geq\mathbf{0}} \quad \langle \mathbf{d}, \mathbf{y}\rangle + \textstyle\sum_{k=1}^K p_i\sigma_i \quad \text{subject to} \quad \sigma_i \geq \xi_i - \langle \boldsymbol{\alpha}_i, \mathbf{y}\rangle, i \in [K].$$

Algorithm 5: Accelerated Stochastic SubGradient Method (ASSG)

```
Input: Initial point $\mathbf{y}_0 \in \mathcal{Y}' = \{\mathbf{y} \geq \mathbf{0} : \|\mathbf{y}\| \leq \frac{\bar{c}}{\underline{d}}\}$, outer iteration count $K$, inner iteration
       count $t$, initial error estimate $\varepsilon_0$, initial diameter $D_1$, Lipschitz constant $G$
Set $\eta_1 = \frac{\varepsilon_0}{3G^2}$
for $k = 1, 2, \ldots, K$ do
    Let $\mathbf{y}_1^k = \mathbf{y}_{k-1}$
    for $\tau = 1, 2, \ldots, t-1$ do
        $\mathbf{y}_{\tau+1}^k = \prod_{\mathcal{Y} \cap \mathcal{B}(\mathbf{y}_{k-1}, D_k)} [\mathbf{y}_\tau^k - \eta_k \mathbf{g}_{\mathbf{y}_\tau^k}]$
    end
    Let $\mathbf{y}_k = \frac{1}{t} \sum_{\tau=1}^{t} \mathbf{y}_\tau^k$
    Let $\eta_{k+1} = \frac{1}{2}\eta_k$ and $D_{k+1} = \frac{1}{2} D_k$
end
Output: $\mathbf{y}_K$
```

Algorithm 6: ASSG with Restart (RASSG)

```
Input: Initial point $\mathbf{y}^0 \in \mathcal{Y}' = \{\mathbf{y} \geq \mathbf{0} : \|\mathbf{y}\| \leq \frac{\bar{c}}{\underline{d}}\}$, outer iteration count $K$, initial distance
       $D_1^{(1)}$, inner iteraion count $t_1$, initial error estimate $\varepsilon_0$ and $\omega \in (0, 1]$, error bound
       parameter $\gamma$, restart round $S$, Lipschitz constant $G$
Set $\varepsilon_0^{(1)} = \varepsilon_0$, $\eta_1 = \frac{\varepsilon_0}{3G^2}$
for $s = 1, 2, \ldots, S$ do
    $\mathbf{y}^{(s)} \leftarrow \mathrm{ASSG}(\mathbf{y}^{(s-1)}, K, t_s, D_1^{(s)}, \varepsilon_0^{(s)})$
    Let $t_{s+1} = t_s 2^{2(1-\gamma^{-1})}$, $D_1^{(s+1)} = D_1^{(s)} 2^{1-\gamma^{-1}}$, and $\varepsilon_0^{(s+1)} = \omega \varepsilon_0^{(s)}$
end
Output: $\mathbf{y}^{(S)}$
```

More compactly, we introduce slack $\boldsymbol{\lambda} \in \mathbb{R}^K$ and define $\mathbf{f} := (\mathbf{d}; \mathbf{p}; \mathbf{0}), \mathbf{z} := (\mathbf{y}; \boldsymbol{\sigma}; \boldsymbol{\lambda}) \geq \mathbf{0}, \mathbf{Q} := (\mathbf{A}^\top, \mathbf{I}, -\mathbf{I})$, $\boldsymbol{\xi} = [\xi_1; \ldots; \xi_K] \in \mathbb{R}^K$. Then, the dual problem can be written in the standard form,

$$\min_{\mathbf{z} \geq \mathbf{0}} \quad \langle \mathbf{f}, \mathbf{z} \rangle \quad \text{s.t.} \quad \mathbf{Q}\mathbf{z} = \boldsymbol{\xi}. \tag{EC.8}$$

When $\mathrm{di}(\mathcal{Y}^\star) > 0$, the result is an application of weak sharp minima to LP (Burke and Ferris 1993). When the primal-dual problems are both non-degenerate, $\mathcal{Y}^\star = \{\mathbf{y}^\star\}$, and applying **Lemma 2**, we get the following error bound in terms of the LP optimal basis.

Lemma EC 5 *Let* (B, N) *denote the optimal basis partition for* (EC.8) *and let* $\mathbf{s}_N$ *denote the dual slack of primal variables* $\mathbf{z}$*, then*

$$\langle \mathbf{f}, \mathbf{z}\rangle - \langle \mathbf{f}, \mathbf{z}^\star\rangle \geq \mu \|\mathbf{z} - \mathbf{z}^\star\|,$$

where $\mu = \frac{\sigma_{\min}(\mathbf{Q}_B)\|\mathbf{s}_N\|_{-\infty}}{\|\mathbf{Q}_N\| + \sigma_{\min}(\mathbf{Q}_B)}$*. Moreover, we have* $f(\mathbf{y}) - f(\mathbf{y}^\star) \geq \mu\|\mathbf{y} - \mathbf{y}^\star\|$*.*

Proof. $\langle \mathbf{f}, \mathbf{z}\rangle - \langle \mathbf{f}, \mathbf{z}^\star\rangle \geq \mu\|\mathbf{z} - \mathbf{z}^\star\|$ follows from **Lemma EC. 2** applied to the compact LP formulation. Next, define $\mathbf{z}_\mathbf{y} := (\mathbf{y}; \boldsymbol{\sigma}_\mathbf{y}; \boldsymbol{\lambda}_\mathbf{y})$ where $\sigma_\mathbf{y} = [\boldsymbol{\xi} - \sum_{k=1}^K \boldsymbol{\alpha}_i y_i]_+$ and $\boldsymbol{\lambda}_\mathbf{y} = \boldsymbol{\sigma}_\mathbf{y} - \boldsymbol{\xi} + \sum_{k=1}^K \boldsymbol{\alpha}_i y_i$. We deduce

$$f(\mathbf{y}) - f(\mathbf{y}^\star) = \langle \mathbf{f}, \mathbf{z}_\mathbf{y}\rangle - \langle \mathbf{f}, \mathbf{z}^\star\rangle \geq \mu\|\mathbf{z}_\mathbf{y} - \mathbf{z}^\star\| \geq \mu\|\mathbf{y} - \mathbf{y}^\star\|,$$

and this completes the proof. □

A.3.3. General growth Given $\mathbf{y}^\star \in \arg\min_\mathbf{y} f(\mathbf{y}) \subseteq \operatorname{int}(\mathcal{Y})$, by optimality condition, $\mathbf{0} = \mathbf{d} - \mathbb{E}[\mathbf{a}\mathbb{I}\{c \geq \langle \mathbf{a}, \mathbf{y}^\star\rangle\}]$ and

$$\mathbf{d} = \int \mathbf{a} \int_{\langle \mathbf{a}, \mathbf{y}^\star\rangle}^{\infty} \mathrm{d}F(c|\mathbf{a})\mathrm{d}F(\mathbf{a}),$$

where $F(c, \mathbf{a})$ denotes the c.d.f. of the distribution of $(c, \mathbf{a})$. Then we deduce that

$$\begin{aligned}
f(\mathbf{y}) - f(\mathbf{y}^\star) &= \langle \mathbf{d}, \mathbf{y} - \mathbf{y}^\star\rangle + \mathbb{E}[[c - \langle \mathbf{a}, \mathbf{y}\rangle]_+ - [c - \langle \mathbf{a}, \mathbf{y}^\star\rangle]_+] \\
&= \int \int_{\langle \mathbf{a}, \mathbf{y}^\star\rangle}^{\infty} \langle \mathbf{a}, \mathbf{y} - \mathbf{y}^\star\rangle \mathrm{d}F(c|\mathbf{a})\mathrm{d}F(\mathbf{a}) + \int \int_{\langle \mathbf{a}, \mathbf{y}\rangle}^{\langle \mathbf{a}, \mathbf{y}^\star\rangle} \mathrm{d}F(c|\mathbf{a})\mathrm{d}F(\mathbf{a}) \\
&= \int \int_{\langle \mathbf{a}, \mathbf{y}\rangle}^{\langle \mathbf{a}, \mathbf{y}^\star\rangle} \mathbb{I}\{c \geq v\}\langle \mathbf{a}, \mathbf{y} - \mathbf{y}^\star\rangle \mathrm{d}v\mathrm{d}F(c, \mathbf{a}) + \int \int_{\langle \mathbf{a}, \mathbf{y}\rangle}^{\langle \mathbf{a}, \mathbf{y}^\star\rangle} \mathrm{d}F(c|\mathbf{a})\mathrm{d}F(\mathbf{a}) \\
&= \int \int_{\langle \mathbf{a}, \mathbf{y}\rangle}^{\langle \mathbf{a}, \mathbf{y}^\star\rangle} \mathbb{I}\{c \geq v\} - \mathbb{I}\{c \geq \langle \mathbf{a}, \mathbf{y}^\star\rangle\}\mathrm{d}v\mathrm{d}F(c, \mathbf{a}).
\end{aligned}$$

Next, we invoke the assumptions and

$$\begin{aligned}
&\int \int_{\langle \mathbf{a}, \mathbf{y}\rangle}^{\langle \mathbf{a}, \mathbf{y}^\star\rangle} \mathbb{I}\{c \geq v\} - \mathbb{I}\{c \geq \langle \mathbf{a}, \mathbf{y}^\star\rangle\} \,\mathrm{d}v\mathrm{d}F(c, \mathbf{a}) \\
&\geq \tfrac{\lambda_5}{2} \int \int_{\langle \mathbf{a}, \mathbf{y}\rangle}^{\langle \mathbf{a}, \mathbf{y}^\star\rangle} |\langle \mathbf{a}, \mathbf{y}^\star\rangle - v|^p \mathrm{d}v\mathrm{d}F(\mathbf{a}) \\
&= \tfrac{\lambda_5}{2(p+1)} \mathbb{E}[|\langle \mathbf{a}, \mathbf{y} - \mathbf{y}^\star\rangle|^{p+1}] \\
&\geq \tfrac{\lambda_5}{2(p+1)} \mathbb{E}[|\langle \mathbf{a}, \mathbf{y} - \mathbf{y}^\star\rangle|]^{p+1} \qquad \text{(EC.9)} \\
&\geq \tfrac{\lambda_4^{p+1}\lambda_5}{2(p+1)} \|\mathbf{y} - \mathbf{y}^\star\|^{p+1},
\end{aligned}$$

where (EC.9) uses $p \geq 0$ and that $\mathbb{E}[|X|^{p+1}] \geq E[|X|]^{p+1}$. Since $\|\mathbf{y} - \mathbf{y}^\star\|^{p+1} > 0$ for $\mathbf{y} \neq \mathbf{y}^\star$, this completes the proof.

