Electronic Companions to "Joint Assortment Planning and Inventory Control under General Choice Models"

EC.1. Proofs in Section 4

Proof of Proposition 1.

Let $\mathbf{q}^* = [q_i^*]_{i\in\mathcal{N}}$ be the optimal solution to Problem (OPT). Throughout this proof, $\mathbf{q}^*$ denotes the initial inventory vector. For any $S \subseteq \mathcal{N}$ and $t \in \mathbb{N}^+$, let $Y_{S,t}$ be a binary indicator variable such that $Y_{S,t} = 1$ if and only if $t \le T$ and the t-th customer sees the assortment S according to the current inventory level. Because each customer sees one assortment and there are T customers, we have

$$\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} Y_{S,t} = T. \tag{EC.1}$$

For any $S \subseteq \mathcal{N}$, let $\bar{f}_S = \sum_{t=1}^{\infty} \mathbb{E}[Y_{S,t}]/\mathbb{E}[T]$. Let $\bar{\mathbf{f}} = [\bar{f}_S]_{S\subseteq\mathcal{N}}$. In the following, we show that $\bar{\mathbf{f}}$ is a feasible solution to Problem (Fluid) with an objective value equal to $\mathbb{E}[R_T(\mathbf{q}^*)]$.

Note that when a customer is offered an assortment S, the probability for this customer to make a purchase is $1 - \phi_0(S)$. Thus, the expected total number of units purchased by all the customers is $\mathbb{E}[\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} Y_{S,t}(1-\phi_0(S))]$, which is at most K. Hence, $\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} \mathbb{E}[Y_{S,t}](1-\phi_0(S)) \le K$, which implies that $\mathbb{E}[T] \sum_{S\subseteq\mathcal{N}} \bar{f}_S(1-\phi_0(S)) \le K$. Therefore, $\bar{\mathbf{f}}$ satisfies constraint (3).

From (EC.1), we have $\mathbb{E}[\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} Y_{S,t}] = \mathbb{E}[T]$. Hence,

$$\sum_{S\subseteq\mathcal{N}} \bar{f}_S = \sum_{S\subseteq\mathcal{N}} \frac{\sum_{t=1}^{\infty} \mathbb{E}[Y_{S,t}]}{\mathbb{E}[T]} = \frac{\mathbb{E}[\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} Y_{S,t}]}{\mathbb{E}[T]} = 1.$$

Therefore, $\bar{\mathbf{f}}$ satisfies constraint (4).

Since $\mathbf{q}^*$ is a feasible solution to Problem (OPT), we have $\sum_{i\in\mathcal{N}} q_i^* \le K$. If the initial inventory vector is $\mathbf{q}^*$, none of the customers will see an assortment that contains more than K products. Hence, $\sum_{t=1}^{\infty} Y_{S,t} = 0$ if $|S| > K$. Thus, $\bar{f}_S = 0$ if $|S| > K$, satisfying constraint (5).

When a customer is offered an assortment S, the expected revenue received from this customer is $\sum_{i\in S} \phi_i(S) r_i$. Thus, the expected total revenue received is $\mathbb{E}[\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} Y_{S,t} \sum_{i\in S} \phi_i(S) r_i]$. Hence,

$$\mathbb{E}[R_T(\mathbf{q}^*)] = \mathbb{E}[\sum_{S\subseteq\mathcal{N}} \sum_{t=1}^{\infty} Y_{S,t} \sum_{i\in S} \phi_i(S) r_i] = \mathbb{E}[T] \sum_{S\subseteq\mathcal{N}} \bar{f}_S \sum_{i\in S} \phi_i(S) r_i,$$

implying that $\bar{\mathbf{f}}$ is a feasible solution to Problem (Fluid) with an objective value equal to $\mathbb{E}[R_T(\mathbf{q}^*)]$. Thus, the optimal solution value of Problem (Fluid) is at least $\mathbb{E}[R_T(\mathbf{q}^*)]$; that is, $z_{\text{fluid}} \ge z_{\text{opt}}$. □

Proof of Proposition 2.

Our proof is based on another alternative process to the dynamic-substitution process (stage 2 in Section 3) to bridge it with the static-allocation process (Section 4.2). We call this alternate process the static-substitution process.

Static-substitution process. This process is similar to the static-allocation process except that it excludes the step of random allocation to units. This process comprises two steps and only considers initial inventory vectors that stock units of products in assortment S'. In particular, in the first step, there are T arrived customers, and each customer independently chooses a product or no-purchase option $i \in S'^{+}$ according to a fixed distribution $\{\pi_i\}_{i\in S'^{+}}$. In the second step, for each product $i \in S'$, we directly use q_i stocked units of this product to fulfill the demand for this product from the first step.

In other words, we still use a Bernoulli random variable Z_{ij} to indicate that the customer j chooses $i \in S'^{+}$ if the customer j arrives. Moreover, we couple the static-substitution process with the static-allocation process by making them share the same T and Z_{ij}. The expected generated revenue in the second step of the static-substitution process if $T = t$ is denoted by

$$R_t^{\mathrm{SS}}(\mathbf{q}) \triangleq \sum_{i\in S'} r_i \mathbb{E}\left[\min\left\{q_i, \sum_{j\le t} Z_{ij}\right\}\right] \qquad \text{for each } \mathbf{q}\in\mathbb{N}^N \text{ and } S_{\mathbf{q}} \subseteq S'. \tag{EC.2}$$

Next, our proof comprises two parts. We first show (i) $R_t(\mathbf{q}) \ge R_t^{\mathrm{SS}}(\mathbf{q})$ for each t, and then prove (ii) $\mathbb{E}[R_T^{\mathrm{SS}}(\mathbf{q})] \ge R^{\mathrm{Static}}(\mathbf{q})$ for any $\mathbf{q}$ with $S_{\mathbf{q}} \subseteq S'$.

Part (i). The proof of the first part is based on induction. Before the induction argument starts, we establish two useful facts. First, for any $t \ge 1$ and any $\mathbf{q}$ with $S_{\mathbf{q}} \subseteq S'$, if $\mathbf{q} \ge \mathbf{e}_i$ holds, then

$$\begin{aligned} R_t^{\mathrm{SS}}(\mathbf{q}) - R_t^{\mathrm{SS}}(\mathbf{q}-\mathbf{e}_i) &= r_i\mathbb{E}[\min(\sum_{j=1}^{t} Z_{ij}, q_i)] - r_i\mathbb{E}[\min(\sum_{j=1}^{t} Z_{ij}, q_i - 1)] = r_i \Pr(\sum_{j=1}^{t} Z_{ij} \ge q_i) \\ &\le r_i. \end{aligned} \tag{EC.3}$$

Second, for any $t \ge 1$ and any $\mathbf{q}$ with $S_{\mathbf{q}} \subseteq S'$, we have the following recursive equation

$$\begin{aligned} R_t^{\mathrm{SS}}(\mathbf{q}) &= \sum_{i\in S'} \Pr(Z_{it}=1)\Big(\mathbf{1}(q_i>0)\big(r_i + r_i\mathbb{E}[\min\{q_i-1, \sum_{j\le t-1} Z_{i'j}\}] + \sum_{i'\in S'|i'\ne i} r_{i'}\mathbb{E}[\min\{q_{i'}, \sum_{j\le t-1} Z_{i'j}\}]\big)\Big) \\ &\quad + \sum_{i\in S'} \Pr(Z_{it}=1)\Big(\mathbf{1}(q_i=0)\big(\sum_{i'\in S'} r_{i'}\mathbb{E}[\min\{q_{i'}, \sum_{j\le t-1} Z_{i'j}\}]\big)\Big) \\ &\quad + \Pr(Z_{0t}=1)\sum_{i'\in S'} r_{i'}\mathbb{E}[\min\{q_{i'}, \sum_{j\le t-1} Z_{i'j}\}] \\ &= \sum_{i\in S'} \pi_i\Big(\mathbf{1}(q_i>0)(r_i + R_{t-1}^{\mathrm{SS}}(\mathbf{q}-\mathbf{e}_i)) + \mathbf{1}(q_i=0)R_{t-1}^{\mathrm{SS}}(\mathbf{q})\Big) + \pi_0 R_{t-1}^{\mathrm{SS}}(\mathbf{q}) \end{aligned} \tag{EC.4}$$

where $\pi_0 = 1 - \sum_{i\in S'}\pi_i$.

We now proceed to use induction to prove part (i): $R_t(\mathbf{q}) \geq R_t^{\text{SS}}(\mathbf{q})$ for any $t = 0, 1, \ldots$ and any $\mathbf{q}$ with $S_{\mathbf{q}} \subseteq S'$. From $R_0(\mathbf{q}) = R_0^{\text{SS}}(\mathbf{q}) = 0$, the required result holds if $t = 0$. Let $t \geq 1$, and assume as the induction hypothesis that the required inequality holds for $t-1$, that is, $R_{t-1}(\mathbf{q}') \geq R_{t-1}^{\text{SS}}(\mathbf{q}')$. for every $\mathbf{q}'$ satisfying $S_{\mathbf{q}'} \subseteq S'$. From the definition of $R_t(\mathbf{q})$ in (2), we have

$$\begin{aligned} R_t(\mathbf{q}) &= \sum_{i \in S_{\mathbf{q}}} \phi_i(S_{\mathbf{q}})(r_i + R_{t-1}(\mathbf{q} - \mathbf{e}_i)) + \phi_0(S_{\mathbf{q}}) R_{t-1}(\mathbf{q}) \\ &\geq \sum_{i \in S_{\mathbf{q}}} \phi_i(S_{\mathbf{q}})(r_i + R_{t-1}^{\text{SS}}(\mathbf{q} - \mathbf{e}_i)) + \phi_0(S_{\mathbf{q}}) R_{t-1}^{\text{SS}}(\mathbf{q}), \end{aligned}$$

where the inequality above follows from the induction hypothesis. Thus, to complete the induction step, it suffices to show that the right-most expression above is at least $R_t^{\text{SS}}(\mathbf{q})$. The difference of these two quantities can be bounded as follows:

