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Appendix EC.1 introduces notation, Appendix EC.2 proves Proposition 2, Appendix EC.3 proves Theorem 2, Appendix EC.4 proves Proposition 4, Appendix EC.5 proves Proposition 3, Appendix EC.6 proves Proposition 5, and Appendix EC.7 proves auxiliary lemmas used in Appendices EC.1–EC.6. Appendix EC.8 contains the MILP formulations of (SP′) under uniform and negative-exponential random utility components. Appendix EC.9 proves Proposition 6. Appendix EC.10 proves Theorem 1 and Proposition 1. Then in Appendix EC.11, we derive the surrogate problem for the random consideration set setting. Finally Appendix EC.12 presents numerical experiments similar to Section 5, but with revenue-dependent utilities.

EC.1. Notation.

We introduce some notation that will be used in the remaining appendices.

EC.1.1. The relaxed limiting problem.

Recall that (LOP) is defined by

$$\begin{aligned}
&\sup_{A\in\mathcal{B}(\mathbb{R}^2)} \quad \pi(A)=\int_A r(1-F_\varepsilon(w(A)-u))f_\mu(u,r)dudr,\\
&\text{s.t.} \quad \int_A f_\mu(u,r)dudr\le\eta,\\
&w(A)=\inf\left\{w:\int_A(1-F_\varepsilon(w-u))f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w-v))f_\nu(v)dv\le\beta\right\}.
\end{aligned}$$

Since a feasible solution $A\in\mathcal{B}(\mathbb{R}^2)$ can be represented as the indicator function $\mathbb{1}_A(x)$, the following problem is a relaxation of (LOP) whose solutions are of the form $\phi:\mathbb{R}^2\to[0,1]$. Namely, we define

$$\sup_{\phi:\mathbb{R}^2\to[0,1]} \quad \pi(\phi)=\int_{\mathbb{R}^2} r(1-F_\varepsilon(w(\phi)-u))\phi(u,r)f_\mu(u,r)dudr, \tag{RLOP}$$

$$\text{s.t.} \quad \int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)dudr\le\eta,$$

$$w(\phi)=\inf\left\{w:\int_{\mathbb{R}^2}(1-F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w-v))f_\nu(v)dv\le\beta\right\}, \tag{EC.1}$$

and denote the optimal objective value by $\widetilde{\pi}^\star$. We say that $\phi:\mathbb{R}^2\to[0,1]$ is an *interior solution* to (RLOP) if the cardinality constraint holds with strict inequality; i.e., $\int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)dudr<\eta$. Interior solutions play an important role in rigorously connecting (OP), (SP), (LOP), and (RLOP), and are used in Appendices EC.2.2 and EC.7.2. Similar to (EC.1), we also define

$$w^+(\phi)=\sup\left\{w:\int_{\mathbb{R}^2}(1-F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w-v))f_\nu(v)dv\ge\beta\right\}, \tag{EC.2}$$

which satisfies $w^+(\phi) \geq w(\phi)$; in case that the set on the right-hand side of (EC.3) is empty, we set $w^+(\phi) = -\infty$. Slightly abusing notation, we write $w(A)$, $w^+(A)$, and $\pi(A)$ for a set $A \in \mathcal{B}(\mathbb{R}^2)$ to denote $w(\mathbb{1}_A)$, $w^+(\mathbb{1}_A)$, and $\pi(\mathbb{1}_A)$, respectively. It is possible that $w(\phi) = -\infty$, but if $w(\phi)$ is finite then so is $w^+(\phi)$. In either case, since both $w \mapsto \int_{\mathbb{R}^2}(1 - F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr$ and $w \mapsto \int_{\mathbb{R}}(1 - F_\xi(w-v))f_\nu(v)dv$ are continuous, we have

$$[w(\phi), w^+(\phi)] = \left\{ w : \int_{\mathbb{R}^2}(1 - F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}}(1 - F_\xi(w-v))f_\nu(v)dv = \beta \right\}. \tag{EC.3}$$

We use (EC.3) throughout the appendix to establish the convergence of $W_{(B+1)}(\cdot)$ to intervals of the form $[w(\cdot), w^+(\cdot)]$. It will also be useful to know that

$$(1 - F_\varepsilon(w^+(\phi) - u))\phi(u,r) = (1 - F_\varepsilon(w(\phi) - u))\phi(u,r) \quad \mu\text{-a.s.} \tag{EC.4}$$

To justify (EC.4), note that

$$\int_{\mathbb{R}^2}(1 - F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr \quad \text{and} \quad \int_{\mathbb{R}}(1 - F_\xi(w-v))f_\nu(v)dv$$

are nonincreasing in w and, in particular, (EC.3) implies that both integrals are constant on $[w(\phi), w^+(\phi)]$. As a consequence,

$$\int_{\mathbb{R}^2}\big[(1 - F_\varepsilon(w(\phi) - u))\phi(u,r) - (1 - F_\varepsilon(w^+(\phi) - u))\phi(u,r)\big] f_\mu(u,r)dudr = 0,$$

and (EC.4) follows because the integrand is nonnegative.

EC.1.2. Extension to continuous assortment.

The following concepts are needed to work with (RSP); e.g., they are used to prove Theorem 3 in Appendix EC.5. We use $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ to denote elements of $\{0,1\}^N$ and $[0,1]^N$, respectively. Recall that $\hat{\pi}_N(\boldsymbol{x})$ and $\hat{\pi}_N(\boldsymbol{x}, w)$ are the objective functions of (SP) and (SP′), respectively, which admit the natural extensions

$$\hat{\pi}_N(\overline{\boldsymbol{x}}, w) = \sum_{i=1}^{N} \overline{x}_i r_i \big(1 - F_\varepsilon(w - u_i)\big),$$

$$\hat{\pi}_N(\overline{\boldsymbol{x}}) = \hat{\pi}_N(\overline{\boldsymbol{x}}, w_{(B+1)}(\overline{\boldsymbol{x}})) = \sum_{i=1}^{N} \overline{x}_i r_i \big(1 - F_\varepsilon(w_{(B+1)}(\overline{\boldsymbol{x}}) - u_i)\big), \qquad \overline{\boldsymbol{x}} \in [0,1]^N,$$

where $w_{(B+1)}(\overline{\boldsymbol{x}})$ extends $w_{(B+1)}(\boldsymbol{x})$, defined in (2), via

$$w_{(B+1)}(\overline{\boldsymbol{x}}) = \inf\left\{ w : \sum_{i=1}^{N} \overline{x}_i \big(1 - F_\varepsilon(w - u_i)\big) + \sum_{j=1}^{N_0} \big(1 - F_\xi(w - v_j)\big) \leq B \right\}, \quad \overline{\boldsymbol{x}} \in [0,1]^N. \tag{EC.5}$$

EC.1.3. Measures and weak convergence.