A.4. Proof of Lemma 1

We verify the conditions in **Lemma EC. 3**.

Condition 1. Take $\mathbf{y}^1 = \mathbf{0} \in \mathcal{Y}$. Then

$$f(\mathbf{y}^1) - f(\mathbf{y}^\star) \leq f(\mathbf{y}^1) = \mathbb{E}[\langle \mathbf{d}, \mathbf{y}^1 \rangle + [c - \langle \mathbf{a}, \mathbf{y}^1 \rangle]_+] = \mathbb{E}[[c]_+] \leq \bar{c},$$

where the first inequality holds since $f(\mathbf{y}^\star) \geq 0$.

Condition 2 holds since $\mathcal{Y}^\star \subseteq \mathcal{Y}$, $\mathcal{Y}^\star$ is closed and $\mathcal{Y}$ is a compact set.

Condition 3 holds since $\mathbf{g}_{\mathbf{y}} = \mathbf{d} - \mathbf{a}\mathbb{I}\{c \geq \langle \mathbf{a}, \mathbf{y} \rangle\}$ and $\|\mathbf{g}\| \leq \sqrt{m}(\bar{a} + \bar{d})$. Hence $G = \sqrt{m}(\bar{a} + \bar{d})$.

Condition 4 holds by the dual error bound condition $f(\mathbf{y}) \geq \mu \cdot \operatorname{dist}(\mathbf{y}, \mathcal{Y}^\star)^\gamma$ with $\lambda = \mu$ and $\theta = 1/\gamma$.

Now invoke **Lemma EC. 3** and we get that, after

$$\begin{aligned} T_\varepsilon &\geq \{\max\{9, 1728\{\log(\tfrac{1}{\delta}) + \log\lceil \log_2(\tfrac{2\bar{c}}{\varepsilon})\rceil\}\}\frac{2^{2(1-\gamma^{-1})}\mu^{-2/\gamma}m(\bar{a}+\bar{d})^2}{\varepsilon^{2(1-\gamma^{-1})}} + 1\}\lceil \log_2(\tfrac{2\bar{c}}{\varepsilon})\rceil \\ &= \mathcal{O}(\varepsilon^{-2(1-\gamma^{-1})}\log(\tfrac{1}{\delta})\log(\tfrac{1}{\varepsilon})) \end{aligned}$$

iterations, the algorithm outputs $\bar{\mathbf{y}}^{T+1}$ such that with probability at least $1 - \delta$,

$$\mu \cdot \operatorname{dist}(\bar{\mathbf{y}}^{T+1}, \mathcal{Y}^\star) \leq f(\bar{\mathbf{y}}^{T+1}) - f(\mathbf{y}^\star) \leq \varepsilon,$$

and this completes the proof.

A.5. Proof of Lemma 2

We verify the conditions in **Lemma EC. 4**.

Condition 1. Since $f(\mathbf{y})$ is convex and has Lipschitz constant $\sqrt{m}(\bar{a} + \bar{d})$, we take $M = 2\sqrt{m}(\bar{a} + \bar{d})$ and deduce that

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) - \langle f'(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle &\leq \sqrt{m}(\bar{a} + \bar{d})\|\mathbf{x} - \mathbf{y}\| + \|f'(\mathbf{y})\| \cdot \|\mathbf{x} - \mathbf{y}\| \\ &\leq 2\sqrt{m}(\bar{a} + \bar{d})\|\mathbf{x} - \mathbf{y}\| \\ &= M\|\mathbf{x} - \mathbf{y}\|. \end{aligned}$$

Condition 2 holds in the stochastic i.i.d. input setting.

Condition 3 holds by taking $\sigma^2 = 4m(\bar{a} + \bar{d})^2$ and notice that

$$\mathbb{E}[\|\mathbf{g}_{\mathbf{y}} - f'(\mathbf{y})\|^2] \leq 2\mathbb{E}[\|\mathbf{g}_{\mathbf{y}}\|^2] + 2[\|f'(\mathbf{y})\|^2] \leq 4m(\bar{a} + \bar{d})^2.$$

Next, we invoke **Lemma EC. 4** and get last-iterate convergence for $T \geq 3$.

$$
\begin{aligned}
\mathbb{E}[f(\mathbf{y}^{T+1}) - f(\mathbf{y})] &\leq \tfrac{\|\mathbf{y}^1 - \mathbf{y}\|^2}{T\alpha} + 2\alpha(M^2 + \sigma^2)(1 + \log T) \\
&\leq \tfrac{\|\mathbf{y}^1 - \mathbf{y}\|^2}{T\alpha} + 16\alpha m(\bar{a} + \bar{d})^2(1 + \log T) \qquad \text{(EC.10)} \\
&\leq \tfrac{\|\mathbf{y}^1 - \mathbf{y}\|^2}{T\alpha} + 32\alpha m(\bar{a} + \bar{d})^2 \log T,
\end{aligned}
$$

where (EC.10) plugs in $M = 2\sqrt{m}(\bar{a} + \bar{d})$ and $\sigma^2 = 4m(\bar{a} + \bar{d})^2$. Taking $\mathbf{y} = \Pi_{\mathcal{Y}^\star}(\mathbf{y}^1)$ completes the proof.

A.6. Proof of Lemma 3

By definition and the fact that $\mathcal{Y}^\star \in \mathcal{Y}$,

$$
\mathbf{y}^\star \in \arg\min_{\mathbf{y} \in \mathcal{Y}} f(\mathbf{y}) \qquad \text{and} \qquad \mathbf{y}_T^\star \in \arg\min_{\mathbf{y} \in \mathcal{Y}} f_T(\mathbf{y}).
$$

According to **A4**, $f(\mathbf{y}^\star) \leq f(\mathbf{y}_T^\star) - \mu \text{dist}(\mathbf{y}_T^\star, \mathcal{Y}^\star)^\gamma$ and

$$
\begin{aligned}
\mu \cdot \text{dist}(\mathbf{y}_T^\star, \mathcal{Y}^\star)^\gamma &\leq f(\mathbf{y}_T^\star) - f(\mathbf{y}^\star) \\
&= f(\mathbf{y}_T^\star) - f_T(\mathbf{y}_T^\star) + f_T(\mathbf{y}_T^\star) - f_T(\mathbf{y}^\star) + f_T(\mathbf{y}^\star) - f(\mathbf{y}^\star). \\
&\leq f(\mathbf{y}_T^\star) - f_T(\mathbf{y}_T^\star) + f_T(\mathbf{y}^\star) - f(\mathbf{y}^\star), \qquad \text{(EC.11)}
\end{aligned}
$$

where (EC.11) uses $f_T(\mathbf{y}_T^\star) - f_T(\mathbf{y}^\star) \leq 0$. Taking expectation and using $\mathbb{E}[f_T(\mathbf{y}^\star)] = f(\mathbf{y}^\star)$, we arrive at

$$
\mu \mathbb{E}[\text{dist}(\mathbf{y}_T^\star, \mathcal{Y}^\star)^\gamma] \leq \mathbb{E}[f(\mathbf{y}_T^\star) - f_T(\mathbf{y}_T^\star)]
$$

and it remains to bound $f(\mathbf{y}_T^\star) - f_T(\mathbf{y}_T^\star)$. For any fixed $\mathbf{y} \in \mathcal{Y}$,

$$
f_T(\mathbf{y}) = \tfrac{1}{T} \textstyle\sum_{t=1}^T \langle \mathbf{d}, \mathbf{y} \rangle + [c_t - \langle \mathbf{a}_t, \mathbf{y} \rangle]_+
$$

and for each t, since $\mathbf{y} \geq \mathbf{0}$,

$$
\begin{aligned}
0 &\leq \langle \mathbf{d}, \mathbf{y} \rangle + [c_t - \langle \mathbf{a}_t, \mathbf{y} \rangle]_+ \\
&\leq \|\mathbf{d}\| \cdot \|\mathbf{y}\| + |c_t| + \|\mathbf{a}_t\| \cdot \|\mathbf{y}\| \\
&\leq \sqrt{m}\bar{d}\tfrac{(\bar{c} + \underline{d})}{\underline{d}} + \bar{c} + \sqrt{m}\bar{a}\tfrac{(\bar{c} + \underline{d})}{\underline{d}} \\
&= \sqrt{m}\tfrac{(\bar{a} + \bar{d})(\bar{c} + \underline{d})}{\underline{d}} + \bar{c}
\end{aligned}
$$

Using **Lemma EC. 1**,

$$
\mathbb{P}\{f(\mathbf{y}) - f_T(\mathbf{y}) \geq \zeta\} \leq \exp\{-\tfrac{2\underline{d}^2 T}{(\sqrt{m}(\bar{a} + \bar{d})(\bar{c} + \underline{d}) + \bar{c}\underline{d})^2}\zeta^2\}.
$$

Recall that $\mathbf{y}_T^\star \in \mathcal{Y}$ by (EC.12), and we construct an ε-net of $\mathcal{Y}$ as follows:

$$\mathcal{Y} \subseteq \mathcal{N}_k := \bigcup_{\{j_i\}_{i=1}^m \in \{0,\ldots,k\}^m} \{\mathbf{y} : \|\mathbf{y} - \textstyle\sum_{i=1}^m \frac{\bar{c}+\underline{d}}{k\underline{d}} j_i \mathbf{e}_i\|_\infty \leq \frac{\bar{c}+\underline{d}}{k\underline{d}}\},$$

where we denote the centers of each net as C_k and $|C_k| = (k+1)^m$. In each member of the net, we have, by Lipschitz continuity of $f(\mathbf{y})$ and $f_T(\mathbf{y})$, that

$$f(\mathbf{y}_1) - f(\mathbf{y}_2) \leq \sqrt{m}(\bar{a}+\bar{d})\|\mathbf{y}_1 - \mathbf{y}_2\| \leq m(\bar{a}+\bar{d})\|\mathbf{y}_1 - \mathbf{y}_2\|_\infty \leq \tfrac{m(\bar{a}+\bar{d})(\bar{c}+\underline{d})}{k\underline{d}}.$$