$$\begin{aligned} &\left\{ \sum_{i \in S_{\mathbf{q}}} \phi_i(S_{\mathbf{q}})(r_i + R_{t-1}^{\text{SS}}(\mathbf{q} - \mathbf{e}_i)) + \phi_0(S_{\mathbf{q}}) R_{t-1}^{\text{SS}}(\mathbf{q}) \right\} - R_t^{\text{SS}}(\mathbf{q}) \\ &= \sum_{i \in S_{\mathbf{q}}} \big(\phi_i(S_{\mathbf{q}}) - \pi_i \mathbf{1}(q_i > 0)\big)\big(r_i + R_{t-1}^{\text{SS}}(\mathbf{q} - \mathbf{e}_i)\big) + \big(\phi_0(S_{\mathbf{q}}) - (\pi_0 + \sum_{i \in S'} \pi_i \mathbf{1}(q_i = 0))\big) R_{t-1}^{\text{SS}}(\mathbf{q}) \\ &= \sum_{i \in S_{\mathbf{q}}} (\phi_i(S_{\mathbf{q}}) - \pi_i)(r_i + R_{t-1}^{\text{SS}}(\mathbf{q} - \mathbf{e}_i)) + \big((1 - \sum_{i \in S_{\mathbf{q}}} \phi_i(S_{\mathbf{q}})) - (1 - \sum_{i \in S_{\mathbf{q}}} \pi_i)\big) R_{t-1}^{\text{SS}}(\mathbf{q}) \\ &= \sum_{i \in S_{\mathbf{q}}} (\phi_i(S_{\mathbf{q}}) - \pi_i)(r_i + R_{t-1}^{\text{SS}}(\mathbf{q} - \mathbf{e}_i) - R_{t-1}^{\text{SS}}(\mathbf{q})) \\ &\geq 0, \end{aligned}$$

where the first equality follows from the expression of $R_t^{\text{SS}}(\mathbf{q})$ in (EC.4), the second equality follows since $\mathbf{1}(q_i > 0) = 1$ holds for any $i \in S_{\mathbf{q}}$ and also $\phi_0(S_{\mathbf{q}}) = 1 - \sum_{i \in S_{\mathbf{q}}} \phi_i(S_{\mathbf{q}})$ holds, and the inequality follows from $\pi_i \leq \phi_i(S_{\mathbf{q}})$ and inequality (EC.3). It completes the induction step, proving part (i).

Part (ii). To show $\mathbb{E}[R_T^{\text{SS}}(\mathbf{q})] \geq R^{\text{Static}}(\mathbf{q})$ for any $\mathbf{q}$ with $S_{\mathbf{q}} \subseteq S'$, we note, for any $t \geq 0$,

$$\begin{aligned} &\min\{\sum_{j=1}^{t} Z_{ij}, q_i\} = \min\{\sum_{j=1}^{t} Z_{ij} (\sum_{k=1}^{\bar{q}_i} Y_{ijk}), q_i\} \geq \min\big\{\sum_{j=1}^{t} Z_{ij} (\sum_{k=1}^{\min\{q_i, \bar{q}_i\}} Y_{ijk}), \min\{q_i, \bar{q}_i\}\big\} \\ &= \min\big\{ \sum_{k=1}^{\min\{q_i, \bar{q}_i\}} (\sum_{j=1}^{t} Z_{ij} Y_{ijk}), \min\{q_i, \bar{q}_i\}\big\} \geq \sum_{k=1}^{\min\{q_i, \bar{q}_i\}} \min\{\sum_{j=1}^{t} Z_{ij} Y_{ijk}, 1\}, \end{aligned}$$

where the first equality follows from $\sum_{k=1}^{\bar{q}_i} Y_{ijk} = 1$. Thus, from (EC.2) and (7)

$$\mathbb{E}[R_T^{\text{SS}}(\mathbf{q})] = \sum_{i \in S'} r_i \mathbb{E}[\min\{q_i, \sum_{j \leq T} Z_{ij}\}] \geq \sum_{i \in S'} r_i \mathbb{E}[\sum_{k=1}^{\min\{q_i, \bar{q}_i\}} \min\{\sum_{j=1}^{T} Z_{ij} Y_{ijk}, 1\}] = R^{\text{Static}}(\mathbf{q}),$$

which completes the proof of part (ii). Combining parts (i) and (ii), we complete the proof. □

Proof of Lemma 2.

Let X be a nonnegative integer-valued IFR random variable. Then, we have $\Pr(X \geq 0) = 1$ and $\Pr(X \geq 1) = \Pr(X \geq 1 | X \geq 0)$. For $i \geq 2$, we can show

$$\Pr(X \geq i) = \prod_{j=1}^{i} \Pr(X \geq j | X \geq j-1) = \prod_{j=1}^{i} (1 - \Pr(X = j-1 | X \geq j-1))$$
$$\leq \prod_{j=1}^{i} (1 - \Pr(X = 0)) = (1 - \Pr(X = 0))^i,$$

where the first equality holds from $\Pr(X \geq 0) = 1$ and the inequality follows from the IFR property that the hazard rate $\Pr(X = i | X \geq i)$ is non-decreasing with i and $\Pr(X = 0 | X \geq 0) = \Pr(X = 0)$.

If $\Pr(X = 0) > 0$, then $\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr(X \geq i) \leq \sum_{i=1}^{\infty} (1 - \Pr(X = 0))^i = \frac{1 - \Pr(X=0)}{\Pr(X=0)}$, which implies

$$\mathbb{E}[\min\{X, 1\}] = 1 - \Pr(X = 0) \geq \frac{\mathbb{E}[X]}{1 + \mathbb{E}[X]}.$$

If $\Pr(X = 0) = 0$, then $\mathbb{E}[\min\{X, 1\}] = 1$ which is at least $\frac{\mathbb{E}[X]}{1+\mathbb{E}[X]}$. Therefore, we conclude that $\mathbb{E}[\min\{X, 1\}] \geq \frac{\mathbb{E}[X]}{1+\mathbb{E}[X]}$ holds. □

EC.2. Proofs in Section 6

In this section, we complete the proof of the Theorem 2, which is in the same spirit as the proof for Theorem 1, but is more involved in algebra. We first present the proof of Theorem 2 and then give the proofs of technical lemmas.

Proof of Theorem 2.

The proof follows the same two-step template as the proof sketch of Theorem 1 in Section 5.1, but uses the stronger lower bound available under deterministic or Poisson arrivals. In particular, Proposition 5 shows that for any inventory vector $\mathbf{q}$ with $S_q \subseteq S'$,

$$\mathbb{E}[R_T(\mathbf{q})] \;\geq\; R^{\mathrm{Pois}}(\mathbf{q}),$$

when T is deterministic or Poisson distributed. Hence, it suffices to analyze $R^{\mathrm{Pois}}(\mathbf{q}')$ and apply the same analysis to $\mathbf{q}''$ by replacing S' with S''. All proofs of the lemmas below are deferred to Appendix EC.2.1.

Step 1: Analysis on $\mathbf{q}'$. In the first step, we compare $R^{\mathrm{Pois}}(\mathbf{q}')$ with $\mathbb{E}[T] f_{S'} \left(\sum_{i \in S'} \phi_i(S') r_i \right)$, which is the contribution of S' to z_{app} in (6). As in Section 5.2, we decompose this quantity into unit-level contributions by defining

$$a_{ik} \triangleq f_{S'} E[T] \pi_i \rho_{ik}, \qquad \text{for} \quad i \in S',\ k \in [\bar{q}_i].$$

Then

$$\mathbb{E}[T] f_{S'}\Big(\sum_{i\in S'} \phi_i(S') r_i\Big) \;=\; \sum_{i\in S'} \sum_{k=1}^{\bar{q}_i} r_i a_{ik}.$$

The key difference from the proof of Theorem 1 is that the unit contribution in $R^{\mathrm{Pois}}(\mathbf{q}')$ now takes the form $r_i\big(1-e^{-a_{ik}/f_{S'}}\big)$, rather than $\frac{r_i a_{ik}/f_{S'}}{1+a_{ik}/f_{S'}}$. To address this, we use a technical lemma as follows.

LEMMA EC.1. *For any $f \in (0,1]$ and any $a \in (0,1]$, the following inequality holds*

$$(1-e^{-a/f}) \;\geq\; \frac{5}{3+2f} \cdot \frac{2a+a^2}{2+2a+a^2}.$$

Therefore, by Lemma EC.1, we have

$$R^{\mathrm{Pois}}(\mathbf{q}') \;\geq\; \frac{5}{3+2f_{S'}} \cdot \max_{\sum_{i\in N} q_i \leq K} \sum_{i\in S'} \sum_{k=1}^{\min\{q_i,\bar{q}_i\}} r_i \frac{a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2}.$$

To relate the right-hand side to $\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} r_i a_{ik}$, we apply another technical lemma. Let $\gamma' \triangleq \sum_{i\in S', 1\leq k\leq \bar{q}_i} \frac{2+2a_{ik}+a_{ik}^2}{2+a_{ik}}/K$.

LEMMA EC.2. *Fix an assortment $S' \subseteq \mathcal{N}$. Let $a_{ik} \in (0,1]$ such that $a_{ik} \geq a_{il}$ holds whenever $k \leq l$. Fix $r_i \geq 0$. Then, there exists $\mathbf{q} \in \mathbb{N}^N$ with $\sum_{i\in\mathcal{N}} q_i \leq K$ such that the following property holds:*

$$\frac{\sum_{i\in S'}\sum_{k=1}^{\min\{q_i,\bar{q}_i\}} \frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2}}{\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} r_i a_{ik}} \;\geq\; \min\left\{\frac{1}{\gamma'}, \frac{3}{5}\right\} = \frac{1}{\max\{\gamma', 5/3\}}. \tag{EC.5}$$

Since $\mathbf{q}'$ maximizes $R^{\mathrm{Pois}}(\cdot)$, we obtain

$$R^{\mathrm{Pois}}(\mathbf{q}') \;\geq\; \frac{5}{3+2f_{S'}} \cdot \frac{1}{\max\{\gamma',\, 5/3\}} \cdot \mathbb{E}[T] f_{S'}\Big(\sum_{i\in S'} \phi_i(S') r_i\Big). \tag{EC.6}$$

Step 2: Analysis on $\hat{\mathbf{q}}$. In the second step, we establish the constant approximation guarantee for $\hat{\mathbf{q}}$, which is either $\mathbf{q}'$ or $\mathbf{q}''$. We separately analyze the cases where $f_{S''}=0$ and $f_{S''}>0$. Define

$$\alpha' \triangleq \frac{\mathbb{E}[T] f_{S'}\big(\sum_{i\in S'} \phi_i(S') r_i\big)}{z_{\mathrm{app}}} \qquad \text{and} \qquad \alpha'' \triangleq \frac{\mathbb{E}[T] f_{S''}\big(\sum_{i\in S''} \phi_i(S'') r_i\big)}{z_{\mathrm{app}}},$$

so that $\alpha', \alpha'' \in [0,1]$ and $\alpha'+\alpha''=1$.