Here, we introduce some concepts around weak convergence of measures. We say that a sequence of finite (signed) Borel measures λ_n on a metric space S converges weakly to a finite (signed) Borel measure λ if for all bounded and continuous $\varphi: S \to \mathbb{R}$,

$$\lim_{N\to\infty} \int_S \varphi(x) d\lambda_n(x) = \int_S \varphi(x) d\lambda(x),$$

and we write $\lambda_n \Rightarrow \lambda$. When $S = \mathbb{R}^d$, weak convergence is equivalent to convergence of distribution functions.

We let μ_N and ν_{N_0} be the measures corresponding to the uniform distributions over $\{(u_i, r_i)\}_{i=1}^N$ and ν_{N_0}, respectively, and let μ and ν denote the measures corresponding to the CDFs $F_\mu(u,r)$ and $F_\nu(v)$ from Assumption 3, respectively. As explained in Chapter 1.3 of Billingsley (2013), the convergence in Assumption 3 is equivalent to

$$\mu_N \Rightarrow \mu \quad \text{and} \quad \nu_{N_0} \Rightarrow \nu. \tag{EC.6}$$

We also say that a Borel set A is a λ-continuity set if $\lambda(\partial A) = 0$, where ∂A denotes the boundary of the set A. Lastly, given a λ-integrable function $\phi: S \to \mathbb{R}$, we let $\phi\lambda$ denote the measure defined by

$$(\phi\lambda)(A) = \int_A \phi(x) d\lambda(x), \quad A \in \mathcal{B}(S). \tag{EC.7}$$

The following auxiliary result is proved in Section EC.7.1.

Lemma EC.1. *If $\{\lambda_n\}_{n=1}^\infty$ and λ are finite nonnegative Borel measures on a separable metric space such that $\lambda_n \Rightarrow \lambda$, then for any Borel set A with $\lambda(\partial A) = 0$,*

$$\mathbb{1}_A \lambda_n \Rightarrow \mathbb{1}_A \lambda. \tag{EC.8}$$

EC.1.4. Convergence to an interval.

Finally, since our proofs relies heavily on lim infs and lim sups, we define the concept of convergence to an interval so as to simplify our exposition. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the set of extended real numbers. Given a sequence of extended real numbers $\{x_n\}_{n=1}^\infty$ and two extended real numbers $a \le b$, we write

$$x_n \to [a,b] \quad \text{a.s. } n \to \infty$$

to express that $a \le \liminf_{N\to\infty} x_n \le \limsup_{N\to\infty} x_n \le b$. When $a = b$, it is equivalent to $x_n \to a$. Similarly, for a sequence of random variables $\{X_n\}_{n=1}^\infty$ that take values in $\overline{\mathbb{R}}$ and two extended real numbers $a \le b$, we write

$$X_n \xrightarrow{p} [a,b]$$

if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\big(X_n \in [a-\epsilon,\, b+\epsilon]\big) = 1,$$

with the conventions $a-\epsilon = -\infty$ when $a = -\infty$ and $b+\epsilon = +\infty$ when $b = +\infty$. In the degenerate cases $a = b = +\infty$ (resp. $a = b = -\infty$), this means that for any $M \in \mathbb{R}$,

$$\mathbb{P}(X_n \geq M) \to 1 \quad (\text{resp. } \mathbb{P}(X_n \leq -M) \to 1).$$

When $a = b \in \mathbb{R}$, this reduces to the usual $X_n \xrightarrow{p} a$.

EC.2. Proofs of Proposition 2.

Recall that $\pi^\star_N$, $\hat{\pi}^\star_N$, $\pi^\star$, and $\widetilde{\pi}^\star$ are the optimal objective values of (OP), (SP), (LOP), and (RLOP), respectively. Proposition 2 is a direct consequence of the following three lemmas, which are proved in Appendices EC.2.1, EC.2.2, and EC.2.3, respectively.

Lemma EC.2. *Under Assumptions 1–3,*

$$\limsup_{n\to\infty} \frac{1}{N}\hat{\pi}^\star_N \leq \widetilde{\pi}^\star \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{N}\pi^\star_N \leq \widetilde{\pi}^\star.$$

Lemma EC.3. *Under Assumptions 1–3,*

$$\liminf_{n\to\infty} \frac{1}{N}\hat{\pi}^\star_N \geq \pi^\star \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{N}\pi^\star_N \geq \pi^\star$$

Lemma EC.4. *Both (LOP) and its relaxation (RLOP) have the same optimal value; i.e., $\widetilde{\pi}^\star = \pi^\star$.*

EC.2.1. Proof of Lemma EC.2

Let μ be the measure corresponding to the limiting distribution $F_\mu(u,r)$ defined in Assumption 3. Given continuous $\overline{\boldsymbol{x}}_N \in [0,1]^N$, we let $\overline{\boldsymbol{x}}_{N,i}$ denote its ith component and define the Borel measure $\mu_{\overline{\boldsymbol{x}}_N}$ via

$$\mu_{\overline{\boldsymbol{x}}_N}(A) = \frac{1}{N}\sum_{i=1}^N \overline{x}_{N,i}\mathbb{1}_A(u_i,r_i) \quad A \in \mathcal{B}(\mathbb{R}^2).$$

To prove Lemma EC.2, we require three auxiliary lemmas.

Lemma EC.5. *Given a sequence $\{\overline{\boldsymbol{x}}_N \in [0,1]^N\}_{N=1}^\infty$, for any subsequence $\{N'\} \subset \{N\}$, there exists a further subsequence $\{N''\} \subset \{N'\}$ such that $\mu_{\overline{\boldsymbol{x}}_{N''}}$ converges weakly to some measure.*

Proof of Lemma EC.5. For any $(u,r) \in \mathbb{R}^2$,

$$\frac{1}{N}\sum_{i=1}^N \overline{x}_{N,i}\mathbb{1}\{(u_i,r_i) \leq (u,r)\} \leq \frac{1}{N}\sum_{i=1}^N \mathbb{1}\{(u_i,r_i) \leq (u,r)\} \to F_\mu(u,r) \tag{EC.9}$$

as $N \to \infty$. Thus, for any $\epsilon > 0$, there exists a rectangle $A_\epsilon \subset \mathbb{R}^2$ and a constant N_ϵ such that

$$\mu_{\overline{\boldsymbol{x}}_N}(A_\epsilon^c) = \frac{1}{N}\sum_{i=1}^N \overline{x}_{N,i}\mathbb{1}_{A_\epsilon^c}(u_i, r_i) \le \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{A_\epsilon^c}(u_i, r_i) < \epsilon, \quad N > N_\epsilon.$$