Next, with union bound,

$$\begin{aligned}\mathbb{P}\{\max_{\mathbf{y}\in C_k} f(\mathbf{y}) - f_T(\mathbf{y}) \geq \zeta\} \leq{} & \textstyle\sum_{\mathbf{z}\in C_k} \mathbb{P}\{f(\mathbf{z}) - f_T(\mathbf{z}) \geq \zeta\} \\ \leq{} & (k+1)^m \exp\{-\tfrac{2\underline{d}^2 T}{(\sqrt{m}(\bar{a}+\bar{d})(\bar{c}+\underline{d})+\bar{c}\underline{d})^2}\zeta^2\}.\end{aligned}$$

Taking $k = \sqrt{T}$, we have

$$\begin{aligned}&\mathbb{P}\{\sup_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{y}) - f_T(\mathbf{y}) \leq \zeta + \tfrac{2m(\bar{a}+\bar{d})(\bar{c}+\underline{d})}{\sqrt{T}\underline{d}}\} \\ \geq{} & \mathbb{P}\{\sup_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{y}) - f_T(\mathbf{y}) \leq \zeta + \tfrac{2m(\bar{a}+\bar{d})(\bar{c}+\underline{d})}{\sqrt{T}\underline{d}} \,|\, \max_{\mathbf{y}\in C_k} f(\mathbf{y}) - f_T(\mathbf{y}) \leq \zeta\} \cdot \mathbb{P}\{\max_{\mathbf{y}\in C_k} f(\mathbf{y}) - f_T(\mathbf{y}) \leq \zeta\} \\ ={} & \mathbb{P}\{\max_{\mathbf{y}\in C_k} f(\mathbf{y}) - f_T(\mathbf{y}) \leq \zeta\} \\ \geq{} & 1 - (\sqrt{T}+1)^m \exp\{-\tfrac{4\underline{d}^2 T}{(\sqrt{m}(\bar{a}+\bar{d})(\bar{c}+\underline{d})+\bar{c}\underline{d})^2}\zeta^2\}.\end{aligned}$$

Taking $\zeta = \sqrt{\frac{3m(\sqrt{m}(\bar{a}+\bar{d})(\bar{c}+\underline{d})+\bar{c}\underline{d})^2}{4\underline{d}^2}\frac{\log T}{T}}$ gives

$$\mathbb{P}\Big\{\sup_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{y}) - f_T(\mathbf{y}) \leq \mathcal{O}\big(\sqrt{\tfrac{\log T}{T}}\big)\Big\} \geq 1 - \tfrac{1}{T}$$

and

$$\mathbb{E}[\text{dist}(\mathbf{y}_T^\star, \mathcal{Y}^\star)]^\gamma \leq \mathbb{E}[\text{dist}(\mathbf{y}_T^\star, \mathcal{Y}^\star)^\gamma] \leq \tfrac{1}{\mu}\mathbb{E}[f(\mathbf{y}_T^\star) - f_T(\mathbf{y}_T^\star)] = \mathcal{O}(\sqrt{\tfrac{\log T}{T}}) = o(1).$$

This completes the proof.

Appendix B: Proof of results in Section 4

B.1. Auxiliary results

Lemma EC 6 (Bounded dual solution (Gao et al. 2023)) *Assume that* **A1** *to* **A3** *hold and suppose* ***Algorithm 1*** *with* $\alpha_t \equiv \alpha$ *starts from* $\mathbf{y}^1$ *and* $\|\mathbf{y}^1\| \leq \frac{\bar{c}}{\underline{d}}$*, then*

$$\|\mathbf{y}^t\| \leq \tfrac{\bar{c}}{\underline{d}} + \tfrac{m(\bar{a}+\bar{d})^2\alpha}{2\underline{d}} + \alpha\sqrt{m}(\bar{a}+\bar{d}) = \mathsf{R}, \textit{ for all } t, \tag{EC.12}$$

almost surely. Moreover, if $\alpha \leq \frac{2\underline{d}}{3m(\bar{a}+\bar{d})^2}$*, then* $\mathbf{y}^t \in \mathcal{Y}$ *for all* t *almost surely.*

Proof. The relation (EC.12) follows immediately from Lemma 5 of Gao et al. (2023). To see $\mathbf{y}^t \in \mathcal{Y}$, we successively deduce, for $\alpha \le \frac{2\underline{d}}{3m(\bar{a}+\bar{d})^2}$, that

$$\frac{m(\bar{a}+\bar{d})^2\alpha}{2\underline{d}} + \alpha\sqrt{m}(\bar{a}+\bar{d}) = \frac{1}{3} + \frac{2\underline{d}}{3\sqrt{m}(\bar{a}+\bar{d})} \le \frac{1}{3} + \frac{2(\bar{a}+\bar{d})}{3\sqrt{m}(\bar{a}+\bar{d})} \le 1$$

and this completes the proof. □

Lemma EC 7 (Subgradient method on strongly convex problems (Rakhlin et al. 2012)) *Let $\delta \in (0, e^{-1})$ and assume $T \ge 4$. Suppose $f(\mathbf{y})$ is μ-strongly convex and $\|\mathbf{g}_\mathbf{y}\| \le G$. Then, the subgradient method with stepsize $\alpha_t = 1/(\mu t)$ satisfies*

$$\|\mathbf{y}^{T+1} - \mathbf{y}^\star\|^2 \le \frac{624\log(\frac{\log T}{\delta}+1)G^2}{\mu^2 T}$$

with probability at least $1-\delta$.

Lemma EC 8 (Subgradient method for $\gamma = 2$) *Suppose* **A1** *to* **A3** *and* **A4** *with $\gamma = 2$ hold. Then, the subgradient method with $\alpha_t = 1/(\mu(t+1))$ outputs $\mathbf{y}^{T+1}$ such that $\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1}, \mathcal{Y}^\star)^2] \le \frac{m(\bar{a}+\bar{d})^2}{\mu^2 T}$.*

Proof. For any $\hat{\mathbf{y}} \in \mathcal{Y}^\star$, we deduce that

$$\begin{aligned} \|\mathbf{y}^{t+1} - \hat{\mathbf{y}}\|^2 &= \|\Pi_{\mathcal{Y}}[\mathbf{y}^t - \alpha_t \mathbf{g}^t] - \hat{\mathbf{y}}\|^2 \\ &\le \|\mathbf{y}^t - \alpha_t \mathbf{g}^t - \hat{\mathbf{y}}\|^2 \qquad \text{(EC.13)} \\ &= \|\mathbf{y}^t - \hat{\mathbf{y}}\|^2 - 2\alpha_t\langle \mathbf{y}^t - \hat{\mathbf{y}}, \mathbf{g}^t\rangle + \alpha_t^2\|\mathbf{g}^t\|^2, \end{aligned}$$

where (EC.13) uses the non-expansiveness of the projection operator. Taking $\hat{\mathbf{y}} = \Pi_{\mathcal{Y}^\star}[\mathbf{y}^t]$ and using $\|\mathbf{g}^t\|^2 \le m(\bar{a}+\bar{d})^2$, we get

$$\|\mathbf{y}^{t+1} - \hat{\mathbf{y}}\|^2 \le \operatorname{dist}(\mathbf{y}^t, \mathcal{Y}^\star)^2 - 2\alpha_t\langle \mathbf{y}^t - \mathbf{y}, \mathbf{g}^t\rangle + \alpha_t^2 m(\bar{a}+\bar{d})^2.$$

Since $\mathbb{E}[\mathbf{g}^t] \in \partial f(\mathbf{y}^t)$, we have, by convexity of f, that

$$-2\langle \mathbf{y}^t - \mathbf{y}^\star, \alpha_t \mathbb{E}[\mathbf{g}^t]\rangle \le -2\alpha_t(f(\mathbf{y}^t) - f(\mathbf{y})).$$

Next, we invoke **A4** to get

$$f(\mathbf{y}^t) - f(\hat{\mathbf{y}}) \ge \mu \cdot \operatorname{dist}(\mathbf{y}^t, \mathcal{Y}^\star)^2.$$

Conditioned on history and taking expectation, we have

$$\begin{aligned}\mathbb{E}[\mathrm{dist}(\mathbf{y}^{t+1},\mathcal{Y}^\star)^2|\mathbf{y}^t] &\le \mathbb{E}[\|\mathbf{y}^{t+1}-\hat{\mathbf{y}}\|^2|\mathbf{y}^t]\\ &\le \mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2-2\alpha_t\mu\mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2+\alpha_t^2m(\bar{a}+\bar{d})^2\\ &=(1-2\alpha_t\mu)\mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2+\alpha_t^2m(\bar{a}+\bar{d})^2. \qquad \text{(EC.14)}\end{aligned}$$

With $\alpha_t=\frac{1}{\mu(t+1)}$, we have

$$\begin{aligned}\mathbb{E}[\mathrm{dist}(\mathbf{y}^{t+1},\mathcal{Y}^\star)^2] &\le (1-2\alpha_t\mu)\mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2+\alpha_t^2m(\bar{a}+\bar{d})^2\\ &=\tfrac{t-1}{t+1}\mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2+\tfrac{m(\bar{a}+\bar{d})^2}{\mu^2(t+1)^2}.\end{aligned}$$

Multiply both sides by $(t+1)^2$ and we get

$$(t+1)^2\mathbb{E}[\mathrm{dist}(\mathbf{y}^{t+1},\mathcal{Y}^\star)^2]\le(t^2-1)\mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2+\tfrac{m(\bar{a}+\bar{d})^2}{\mu^2} \qquad \text{(EC.15)}$$

$$4\mathbb{E}[\mathrm{dist}(\mathbf{y}^2,\mathcal{Y})^2]\le\tfrac{m(\bar{a}+\bar{d})^2}{\mu^2} \qquad \text{(EC.16)}$$

Re-arranging the terms, we arrive at

$$(t+1)^2\mathbb{E}[\mathrm{dist}(\mathbf{y}^{t+1},\mathcal{Y}^\star)^2]-t^2\mathrm{dist}(\mathbf{y}^t,\mathcal{Y}^\star)^2\le\tfrac{m(\bar{a}+\bar{d})^2}{\mu^2}.$$

Taking expectation over all the randomness and telescoping from $t=2$ to T, with (EC.16) added, gives

$$\mathbb{E}[\mathrm{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)^2]\le\tfrac{m(\bar{a}+\bar{d})^2T}{\mu^2(T+1)^2}\le\tfrac{m(\bar{a}+\bar{d})^2}{\mu^2T}$$

and this completes the proof. □

Lemma EC 9 (Subgradient with constant stepsize) *Under the same assumptions as* ***Lemma EC. 8****, if $\alpha_t\equiv\alpha<1/(2\mu)$, then*

$$\mathbb{E}[\mathrm{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)^2]\le\tfrac{\Delta^2}{\mu\alpha T}+\tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha,$$

where $\Delta=\mathrm{dist}(\mathbf{y}_1,\mathcal{Y}^\star)$.