Case $f_{S''}=0$. If $f_{S''}=0$, then $\hat{\mathbf{q}}=\mathbf{q}'$ and $\alpha'=1$. We provide an upper bound on γ'.

LEMMA EC.3. *Suppose $f_{S''}=0$. Fix an assortment $S' \in \mathcal{N}$, with the definition $a_{ik} \triangleq f_{S'}\mathbb{E}[T]\pi_i\rho_{ik}$ for unit $k \in \{1,\ldots,\bar{q}_i\}$ of product $i \in S'$ in (13), the following property holds:*

$$\gamma' \;=\; \frac{1}{K}\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i}\left(\frac{2+2a_{ik}+a_{ik}^2}{2+a_{ik}}\right) \;\leq\; \frac{8}{3}. \tag{EC.7}$$

Thus, using $f_{S'} \leq 1$ and $\gamma' \leq 8/3$, we obtain

$$\frac{\mathbb{E}[R_T(\hat{\mathbf{q}})]}{z_{\mathrm{opt}}} \;\geq\; \kappa_1 \cdot \frac{5}{3+2f_{S'}} \cdot \frac{1}{\max\{\gamma',\, 5/3\}} \;\geq\; \frac{3\kappa_1}{8} \;=\; 0.375\,\kappa_1,$$

which is the required result when $f_{S''}=0$. Thus, we proceed assuming otherwise.

Case $f_{S''} > 0$. By applying the same analysis for $\mathbf{q}''$, we obtain that $\tilde{R}^{\text{Pois}}(\mathbf{q}'')$ is a lower bound for $\mathbb{E}[R_T(\mathbf{q}'')]$. Furthermore, define $\tilde{q}_i \triangleq \lceil \mathbb{E}[T] f_{S''} \phi_i(S'') \rceil$ for any $i \in S''$, and

$$\tilde{a}_{ik} = \begin{cases} 1 & \text{if } 1 \le k \le \mathbb{E}[T] f_{S''} \phi_i(S'') \\ \mathbb{E}[T] f_{S''} \phi_i(S'') - \lfloor \mathbb{E}[T] f_{S''} \phi_i(S'') \rfloor & \text{if } \mathbb{E}[T] f_{S''} \phi_i(S'') < k = \tilde{q}_i \\ 0 & \text{otherwise} \end{cases}$$

for any $i \in S''$. Define $\gamma'' \triangleq \sum_{i \in S'', 1 \le k \le \tilde{q}_i} \frac{2+2\tilde{a}_{ik}+\tilde{a}_{ik}^2}{2+\tilde{a}_{ik}} / K$. Then, as Eq. (EC.6), we obtain

$$\frac{\tilde{R}^{\text{Pois}}(\mathbf{q}'')}{\mathbb{E}[T] f_{S''} (\sum_{i \in S''} \phi_i(S'') r_i)} \ge \frac{5}{3+2f_{S''}} \cdot \frac{1}{\max\{\gamma'', 5/3\}}. \tag{EC.8}$$

Also, we have $\gamma'' \le 8/3$.

Since $\hat{\mathbf{q}}$ is chosen as the one in $\{\mathbf{q}', \mathbf{q}''\}$ that has the larger lower bound, and $z_{\text{app}} \ge \kappa_1 z_{\text{opt}}$, we have

$$\frac{\mathbb{E}[R_T(\hat{\mathbf{q}})]}{z_{\text{opt}}} \ge \kappa_1 \max\left\{ \alpha' \cdot \frac{5}{3+2f_{S'}} \cdot \frac{1}{\max\{\gamma', 5/3\}}, \alpha'' \cdot \frac{5}{3+2f_{S''}} \cdot \frac{1}{\max\{\gamma'', 5/3\}} \right\}.$$

Below, we will argue that the right-side of the above inequality is bounded below by $3\kappa_1/11$. To do this, we first provide an upper bound on $\gamma' + \gamma''$ and a technical lemma as follows.

LEMMA EC.4. *With the definition* $\tilde{a}_{ik}$ *and* $a_{ik} \triangleq f_{S'} \mathbb{E}[T] \pi_i \rho_{ik}$ *for unit* $k \in \{1, \ldots, \bar{q}_i\}$ *of product* $i \in S'$ *in (13), suppose that* $\{f_{S'}, f_{S''}\}$ *is a feasible solution to Problem (Fluid). Then, the following property holds:*

$$\gamma' + \gamma'' \le \frac{11}{3}. \tag{EC.9}$$

LEMMA EC.5. *Given any* $f_1, f_2 \in [0,1]$ *with* $f_1 + f_2 \le 1$, *any* $\gamma_1, \gamma_2 \in [0, 8/3]$ *with* $\gamma_1 + \gamma_2 \le \frac{11}{3}$, *and any* $\alpha_1, \alpha_2 \in [0,1]$ *with* $\alpha_1 + \alpha_2 = 1$, *we have*

$$\max\left\{ \alpha_1 \cdot \frac{5}{3+2f_1} \cdot \frac{1}{\max\{\gamma_1, 5/3\}}, \alpha_2 \cdot \frac{5}{3+2f_2} \cdot \frac{1}{\max\{\gamma_2, 5/3\}} \right\} \ge \frac{3}{11}.$$

Then, combining Lemmas EC.4 and EC.5 gives

$$\max\left\{ \alpha' \cdot \frac{5}{3+2f_{S'}} \cdot \frac{1}{\max\{\gamma', 5/3\}}, \alpha'' \cdot \frac{5}{3+2f_{S''}} \cdot \frac{1}{\max\{\gamma'', 5/3\}} \right\} \ge \frac{3}{11}.$$

Consequently, we obtain

$$\mathbb{E}[R_T(\hat{\mathbf{q}})] \ge \frac{3\kappa_1}{11} z_{\text{opt}},$$

which implies the stated bound $\mathbb{E}[R_T(\hat{\mathbf{q}})] \ge 0.272\, \kappa_1 z_{\text{opt}}$. □

EC.2.1. Omitted Proofs in Section EC.2

Proof of Lemma EC.1.

This inequality is the Poisson counterpart of the per-unit bound used in the IFR analysis, and its proof is purely analytic. For each fixed f, the minimum of $h_f(a) = (1-e^{-a/f})\frac{2+2a+a^2}{2a+a^2}$ over $a \in (0,1]$ must occur at an interior stationary point or at $a=1$, or be approached as $a \to 0$. We show that $h_f(a) \geq 5/(3+2f)$ in each of these cases. For an interior stationary point, the desired inequality reduces to a polynomial inequality that is linear in f and can therefore be verified at the endpoints $f=0$ and $f=1$.

Fix $f \in (0,1]$ and define

$$h_f(a) = \left(1-e^{-a/f}\right) v(a), \qquad \text{where} \qquad v(a) = \frac{2+2a+a^2}{2a+a^2}.$$

Let $u(a) = 1-e^{-a/f}$. Then

$$\begin{aligned} u'(a) &= \frac{1}{f}e^{-a/f} && \text{and} \\ v'(a) &= \frac{(2+2a)(2a+a^2)-(2+2a+a^2)(2+2a)}{(2a+a^2)^2} && -\frac{2(2+2a)}{(2a+a^2)^2}. \end{aligned}$$

Combining the above expressions, the derivative of $h_f(a)$ is

$$\frac{\partial h_f}{\partial a} = \frac{1}{f}e^{-a/f}\frac{2+2a+a^2}{2a+a^2} - \left(1-e^{-a/f}\right)\frac{2(2+2a)}{(2a+a^2)^2}.$$

Setting $\partial h_f/\partial a = 0$ and rearranging terms gives

$$\frac{1}{f}e^{-a/f}\frac{2+2a+a^2}{2a+a^2} = \left(1-e^{-a/f}\right)\frac{2(2+2a)}{(2a+a^2)^2}.$$

Dividing both sides by $1-e^{-a/f} > 0$ and simplifying, we obtain the equivalent equation

$$\frac{e^{-a/f}}{1-e^{-a/f}} = \frac{2f(2+2a)}{(2a+a^2)(2+2a+a^2)}.$$

Therefore, when $\partial h_f/\partial a = 0$, we have

$$\begin{aligned} h_f(a) &= \left(1-e^{-a/f}\right)\frac{2+2a+a^2}{2a+a^2} \\ &= \frac{2f(2+2a)+(2a+a^2)(2+2a+a^2)-2f(2+2a)}{2f(2+2a)+(2a+a^2)(2+2a+a^2)} \cdot \frac{2+2a+a^2}{2a+a^2} \\ &= \frac{(2+2a+a^2)^2}{2f(2+2a)+(2a+a^2)(2+2a+a^2)} \\ &\geq \frac{5}{3+2f}. \end{aligned}$$

Since all quantities are positive, it suffices to verify that the following inequality holds:

$$(3+2f)(2+2a+a^2)^2 \geq 5\Big(2f(2+2a)+(2a+a^2)(2+2a+a^2)\Big).$$

Let

$$A(a) = a^4 + 4a^3 + 8a^2 - 2a - 6, \qquad \text{and} \qquad B(a) = 6 + 2a - 3a^2 - 4a^3 - a^4.$$

Expanding both sides of the above inequality and simplifying, the difference between the left- and right-hand sides can be written as

$$2\big[f\,A(a) + B(a)\big],$$

Then the inequality holds if and only if this expression is nonnegative. Note that this expression is linear in f. Therefore, for any fixed $a \in (0,1]$, its minimum over $f \in [0,1]$ is attained at one of the endpoints. At $f = 1$, we have $A(a) + B(a) = 5a^2 \geq 0$. At $f = 0$, we have $B(a) = (1-a)(a+3)(a^2+2a+2) \geq 0$ for all $a \in (0,1]$. Hence $fA(a) + B(a) \geq 0$ for all $f \in [0,1]$ and $a \in (0,1]$, which establishes the inequality.