Thus, $\{\mu_{\overline{\boldsymbol{x}}_N}\}_{N=1}^\infty$ is a tight sequence of measures, and the result follows from Prokhorov's Theorem (Theorem 8.6.2, Bogachev and Ruas (2007). $\square$

Lemma EC.6. *Given a sequence $\{\overline{\boldsymbol{x}}_N \in [0,1]^N\}_{N=1}^\infty$, any weakly convergent subsequence of $\{\mu_{\overline{\boldsymbol{x}}_N}\}$ converges to a limit of the form $\phi\mu$ for some function $\phi : \mathbb{R}^2 \to [0,1]$, which may depend on the subsequence, where $\phi\mu$ is as in (EC.7).*

Proof of Lemma EC.6. It follows from (EC.6) and (EC.9) that $\{\mu_{N'} - \mu_{\overline{\boldsymbol{x}}_{N'}}\}_{N'=1}^\infty$ is a weakly convergent sequence of nonnegative measures that converges to $\mu - \bar{\mu}$, where we recall from Appendix EC.1.3 that $\mu_{N'}$ is the uniform measure over $\{(u_i, r_i)\}_{i=1}^{N'}$. We claim that

$$\mu(A) - \bar{\mu}(A) \ge 0, \quad A \in \mathcal{B}(\mathbb{R}^2).$$

Postponing the proof of this claim, the Radon-Nikodym theorem (Rudin 1987, Theorem 6.10) implies that $\bar{\mu} = \varphi\mu$ for some nonnegative $\varphi : \mathbb{R}^2 \to \mathbb{R}$. Since $\bar{\mu} = \varphi\mu \le \mu$, the function $\varphi(x)$ is bounded by one except for a set of μ-measure zero, and therefore $\phi(x) = \varphi(x) \wedge 1$ satisfies $\bar{\mu} = \phi\mu$.

To show that $\mu - \bar{\mu}$ is a nonnegative measure, consider the linear functional $\varphi \mapsto \int_{\mathbb{R}^2} \varphi(x) d(\mu - \bar{\mu})(x)$ that maps $C_c(\mathbb{R}^2)$ functions to real values, where $C_c(\mathbb{R}^2)$ is the set of all the continuous functions on $\mathbb{R}^2$ with compact support. Since $\mu_{N'} - \mu_{\overline{\boldsymbol{x}}_{N'}} \Rightarrow \mu - \bar{\mu}$, it follows that this linear functional is nonnegative, in the sense that it maps nonnegative functions in $C_c(\mathbb{R}^2)$ to nonnegative real values. We conclude from the Riesz representation for positive linear functionals (Rudin 1987, Theorem 2.14) that $\mu - \bar{\mu}$ is a nonnegative Borel measure. $\square$

The next lemma is a continuity result. Its proof is deferred to Appendix EC.2.4.

Lemma EC.7. *Suppose $\{\overline{\boldsymbol{x}}_N \in [0,1]^N\}_{N=1}^\infty$ is such that $\mu_{\overline{\boldsymbol{x}}_N} \Rightarrow \phi\mu$ for some function $0 \le \phi(x) \le 1$. Then*

$$w_{(B+1)}(\overline{\boldsymbol{x}}_N) \to [w(\phi), w^+(\phi)] \quad \textit{and} \quad \lim_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N) = \pi(\phi).$$

Similarly, if $\{\boldsymbol{x}_N \in \{0,1\}^N\}_{N=1}^\infty$ is such that $\mu_{\boldsymbol{x}_N} \Rightarrow \phi\mu$ for some function $0 \le \phi(x) \le 1$, then

$$W_{(B+1)}(\boldsymbol{x}_N) \xrightarrow{p} [w(\phi), w^+(\phi)] \quad \textit{and} \quad \lim_{N\to\infty} \frac{1}{N}\pi_N(\boldsymbol{x}_N) = \pi(\phi).$$

We are now ready to prove Lemma EC.2.

Proof of Lemma EC.2. Let $\boldsymbol{x}_N$ be the optimal solution to (OP). Then we can find a subsequence $\{N'\} \subset \{N\}$ such that $\limsup_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N) = \lim_{N\to\infty}(1/N')\pi_{N'}(\boldsymbol{x}_{N'})$. By Lemmas EC.5 and EC.6, we can find a further subsequence $\{N''\} \subset \{N'\}$ such that $\mu_{\boldsymbol{x}_{N''}}$ converges weakly to $\phi\mu$ for some function $\phi : \mathbb{R}^2 \to [0,1]$. Since ϕ is feasible to (RLOP) (we show at the end), Lemma EC.7 then implies that

$$\limsup_{N\to\infty} \frac{1}{N}\pi_N(\boldsymbol{x}_N) = \lim_{N\to\infty} \frac{1}{N''}\pi_{N''}(\boldsymbol{x}_{N''}) = \pi(\phi) \le \widetilde{\pi}^\star,$$

where we recall that $\widetilde{\pi}^\star$ is the optimal objective value of (RLOP). Applying the same argument to (SP) gives $\limsup_{N\to\infty}(1/N)\hat{\pi}_N^\star \le \widetilde{\pi}^\star$.

It remains to show that ϕ is feasible to (RLOP). Since the optimal $\boldsymbol{x}_N$ for (OP) satisfies the cardinality constraint $\|x_N\|_1 \le \eta N$, and $\mu_{\boldsymbol{x}_{N''}} \Rightarrow \phi\mu$, we have

$$\int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r) du dr = \lim_{N\to\infty} \frac{1}{N''} \|\boldsymbol{x}_{N''}\|_1 \le \eta,$$

which means that ϕ is feasible to (RLOP). □

EC.2.2. Proof of Lemma EC.3.

Recall from Section EC.1.1 that an interior solution to (RLOP) is one that satisfies $\int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r) du dr < \eta$. The following auxiliary lemma is proved in Section EC.7.2.

LEMMA EC.8. *For any feasible solution $\phi : \mathbb{R}^2 \to [0,1]$ to (RLOP) and any $\epsilon > 0$, there exists an interior solution to (RLOP) of the form $\mathbb{1}_A(x)$ for some F_μ-continuity set A, such that*

$$|\pi(\phi) - \pi(A)| \le \epsilon \quad \text{ and } \quad w(\phi) - \epsilon \le w(A) \le w^+(A) \le w^+(\phi) + \epsilon.$$

In particular, A is also a feasible solution to (LOP).