Proof. Taking $\alpha_t\equiv\alpha<1/(2\mu)$ and unrolling the recursion from (EC.14) till $\mathbf{y}^1$, we have

$$\begin{aligned}\mathbb{E}[\mathrm{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)^2] &\le(1-2\mu\alpha)\mathbb{E}[\mathrm{dist}(\mathbf{y}^T,\mathcal{Y}^\star)^2]+\alpha^2m(\bar{a}+\bar{d})^2\\ &\le(1-2\mu\alpha)^T\mathrm{dist}(\mathbf{y}^1,\mathcal{Y}^\star)^2+\textstyle\sum_{j=0}^{T-1}\alpha^2m(\bar{a}+\bar{d})^2(1-2\mu\alpha)^j\\ &\le(1-2\mu\alpha)^T\mathrm{dist}(\mathbf{y}^1,\mathcal{Y}^\star)^2+\tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha \qquad \text{(EC.17)}\\ &\le\tfrac{1}{\mu\alpha T}\mathrm{dist}(\mathbf{y}^1,\mathcal{Y}^\star)^2+\tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha \qquad \text{(EC.18)}\\ &=\tfrac{\Delta^2}{\mu\alpha T}+\tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha,\end{aligned}$$

where (EC.17) uses the relation $\sum_{j=0}^{T-1}(1-2\mu\alpha)^j = \frac{1-(1-2\mu\alpha)^T}{2\mu\alpha} \le \frac{1}{\mu\alpha}$ and (EC.18) is by $(1-2\mu\alpha)^T \le \frac{1}{1+2\mu\alpha T} \le \frac{1}{\mu\alpha T}$. This completes the proof. □

B.2. Proof of Lemma 4

By the definition of regret, we deduce that

$$
\begin{aligned}
\mathbb{E}[r(\hat{\mathbf{x}}_T)] &= \mathbb{E}[\langle \mathbf{c}, \mathbf{x}_T^\star\rangle - \langle \mathbf{c}, \hat{\mathbf{x}}_T\rangle] \\
&= \mathbb{E}[T f_T(\mathbf{y}_T^\star) - \langle \mathbf{c}, \hat{\mathbf{x}}_T\rangle] \qquad \text{(EC.19)} \\
&\le \mathbb{E}[T f_T(\mathbf{y}^\star) - \langle \mathbf{c}, \hat{\mathbf{x}}_T\rangle] \qquad \text{(EC.20)} \\
&= T f(\mathbf{y}^\star) - \mathbb{E}[\langle \mathbf{c}, \hat{\mathbf{x}}_T\rangle] \qquad \text{(EC.21)} \\
&\le \mathbb{E}[\textstyle\sum_{t=1}^T f(\mathbf{y}^t) - \langle \mathbf{c}, \hat{\mathbf{x}}_T\rangle] \\
&= \textstyle\sum_{t=1}^T \mathbb{E}[\langle \mathbf{d}, \mathbf{y}^t\rangle + [c_t - \langle \mathbf{a}_t, \mathbf{y}^t\rangle]_+ - c_t x^t] \qquad \text{(EC.22)} \\
&= \textstyle\sum_{t=1}^T \mathbb{E}[\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle],
\end{aligned}
$$

where (EC.19) uses strong duality of LP; (EC.20) uses the fact $\mathbf{y}^\star$ is a feasible solution and that $\mathbf{y}_T^\star$ is the optimal solution to the sample LP; (EC.22) uses the definition of $f(\mathbf{y})$ and that $(c_t, \mathbf{a}_t)$ are i.i.d. generated. Then we have

$$
\begin{aligned}
\|\mathbf{y}^{t+1}\|^2 - \|\mathbf{y}^t\|^2 &= \|[\mathbf{y}^t - \alpha(\mathbf{d} - \mathbf{a}_t x^t)]_+\|^2 - \|\mathbf{y}^t\|^2 \\
&\le \|\mathbf{y}^t - \alpha(\mathbf{d} - \mathbf{a}_t x^t)\|^2 - \|\mathbf{y}^t\|^2 \qquad \text{(EC.23)} \\
&= -2\alpha\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle + \alpha^2\|\mathbf{d} - \mathbf{a}_t x^t\|^2 \\
&\le -2\alpha\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle + m(\bar{a} + \bar{d})^2\alpha^2, \qquad \text{(EC.24)}
\end{aligned}
$$

where (EC.23) uses $\|[\mathbf{x}]_+\| \le \|\mathbf{x}\|$ and (EC.24) uses **A2**, **A3**. A simple rearrangement gives

$$
\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle \le \frac{m(\bar{a}+\bar{d})^2\alpha}{2} + \frac{\|\mathbf{y}^t\|^2 - \|\mathbf{y}^{t+1}\|^2}{2\alpha}. \qquad \text{(EC.25)}
$$

Next, we telescope the relation (EC.25) from $t = 1$ to T and get

$$
\begin{aligned}
\mathbb{E}[r(\hat{\mathbf{x}}_T)] &= \textstyle\sum_{t=1}^T \mathbb{E}[\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle] \\
&\le \frac{m(\bar{a}+\bar{d})^2\alpha}{2}T + \sum_{t=1}^T \frac{\mathbb{E}[\|\mathbf{y}^t\|^2] - \mathbb{E}[\|\mathbf{y}^{t+1}\|^2]}{2\alpha} \qquad \text{(EC.26)} \\
&= \frac{m(\bar{a}+\bar{d})^2\alpha}{2}T + \frac{\mathbb{E}[\|\mathbf{y}^1\|^2] - \mathbb{E}[\|\mathbf{y}^{T+1}\|^2]}{2\alpha} \\
&= \frac{m(\bar{a}+\bar{d})^2\alpha}{2}T + \frac{\mathbb{E}[\langle \mathbf{y}^1 + \mathbf{y}^{T+1}, \mathbf{y}^1 - \mathbf{y}^{T+1}\rangle]}{2\alpha} \qquad \text{(EC.27)} \\
&\le \frac{m(\bar{a}+\bar{d})^2\alpha}{2}T + \frac{\mathsf{R}}{\alpha}\mathbb{E}[\|\mathbf{y}^1 - \mathbf{y}^{T+1}\|] \qquad \text{(EC.28)} \\
&\le \frac{m(\bar{a}+\bar{d})^2\alpha}{2}T + \frac{\mathsf{R}}{\alpha}\mathbb{E}[\|\mathbf{y}^1 - \mathbf{y}^\star\| + \|\mathbf{y}^{T+1} - \mathbf{y}^\star\|], \qquad \text{(EC.29)}
\end{aligned}
$$

where (EC.26) again uses relation (EC.25); (EC.28) uses Cauchy's inequality

$$\langle \mathbf{y}^1 + \mathbf{y}^1, \mathbf{y}^1 - \mathbf{y}^{T+1} \rangle \leq \|\mathbf{y}^1 + \mathbf{y}^{T+1}\| \cdot \|\mathbf{y}^1 - \mathbf{y}^{T+1}\|$$

and almost sure boundedness of iterations derived from **Lemma 6**:

$$\|\mathbf{y}^1 + \mathbf{y}^{T+1}\| \leq \|\mathbf{y}^{T+1}\| + \|\mathbf{y}^1\| \leq 2\mathsf{R}.$$

Finally (EC.29) is obtained from the triangle inequality

$$\|\mathbf{y}^1 - \mathbf{y}^{T+1}\| = \|\mathbf{y}^1 - \mathbf{y}^\star + \mathbf{y}^\star - \mathbf{y}^{T+1}\| \leq \|\mathbf{y}^1 - \mathbf{y}^\star\| + \|\mathbf{y}^{T+1} - \mathbf{y}^\star\|$$

and this completes the proof.

B.3. Proof of Lemma 5

For constraint violation, recall that

$$\mathbb{E}[v(\hat{\mathbf{x}}_T)] = \mathbb{E}[\|[\mathbf{A}\hat{\mathbf{x}}_T - \mathbf{b}]_+\|] = \mathbb{E}\Big[\Big\|\Big[\textstyle\sum_{t=1}^T (\mathbf{a}_t x^t - \mathbf{d})\Big]_+\Big\|\Big]$$

and that

$$\mathbf{y}^{t+1} = [\mathbf{y}^{t+1} - \alpha(\mathbf{d} - \mathbf{a}_t x^t)]_+ \geq \mathbf{y}^t - \alpha(\mathbf{d} - \mathbf{a}_t x^t).$$

A re-arrangement gives

$$\mathbf{a}_t x^t \leq \mathbf{d} + \tfrac{1}{\alpha}(\mathbf{y}^{t+1} - \mathbf{y}^t). \tag{EC.30}$$

and that

$$\begin{aligned} \textstyle\sum_{t=1}^T (\mathbf{a}_t x^t - \mathbf{d}) &\leq \tfrac{1}{\alpha} \textstyle\sum_{t=1}^T (\mathbf{y}^{t+1} - \mathbf{y}^t) \qquad \text{(EC.31)} \\ &= \tfrac{1}{\alpha}(\mathbf{y}^{T+1} - \mathbf{y}^1) \end{aligned}$$

where (EC.31) uses (EC.30). Now, we apply the triangle inequality again:

$$\begin{aligned} \mathbb{E}[\|[\mathbf{A}\hat{\mathbf{x}}_T - \mathbf{b}]_+\|] &\leq \tfrac{1}{\alpha}\mathbb{E}[\|\mathbf{y}^{T+1} - \mathbf{y}^1\|] \\ &\leq \tfrac{1}{\alpha}\mathbb{E}[\|\mathbf{y}^1 - \mathbf{y}^\star\| + \|\mathbf{y}^{T+1} - \mathbf{y}^\star\|], \qquad \text{(EC.32)} \end{aligned}$$

and this completes the proof.