Also, $h_f(1) = \left(1 - e^{-1/f}\right) \cdot \frac{2+2+1}{2+1} = \frac{5}{3}(1 - e^{-1/f})$. As $a \to 0$, $1 - e^{-a/f} \sim a/f$ and $\frac{2+2a+a^2}{a(a+2)} \sim \frac{1}{a}$, hence

$$\lim_{a \to 0} h_f(a) = \frac{1}{f}$$

Since $\frac{1}{f} \geq \frac{5}{3+2f}$ and $\frac{5}{3}(1-e^{-1/f}) \geq \frac{5}{3+2f}$ for any $f \in (0,1]$, we conclude that for any $f \in (0,1]$ and $a \in (0,1]$,

$$(1 - e^{-a/f}) \cdot \frac{2 + 2a + a^2}{2a + a^2} \geq \frac{5}{3 + 2f}.$$

which proves Lemma EC.1. □

Proof of Lemma EC.2.

This is a factor-revealing exchange argument and the Poisson analogue of Lemma 3. If the total number of candidate units does not exceed K, selecting all units establishes the result directly. Otherwise, an optimal solution selects exactly K units. We let τ be the smallest value of $\frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2}$ among the selected units. Optimality implies that the corresponding value of every unselected unit is at most τ. We use this inequality for the unselected units and an exact decomposition around τ for the selected units. Summing these expressions yields the stated ratio, where the constant $5/3$ follows from $\frac{2+a}{2+2a+a^2} \geq 3/5$ for $a \in (0,1]$.

Let $\mathbf{q}^*$ denote the solution that maximizes the left-side expression in (EC.5). Suppose $\sum_{i \in S'} \bar{q}_i \leq K$. Then, $\bar{\mathbf{q}}$ is indeed an optimal solution to the problem in (EC.5). From $a_{ik} \leq 1$, the left-side expression in (EC.5) is at least $3/5$. Thus, the required inequality holds.

Next, we consider the case $\sum_{i \in S'} \bar{q}_i > K$. In this case, we must have $\sum_{i \in S'} q_i^* = K$. Denote by τ the smallest value of $\frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2}$ over all possible (i,k) pairs satisfying $i \in S'$ and $1 \leq k \leq q_i^*$. Fix $i \in S'$. Since $\mathbf{q}^*$ is an optimal solution and $a_{ik} \geq a_{il}$ holds whenever $k \leq l$, it follows that $\tau \geq \frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2}$ if $k > q_i^*$. Thus, for fixed k satisfying $q_i^* < k \leq \bar{q}_i$, we have

$$r_i a_{ik}(2 + a_{ik}) \;\leq\; \tau(2 + 2a_{ik} + a_{ik}^2). \tag{EC.10}$$

In addition, for any k satisfying $1 \le k \le q_i^*$,

$$r_i a_{ik}(2+a_{ik}) \;=\; \tau(2+2a_{ik}+a_{ik}^2) + \left(\frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2} - \tau\right)(2+2a_{ik}+a_{ik}^2). \tag{EC.11}$$

Combining inequalities (EC.10) and (EC.11) and summing them over $i \in S'$ and k satisfying $1 \le k \le \bar{q}_i$, we get

$$\begin{aligned}\sum_{i\in S', 1\le k\le \bar{q}_i} r_i a_{ik} &= \sum_{i\in S', 1\le k\le \bar{q}_i} \frac{r_i a_{ik}(2+a_{ik})}{2+a_{ik}} \\ &\le \tau \sum_{i\in S', 1\le k\le \bar{q}_i} \frac{2+2a_{ik}+a_{ik}^2}{2+a_{ik}} + \sum_{i\in S', 1\le k\le q_i^*} \left(\frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2} - \tau\right)\cdot \frac{2+2a_{ik}+a_{ik}^2}{2+a_{ik}}.\end{aligned}$$

Also, since $\sum_{i\in S'} q_i^* = K$, we have

$$\sum_{i\in S', 1\le k\le q_i^*} \frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2} \;=\; \tau K + \sum_{i\in S', 1\le k\le q_i^*} \left(\frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2} - \tau\right).$$

Thus, putting together the above two results, it follows

$$\begin{aligned}\frac{\sum_{i\in S', 1\le k\le q_i^*} \frac{r_i a_{ik}(2+a_{ik})}{2+2a_{ik}+a_{ik}^2}}{\sum_{i\in S', 1\le k\le \bar{q}_i} r_i a_{ik}} &\ge \min\left\{\frac{K}{\sum_{i\in S', 1\le k\le \bar{q}_i} \frac{2+2a_{ik}+a_{ik}^2}{2+a_{ik}}}, \min_{\substack{i\in S'\\ 1\le k\le q_i^*}} \frac{2+a_{ik}}{2+2a_{ik}+a_{ik}^2}\right\} \\ &\ge \frac{1}{\max\{\gamma', 5/3\}},\end{aligned}$$

where the last inequality follows from $\frac{2+a_{ik}}{2+2a_{ik}+a_{ik}^2} \ge \frac{3}{5}$ for $a_{ik} \in (0,1]$. We prove the required result. □

Proof of Lemma EC.3.

We first give an overview of the proof. Let $x_i \triangleq \mathbb{E}[T] f_{S'} \phi_i(S')$. By construction, $\{a_{ik}\}_{k=1}^{\bar{q}_i}$ contains $\lfloor x_i \rfloor$ entries equal to one and, when x_i is nonintegral, one entry equal to its fractional part θ_i. The proof writes each summand as $g(a) = \frac{2+2a+a^2}{2+a} = a + \frac{2}{a+2}$ and decomposes $K\gamma'$ into $\frac{5}{3}\sum_i x_i$ plus a residual contribution from the fractional entries. The capacity bound $\sum_i x_i \le K$, together with the fact that there are at most $|S'| \le K$ fractional entries, then yields $\gamma' \le 8/3$. The details follow below.

Define $g(a) := \frac{2+2a+a^2}{2+a} = a + \frac{2}{a+2}$, where $a \in [0,1]$. Then the target quantity equals

$$\frac{1}{K}\sum_{i\in S'} \sum_{k=1}^{\bar{q}_i} g(a_{ik}).$$

Write $x_i = n_i + \theta_i$ where $n_i := \lfloor x_i \rfloor \in \mathbb{Z}_{\ge 0}$ and $\theta_i := x_i - \lfloor x_i \rfloor \in [0,1)$. By construction, among $\{a_{ik}\}_{k=1}^{\bar{q}_i}$, there are exactly n_i entries equal to 1 and (if $\theta_i > 0$) one entry equal to θ_i. Hence

$$\sum_{k=1}^{\bar{q}_i} g(a_{ik}) \;=\; n_i\, g(1) + \mathbf{1}(\theta_i > 0)\, g(\theta_i) \;=\; \frac{5}{3} n_i + \mathbf{1}(\theta_i > 0)\, g(\theta_i).$$

Therefore,

$$\frac{1}{K}\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) = \frac{1}{K}\left(\frac{5}{3}\sum_{i\in S'} n_i + \sum_{i:\,\theta_i>0} g(\theta_i)\right).$$

Add and subtract $\frac{5}{3}\sum_{i:\theta_i>0}\theta_i$ to obtain

$$\frac{1}{K}\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) \;=\; \frac{5}{3K}\sum_{i\in S'}(n_i+\theta_i) + \frac{1}{K}\sum_{i:\,\theta_i>0}\left(g(\theta_i)-\frac{5}{3}\theta_i\right) \;=\; \frac{5}{3K}\sum_{i\in S'} x_i + \frac{1}{K}\sum_{i:\,\theta_i>0} h(\theta_i),$$

where

$$h(\theta) := g(\theta) - \frac{5}{3}\theta = \frac{2}{\theta+2} - \frac{2}{3}\theta.$$

We have $h'(\theta) = -\frac{2}{(\theta+2)^2} - \frac{2}{3} < 0$, $\forall \theta \in [0,1)$, so h is decreasing on $[0,1)$ and thus $h(\theta) \le h(0) = 1$ for all $\theta \in [0,1)$. Let $m := |\{i \in S' : \theta_i > 0\}|$ denote the number of non-integer x_i's. Then,

$$\frac{1}{K}\sum_{i:\,\theta_i>0} h(\theta_i) \;\le\; \frac{m}{K} \;\le\; \frac{|S'|}{K} \;\le\; 1.$$

Combining with $\sum_{i\in S'} x_i \le K$, we conclude

$$\frac{1}{K}\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) \;\le\; \frac{5}{3K}\cdot K + 1 \;=\; \frac{8}{3},$$

as required. □

Proof of Lemma EC.4.

Proof overview. This result extends Lemma EC.3 to two assortments. We apply the same decomposition to the units associated with S' and S''. The joint fluid-capacity constraint bounds the combined demand mass by K, while $|S'| + |S''| \le 2K$ bounds the total number of possible fractional entries. Consequently, $K(\gamma' + \gamma'') \le \frac{5}{3}K + 2K = \frac{11}{3}K$, which gives $\gamma' + \gamma'' \le 11/3$. The details follow below.