Proof of Lemma EC.3. Since $\pi^\star$ is the supremum over all feasible solutions to (LOP), for any $\epsilon > 0$, we can find a feasible solution $A_{\epsilon/2}$ to (LOP) that is $(\epsilon/2)$-optimal, in the sense that $\pi(A_{\epsilon/2}) \ge \pi^\star - \epsilon/2$. Applying Lemma EC.8 with $\mathbb{1}_{A_{\epsilon/2}}(x)$, there exists an F_μ-continuity set A that is an interior solution and such that $|\pi(A) - \pi(A_{\epsilon/2})| < \epsilon/2$, implying that $|\pi(A) - \pi^\star| < \epsilon$. Let

$$\boldsymbol{x}_N = \big(\mathbb{1}_A(u_1, r_1), \ldots, \mathbb{1}_A(u_N, r_N)\big), \quad N \ge 1,$$

and observe that $\mu_{\boldsymbol{x}_N}(B) = \mu_N(A \cap B)$ for any Borel measurable set B, where we recall from Appendix EC.1.3 that μ_N is the discrete uniform distribution over $\{(u_i, r_i)\}_{i=1}^N$. Since $\mu_N \Rightarrow \mu$ by (EC.6), we have

$$\lim_{N\to\infty} \frac{1}{N} ||\boldsymbol{x}_N||_1 = \int_A \phi(u,r) f_\mu(u,r) du dr < \eta,$$

which means that $\boldsymbol{x}_N$ is feasible to (OP) and (SP) for large-enough N. It then follows from Lemma EC.1 that $\mu_{\boldsymbol{x}_N} \Rightarrow \mathbb{1}_A\mu$, and from Lemma EC.7 we get

$$\lim_{N\to\infty} \frac{1}{N}\pi_N^\star \ge \lim_{N\to\infty} \frac{1}{N}\pi_N(\boldsymbol{x}_N) = \pi(A) \ge \pi^\star - \epsilon,$$
$$\lim_{N\to\infty} \frac{1}{N}\hat{\pi}_N^\star \ge \lim_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N) = \pi(A) \ge \pi^\star - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude. $\square$

EC.2.3. Proof of Lemma EC.4.

The inequality $\widetilde{\pi}^\star \ge \pi^\star$ is immediate because (RLOP) is a relaxation of (LOP). To show the other direction, for any $\epsilon > 0$, there exists a feasible solution ϕ to (RLOP) such that $\pi(\phi) \ge \widetilde{\pi}^\star - \epsilon/2$. By Lemma EC.8, we can find a set A that is feasible to (LOP) such that $|\pi(A) - \pi(\phi)| < \epsilon/2$, implying that $\pi^\star \ge \pi(A) \ge \pi(\phi) - \epsilon/2 \ge \widetilde{\pi}^\star - \epsilon$. Since ϵ is arbitrary, we conclude. $\square$

EC.2.4. Proof of Lemma EC.7.

We first state and prove an auxiliary lemma and then prove Lemma EC.7.

Lemma EC.9. *Suppose that $\{\overline{\boldsymbol{x}}_N \in [0,1]^N\}_{N=1}^\infty$ is such that $\mu_{\overline{\boldsymbol{x}}_N} \Rightarrow \phi\mu$ for some $\phi : \mathbb{R}^2 \to [0,1]$. Then for any $w \in \overline{\mathbb{R}}$,*

$$\frac{1}{N}\sum_{i=1}^N \mathbb{1}(U_i > w)\overline{x}_{N,i} \xrightarrow{p} \int_{\mathbb{R}^2} (1 - F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr, \tag{EC.10}$$
$$\frac{1}{N_0}\sum_{j=1}^{N_0} \mathbb{1}(V_j > w) \xrightarrow{p} \int_{\mathbb{R}} (1 - F_\xi(w-v))f_\nu(v)dv, \tag{EC.11}$$

as $N \to \infty$, where $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$ are as in Assumption 2.

Proof of Lemma EC.9. Observe that

$$Y_N = \frac{1}{N}\sum_{i=1}^N \big(\mathbb{1}(U_i > w) - \mathbb{P}(U_i > w)\big)\overline{x}_{N,i}$$

is a mean-zero random variable. Since $U_i = u_i + \varepsilon_i$, and $\varepsilon 1, \dots, \varepsilon N$ are independent random variables, it follows that

$$\mathrm{Var}(Y_N) = \frac{1}{N^2}\sum_{i=1}^N \overline{x}_{N_i}\mathrm{Var}\Big(1(U_i > w) - \mathbb{P}(U_i > w)\Big) \le \frac{1}{N}.$$

Applying Chebyshev's inequality yields $Y_N \xrightarrow{p} 0$. Since $\mathbb{P}(U_i > w) = 1 - F_\varepsilon(w - u_i)$ is a continuous and bounded function of u_i and $\mu_{\boldsymbol{x}_N} \Rightarrow \phi\mu$, it follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \mathbb{P}(U_i > w)\overline{x}_{N,i} = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N (1 - F_\varepsilon(w-u_i))\overline{x}_{N,i} = \int_{\mathbb{R}^2}(1 - F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr,$$

proving (EC.10). The proof of (EC.11) is identical. $\square$

Proof of Lemma EC.7. We first prove by contradiction that $w(\phi) \leq \liminf_{N\to\infty} w_{(B+1)}(\overline{\boldsymbol{x}}_N)$. A similar argument yields $w^+(\phi) \geq \limsup_{N\to\infty} w_{(B+1)}(\overline{\boldsymbol{x}}_N)$, implying that $w_{(B+1)}(\overline{\boldsymbol{x}}_N) \to [w(\phi), w^+(\phi)]$ as $N \to \infty$.

Suppose that $w(\phi) > \liminf_{N\to\infty} w_{(B+1)}(\overline{\boldsymbol{x}}_N)$, which implies, in particular, that $w(\phi) > -\infty$. Recalling from Assumption 1 that B depends on N, there exists a subsequence $\{w_{(B'+1)}(\boldsymbol{x}_{N'})\}$ and some $\epsilon > 0$ such that $w_{(B'+1)}(\overline{\boldsymbol{x}}_{N'}) < w(\phi) - \epsilon$ for all N'. For notational convenience, let us write N and B instead of N' and B', respectively. From the definition of $w_{(B+1)}(\cdot)$ in (2), it follows from the continuity of $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$ that

$$\frac{1}{N}\sum_{i=1}^{N}(1 - F_\varepsilon(w_{(B+1)}(\overline{\boldsymbol{x}}_N) - u_i))\overline{x}_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0}(1 - F_\xi(w_{(B+1)}(\overline{\boldsymbol{x}}_N) - v_i)) = \frac{\min(B, ||\overline{\boldsymbol{x}}_N||_1 + N_0)}{N}.$$