B.4. Proof of Lemma 6

Similar to the proof of **Lemma 4** and **Lemma 5**, we deduce that

$$\begin{aligned}\mathbb{E}[r(\hat{\mathbf{x}}_T)] &\le Tf(\mathbf{y}^\star) - \mathbb{E}[\langle \mathbf{c}, \hat{\mathbf{x}}_T\rangle] \qquad \text{(EC.33)}\\ &= T_e f(\mathbf{y}^\star) - \mathbb{E}[\textstyle\sum_{t=1}^{T_e} c_t x^t] + \sum_{t=T_e+1}^{T} \mathbb{E}[f(\mathbf{y}^\star) - c_t x^t]\\ &\le T_e f(\mathbf{y}^\star) - \mathbb{E}[\textstyle\sum_{t=1}^{T_e} c_t x^t] + \sum_{t=T_e+1}^{T} \mathbb{E}[f(\mathbf{y}^t) - c_t x^t]\\ &= T_e f(\mathbf{y}^\star) - \mathbb{E}[\textstyle\sum_{t=1}^{T_e} c_t x^t] + \sum_{t=T_e+1}^{T} \mathbb{E}[\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle],\end{aligned}$$

where (EC.33) is directly obtained from (EC.21). Next, we analyze $\sum_{t=T_e+1}^{T} \mathbb{E}[\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle]$. Using (EC.25),

$$\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle \le \tfrac{m(\bar{a}+\bar{d})^2\alpha}{2} + \tfrac{\|\mathbf{y}^t\|^2 - \|\mathbf{y}^{t+1}\|^2}{2\alpha},$$

and we deduce that

$$\begin{aligned}\textstyle\sum_{t=T_e+1}^{T} \mathbb{E}[\langle \mathbf{d} - \mathbf{a}_t x^t, \mathbf{y}^t\rangle] &\le \textstyle\sum_{t=T_e+1}^{T} \left[\tfrac{m(\bar{a}+\bar{d})^2\alpha}{2} + \tfrac{1}{2\alpha}\mathbb{E}[\|\mathbf{y}^t\|^2 - \|\mathbf{y}^{t+1}\|^2]\right]\\ &= \tfrac{m(\bar{a}+\bar{d})^2\alpha}{2} T_p + \tfrac{1}{2\alpha}\mathbb{E}[\|\mathbf{y}^{T_e+1}\|^2 - \|\mathbf{y}^{T+1}\|^2]\\ &\le \tfrac{m(\bar{a}+\bar{d})^2\alpha}{2} T_p + \tfrac{\mathsf{R}}{\alpha}\mathbb{E}[\|\mathbf{y}^{T_e+1} - \mathbf{y}^\star\| + \|\mathbf{y}^{T+1} - \mathbf{y}^\star\|], \qquad \text{(EC.34)}\end{aligned}$$

where (EC.34) uses triangle inequality as in (EC.28). Next, we consider constraint violation, and we have

$$\begin{aligned}\mathbb{E}[v(\hat{\mathbf{x}}_T)] &= \mathbb{E}[\|[\mathbf{A}\hat{\mathbf{x}}_T - \mathbf{b}]_+\|]\\ &= \mathbb{E}[\|[\textstyle\sum_{t=1}^{T_e}(\mathbf{a}_t x^t - \mathbf{d}) + \sum_{t=T_e+1}^{T}(\mathbf{a}_t x^t - \mathbf{d})]_+\|]\\ &\le \mathbb{E}[\|[\textstyle\sum_{t=1}^{T_e}(\mathbf{a}_t x^t - \mathbf{d})]_+\|] + \mathbb{E}[\|[\sum_{t=T_e+1}^{T}(\mathbf{a}_t x^t - \mathbf{d})]_+\|], \qquad \text{(EC.35)}\end{aligned}$$

where (EC.35) is by $\|[\mathbf{x}+\mathbf{y}]_+\| \le \|[\mathbf{x}]_+\| + \|[\mathbf{y}]_+\|$ and we bound

$$\mathbb{E}[\|[\textstyle\sum_{t=T_e+1}^{T}(\mathbf{a}_t x^t - \mathbf{d})]_+\|] \le \tfrac{1}{\alpha}\mathbb{E}[\|\mathbf{y}^{T_e+1} - \mathbf{y}^\star\| + \|\mathbf{y}^{T+1} - \mathbf{y}^\star\|]$$

with the same argument as (EC.32). Putting two relations together and using

$$V(T_e) = \mathbb{E}[\|[\textstyle\sum_{t=1}^{T_e}(\mathbf{a}_t x^t - \mathbf{d})]_+\| + \sum_{t=1}^{T_e} f(\mathbf{y}^\star) - c_t x^t].$$

We arrive at

$$\begin{aligned}&\mathbb{E}[r(\hat{\mathbf{x}}_T) + v(\hat{\mathbf{x}}_T)]\\ &\le V(T_e) + \tfrac{m(\bar{a}+\bar{d})^2\alpha}{2} T_p + \tfrac{\mathsf{R}+1}{\alpha}\mathbb{E}[\|\mathbf{y}^{T_e+1} - \mathbf{y}^\star\| + \|\mathbf{y}^{T+1} - \mathbf{y}^\star\|]\\ &\le V(T_e) + \tfrac{m(\bar{a}+\bar{d})^2\alpha}{2} T_p + \tfrac{\mathsf{R}+1}{\alpha}\mathbb{E}[\mathrm{dist}(\mathbf{y}^{T_e+1}, \mathcal{Y}^\star) + \mathrm{dist}(\mathbf{y}^{T+1}, \mathcal{Y}^\star) + 2\mathrm{di}(\mathcal{Y}^\star)], \qquad \text{(EC.36)}\end{aligned}$$

where (EC.36) uses

$$\|\mathbf{y}-\mathbf{y}^\star\| = \|\mathbf{y}-\Pi_{\mathcal{Y}^\star}[\mathbf{y}]+\Pi_{\mathcal{Y}^\star}[\mathbf{y}]-\mathbf{y}^\star\| \le \operatorname{dist}(\mathbf{y},\mathcal{Y}^\star)+\operatorname{di}(\mathcal{Y}^\star)$$

for all $\mathbf{y}$ and it remains to analyze $\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)+\operatorname{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)]$.

By **Lemma 1**, we have with probability $1-1/T^{2\gamma}$ that

$$\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)\le\Delta.$$

Conditioned on the event $\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)\le\Delta$, we deduce that

$$\begin{aligned}
\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)] &= \mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)|\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)\le\Delta]\cdot\mathbb{P}\{\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)\le\Delta\}\\
&\quad+\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)|\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)>\Delta]\cdot\mathbb{P}\{\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)>\Delta\}\\
&\le\Delta+\frac{\mathsf{R}}{T^{2\gamma}}, &\text{(EC.37)}\\
\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)^2] &= \mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)^2|\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)\le\Delta]\cdot\mathbb{P}\{\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)\le\Delta\}\\
&\quad+\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)^2|\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)>\Delta]\cdot\mathbb{P}\{\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)>\Delta\}\\
&\le\Delta^2+\frac{\mathsf{R}^2}{T^{2\gamma}}, &\text{(EC.38)}
\end{aligned}$$

where both (EC.37) and (EC.38) use the fact that $\mathbf{y}^{T_e+1}\in\mathcal{Y}$ imposed by **Algorithm 3**. Using **Lemma 2**, we have, conditioned on $\mathbf{y}^{T_e+1}$, that

$$\begin{aligned}
\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)]^\gamma &\le \mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)^\gamma] &\text{(EC.39)}\\
&=\mathbb{E}[\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)^\gamma]|\mathbf{y}^{T_e+1}]\\
&\le\tfrac{1}{\mu}\mathbb{E}[\tfrac{1}{\alpha T_p}\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)^2+32m(\bar{a}+\bar{d})^2\alpha\log T_p] &\text{(EC.40)}\\
&\le\tfrac{1}{\mu}[\tfrac{1}{\alpha T_p}\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1},\mathcal{Y}^\star)^2]+32m(\bar{a}+\bar{d})^2\alpha\log T] &\text{(EC.41)}\\
&\le\tfrac{1}{\mu}[\tfrac{1}{\alpha T_p}(\Delta^2+\tfrac{\mathsf{R}^2}{T^{2\gamma}})+32m(\bar{a}+\bar{d})^2\alpha\log T], &\text{(EC.42)}
\end{aligned}$$

where (EC.39) uses $\mathbb{E}[X]^\gamma\le\mathbb{E}[X^\gamma]$ for nonnegative random variable X; (EC.40) invokes **Lemma 2**; (EC.41) uses $T_p\le T$ and (EC.42) plugs in (EC.38). Putting the results together, we get

$$\begin{aligned}
\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1},\mathcal{Y}^\star)] &\le(\tfrac{1}{\mu\alpha T_p}(\Delta^2+\tfrac{\mathsf{R}^2}{T^{2\gamma}})+\tfrac{32m(\bar{a}+\bar{d})^2\alpha}{\mu}\log T)^{1/\gamma}\\
&\le(\tfrac{1}{\mu})^{1/\gamma}\tfrac{1}{\alpha^{1/\gamma}T_p^{1/\gamma}}[\Delta^{2/\gamma}+\tfrac{\mathsf{R}^{2/\gamma}}{T^2}]+(\tfrac{32m(\bar{a}+\bar{d})^2}{\mu})^{1/\gamma}\alpha^{1/\gamma}(\log T)^{1/\gamma}, &\text{(EC.43)}
\end{aligned}$$