Let

$$x_i := \mathbb{E}[T]\, f_{S'}\, \phi_i(S') \quad (i \in S'), \qquad \text{and} \qquad \tilde{x}_i := \mathbb{E}[T]\, f_{S''}\, \phi_i(S'') \quad (i \in S'').$$

We have

$$\sum_{i\in S'} x_i + \sum_{i\in S''} \tilde{x}_i \le K, \qquad |S'| \le K, \qquad \text{and} \qquad |S''| \le K,$$

and recall $\bar{q}_i = \lceil x_i \rceil$ and $\tilde{q}_i = \lceil \tilde{x}_i \rceil$. Define

$$g(a) := \frac{2}{2+a} + \frac{2a}{2+a} + \frac{a^2}{2+a} = \frac{2+2a+a^2}{2+a} = a + \frac{2}{a+2}, \qquad a \in [0,1].$$

Then the quantity of interest equals $\frac{1}{K}\left(\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) + \sum_{i\in S''}\sum_{k=1}^{\tilde{q}_i} g(\tilde{a}_{ik})\right)$. Write $x_i = n_i + \theta_i$ where $n_i := \lfloor x_i \rfloor \in \mathbb{Z}_{\ge 0}$ and $\theta_i := x_i - \lfloor x_i \rfloor \in [0,1)$, and similarly $\tilde{x}_i = \tilde{n}_i + \tilde{\theta}_i$. By the construction of $\{a_{ik}\}_{k=1}^{\bar{q}_i}$, there are exactly n_i indices with $a_{ik} = 1$ and (if $\theta_i > 0$) one index with $a_{ik} = \theta_i$; hence

$$\sum_{k=1}^{\bar{q}_i} g(a_{ik}) = n_i g(1) + \mathbf{1}(\theta_i > 0)\, g(\theta_i) = \frac{5}{3} n_i + \mathbf{1}(\theta_i > 0)\, g(\theta_i).$$

Define $h(\theta) := g(\theta) - \frac{5}{3}\theta = \frac{2}{\theta+2} - \frac{2}{3}\theta$. We have $h'(\theta) = -\frac{2}{(\theta+2)^2} - \frac{2}{3} < 0, \forall \theta \in [0,1)$. Thus, h is decreasing on $[0,1)$ and thus $h(\theta) \le h(0) = 1$ for all $\theta \in [0,1)$. Let $m' := |\{i \in S' : \theta_i > 0\}|$ denote the number of non-integer x_i's. Then

$$\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) \;=\; \frac{5}{3}\sum_{i\in S'} x_i + \sum_{i:\theta_i>0} h(\theta_i) \;\le\; \frac{5}{3}\sum_{i\in S'} x_i + m'.$$

Similarly, letting $m'' := |\{i \in S'' : \tilde{\theta}_i > 0\}|$, we have

$$\sum_{i\in S''}\sum_{k=1}^{\tilde{q}_i} g(\tilde{a}_{ik}) \;\le\; \frac{5}{3}\sum_{i\in S''} \tilde{x}_i + m''.$$

Therefore,

$$\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) + \sum_{i\in S''}\sum_{k=1}^{\tilde{q}_i} g(\tilde{a}_{ik}) \;\le\; \frac{5}{3}\Big(\sum_{i\in S'} x_i + \sum_{i\in S''} \tilde{x}_i\Big) + (m'+m'') \;\le\; \frac{5}{3}K + (m'+m'').$$

Dividing by K yields the refined bound

$$\frac{1}{K}\left(\sum_{i\in S'}\sum_{k=1}^{\bar{q}_i} g(a_{ik}) + \sum_{i\in S''}\sum_{k=1}^{\tilde{q}_i} g(\tilde{a}_{ik})\right) \;\le\; \frac{5}{3} + \frac{m'+m''}{K} \;\le\; \frac{5}{3} + \frac{|S'|+|S''|}{K} \;\le\; \frac{11}{3},$$

as required. □

Proof of Lemma EC.5. Let τ be the left-side of the required inequality in the statement of the lemma. Then, $\frac{\alpha_1}{\tau} \le \max\{\gamma_1, 5/3\} \cdot \frac{3+2f_1}{5}$ and $\frac{\alpha_2}{\tau} \le \max\{\gamma_2, 5/3\} \cdot \frac{3+2f_2}{5}$. Thus,

$$\frac{1}{\tau} \;=\; \frac{\alpha_1+\alpha_2}{\tau} \;\le\; \max\{\gamma_1, 5/3\} \cdot \frac{3+2f_1}{5} + \max\{\gamma_2, 5/3\} \cdot \frac{3+2f_2}{5} \;\le\; \frac{11}{3}.$$

Above, the last inequality holds with equality when $f_1 = 1$ and $\gamma_1 = 8/3$. □

EC.3. Proofs in Section 7

Proof of Proposition 7.

We show that the program (26) attains a normalized objective value of at least $(\kappa_2 - \epsilon)\tilde{z}_{\text{fluid}}$, which by Section 7.2 is equivalent to a $(\kappa_2 - \epsilon)$-approximation for (Fluid). Using that an optimal fluid solution is supported on at most two assortments, we split into two cases. In Case 1, the assortment $\underline{S}$ found at $\mu = 0$ is already capacity-feasible, so offering it alone is a κ_2-approximation. In Case 2, the bisection returns two assortments $\underline{S}$ and $\bar{S}$ whose per-customer consumptions $1 - \phi_0(\cdot)$ straddle the normalized capacity $\kappa_2 K/\mathbb{E}[T]$; a Lagrangian bound (inequality (EC.13)) certifies that each is near-optimal for the μ-penalized objective, so a suitable convex combination of them, which is feasible for (26), achieves at least $\kappa_2\tilde{z}_{\text{fluid}}$, up to an error proportional to the terminal interval width $\bar{\mu} - \underline{\mu} \le \zeta$ that the stopping rule keeps below $\epsilon\tilde{z}_{\text{fluid}}$.

Let $\mathbf{f}^*$ be an optimal solution to Problem (Fluid). As explained in Section 7.2, it is convenient to work in normalized units: we write $\tilde{z}_{\text{fluid}} \triangleq z_{\text{fluid}}/\mathbb{E}[T]$ for the optimal value of the normalized fluid program, so

that $\sum_S f^*_S \sum_{i\in S} \phi_i(S) r_i = \tilde{z}_{\text{fluid}}$ and $z_{\text{fluid}} = \mathbb{E}[T]\, \tilde{z}_{\text{fluid}}$. Since Problem (Fluid) has only two constraints (3) and (4) in addition to nonnegative constraints, there exists an optimal solution supported on at most two assortments. Accordingly, let $\tilde{S}'$ and $\tilde{S}''$ be two assortments such that $f^*_{\tilde{S}'} > 0$, $f^*_{\tilde{S}''} \geq 0$, and

$$f^*_{\tilde{S}'} + f^*_{\tilde{S}''} = \sum_{\tilde{S}\subseteq\mathcal{N}:|\tilde{S}|\leq K} f^*_{\tilde{S}}.$$

We consider two cases.

Case 1: The condition in Line 6 of Algorithm 1, $1 - \phi_0(\underline{S}) \leq \frac{K}{\mathbb{E}[T]}$, holds. Then, the algorithm returns $f_{\underline{S}} = 1$, which is feasible for Problem (Fluid). Moreover, since $\underline{S}$ is obtained by intersecting the set of products with nonzero prices with a κ_2-approximate solution to Problem (Asmt) under revenues $r'_i = r_i$, we have $\sum_{i\in\underline{S}} \phi_i(\underline{S}) r_i \geq \kappa_2 \sum_{i\in S} \phi_i(S) r_i$ for any feasible assortment S (in particular, for $S = \tilde{S}'$ and $S = \tilde{S}''$). Therefore,

$$\begin{aligned}\sum_{i\in\underline{S}} \phi_i(\underline{S}) r_i &\geq (f^*_{\tilde{S}'} + f^*_{\tilde{S}''}) \sum_{i\in\underline{S}} \phi_i(\underline{S}) r_i \\ &\geq \kappa_2 \left(f^*_{\tilde{S}'} \sum_{i\in\tilde{S}'} \phi_i(\tilde{S}') r_i + f^*_{\tilde{S}''} \sum_{i\in\tilde{S}''} \phi_i(\tilde{S}'') r_i \right) = \kappa_2 \tilde{z}_{\text{fluid}}.\end{aligned}$$

Thus, in this case, the returned solution is a κ_2-approximate solution to Problem (Fluid).

Case 2: The condition in Line 6 is violated. Algorithm 1 then runs the bisection loop until $\bar{\mu} - \underline{\mu} \leq \zeta$. We first note that before the start of the bisection, we have $1 - \phi_0(\underline{S}) > \frac{K}{\mathbb{E}[T]}$ and $\sum_{i\in\underline{S}} \phi_i(\underline{S}) r_i \geq (1 - \phi_0(\underline{S})) \min_{i\in\mathcal{N}:r_i>0}\{r_i\}$. Thus, $\mathbf{f}$ with $f_{\underline{S}} = \frac{K}{\mathbb{E}[T](1-\phi_0(\underline{S}))}$ is a feasible solution to (Fluid), and thus,

$$\tilde{z}_{\text{fluid}} \;\geq\; \frac{K}{\mathbb{E}[T](1-\phi_0(\underline{S}))} \sum_{i\in\underline{S}} \phi_i(\underline{S}) r_i \;\geq\; \frac{K \cdot \min_{i\in\mathcal{N}:r_i>0}\{r_i\}}{\mathbb{E}[T]}. \tag{EC.12}$$

Next, we lower bound the objective value of our solution. By construction of the updates, at termination we have

$$1 - \phi_0(\underline{S}) \;>\; \kappa_2 \frac{K}{\mathbb{E}[T]} \qquad \text{and} \qquad 1 - \phi_0(\bar{S}) \;\leq\; \kappa_2 \frac{K}{\mathbb{E}[T]}.$$

We first show that, for $(S, \mu) = (\underline{S}, \underline{\mu})$ or $(S, \mu) = (\bar{S}, \bar{\mu})$,

$$\sum_{i\in S} \phi_i(S) r_i \;\geq\; \kappa_2 \tilde{z}_{\text{fluid}} + \mu \left(\sum_{i\in S} \phi_i(S) - \kappa_2 \sum_{\tilde{S}\in\{\tilde{S}',\tilde{S}''\}} f^*_{\tilde{S}} \big(1 - \phi_0(\tilde{S})\big) \right). \tag{EC.13}$$

To prove (EC.13), write

$$\sum_{i\in S} \phi_i(S) r_i \;=\; \sum_{i\in S} \phi_i(S)(r_i - \mu) + \mu \sum_{i\in S} \phi_i(S).$$