Observe that

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r)dudr \geq & \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - u))\phi(u,r) f_\mu(u,r)dudr \\ =& \beta - \frac{\beta}{\gamma}\int_{\mathbb{R}}(1 - F_\xi(w(\phi) - v)) f_\nu(v)dv \geq \beta - \frac{\beta}{\gamma},\end{aligned}$$

where the equality is true because $w(\phi) > -\infty$. It follows from $\mu_{\overline{\boldsymbol{x}}_N} \Rightarrow \phi\mu$ that $||\overline{\boldsymbol{x}}_N||_1/N = \frac{1}{N}\overline{x}_{N,i} \to \int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r)dudr$. Combined with the facts that $B/N \to \beta$ and $N_0/N \to \beta/\gamma$ by Assumption 1, it follows that

$$\frac{\min(B, ||\overline{\boldsymbol{x}}_N||_1 + N_0)}{N} \to \min\left(\beta, \int_{\mathbb{R}^2} |\phi(u,r)| f_\mu(u,r)dudr + \frac{\beta}{\gamma}\right) = \beta.$$

We conclude that

$$\begin{aligned}\beta =& \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}\left(1 - F_\varepsilon\left(w_{(B+1)}(\overline{\boldsymbol{x}}_N) - u_i\right)\right)\overline{x}_{N,i} + \frac{N_0}{N}\frac{1}{N_0}\sum_{j=1}^{N_0}\left(1 - F_\xi\left(w_{(B+1)}(\overline{\boldsymbol{x}}_N) - v_j\right)\right)\right] \\ \geq& \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}\left(1 - F_\varepsilon\left(w(\phi) - \epsilon - u_i\right)\right)\overline{x}_{N,i} + \frac{N_0}{N}\frac{1}{N_0}\sum_{j=1}^{N_0}\left(1 - F_\xi\left(w(\phi) - \epsilon - v_j\right)\right)\right] \\ =& \int_{\mathbb{R}^2}\left(1 - F_\varepsilon\left(w(\phi) - \epsilon - u\right)\right)\phi(u,r) f_\mu(u,r)\,du\,dr + \frac{\beta}{\gamma}\int_{\mathbb{R}}\left(1 - F_\xi\left(w(\phi) - \epsilon - v\right)\right) f_\nu(v)\,dv > \beta,\end{aligned}$$

where the first inequality holds because $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$ are nonincreasing, the second equality is due to $\nu_{N_0} \Rightarrow \nu$ (see (EC.6)) and $\mu_{\overline{\boldsymbol{x}}_N} \Rightarrow \phi\mu$, and the final inequality is by definition of $w(\phi)$ in (EC.1). This yields a contradiction and we conclude that $w_{(B+1)}(\overline{\boldsymbol{x}}_N) \to [w(\phi), w^+(\phi)]$.

Next, we prove that $W_{(B+1)}(\boldsymbol{x}_N) \xrightarrow{p} [w(\phi), w^+(\phi)]$ as $N \to \infty$. We do so by first showing that when $-\infty < w(\phi)$, then for any $\epsilon > 0$,

$$\begin{aligned}\lim_{N\to\infty}\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) < w(\phi) - \epsilon) =& 0 \quad \text{if} \quad w(\phi) > -\infty, \text{ and} \\ \lim_{N\to\infty}\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) > w^+(\phi) + \epsilon) =& 0 \quad \textit{if} \quad w^+(\phi) > -\infty.\end{aligned}$$

Then, when $w^+(\phi) = -\infty$, we show that for any $M \in \mathbb{R}$, $\lim_{N\to\infty} \mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) > M) = 0$. Suppose that $-\infty < w(\phi)$ and fix $\epsilon > 0$. We now argue that $\lim_{N\to\infty} \mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) < w(\phi) - \epsilon) = 0$. Observe that

$$\begin{aligned}\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) < w(\phi) - \epsilon) &\le \mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) \le w(\phi) - \epsilon)\\ &= \mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(U_i > w(\phi) - \epsilon)\, x_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0} \mathbb{1}(V_j > w(\phi) - \epsilon) \le \frac{B}{N}\right),\end{aligned}$$

where the equality is due to the definition of $W_{(B+1)}(\boldsymbol{x}_N)$ in (1). The right-hand side converges to zero as $N \to \infty$. To see this, we apply Lemma EC.9 to get

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(U_i > w(\phi) - \epsilon)\, x_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0} \mathbb{1}(V_j > w(\phi) - \epsilon) - \frac{B}{N}\\ &\xrightarrow{p} \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - \epsilon - u))\phi(u,r) f_\mu(u,r)\,du\,dr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi) - \epsilon - v)) f_\nu(v)\,dv - \beta > 0,\end{aligned}$$

as $N \to \infty$, where the last inequality follows from the definition of $w(\phi)$ in (EC.1) and because $w(\phi) > -\infty$. A similar argument yields $\lim_{N\to\infty} \mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) \ge w^+(\phi) + \epsilon) = 0$ when $w^+(\phi) > -\infty$.

Finally, when $w^+(\phi) = -\infty$, for any $M \in \mathbb{R}$, we have

$$\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N) > M) = \mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(U_i > M)\, x_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0} \mathbb{1}(V_j > M) > \frac{B}{N}\right).$$

A similar argument can be applied to show that the right hand side converges to 0 as $N \to \infty$.

For the final part of the proof, we first show that $\lim_{N\to\infty}(1/N)\hat{\pi}_N(\overline{\boldsymbol{x}}_N) = \pi(\phi)$, and then we prove that $\lim_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N) = \pi(\phi)$. For any $\epsilon > 0$,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N) &= \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} r_i(1 - F_\varepsilon(w_{(B+1)}(\overline{\boldsymbol{x}}_N) - u_i))\overline{x}_{N,i}\\ &\le \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} r_i(1 - F_\varepsilon(w(\phi) - \epsilon - u_i))\overline{x}_{N,i}\\ &= \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - \epsilon - u))\phi(u,r) f_\mu(u,r)\,du\,dr.\end{aligned}$$

The inequality follows from $w_{(B+1)}(\overline{\boldsymbol{x}}_N) \to [w(\phi), w^+(\phi)]$ and the fact that $F_\varepsilon(\cdot)$ is nonincreasing, and the second equality follows from $\mu_{\overline{\boldsymbol{x}}_N} \Rightarrow \phi\mu$, the continuity of $F_\varepsilon(\cdot)$, and our assumption that $r \le r_{\max}$. Taking $\epsilon \to 0$, the dominate convergence theorem yields

$$\limsup_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N) \le \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - u))\phi(u,r) f_\mu(u,r)\,du\,dr = \pi(\phi).$$