where (EC.43) recursively applies $(a+b)^{1/\gamma} \le a^{1/\gamma} + b^{1/\gamma}$ and we arrive at

$$\begin{aligned}
&\mathbb{E}[r(\hat{\mathbf{x}}_T) + v(\hat{\mathbf{x}}_T)] \\
&\le V(T_e) + \tfrac{m(\bar{a}+\bar{d})^2}{2}\alpha T_p \\
&\quad + \tfrac{\mathsf{R}+1}{\alpha}[\Delta + \tfrac{\mathsf{R}}{T^{2\gamma}} + (\tfrac{1}{\mu})^{1/\gamma}\tfrac{1}{\alpha^{1/\gamma}T_p^{1/\gamma}}(\Delta^{2/\gamma} + \tfrac{\mathsf{R}^{2/\gamma}}{T^2}) + (\tfrac{32m(\bar{a}+\bar{d})^2}{\mu})^{1/\gamma}\alpha^{1/\gamma}(\log T)^{1/\gamma} + 2\mathrm{di}(\mathcal{Y}^\star)] \\
&= V(T_e) + \tfrac{m(\bar{a}+\bar{d})^2}{2}\alpha T_p + (\mathsf{R}+1)[\tfrac{\Delta}{\alpha} + (\tfrac{1}{\mu})^{1/\gamma}(\tfrac{\Delta^{2/\gamma}}{\alpha^{1/\gamma+1}T_p^{1/\gamma}}) + (\tfrac{32m(\bar{a}+\bar{d})^2}{\mu})^{1/\gamma}\alpha^{1/\gamma-1}(\log T)^{1/\gamma}] \\
&\quad + (\mathsf{R}+1)[\tfrac{\mathsf{R}}{\alpha T^{2\gamma}} + (\tfrac{1}{\mu})^{1/\gamma}\tfrac{\mathsf{R}^{2/\gamma}}{\alpha^{1/\gamma+1}T_p^{1/\gamma}T^2}] + \tfrac{2(\mathsf{R}+1)}{\alpha}\mathrm{di}(\mathcal{Y}^\star) \\
&= V(T_e) + O(\alpha T_p + \tfrac{\Delta}{\alpha} + \tfrac{\Delta^{2/\gamma}}{\alpha^{1/\gamma+1}T_p^{1/\gamma}} + \alpha^{1/\gamma-1}(\log T)^{1/\gamma} + \tfrac{1}{\alpha}\mathrm{di}(\mathcal{Y}^\star) + \tfrac{1}{\alpha T^{2\gamma}} + \tfrac{1}{\alpha^{1/\gamma+1}T_p^{1/\gamma}T^2})
\end{aligned}$$

and this completes the proof. Here, the explicit expression of T_e can be obtained from **Lemma 1**:

$$T_e = \tfrac{1}{\mu}\{\max\{9, 1728\{2\gamma\log T + \log\lceil\log_2(\tfrac{2\bar{c}}{\Delta^\gamma})\rceil\}\}\tfrac{\mu^{-2/\gamma}m(\bar{a}+\bar{d})^2}{\Delta^{2(\gamma-1)}} + 1\}\lceil\log_2(\tfrac{2\bar{c}}{\Delta^\gamma})\rceil.$$

B.5. Proof of Lemma 7

Using **Lemma EC. 7**, it suffices to verify that the distributional dual objective is strongly convex:

$$f(y) = \tfrac{1}{2}y + \mathbb{E}_c[[c-y]_+] = \tfrac{1}{2}y + \int_y^1 (c-y)\mathrm{d}c = \tfrac{1}{2}y^2 - \tfrac{1}{2}y + \tfrac{1}{2}.$$

and indeed, $f(y)$ is 1-strongly convex.

B.6. Proof of Lemma 8

First, we establish the update rule formula for $\mathbb{E}[y^{t+1}]$ in terms of $\mathbb{E}[y^t]$. Specifically, we have

$$\begin{aligned}
\mathbb{E}[y^{t+1}] &= \mathbb{E}[[y^t - \tfrac{1}{t}(\tfrac{1}{2} - \mathbb{I}\{c_t > y^t\})]_+] && \text{(EC.44)} \\
&\ge \mathbb{E}[y^t - \tfrac{1}{t}(\tfrac{1}{2} - \mathbb{I}\{c_t > y^t\})] && \text{(EC.45)} \\
&\ge \mathbb{E}[y^t - \tfrac{1}{t}y^t + \tfrac{1}{2t}] && \text{(EC.46)}
\end{aligned}$$

where (EC.44) is obtained by the update rule of subgradient, (EC.45) uses Jensen's inequality, and (EC.46) is obtained by the fact that c_t is independent of y^t and it is drawn uniformly from $[0,1]$. Indeed, we have

$$\mathbb{E}[\mathbb{I}\{c_t > y^t\}] = \mathbb{E}[\mathbb{E}[\mathbb{I}\{c_t > y^t\}|y^t]] = \mathbb{E}[\int_0^1 \mathbb{I}\{c > y^t\}\mathrm{d}c|y^t] = \mathbb{E}[1 - y^t].$$

Subtracting $t/2$ from both sides and multiplying both sides the the inequality by t, we have

$$t(\mathbb{E}[y^{t+1}] - \tfrac{1}{2}) \ge (t-1)(\mathbb{E}[y^t] - \tfrac{1}{2}), \quad \text{for all } t = 1, \ldots, T.$$

Next we condition on the value of y^{t_0} and

$$t(\mathbb{E}[y^{t+1}|y^{t_0}]-\tfrac{1}{2})\geq(t_0-1)(y^{t_0}-\tfrac{1}{2}). \tag{EC.47}$$

Thus, given $y^{t_0}>y^\star+\frac{1}{\sqrt{T}}=\frac{1}{2}+\frac{1}{\sqrt{T}}$ for some t_0, we have

$$t(\mathbb{E}[y^{t+1}|y^{t_0}]-\tfrac{1}{2})\geq(t_0-1)(y^{t_0}-\tfrac{1}{2})\geq\tfrac{t_0-1}{\sqrt{T}}, \tag{EC.48}$$

As a result, when $t_0\geq\frac{T}{10}+1$, (EC.48) implies

$$\mathbb{E}[y^{t+1}|y^{t_0}]\geq\tfrac{1}{2}+\tfrac{t_0-1}{t\times\sqrt{T}}\geq\tfrac{1}{2}+\tfrac{1}{10\sqrt{T}},$$

since we assume $t_0\geq T/10+1$. This completes the proof.

B.7. Proof of Proposition 1

Based on Rakhlin et al. (2012), there exists some universal constant $c>0$ such that with probability no less than $1-1/T^4$, $|y^t-y^\star|\leq c\log T/\sqrt{T}$ for all $t\geq t_0$, where $y^\star=\frac{1}{2}$ and $t_0=O(\log T)$. Thus, without loss of generality, we assume

$$y^t\in[\tfrac{1}{4},\tfrac{3}{4}],\text{ and }y^{t+1}=y^t-\tfrac{1}{t}(\tfrac{1}{2}-\mathbb{I}\{c_t>y^t\}) \tag{EC.49}$$

for all $t\geq t_0$ by setting a new random initialization $y^{t_0}\in[1/4,3/4]$ and ignoring the all decision steps before the t_0 step. In the following, we show that SGM using $O(1/(\mu t))$ stepsize must have $\Omega(T^{1/2})$ regret or constraint violation for any initialization y^{t_0}. We first calculate $\mathbb{E}[y^t-\frac{1}{2}]$ and $\mathbb{E}[(y^t-\frac{1}{2})^2]$ similar to the proof of **Lemma 8**. Specifically, for $\mathbb{E}[y^t-1/2]$, we have

$$\mathbb{E}[y^{t+1}|y^t]=\big(1-\tfrac{1}{t}\big)y^t+\tfrac{1}{2t},$$

which implies

$$\mathbb{E}[y^{t+1}-\tfrac{1}{2}|y^{t_0}]=\tfrac{t_0-1}{t}(y^1-\tfrac{1}{2})+\tfrac{1}{2}, \tag{EC.50}$$

Also, similarly, for $\mathbb{E}[(y^t-1/2)^2]$ we have under assumption (EC.49)

$$\begin{aligned}\mathbb{E}[(y^{t+1}-\tfrac{1}{2})^2|y^t]&=\mathbb{E}[(y^t-\tfrac{1}{t}(\tfrac{1}{2}-\mathbb{I}\{c_t>y^t\})-\tfrac{1}{2})^2|y^t]\\&=(1-\tfrac{1}{t})^2(y^t-\tfrac{1}{2})^2+\tfrac{1}{4t^2}-\tfrac{1}{t^2}(y^t-\tfrac{1}{2})^2\\&\geq(1-\tfrac{1}{t})^2(y^t-\tfrac{1}{2})^2+\tfrac{1}{4t^2}-\tfrac{c}{t^3},\end{aligned}$$

which implies

$$\mathbb{E}[(y^{t+1}-\tfrac{1}{2})^2|y^t] \geq \tfrac{(t_0-1)^2}{t^2}(y^{t_0}-\tfrac{1}{2})^2+\tfrac{1}{4t}-\tfrac{c\log t+t_0}{t^2}. \tag{EC.51}$$

Combining (EC.50) and (EC.51), we then can compute

$$\begin{aligned}
&\mathbb{E}[(\textstyle\sum_{t=t_0}^{T}\mathbb{I}\{c_t>y^t\}-\tfrac{T-t_0+1}{2})^2] && \text{(EC.52)}\\
&=\textstyle\sum_{t=t_0}^{T}\mathbb{E}[(\mathbb{I}\{c_t>y^t\}-\tfrac{1}{2})^2]+2\sum_{t_0\leq i<j\leq T}\mathbb{E}[(\mathbb{I}\{c_j>y^j\}-\tfrac{1}{2})(\mathbb{I}\{c_i>y^i\}-\tfrac{1}{2})]\\
&=\tfrac{T-t_0}{4}+2\textstyle\sum_{t_0\leq i<j\leq T}\mathbb{E}[(\mathbb{I}\{c_j>y^j\}-\tfrac{1}{2})(\mathbb{I}\{c_i>y^i\}-\tfrac{1}{2})]\\
&=\tfrac{T-t_0}{4}+2\textstyle\sum_{t_0\leq i<j\leq T}\tfrac{i-1}{j-1}\mathbb{E}[(y^i-\tfrac{1}{2})^2]-\tfrac{i-1}{4i(j-1)} && \text{(EC.53)}\\
&\geq\tfrac{T-t_0}{4}-2\textstyle\sum_{t_0\leq i<j\leq T}\tfrac{c\log T+t_0}{(i-1)^2}\\
&=\Omega(T).
\end{aligned}$$

In addition, since $|y^t-\frac{1}{2}|\leq\frac{c}{\sqrt{T}}$, by **Lemma EC. 1**, we have with probability no less than $1-\frac{1}{T^2}$

$$\big|\textstyle\sum_{t=t_0}^{T}\mathbb{I}\{c_t>y^t\}-\tfrac{T-t_0+1}{2}\big|=O(\sqrt{T}\log T).$$

Consequently, by (EC.52), we have

$$\mathbb{E}\big[\big|\textstyle\sum_{t=t_0}^{T}\mathbb{I}\{c_t>y^t\}-\tfrac{T-t_0+1}{2}\big|\big]=\Omega(\tfrac{\sqrt{T}}{\log T}). \tag{EC.54}$$

This is the summation of constraint violation and constraint (resource) leftover, and thus, the summation of constraint violation and the regret must be no less than $\Omega(\sqrt{T}/\log T)$.