If $\mathbf{f}^*$ has a single nonzero variable (i.e., $f^*_{\tilde{S}'} > 0$), then by the κ_2-approximation guarantee applied to revenues $r'_i = r_i - \mu$, and using $0 \le f^*_{\tilde{S}'} \le 1$,

$$\begin{aligned}\sum_{i\in S}\phi_i(S)(r_i-\mu) &\ge \max\Big\{0,\ \kappa_2\sum_{i\in\tilde{S}'}\phi_i(\tilde{S}')(r_i-\mu)\Big\}\\ &\ge f^*_{\tilde{S}'}\max\Big\{0,\ \kappa_2\sum_{i\in\tilde{S}'}\phi_i(\tilde{S}')(r_i-\mu)\Big\}\\ &\ge \kappa_2 f^*_{\tilde{S}'}\sum_{i\in\tilde{S}'}\phi_i(\tilde{S}')(r_i-\mu),\end{aligned}$$

which yields (EC.13). If $\mathbf{f}^*$ has two nonzero variables, then applying the same κ_2-approximation guarantee to each $\tilde{S}\in\{\tilde{S}',\tilde{S}''\}$ gives, for $(S,\mu)=(\underline{S},\underline{\mu})$ or $(S,\mu)=(\bar{S},\bar{\mu})$,

$$\begin{aligned}\sum_{i\in S}\phi_i(S)r_i &= \sum_{i\in S}\phi_i(S)(r_i-\mu)+\mu\sum_{i\in S}\phi_i(S)\\ &\ge \kappa_2\sum_{i\in\tilde{S}}\phi_i(\tilde{S})(r_i-\mu)+\mu\sum_{i\in S}\phi_i(S)\\ &= \kappa_2\sum_{i\in\tilde{S}}\phi_i(\tilde{S})r_i+\mu\left(\sum_{i\in S}\phi_i(S)-\kappa_2\sum_{i\in\tilde{S}}\phi_i(\tilde{S})\right).\end{aligned}\tag{EC.14}$$

Then, multiplying (EC.14) by $f^*_{\tilde{S}}$ and summing over $\tilde{S}\in\{\tilde{S}',\tilde{S}''\}$ establishes (EC.13) (using $f^*_{\tilde{S}'}+f^*_{\tilde{S}''}=1$, which follows from the fact that constraints (3) and (4) are tight if the optimal solution has two nonzero variables).

We next lower bound the optimal value of (26). Since $1-\phi_0(\underline{S}) > \kappa_2\frac{K}{\mathbb{E}[T]}$ and $1-\phi_0(\bar{S}) \le \kappa_2\frac{K}{\mathbb{E}[T]}$, there exist coefficients $\hat{f}_{\underline{S}}, \hat{f}_{\bar{S}} \in [0,1]$ such that

$$\hat{f}_{\underline{S}}\big(1-\phi_0(\underline{S})\big)+\hat{f}_{\bar{S}}\big(1-\phi_0(\bar{S})\big)=\kappa_2\frac{K}{\mathbb{E}[T]} \qquad \text{and} \qquad \hat{f}_{\underline{S}}+\hat{f}_{\bar{S}}=1.$$

Using (EC.13) for $(\underline{S},\underline{\mu})$ and $(\bar{S},\bar{\mu})$ and combining the two inequalities gives

$$\begin{aligned}&\hat{f}_{\underline{S}}\sum_{i\in\underline{S}}\phi_i(\underline{S})r_i+\hat{f}_{\bar{S}}\sum_{i\in\bar{S}}\phi_i(\bar{S})r_i\\ &\quad\ge \kappa_2\tilde{z}_{\text{fluid}}+\underline{\mu}\left(\kappa_2\frac{K}{\mathbb{E}[T]}-\kappa_2\sum_{\tilde{S}\in\{\tilde{S}',\tilde{S}''\}}f^*_{\tilde{S}}\big(1-\phi_0(\tilde{S})\big)\right)\\ &\qquad+\hat{f}_{\bar{S}}(\bar{\mu}-\underline{\mu})\left(\sum_{i\in\bar{S}}\phi_i(\bar{S})-\kappa_2\sum_{\tilde{S}\in\{\tilde{S}',\tilde{S}''\}}f^*_{\tilde{S}}\big(1-\phi_0(\tilde{S})\big)\right).\end{aligned}$$

From $\sum_{\tilde{S}} f^*_{\tilde{S}}\big(1-\phi_0(\tilde{S})\big) \le \frac{K}{\mathbb{E}[T]}$, the second term on the right side of the above inequality is nonnegative. Moreover, we have

$$\left|\sum_{i\in\bar{S}}\phi_i(\bar{S})-\kappa_2\sum_{\tilde{S}\in\{\tilde{S}',\tilde{S}''\}}f^*_{\tilde{S}}\big(1-\phi_0(\tilde{S})\big)\right|\le 1,$$

and the stopping criterion ensures $|\bar{\mu} - \underline{\mu}| \leq \zeta$. Hence,

$$\hat{f}_{\underline{S}} \sum_{i \in \underline{S}} \phi_i(\underline{S}) r_i + \hat{f}_{\bar{S}} \sum_{i \in \bar{S}} \phi_i(\bar{S}) r_i \;\geq\; \kappa_2 \tilde{z}_{\text{fluid}} - \zeta \;\geq\; (\kappa_2 - \epsilon)\tilde{z}_{\text{fluid}},$$

where the last inequality follows from (EC.12) and the definition of ζ. Therefore, the optimal objective value of (26) is at least $(\kappa_2 - \epsilon)\tilde{z}_{\text{fluid}}$. Equivalently, in the original scaling, its objective value in (Fluid) is at least $(\kappa_2 - \epsilon) z_{\text{fluid}}$. Hence, Algorithm 1 returns a $(\kappa_2 - \epsilon)$-approximate solution supported on at most two assortments. □

EC.4. Computational Experiments

In this section, we evaluate the numerical performance of our algorithm under two ground-truth choice models: the Markov chain choice model and the MNL. The Markov chain experiments test the effectiveness of our method in settings with richer substitution patterns, while the MNL experiments remove model misspecification and compare our method directly with MNL-based benchmark algorithms. In all experiments, the number of arriving customers follows a Poisson distribution.

EC.4.1. Experimental Setup

We consider two families of instances. The first family uses a general Markov chain choice model (Blanchet et al. 2016, Feldman and Topaloglu 2017). The Markov chain choice model is specified by initial arrival probabilities λ_i for each product $i \in \mathcal{N}^+$ and transition probabilities ρ_{ij} for any pair $(i,j) \in \mathcal{N} \times \mathcal{N}^+$. Here, λ_i denotes the probability that the customer initially prefers product i over all other products, and ρ_{ij} denotes the probability of substituting from product i to product j if product i is not available. The customer's choice behavior can be described as follows. For any offered assortment $S \subseteq \mathcal{N}$, the customer initially prefers product i with probability λ_i . If $i \in S$, the customer chooses i; otherwise, the customer substitutes to product j with probability ρ_{ij}. This substitution process continues until the customer substitutes to and chooses a product in the assortment or eventually reaches state 0. For each instance, we generate a Markov chain choice model by randomly selecting the parameters$\{\lambda_j, \rho_{ij} | i \in \mathcal{N}, j \in \mathcal{N}^+\}$. In particular, the groups of parameters, $\{\lambda_j | j \in \mathcal{N}^+\}$ or $\{\rho_{ij} | j \in \mathcal{N}^+\}$, are generated using the same distribution as follows. First, we sample ω_j from a uniform distribution in the interval $[0,1]$ for each $j \in \mathcal{N}^+$. Then, we normalize these values to ensure their sum equals one, and assign each normalized value to the corresponding parameter. Prices r_i are i.i.d. random samples taken from a standard log-normal distribution with a mean of 0 and a standard deviation of 0.3. For the Markov chain experiments, we consider $N \in \{25, 50, 100\}$ and, for each N, independently vary K and $\mathbb{E}[T]$ over $\{N, 2N, 4N\}$. For each configuration $(N, K, \mathbb{E}[T])$, we generate 100 independent instances, with the number of arriving customers distributed as $T \sim \text{Poisson}(\mathbb{E}[T])$. We restrict attention to the regime $K \geq N$, since in this case solving the fluid relaxation (Fluid) only requires

solving an unconstrained assortment optimization problem under the Markov chain choice model, which can be solved in polynomial time via linear programming (Feldman and Topaloglu 2017).

The second family uses an MNL choice model as the ground-truth choice model. For any offered assortment $S \subseteq \mathcal{N}$, the purchase probability of product i is $\phi_i(S) = \mathbb{1}[i \in S]v_i/(v_0 + \sum_{j\in S} v_j)$. We set the outside-option attraction to $v_0 = 1$. We fix $N = 20$ products and consider $K \in \{25, 50, 100\}$ and $\mathbb{E}[T] \in \{25, 50, 100\}$, where the number of arriving customers follows a Poisson distribution $T \sim \text{Poisson}(\mathbb{E}[T])$. The product prices are sampled independently from a lognormal distribution with parameters $\mu = 0$ and $\sigma = 2$. Given the prices, we generate the MNL preference weights according to

$$v_i = r_i^\alpha \exp(\epsilon_i), \qquad \epsilon_i \sim \mathcal{N}(0, 0.6^2), \qquad v_0 = 1,$$

where r_i is the price of product i. The parameter α controls the strength and direction of price-preference correlation. We use $\alpha \in \{-2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2\}$, and report the normalized coefficient $\alpha/2$ on the horizontal axis of Figure EC.1, so that the plotted range is $[-1, 1]$. Negative values correspond to settings in which higher-priced products tend to have lower attraction weights, while positive values correspond to settings in which higher-priced products tend to be more preferred. For each parameter combination, we generate 100 independent random instances.

EC.4.2. Benchmarks

We compare our approach with two benchmark algorithms originally developed for MNL choice models: the greedy-like algorithm (GLA) inspired by Aouad et al. (2018) and the threshold-based algorithm (THR) of Sun et al. (2024). In the Markov chain experiments, these MNL-based benchmarks are implemented using an MNL approximation to the true Markov chain model. In the MNL experiments, the benchmarks and our method are all run under the correctly specified MNL choice model.