Similarly,

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N) &= \liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i(1-F_\varepsilon(w_{(B+1)}(\overline{\boldsymbol{x}}_N)-u_i))\overline{x}_{N,i}\\ &\geq \liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i(1-F_\varepsilon(w^+(\phi)+\epsilon-u_i))\overline{x}_{N,i}\\ &= \int_{\mathbb{R}^2} r(1-F_\varepsilon(w^+(\phi)+\epsilon-u))\phi(u,r)f_\mu(u,r)dudr,\end{aligned}$$

implying that

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N) &\geq \int_{\mathbb{R}^2} r(1-F_\varepsilon(w^+(\phi)-u))\phi(u,r)f_\mu(u,r)dudr\\ &= \int_{\mathbb{R}^2} r(1-F_\varepsilon(w(\phi)-u))\phi(u,r)f_\mu(u,r)dudr = \pi(\phi),\end{aligned}$$

where the first equality follows from (EC.4). This proves that $\lim_{N\to\infty}(1/N)\hat{\pi}_N(\overline{\boldsymbol{x}}_N)=\pi(\phi)$, and we move on to show that $\lim_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N)=\pi(\phi)$ by showing that $\limsup_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N)\leq \pi(\phi)$ and $\liminf_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N)\geq\pi(\phi)$. We first show that $\limsup_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N)\leq\pi(\phi)$. Assume without loss of generality that $w(\phi)>-\infty$, otherwise

$$\limsup_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N)\leq\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i x_{N,i} = \int_{\mathbb{R}^2} r\phi(u,r)f_\mu(u,r)dudr=\pi(\phi)$$

follows trivially. Fix $\epsilon>0$ and observe that for any $i\in[N]$,

$$\begin{aligned}&\mathbb{P}(U_i>W_{(B+1)}(\boldsymbol{x}_N))\\ &=\mathbb{P}(U_i>W_{(B+1)}(\boldsymbol{x}_N),W_{(B+1)}(\boldsymbol{x}_N)>w(\phi)-\epsilon)+\mathbb{P}(U_i>W_{(B+1)}(\boldsymbol{x}_N),W_{(B+1)}(\boldsymbol{x}_N)\leq w(\phi)-\epsilon)\\ &\leq\mathbb{P}(U_i>w(\phi)-\epsilon)+\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N)\leq w(\phi)-\epsilon),\end{aligned}$$

implying that

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N) &= \limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\mathbb{P}(U_i>W_{(B+1)}(\boldsymbol{x}_N))x_{N,i}\\ &\leq \limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\big[\mathbb{P}(U_i>w(\phi)-\epsilon)+\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N)\leq w(\phi)-\epsilon)\big]x_{N,i}\\ &= \limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i(1-F_\varepsilon(w(\phi)-\epsilon-u_i))x_{N,i}\\ &= \int_{\mathbb{R}^2} r(1-F_\varepsilon(w(\phi)-\epsilon-u))\phi(u,r)f_\mu(u,r)dudr,\end{aligned}$$

where the second-to-last equality is due to $W_{(B+1)}(\boldsymbol{x}_N)\xrightarrow{p}[w(\phi),w^+(\phi)]$ and the last equality follows from $\mu_{\boldsymbol{x}_N}\Rightarrow\phi\mu$; we used the fact that $r_i\leq r_{\max}$ in both equations. Letting $\epsilon\to 0$ yields

$$\limsup_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N)\leq\int_{\mathbb{R}^2} r(1-F_\varepsilon(w(\phi)-u))\phi(u,r)f_\mu(u,r)dudr=\pi(\phi).$$

We now proceed to show that $\liminf_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N)\geq\pi(\phi)$. If $w^+(\phi)>-\infty$, we have

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N)&=\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\mathbb{P}(U_i>W_{(B+1)}(\boldsymbol{x}_N))x_{N,i}\\&\geq\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\mathbb{P}(U_i>w^+(\phi)+\epsilon)x_{N,i}\\&=\int_{\mathbb{R}^2}r(1-F_\varepsilon(w^+(\phi)+\epsilon-u))\phi(u,r)f_\mu(u,r)dudr\end{aligned}$$

Taking $\epsilon\to 0$, and using (EC.4) we can similarly show that

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N)&\geq\int_{\mathbb{R}^2}r(1-F_\varepsilon(w^+(\phi)-u))\phi(u,r)\,f_\mu(u,r)dudr\\&=\int_{\mathbb{R}^2}r(1-F_\varepsilon(w(\phi)-u))\phi(u,r)f_\mu(u,r)dudr=\pi(\phi).\end{aligned}$$

If, otherwise, $w^+(\phi)=-\infty$ (meaning that $w(\phi)=-\infty$ as well). Then we know that for any $M>0$, $\lim_{N\to\infty}\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N)>-M)=0$. Therefore

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N)&=\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\mathbb{P}(U_i>W_{(B+1)}(\boldsymbol{x}_N))x_{N,i}\\&\geq\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\big(\mathbb{P}(U_i>-M)-\mathbb{P}(W_{(B+1)}(\boldsymbol{x}_N)>-M)\big)x_{N,i}\\&=\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N r_i\mathbb{P}(U_i>-M)x_{N,i}\\&=\int_{\mathbb{R}^2}r(1-F_\varepsilon(-M-u))\phi(u,r)f_\mu(u,r)dudr\end{aligned}$$

Take $M\to\infty$, by the dominate convergence theorem we have

$$\liminf_{N\to\infty}\frac{1}{N}\pi_N(\boldsymbol{x}_N)\geq\int_{\mathbb{R}^2}r\phi(u,r)f_\mu(u,r)dudr=\pi(\phi),$$

where the last equality is because $w(\phi)=-\infty$.

□

EC.3. Proof of Theorem 2.