B.8. Proof of Theorem 2

First note that for sufficiently large T, the condition $\alpha_e\leq\frac{2\underline{d}}{3m(\bar{a}+\bar{d})^2}$ from **Lemma EC. 6** will be satisfied and all the dual iterates $\{\mathbf{y}^t\}_{t=T_e+1}^{T}$ will stay in $\mathcal{Y}$ almost surely. When $\mathrm{di}(\mathcal{Y}^\star)=0$, we consider

$$\frac{1}{\alpha_e}+\alpha_eT_e+\alpha_pT_p+\frac{\Delta}{\alpha_p}+\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1}T_p^{1/\gamma}}+\alpha_p^{1/\gamma-1}(\log T)^{1/\gamma}+\frac{1}{\alpha_pT^{2\gamma}}+\frac{1}{\alpha_p^{1/\gamma+1}T_p^{1/\gamma}T^2}.$$

Since α_e only appears in $\frac{1}{\alpha_e}+\alpha_eT_e$, we let $\alpha_e=O(1/\sqrt{T_e})$ to optimize the trade-off. Hence, it suffices to consider

$$\sqrt{T_e}+\alpha_pT_p+\frac{\Delta}{\alpha_p}+\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1}T_p^{1/\gamma}}+\alpha_p^{1/\gamma-1}(\log T)^{1/\gamma}+\frac{1}{\alpha_pT^{2\gamma}}+\frac{1}{\alpha_p^{1/\gamma+1}T_p^{1/\gamma}T^2}.$$

Taking $\Delta = O(T^{-\beta})$ and $\alpha_p = O(T^{-\lambda})$ with $(\beta, \lambda) \geq 0$, we have $T_e = O(T^{2\beta(\gamma-1)} \log^2 T)$ according to **Lemma 1** and (7), and $\sqrt{T_e} = O(T^{\beta\gamma-\beta} \log T)$. Moreover, we have, using $\cong$ to denote equivalence under $O(\cdot)$ notation, that

$$\begin{aligned}
\alpha_p T_p &\cong T^{1-\lambda} \\
\frac{\Delta}{\alpha_p} &\cong T^{\lambda-\beta} \\
\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma}} &\cong \frac{T^{-2\beta/\gamma}}{T^{-\lambda/\gamma-\lambda} T^{1/\gamma} (1-T^{2\beta(\gamma-1)-1} \log^2 T)^{1/\gamma}} = \frac{T^{\frac{-2\beta+\lambda-1}{\gamma}+\lambda}}{(1-T^{2\beta(\gamma-1)-1} \log^2 T)^{1/\gamma}} \\
\alpha_p^{1/\gamma-1} (\log T)^{1/\gamma} &\cong T^{\lambda-\lambda/\gamma} (\log T)^{1/\gamma} \\
\frac{1}{\alpha_p T^{2\gamma}} &= O(1) \\
\frac{1}{\alpha_p^{1/\gamma-1} T_p^{1/\gamma} T^2} &= O(1).
\end{aligned}$$

Suppose $2\beta(\gamma-1) - 1 < 0$. Then $\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma}} \cong T^{\frac{-2\beta+\lambda-1}{\gamma}+\lambda}$ and

$$\begin{aligned}
&\sqrt{T_e} + \alpha_p T_p + \frac{\Delta}{\alpha_p} + \frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma}} + \alpha_p^{1/\gamma-1} (\log T)^{1/\gamma} + \frac{1}{\alpha_p T^{2\gamma}} + \frac{1}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma} T^2} \\
&\cong T^{\beta\gamma-\beta} \log T + T^{1-\lambda} + T^{\lambda-\beta} + T^{\frac{-2\beta+\lambda-1}{\gamma}+\lambda} + T^{\lambda-\lambda/\gamma} (\log T)^{1/\gamma} \\
&\lesssim [T^{\beta\gamma-\beta} + T^{1-\lambda} + T^{\lambda-\beta} + T^{\frac{-2\beta+\lambda-1}{\gamma}+\lambda} + T^{\lambda-\lambda/\gamma}] \log T, \qquad \text{(EC.55)}
\end{aligned}$$

where (EC.55) uses $\gamma \geq 1$ and that $(\log T)^{1/\gamma} \leq \log T$. To find the optimal trade-off, we solve the following optimization problem

$$\begin{aligned}
&\min_{\lambda,\beta} \max\{\beta\gamma - \beta, 1 - \lambda, \lambda - \beta, \frac{-2\beta+\lambda-1}{\gamma} + \lambda, \lambda - \frac{\lambda}{\gamma}\} \\
&\text{s.t.} (\lambda, \beta) \geq 0.
\end{aligned}$$

The solution yields $\lambda^\star = \frac{\gamma}{2\gamma-1}$ and $\beta^\star = \frac{1}{2\gamma-1}$ and

$$2\beta^\star(\gamma-1) - 1 = \frac{2\gamma-2}{2\gamma-1} - 1 = -\frac{1}{2\gamma-1} < 0$$

always holds. Hence

$$T^{\beta^\star\gamma-\beta^\star} + T^{1-\lambda^\star} + T^{\lambda^\star-\beta^\star} + T^{\frac{-2\beta^\star+\lambda^\star-1}{\gamma}+\lambda^\star} + T^{\lambda^\star-\lambda^\star/\gamma} = O(T^{\frac{\gamma-1}{2\gamma-1}} \log T)$$

and this completes the proof for $\mathrm{di}(\mathcal{Y}^\star) = 0$.

Next, consider the case $\mathrm{di}(\mathcal{Y}^\star) > 0$. In this case we need to consider the trade-off:

$$\frac{1}{\alpha_e} + \alpha_e T_e + \alpha_p T_p + \frac{\Delta}{\alpha_p} + \frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma-1} T_p^{1/\gamma}} + \alpha_p^{1/\gamma-1}(\log T)^{1/\gamma} + \frac{\mathrm{di}(\mathcal{Y}^\star)}{\alpha_p} + \frac{1}{\alpha_p T^{2\gamma}} + \frac{1}{\alpha_p^{1/\gamma-1} T_p^{1/\gamma} T^2}.$$

Note that $\frac{1}{\alpha_e} + \alpha_e T_e + \alpha_p T_p + \frac{\mathrm{di}(\mathcal{Y}^\star)}{\alpha_p} \geq 2\sqrt{T_e} + 2\sqrt{T_p \mathrm{di}(\mathcal{Y}^\star)}$ and that $T_e + T_p = T$ make it impossible to achieve better than $O(\sqrt{T})$ regret. Hence, we consider improving the constant associated with $\sqrt{T}$.

Using $\mathsf{R} = \frac{\bar{c}}{\underline{d}} + O(\max\{\alpha_e, \alpha_p\})$ and suppose α_e, α_p are of the same order with respect to T,

$$\begin{aligned}
\mathbb{E}[r(\hat{\mathbf{x}}_T) + v(\hat{\mathbf{x}}_T)] &\leq \frac{m(\bar{a}+\bar{d})^2}{2}(\alpha_e T_e + \alpha_p T_p) + \frac{\mathsf{R}}{\alpha_e} + \frac{2(\mathsf{R}+1)}{\alpha_p}\mathrm{di}(\mathcal{Y}^\star) \\
&\quad + (\mathsf{R}+1)\Big[\frac{\Delta}{\alpha_p} + \big(\frac{1}{\mu}\big)^{1/\gamma}\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma}} + \big(\frac{32m(\bar{a}+\bar{d})^2}{\mu}\big)^{1/\gamma}\alpha_p^{1/\gamma}(\log T)^{1/\gamma}\Big] + O(1) \\
&= \frac{m(\bar{a}+\bar{d})^2}{2}(\alpha_e T_e + \alpha_p T_p) + \frac{\bar{c}}{\underline{d}}\frac{1}{\alpha_e} + \frac{\bar{c}}{\underline{d}}\frac{2\mathrm{di}(\mathcal{Y}^\star)}{\alpha_p} \\
&\quad + (\mathsf{R}+1)\Big[\frac{\Delta}{\alpha_p} + \big(\frac{1}{\mu}\big)^{1/\gamma}\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma}} + \big(\frac{32m(\bar{a}+\bar{d})^2}{\mu}\big)^{1/\gamma}\alpha_p^{1/\gamma}(\log T)^{1/\gamma}\Big] + O(1).
\end{aligned}$$