Our approach. Because T follows a Poisson distribution and Problem (Asmt) is polynomial-time solvable under the MNL model, as well as under the Markov chain choice model when $K \geq N$, we have $\kappa_2 = 1$ in all our experimental settings. Consequently, our framework provides a 0.272-approximation guarantee for these settings by Corollary 1. However, in the implementation, we adopt the $(0.25 - \epsilon)$-approximation algorithm provided in Corollary 2. There are two main differences between our implementation and the 0.272-approximation algorithm.

First, we solve (Fluid) using Algorithm 1 instead of the ellipsoid method. Although the ellipsoid method provides a slightly stronger theoretical guarantee, the improvement is negligible in practice when the tolerance parameter ϵ in Algorithm 1 is chosen sufficiently small (we set $\epsilon = 0.01$ in our experiments). Moreover, the ellipsoid method is considerably more complex to implement, whereas Algorithm 1 is much simpler: it only requires a standard bisection procedure with $O(\log_2(1/\epsilon))$ iterations, and each iteration solves an unconstrained assortment optimization problem.

Second, although the Poisson distribution of T can be explicitly incorporated into the rounding procedure to further improve the approximation ratio (see Section 6), we instead use the rounding method developed for general IFR distributions (see Section 4.2), which requires only the input $\mathbb{E}[T]$. This choice allows us to evaluate an implementation that uses only $\mathbb{E}[T]$ and does not exploit the full Poisson arrival distribution. In addition, one of our benchmark algorithms (Sun et al. 2024) does not rely on precise knowledge of the distribution of T, and thus this implementation ensures a fair comparison.

Lastly, since our approach may produce an inventory vector that does not fully utilize the capacity K, we apply a practical scaling-up procedure to allocate the remaining units while keeping the polynomial time complexity. Specifically, let K_{remain} denote the unused capacity and let $\hat{\mathbf{q}}$ be the current inventory vector. We proportionally distribute the remaining capacity according to the current inventory levels. In particular, we first allocate $\left\lfloor \frac{\hat{q}_i}{\sum_{j=1}^{N} \hat{q}_j} \cdot K_{\text{remain}} \right\rfloor$ additional units to each product i. The leftover units (due to rounding down) are then assigned greedily to the products with the largest fractional remainders, until the full capacity K is utilized.

Greedy-like Algorithm (GLA). Aouad et al. (2018) propose a constant-factor approximation algorithm for Problem (OPT) under the MNL model. Their method restricts attention to "expensive" products, defined as those with prices no less than the expected revenue of the static assortment optimization. The algorithm constructs two candidate inventory vectors: one obtained via a greedy selection procedure among the expensive products, and the other derived by maximizing a newsvendor-like lower bound. It then selects the inventory vector with the higher expected revenue. Their approach requires knowledge of the full distribution of T and relies on sampling to evaluate the expected revenues of candidate inventory vectors. To benchmark against this method, we proceed as follows.

First, in the Markov chain experiments, we implement GLA under an MNL approximation to the ground-truth Markov chain choice model. Specifically, we set $v_i = \lambda_i/\lambda_0$, so the benchmark captures the initial preference probabilities but not the full substitution structure of the Markov chain. In the MNL experiments, we instead provide GLA with the true MNL attraction weights v_i. Second, since our approach requires only the knowledge of $\mathbb{E}[T]$, we also use a misspecified distribution of T for GLA to ensure a fair comparison. In particular, we assume that T is deterministic and equal to its true mean. Third, GLA requires evaluating $\mathbb{E}[R_T(\mathbf{q})]$ (under the misspecified distribution of T and the choice model) for intermediate inventory vectors during the greedy selection process. We approximate this expectation via simulation. Given an inventory vector $\mathbf{q}$, we perform 100 independent simulations of the dynamic-substitution process. In each simulation, we generate T customers (with T deterministic for GLA) and simulate their choices under this MNL approximation. We then use the average revenue across the 100 simulations as an estimate of $\mathbb{E}[R_T(\mathbf{q})]$.

Threshold-based algorithm (THR). Sun et al. (2024) propose a $(0.195 - \epsilon)$-approximation algorithm for Problem (OPT) when the distribution of T is IFR and the choice model follows the MNL model. Their method requires only the input $\mathbb{E}[T]$, rather than the full distribution of T, and runs in time polynomial in N,

$\log_2 K$, and $\log_2 \mathbb{E}[T]$. Prior to our work, this represents the state-of-the-art approximation guarantee under the MNL model. To benchmark against this method, we implement it with several practical modifications.

Similarly, in the Markov chain experiments, THR is run under the same MNL approximation based on $v_i = \lambda_i/\lambda_0$, whereas in the MNL experiments it is run using the true MNL attraction weights. We also introduce a practical speed-up to their implementation without affecting the theoretical performance guarantee. Specifically, under the MNL choice model $\phi_i'(S) = \mathbb{1}[i \in S]\frac{v_i}{1+\sum_{j\in S} v_j}$, their approach optimizes the following fluid relaxation:

$$\max_{\mathbf{q}\in\mathbb{N}^N:\sum_{i=1}^N q_i \le K} \quad \mathrm{FR}(\mathbf{q}) \triangleq \max_{y_i\ge 0,\, 0\le i\le N} \left\{ \sum_{i=1}^N r_i y_i \;\middle|\; y_i \le \min\{q_i, v_i y_0/v_0\},\ \forall i\in\mathcal{N},\ \sum_{i=0}^N y_i = \mathbb{E}[T] \right\}.$$

Here, $\mathrm{FR}(\mathbf{q})$ serves as an upper bound on $\mathbb{E}[R_T(\mathbf{q})]$ under the misspecified MNL model $\{\phi_i'(S)\}$, and y_i can be interpreted as the expected sales of product i. However, $\mathrm{FR}(\mathbf{q})$ is not necessarily submodular and is therefore difficult to optimize directly. To address this, THR enumerates candidate values of y_0 and, for each fixed y_0, solves

$$\max_{\mathbf{q}\in\mathbb{N}^N:\sum_{i=1}^N q_i \le K} \quad \mathrm{FR}(\mathbf{q}, y_0) \triangleq \max_{y_i\ge 0,\, 1\le i\le N} \left\{ \sum_{i=1}^N r_i y_i \;\middle|\; y_i \le \min\{q_i, v_i y_0/v_0\},\ \forall i\in\mathcal{N},\ \sum_{i=0}^N y_i = \mathbb{E}[T] \right\}.$$

For fixed y_0, $\mathrm{FR}(\mathbf{q}, y_0)$ is DR-submodular (Theorem 4.3 in Bai et al. 2025). Accordingly, THR maximizes $\mathrm{FR}(\mathbf{q}, y_0)$ over $\{\mathbf{q} : \sum_{i=1}^N q_i \le K\}$ with approximation ratio $1 - 1/e - \epsilon$ in time polynomial in N and $\log_2 K$ (Soma and Yoshida 2018). In our experimental scale, however, the classical greedy algorithm of Nemhauser et al. (1978) achieves substantially faster running time in practice. Therefore, we use the greedy method to optimize $\mathrm{FR}(\mathbf{q}, y_0)$, which maintains the $1 - 1/e$ approximation guarantee, although its theoretical running time is polynomial in K. Finally, since THR may produce inventory vectors that do not fully utilize the capacity K, we apply the same scaling-up procedure used in our approach to ensure a fair comparison.

EC.4.3. Numerical Results

EC.4.3.1. Ground-Truth Model: Markov Chain Choice Model We compare our approach with the two benchmark algorithms in terms of (i) the achieved percentage of the fluid optimum and (ii) relative performance. For each solution with inventory vector $\mathbf{q}$, we estimate its expected revenue using a sampling procedure similar to that adopted in the GLA benchmark. Specifically, given $\mathbf{q}$, we perform 100 independent simulations of the dynamic substitution process. In each simulation, we first sample T and then sequentially simulate the choices of T customers. The average revenue across the 100 simulations is used as an estimate of $\mathbb{E}[R_T(\mathbf{q})]$. For a given instance specified by N, K, $\mathbb{E}[T]$, and the choice model, the achieved percentage of the fluid optimum is defined as the ratio between the estimated expected revenue of the solution and the fluid upper bound z_{fluid} obtained from (Fluid). To measure relative performance, we compute, for each instance, the ratio between the estimated expected revenue of a given approach and the highest estimated expected

Table EC.1 Performance of Tested Algorithms under Markov chain choice models.