Proof of Theorem 2. We first show the equivalence of (12) and (13). Let $\{\boldsymbol{x}_N\}$ be a sequence of solutions and ϵ such that (13) holds; i.e.,

$$\limsup_{N\to\infty}\frac{1}{N}(\hat{\pi}_N(\boldsymbol{x}_N^{SP\star})-\hat{\pi}_N(\boldsymbol{x}_N))\leq\epsilon. \tag{EC.12}$$

We now show that (12) holds. The proof of the other direction is identical. Since $(1/N)\hat{\pi}_N(\boldsymbol{x}_N^{SP\star})\to\pi^\star$ by Proposition 2, it follows by (EC.12) and the fact that $\limsup_{N\to\infty}(1/N)(-\hat{\pi}_N(\boldsymbol{x}_N))=-\liminf_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{x}_N)$ that

$$\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N)\geq\pi^\star-\epsilon.$$

We prove by contradiction that $\liminf_{N\to\infty} \frac{1}{N}\pi_N(\boldsymbol{x}_N) \geq \pi^\star - \epsilon$, which, by Proposition 2, is equivalent to

$$\limsup_{N\to\infty} \frac{1}{N}(\pi_N(\boldsymbol{x}_N^{SP\star}) - \pi_N(\boldsymbol{x}_N)) \leq \epsilon.$$

Suppose that $\liminf_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N) < \pi^\star - \epsilon$. Then there is a subsequence $\{N'\} \subset \{N\}$ such that $\lim_{N\to\infty}(1/N')\pi_{N'}(\boldsymbol{x}_{N'}) = \liminf_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N) < \pi^\star - \epsilon$. By Lemma EC.5 and EC.6, we can find a further subsequence $\{N''\} \subset \{N'\}$ such that $\mu_{\boldsymbol{x}_{N''}}$ converges weakly to $\phi\mu$ for some $0 \leq \phi(x) \leq 1$. Lemma EC.7 then yields

$$\pi(\phi) = \lim_{N\to\infty} \frac{1}{N''}\hat{\pi}_{N''}(\boldsymbol{x}_{N''}) \geq \liminf_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N) \geq \pi^\star - \epsilon,$$
$$\pi(\phi) = \lim_{N\to\infty} \frac{1}{N''}\pi_{N''}(\boldsymbol{x}_{N''}) = \liminf_{N\to\infty} \frac{1}{N}\pi_N(\boldsymbol{x}_N) < \pi^\star - \epsilon,$$

which is a contradiction, implying that $\liminf_{N\to\infty}(1/N)\pi_N(\boldsymbol{x}_N) \geq \pi^\star - \epsilon$. Now we prove the equivalence of (12) and (14). Note that by Proposition 2, (14) is equivalent to

$$\pi^\star - \liminf_{N\to\infty} \frac{1}{N}\pi_N(\boldsymbol{x}_N)) \leq \epsilon \quad \text{and} \quad \pi^\star - \liminf_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N) \leq \epsilon,$$

where, by Proposition 2 again, the former is equivalent to (12) and the latter is equivalent to (13). We conclude by the fact that (12) and (13) are equivalent. □

EC.4. Proof of Proposition 4.

Recall from Lemma EC.4 that $\widetilde{\pi}^\star = \pi^\star$, and from (EC.1) that

$$w(\phi) = \inf\left\{w : \int_{\mathbb{R}^2}(1 - F_\varepsilon(w-u))\phi(u,r)f_\mu(u,r)dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}}(1 - F_\xi(w-v))f_\nu(v)dv \leq \beta\right\}.$$

The following auxiliary lemma is proved at the end of this section, after the proof of Proposition 4.

Lemma EC.10. *There exists an optimal solution $\phi^\star : \mathbb{R}^2 \to [0,1]$ to (RLOP). Namely, $\pi(\phi^\star) = \pi^\star$.*

Proof of Proposition 4. Let $\phi^\star : \mathbb{R}^2 \to [0,1]$ be an optimal solution to (RLOP) given by Lemma EC.10. We claim that any optimal solution (if it exists) to the optimization problem

$$\begin{aligned} \sup_{\phi:\mathbb{R}^2\to[0,1]} \quad & \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi^\star) - u))\phi(u,r)f_\mu(u,r)dudr \\ \text{s.t.} \quad & \int_{\mathbb{R}^2}(1 - F_\varepsilon(w(\phi^\star) - u))\phi(u,r)f_\mu(u,r)dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}}(1 - F_\xi(w(\phi^\star) - v))f_\nu(v)dv \leq \beta, \end{aligned} \tag{EC.13}$$

must be an optimal solution to (RLOP) as well. To see this, let $\varphi : \mathbb{R}^2 \to [0,1]$ be any optimal solution to (EC.13). Since it satisfies the constraint

$$\int_{\mathbb{R}^2}(1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r)f_\mu(u,r)dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}}(1 - F_\xi(w(\phi^\star) - v))f_\nu(v)dv \leq \beta,$$

the definition of $w(\varphi)$ implies that $w(\varphi) \le w(\phi^\star)$. Therefore

$$\begin{aligned}\pi^\star = \pi(\phi^\star) &\le \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r)\,dudr \\ &\le \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\varphi) - u))\varphi(u,r) f_\mu(u,r)\,dudr = \pi(\varphi) \le \pi^\star,\end{aligned}$$

where the second inequality is due to the monotonicity of $1 - F_\varepsilon(\cdot)$. Thus, φ is an optimal solution to (RLOP). Next, we show by contradiction that

$$\int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi^\star) - u))\phi^\star(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v)\,dv = \beta. \tag{EC.14}$$

If the equality in (EC.14) does not hold, then $w(\phi^\star) = -\infty$ (from the definition of $w(\phi)$) and, therefore,

$$\begin{aligned}&\int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi^\star) - u))\phi^\star(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v)\,dv \\ &= \int_{\mathbb{R}^2} \phi^\star(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma} < \beta\end{aligned} \tag{EC.15}$$

Then (EC.13) becomes

$$\begin{aligned}\sup_{\phi:\mathbb{R}^2 \to [0,1]} \quad & \int_{\mathbb{R}^2} r\phi(u,r) f_\mu(u,r)\,dudr \\ \text{s.t.} \quad & \int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma} \le \beta,\end{aligned} \tag{EC.16}$$

and combining the inequality in (EC.15) with the fact that $\beta \le 1$ yields $\int_{\mathbb{R}^2} \big(1 - \phi^\star(u,r)\big) f_\mu(u,r)\,dudr > 0$, and, therefore,

$$\int_{\mathbb{R}^2} r\big(1 - \phi^\star(u,r)\big) f_\mu(u,r)\,dudr > 0. \tag{EC.17}$$

For any $\alpha > 0$ consider $\varphi_\alpha = \frac{\phi^\star + \alpha}{1+\alpha}$ as a function from $\mathbb{R}^2$ to $[0,1]$. Then

$$\begin{aligned}\int_{\mathbb{R}^2} r\varphi_\alpha(u,r) f_\mu(u,r)\,dudr &= \frac{1}{1+\alpha}\pi(\phi^\star) + \frac{\alpha}{1+\alpha}\int_{\mathbb{R}^2} r f_\mu(u,r)\,dudr \\ &> \frac{1}{1+\alpha}\pi(\phi^\star) + \frac{\alpha}{1+\alpha}\int_{\mathbb{R}^2} r\phi^\star(u,r) f_\mu(u,r)\,dudr = \pi(\phi^\star),\end{aligned} \tag{EC.18}$$

where the inequality is due to (EC.17). Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi_\alpha(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma} &= \frac{1}{1+\alpha}\int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma} + \frac{\alpha}{1+\alpha} \\ &\to \int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r)\,dudr + \frac{\beta}{\gamma} < \beta.\end{aligned}$$

where the convergence is as $\alpha \downarrow 0$. This means that there exists $\alpha_0 > 0$ such that φ_{α_0} is feasible to (EC.16), but achieves a higher objective value than $\phi^\star$, leading to the contradiction and concluding that (EC.14) holds.