Suppose we take $T_e = \theta T$ and $T_p = (1-\theta)T$ for $\theta \in (0,1)$ and we let $\alpha_e = \frac{\beta_e}{\sqrt{T_e}} = \frac{\beta_e}{\sqrt{\theta T}}, \alpha_p = \frac{\beta_p}{\sqrt{T_p}} = \frac{\beta_p}{\sqrt{(1-\theta)T}}$. Then, $\Delta = o(1)$ and

$$(\mathsf{R}+1)\Big[\frac{\Delta}{\alpha_p} + \big(\frac{1}{\mu}\big)^{1/\gamma}\frac{\Delta^{2/\gamma}}{\alpha_p^{1/\gamma+1} T_p^{1/\gamma}} + \big(\frac{32m(\bar{a}+\bar{d})^2}{\mu}\big)^{1/\gamma}\alpha_p^{1/\gamma}(\log T)^{1/\gamma}\Big] = o(\sqrt{T}).$$

Hence, it suffices to consider

$$\begin{aligned}
&\frac{m(\bar{a}+\bar{d})^2}{2}(\alpha_e T_e + \alpha_p T_p) + \frac{\bar{c}}{\underline{d}}\frac{1}{\alpha_e} + \frac{\bar{c}}{\underline{d}}\frac{2\mathrm{di}(\mathcal{Y}^\star)}{\alpha_p} \\
&= \frac{m(\bar{a}+\bar{d})^2}{2}\beta_e\sqrt{\theta T} + \frac{m(\bar{a}+\bar{d})^2}{2}\beta_p\sqrt{(1-\theta)T} + \frac{\bar{c}}{\underline{d}}\frac{\sqrt{\theta T}}{\beta_e} + \frac{\bar{c}}{\underline{d}}\frac{2\mathrm{di}(\mathcal{Y}^\star)}{\beta_p}\sqrt{(1-\theta)T} \\
&= \Big[\frac{m(\bar{a}+\bar{d})^2}{2}\beta_e + \frac{\bar{c}}{\underline{d}\beta_e}\Big]\sqrt{\theta T} + \Big[\frac{m(\bar{a}+\bar{d})^2}{2}\beta_p + \frac{2\mathrm{di}(\mathcal{Y}^\star)\bar{c}}{\underline{d}\beta_p}\Big]\sqrt{(1-\theta)T}.
\end{aligned}$$

Taking $\beta_e = \sqrt{\frac{2}{m(\bar{a}+\bar{d})^2}\cdot\frac{\bar{c}}{\underline{d}}}$ and $\beta_p = \sqrt{\frac{2}{m(\bar{a}+\bar{d})^2}\cdot\frac{2\mathrm{di}(\mathcal{Y}^\star)\bar{c}}{\underline{d}}}$ to optimize the two trade-offs, we get

$$\mathbb{E}[r(\hat{\mathbf{x}}_T) + v(\hat{\mathbf{x}}_T)] \leq 2\sqrt{\frac{m\bar{c}}{2\underline{d}}}(\bar{a}+\bar{d})\sqrt{\theta T} + 2\sqrt{2}\sqrt{\frac{m\bar{c}}{2\underline{d}}}(\bar{a}+\bar{d})\sqrt{\mathrm{di}(\mathcal{Y}^\star)}\sqrt{(1-\theta)T}$$

With $\theta = \frac{2\mathrm{di}(\mathcal{Y}^\star)}{2\mathrm{di}(\mathcal{Y}^\star)+1}$, we have

$$\mathbb{E}[r(\hat{\mathbf{x}}_T) + v(\hat{\mathbf{x}}_T)] \leq 4\sqrt{\frac{m\bar{c}}{2\underline{d}}}\sqrt{\frac{2\mathrm{di}(\mathcal{Y}^\star)}{2\mathrm{di}(\mathcal{Y}^\star)+1}}(\bar{a}+\bar{d})\sqrt{T}.$$

Since $\mathrm{di}(\mathcal{Y}^\star) \geq 0$, this completes the proof.

B.9. Removing additional $\log T$ when $\gamma = 2$

When $\gamma = 2$, the dual error bound condition reduces to quadratic growth, and it is possible to remove the $\log T$ factor in the regret result. Recall that $\log T$ terms appear when bounding $\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1}, \mathcal{Y}^\star)]$ and $\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1}, \mathcal{Y}^\star)^2]$. For $\gamma = 2$, using a tailored analysis, **Lemma EC. 8** guarantees

$$\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1}, \mathcal{Y}^\star)] \leq \sqrt{\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1}, \mathcal{Y}^\star)^2]} = \mathcal{O}(\tfrac{1}{\sqrt{T}}).$$

Moreover, using **Lemma EC. 9**, we can directly bound the expectation

$$\begin{aligned}
\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1}, \mathcal{Y}^\star)^2] &= \mathbb{E}\big[\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T+1}, \mathcal{Y}^\star)^2 | \mathbf{y}^{T_e+1}]\big] \\
&\leq \mathbb{E}\big[\tfrac{\operatorname{dist}(\mathbf{y}^{T_e+1}, \mathcal{Y}^\star)^2}{\mu\alpha T} + \tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha\big] \\
&= \tfrac{\mathbb{E}[\operatorname{dist}(\mathbf{y}^{T_e+1}, \mathcal{Y}^\star)^2]}{\mu\alpha T} + \tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha \\
&\leq \tfrac{\Delta^2}{\mu\alpha T} + \tfrac{m(\bar{a}+\bar{d})^2}{\mu}\alpha.
\end{aligned}$$

Therefore, $\log T$ terms can be removed from the analysis.

B.10. Removing the constraint violation

This subsection introduces a mechanism to address the constraint violation in **Theorem 2**. In particular, we introduce an add-on strategy that, with high probability, results in no constraint violation. This strategy can be further generalized to convert constraint violation into regret for any OLP algorithm when the resource budget satisfies $B = \Theta(T)$ and some boundedness assumptions. Below, we first state the result, then describe the method, and finally sketch the proof. Consider any online LP algorithm whose constraint violation satisfies

$$v(\hat{\mathbf{x}}_T) \leq k\, f(T) \qquad \text{with probability at least } 1 - \delta(T),$$

for some constant $k > 0$ and functions $f(T)$ and $\delta(T)$, where we assume $f(T) = \mathcal{O}(T)$. Then we can use our add-on mechanism to construct a modified algorithm that guarantees:

- **no constraint violation**, i.e., $v(\hat{\mathbf{x}}'_T) = 0$, and
- **regret at most**

$$r(\hat{\mathbf{x}}'_T) \leq r(\hat{\mathbf{x}}_T) + kmf(T) + \delta(T)T \tag{EC.56}$$

with high probability, for some constant $m > 0$. For brevity, we illustrate the mechanism for $\gamma = 2$. In our setting, we can show (using a standard high-probability analysis for SGD as in (Rakhlin et al. 2012)) that Algorithm 4 achieves

$$v(\hat{\mathbf{x}}_T) = \mathcal{O}(T^{1/3}) \qquad \text{with probability at least } 1 - \tfrac{1}{T^2}. \tag{EC.57}$$

Applying the above transformation, we obtain a parallel version of Algorithm 4 that achieves $O(T^{1/3})$ regret while ensuring zero constraint violation.

The add-on mechanism consists of two parts:

(1) **Early rejection buffer:** For each resource $i = 1, \dots, m$, automatically reject the first

$$\tfrac{2k}{\underline{d}} f(T)$$

order for which $a_{ti} > \underline{d}/2$ and the original algorithm would have chosen $\hat{x}^t = 1$.

(2) **Same update rule as the original algorithm:** The dual variables are updated in exactly the same way as in Algorithm 4. For those orders rejected in Step (1), the mechanism still updates the dual variables as if the orders had been accepted.

We now explain why this mechanism guarantees feasibility while keeping regret small. First, note that when $v(\hat{\mathbf{x}}_T) < f(T)$, the buffer created in Step (1) is sufficient to eliminate all constraint violations. To see this, fix any resource j that becomes fully used. If fewer than $\frac{\underline{d}T}{2\bar{a}}$ accepted orders have $a_{tj} > \underline{d}/2$, then the total amount of resource j consumed by all T arrivals is at most

$$\bar{a} \cdot \tfrac{\underline{d}T}{2\bar{a}} + \tfrac{\underline{d}}{2} \cdot (T - \tfrac{\underline{d}T}{2\bar{a}}) < \underline{d}T,$$

which contradicts the assumption that this resource is exhausted.

Therefore, whenever a resource is used up, there must exist at least $\frac{\underline{d}T}{2\bar{a}}$ accepted orders with $a_{tj} > \underline{d}/2$. For sufficiently large T, Step (1) allows us to reject $\frac{2k}{\underline{d}} f(T)$ such orders in advance, thereby leaving $kf(T)$ units of unused capacity for resource j as a safety buffer. Consequently, whenever $v(\hat{\mathbf{x}}_T) < f(T)$, this buffer guarantees that the mechanism prevents any constraint violation.

Regarding the regret, Step (1) rejects at most $\frac{2mk}{\underline{d}} f(T)$ orders. Since each rejected bid has reward at most $\bar{c}$, the additional regret incurred by this rejection is therefore bounded above by $\frac{2\bar{c}mk}{\underline{d}} f(T)$.

B.11. Learning with unknown parameters

It is possible that γ and μ are unknown in practice. When μ is unknown, it is possible to run parameter-free variants of first-order methods **Algorithm 6**, which is slightly more complicated. In terms of γ, in the finite-support setting, the LP polyhedral error bound always guarantees $\gamma = 1$. In the continuous support setting, it suffices to know an upper bound on γ: if **A4** holds for some $\gamma > 0$, then given $\theta > 0$,

$$\begin{aligned} f(\mathbf{y}) &\geq f(\mathbf{y}^\star) + \mu \mathrm{dist}(\mathbf{y}, \mathcal{Y}^\star)^\gamma \\ &= f(\mathbf{y}) - f(\mathbf{y}^\star) + \mu \tfrac{\mathrm{dist}(\mathbf{y}, \mathcal{Y}^\star)^{\gamma+\theta}}{\mathrm{dist}(\mathbf{y}, \mathcal{Y}^\star)^\theta} \\ &\geq f(\mathbf{y}) - f(\mathbf{y}^\star) + \tfrac{\mu}{\mathrm{diam}(\mathcal{Y})^\theta} \mathrm{dist}(\mathbf{y}, \mathcal{Y}^\star)^{\gamma+\theta}. \end{aligned}$$

and **A4** also holds for $\gamma' > \gamma$.