Parameters			Achieved Percentages of Fluid Optimum (%)			Relative Performance (%)		
N	K	$\mathbb{E}[T]$	Ours	GLA	THR	Ours	GLA	THR
25	25	25	**94.73(90.56)**	89.91(70.14)	89.80(63.88)	**99.34(94.99)**	94.28(74.62)	94.15(68.20)
		50	**92.21(82.96)**	91.61(69.17)	91.80(74.60)	**98.18(86.92)**	97.52(75.21)	97.72(77.35)
		100	**100.00(99.84)**	98.55(78.80)	99.08(80.80)	**100.00(100.00)**	98.55(78.80)	99.08(80.80)
	50	25	**99.84(95.19)**	95.93(65.39)	94.36(66.69)	**99.14(92.43)**	95.23(65.04)	93.69(66.93)
		50	**97.43(93.33)**	91.74(66.97)	91.31(66.24)	**99.79(97.15)**	93.94(69.90)	93.52(66.90)
		100	**94.34(84.57)**	92.86(69.14)	93.42(75.15)	**98.69(87.92)**	97.16(72.59)	97.73(78.46)
	100	25	**99.52(94.07)**	96.24(68.16)	95.69(66.79)	**98.83(91.37)**	95.56(67.39)	94.99(66.95)
		50	**99.96(96.15)**	96.04(66.55)	94.26(63.85)	**99.46(95.59)**	95.55(64.54)	93.78(63.92)
		100	**98.98(96.08)**	92.50(68.61)	92.06(65.12)	**99.97(98.39)**	93.43(69.73)	92.98(66.06)
50	50	50	**96.90(93.56)**	90.79(60.80)	90.46(63.34)	**99.88(97.55)**	93.58(64.37)	93.25(65.70)
		100	93.62(**83.31**)	93.16(67.64)	**93.85**(70.22)	98.00(**86.12**)	97.50(70.20)	**98.21**(72.88)
		200	**100.00(100.00)**	98.57(71.75)	99.02(75.72)	**100.00(100.00)**	98.57(71.75)	99.02(75.72)
	100	50	**100.48(95.41)**	95.60(61.31)	94.30(63.33)	**99.77(96.41)**	94.92(61.53)	93.63(63.56)
		100	**98.33(95.64)**	91.98(61.65)	91.56(62.08)	**99.86(97.07)**	93.41(63.54)	92.99(63.99)
		200	**95.17(85.27)**	93.64(65.83)	94.91(70.50)	**98.50(87.94)**	96.90(68.58)	98.21(73.45)
	200	50	**100.03(94.72)**	95.80(60.44)	95.32(62.48)	**99.47(96.04)**	95.26(60.52)	94.79(62.55)
		100	**99.83(97.04)**	96.00(61.41)	94.56(62.77)	**99.66(96.54)**	95.81(61.71)	94.38(63.07)
		200	**99.33(97.48)**	92.20(63.56)	91.93(60.86)	**99.98(99.20)**	92.81(63.27)	92.54(60.58)
100	100	100	**98.27(95.75)**	92.67(68.56)	92.20(66.44)	**99.92(98.13)**	94.25(68.27)	93.76(66.16)
		200	**95.59(89.65)**	93.26(68.65)	95.02(77.82)	**98.93(90.83)**	96.49(71.70)	98.31(81.28)
		400	**100.00(100.00)**	98.63(72.63)	99.14(82.13)	**100.00(100.00)**	98.63(72.63)	99.14(82.13)
	200	100	**99.90(96.31)**	96.55(66.25)	95.34(68.57)	**99.59(96.11)**	96.24(66.66)	95.04(69.00)
		200	**99.28(97.50)**	93.15(67.07)	92.80(67.84)	**99.96(99.05)**	93.80(67.96)	93.44(68.75)
		400	**96.70(90.07)**	93.43(69.05)	95.65(78.53)	**99.25(91.17)**	95.86(71.41)	98.15(81.20)
	400	100	**99.80(95.58)**	96.80(68.60)	96.43(68.63)	**99.52(96.69)**	96.52(68.60)	96.15(68.60)
		200	**99.93(98.55)**	96.66(67.73)	95.50(67.07)	**99.77(97.86)**	96.51(67.48)	95.34(66.81)
		400	**99.70(98.36)**	93.35(68.06)	92.91(68.03)	**99.98(99.16)**	93.62(67.47)	93.19(67.57)

Table EC.2 Running Time of Tested Algorithms under Markov chain choice models.

Parameters			Running Time (seconds)		
N	K	$\mathbb{E}[T]$	Ours	GLA	THR
25	25	25	**0.012(0.073)**	0.118(0.307)	0.076(0.128)
		50	**0.057(0.173)**	0.165(0.445)	0.075(0.165)
		100	**0.067(0.177)**	0.254(0.516)	0.081(0.151)
	50	25	**0.022(0.373)**	0.365(0.869)	0.118(0.179)
		50	**0.018(0.295)**	0.505(1.298)	0.124(0.195)
		100	**0.084(0.421)**	0.638(1.919)	0.136(0.218)
	100	25	**0.027(0.348)**	1.165(2.392)	0.268(0.371)
		50	**0.023(0.155)**	1.817(4.613)	0.278(0.497)
		100	**0.085(0.784)**	2.472(7.054)	0.267(0.584)
50	50	50	**0.111(1.587)**	0.969(2.972)	0.412(0.604)
		100	**0.421(2.081)**	1.279(3.355)	0.584(1.611)
		200	**0.343(1.820)**	1.701(6.386)	0.553(1.456)
	100	50	**0.205(1.369)**	2.711(6.921)	1.029(2.092)
		100	**0.206(1.307)**	3.585(14.898)	1.222(2.989)
		200	**0.346(1.937)**	4.816(18.977)	1.553(3.092)
	200	50	**0.191(1.961)**	6.181(18.236)	2.220(4.951)
		100	**0.113(2.745)**	10.069(36.898)	2.141(5.889)
		200	**0.074(0.957)**	14.622(51.624)	2.352(6.187)
100	100	100	**0.213(1.877)**	6.597(21.422)	4.995(9.632)
		200	**0.622(1.910)**	9.372(36.163)	5.924(11.478)
		400	**0.603(1.300)**	13.827(44.977)	6.042(10.969)
	200	100	**0.112(0.413)**	18.313(79.504)	8.379(18.607)
		200	**0.133(1.153)**	27.127(88.844)	9.762(18.130)
		400	**0.560(1.525)**	39.889(130.483)	10.900(19.282)
	400	100	**0.099(0.943)**	39.252(127.053)	14.417(31.451)
		200	**0.085(0.243)**	69.062(227.588)	17.518(36.239)
		400	**0.098(1.217)**	112.045(317.082)	15.810(33.187)

revenue among all tested approaches. For example, if our algorithm achieves an estimated expected revenue of 1 for a particular instance, while all competing heuristics achieve 0.8, then the relative performance is 1 for our algorithm and 0.8 for the others.

Table EC.1 reports the results when customer choices follow the more general Markov chain choice model, under which the MNL-based benchmarks are subject to model misspecification. The corresponding running times are summarized in Table EC.2 for the Markov chain experiments. For each configuration, the reported value is the average over 100 independently generated instances, and the value in parentheses is the minimum over those instances. All algorithms were implemented in C++ and executed on a Linux server equipped with Intel Xeon E5-2650 v4 processors.

Our approach exhibits a pronounced performance advantage in the Markov chain experiments, where it uses the ground-truth Markov chain choice model while the MNL-based benchmarks rely on an MNL approximation. As shown in Table EC.1, our approach achieves the highest average performance under both metrics in 26 of the 27 tested configurations. The only exception is $(N, K, \mathbb{E}[T]) = (50, 50, 100)$, where THR performs slightly better on average. Even in this configuration, our approach achieves a substantially higher minimum performance over the 100 generated instances. More generally, our method provides the highest minimum performance in every tested Markov chain configuration. In many configurations, its estimated expected revenue is also close to the fluid-relaxation optimum, suggesting near-optimal performance in practice.

When the true model is a Markov chain choice model, explicitly accounting for the more general substitution structure yields a clear and robust performance advantage over the MNL-based benchmarks. Compared with THR, which is algorithmically similar to ours in that it runs in polynomial time and does not rely on sampling but is designed specifically for the MNL model, our method improves expected revenue by 5.1% on average in the Markov chain experiments. This gap illustrates the practical value of using a richer choice model when customer behavior deviates from the MNL structure.

We also observe that the average achieved percentage of the fluid optimum occasionally exceeds 100 in Table EC.1. This phenomenon is caused by sampling error in the estimation of $\mathbb{E}[R_T(\mathbf{q})]$, which may cause the estimated revenue to slightly exceed the fluid upper bound z_{fluid}. These deviations are small and do not affect the overall performance comparisons.

Table EC.2 demonstrates that our approach is also highly computationally efficient. In the Markov chain experiments, its average running time never exceeds 0.622 seconds. Our approach is faster than GLA and THR in every tested configuration.

We provide further explanation for this improvement below. GLA relies on a greedy selection procedure that iteratively adds inventory units over K steps. At each step, it enumerates all possible additions among the expensive products and evaluates the corresponding expected revenue using sampling-based estimation. This repeated simulation makes the algorithm computationally expensive. In contrast, our approach is fully

deterministic and runs in time polynomial in $\log_2 K$. It does not rely on sampling, leading to substantially lower computational overhead. THR is considerably faster than GLA, as it also avoids sampling and enjoys a similar theoretical time complexity to ours. However, it remains slower than our method in practice. This is because THR enumerates candidate values of y_0 and, for each guess, solves a DR-submodular maximization problem

$$\max_{\mathbf{q}\geq\mathbf{0}:\sum_{i=1}^{N} q_i\leq K} \mathrm{FR}(\mathbf{q}, y_0).$$

Although this enumeration has complexity comparable to our bisection procedure, the subproblem it solves at each step is computationally more demanding than the assortment optimization problem solved in our framework.

EC.4.3.2. Ground-Truth Model: MNL Figure EC.1 shows how the relative performance of the algorithms varies with the normalized price-preference coefficient. When the correlation is negative or weak, GLA and THR remain competitive, and in several configurations GLA slightly outperforms our approach. This is intuitive: when high-price products are less likely to be chosen, the value of more refined inventory construction is reduced, since revenue opportunities are less concentrated among highly attractive high-price products.

The pattern changes as α becomes positive. When prices and preference weights are positively correlated, our method becomes consistently stronger. Averaging across all K and $\mathbb{E}[T]$ configurations, the average relative performances of our method are about 98.24%, 97.97%, and 97.91% for normalized coefficients 0.5, 0.75, and 1, respectively, whereas the corresponding averages for THR are about 97.01%, 95.45%, and 94.87%. The improvement is most visible when inventory is relatively ample compared with expected demand, such as $(K, \mathbb{E}[T]) = (50, 25)$, $(100, 25)$, and $(100, 50)$. For example, at the largest positive correlation, our method reaches approximately 99.37% for $(K, \mathbb{E}[T]) = (100, 25)$, compared with 94.09% for GLA and 91.56% for THR.

The observed effect of α has an intuitive economic interpretation. Under the specification $v_i = r_i^{\alpha}\exp(\epsilon_i)$, a positive α makes higher-priced products tend to have larger attraction weights. Such a positive association is economically meaningful in markets where products commanding higher prices also possess greater underlying quality, brand appeal, or baseline customer preference. It should not be interpreted as implying that increasing a product's price causally increases its demand. Rather, it represents cross-product heterogeneity in which products that command higher prices are also intrinsically more attractive to customers. In such settings, revenue opportunities are concentrated among products that are both valuable and attractive, making accurate assortment selection and inventory allocation particularly important. This provides intuition for why our approach performs particularly well as α becomes positive.

Overall findings. Taken together, the two experiment families show that our approach provides solution quality comparable to algorithms specialized for the MNL model when all methods use the correctly specified MNL choice parameters, and superior solution quality when customer choices follow the more general Markov chain model. In addition, the running-time results for the Markov chain experiments show that our approach is substantially more computationally efficient than both benchmarks.

Figure EC.1 Achieved percentages of the fluid optimum under price-correlated MNL preferences.