We now construct an optimal solution to (EC.13). Define

$$r_{LOP} = \inf\left\{r' \geq 0 : \int_{r\geq r'} (1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v) dv \leq \beta\right\}$$

and $A^\star = \{(u,r) : r \geq r_{LOP}\}$. We know that r_{LOP} is well defined (although it could be infinite) because

$$\begin{aligned}
&\lim_{r'\to\infty} \int_{r\geq r'} (1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v) dv \\
&= \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v) dv \leq \beta
\end{aligned}$$

where the inequality is due to (EC.3). Since $\mathbb{1}_{A^\star}(u,r)$ is a feasible solution to (EC.13), we now argue that it is also an optimal solution by showing that it maximizes the respective objective value. Note that if $r_{LOP} = +\infty$, then A^* is an empty set, so that $\mathbb{1}_{r>c}(u,r)$ is optimal to (LOP) for any $c \geq 0$. Given any feasible solution $\varphi : \mathbb{R}^2 \to [0,1]$, note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr \\
&\leq \int_{r\geq r_{LOP}} r(1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr + r_{LOP}\int_{r<r_{LOP}} (1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr \\
&\leq \int_{r\geq r_{LOP}} r(1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr \\
&\quad + r_{LOP}\left(\beta - \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v) dv - \int_{r\geq r_{LOP}} (1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr\right),
\end{aligned} \tag{EC.19}$$

where the last inequality follows from the constraint in (EC.13). By (EC.14), the equality in the definition of r_{LOP} can be achieved, i.e.,

$$\beta - \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi^\star) - v)) f_\nu(v) dv = \int_{r\geq r_{LOP}} (1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr,$$

meaning that the right-hand side of (EC.19) equals

$$\begin{aligned}
&\int_{r\geq r_{LOP}} r(1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr \\
&\quad + r_{LOP}\left(\int_{r\geq r_{LOP}} (1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr - \int_{r\geq r_{LOP}} (1 - F_\varepsilon(w(\phi^\star) - u))\varphi(u,r) f_\mu(u,r) du dr\right) \\
&= \int_{r\geq r_{LOP}} (r\varphi(u,r) + r_{LOP} - r_{LOP}\varphi(u,r))(1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr \\
&\leq \int_{r\geq r_{LOP}} r(1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr = \int_{A^\star} r(1 - F_\varepsilon(w(\phi^\star) - u)) f_\mu(u,r) du dr,
\end{aligned}$$

where the last inequality is because when $r \geq r_{LOP}$,

$$r\varphi_1(u,r) + r_{LOP} - r_{LOP}\varphi_1(u,r) = (r - r_{LOP})\varphi_1(u,r) + r_{LOP} \leq (r - r_{LOP}) + r_{LOP} = r.$$

Thus, $\mathbb{1}_{A^\star}(u,r)$ is an optimal solution to (EC.13), implying that $\pi(A^\star) = \widetilde{\pi}^\star = \pi^\star$, where the last equality is due to Lemma EC.4. Let $\boldsymbol{y}_N$ be the policy of offering all items $i \in [N]$ such that $r_i \geq r_{LOP}$. Recalling the definitions of $\mu_{\boldsymbol{y}_N}$ and $\mathbb{1}_{A^\star}\mu$ from the discussion preceding (EC.7), since $A^\star$ is a continuity set with respect to the measure induced by F_μ, Lemma EC.1 implies that $\mu_{\boldsymbol{y}_N} \Rightarrow \mathbb{1}_{A^\star}\mu$. Then, $\lim_{N\to\infty}(1/N)\pi_N(\boldsymbol{y}_N) = \pi(A^\star) = \pi^\star$ by Lemma EC.7. □

Proof of Lemma EC.10. Let $\boldsymbol{x}_N$ be the optimal solution to (OP). It follows from Lemmas EC.5 and EC.6 that there exists a subsequence $\{\boldsymbol{x}_{N'}\}$ and $\phi : \mathbb{R}^2 \to [0,1]$ such that $\mu_{\boldsymbol{x}_{N'}} \Rightarrow \phi\mu$. We now show that $\phi(x)$ is an optimal solution to (RLOP). Note that

$$\int_{\mathbb{R}^2} \phi(u,r) f_\mu(u,r)dudr = \lim_{N\to\infty} \frac{1}{N'} \sum_{i=1}^{N'} \boldsymbol{x}_{N',i} = \lim_{N\to\infty} \frac{1}{N'} ||\boldsymbol{x}_{N'}||_1 \leq \eta,$$

where the first equality is due to $\mu_{\boldsymbol{x}_{N'}} \Rightarrow \phi\mu$ and the last equality is due to the feasibility of $\boldsymbol{x}_{N'}$, implying that $\phi(x)$ is feasible to (RLOP). Furthermore,

$$\pi(\phi) = \lim_{N\to\infty} \frac{1}{N} \pi_N(\boldsymbol{x}_{N'}) = \pi^\star = \widetilde{\pi}^\star,$$

where the first equality follows from Lemma EC.7, the second equality from Proposition 2, and the third equality from Lemma EC.4, implying the optimality of $\phi(x)$. □

EC.5. Proof of Proposition 3.

In this section we use $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ to denote elements of $\{0,1\}^N$ and $[0,1]^N$, respectively. Define the set of feasible solutions to RSP(w) by

$$\mathcal{F}_N(w) = \left\{ \overline{\boldsymbol{x}} \in [0,1]^N : \sum_{i=1}^{N} \overline{x}_i \big(1 - F_\varepsilon(w - u_i)\big) + \sum_{j=1}^{N_0} \big(1 - F_\xi(w - v_j)\big) \leq B, \; \sum_{i=1}^{N} \overline{x}_i \leq \eta N \right\}. \quad \text{(EC.20)}$$

Recall the extension of $\hat{\pi}_N(\overline{\boldsymbol{x}}), \hat{\pi}_N(\overline{\boldsymbol{x}}, w)$ and $w_{(B+1)}(\overline{\boldsymbol{x}})$ to continuous $\overline{\boldsymbol{x}} \in [0,1]^N$ defined in Section EC.1.2. Also recall that the optimal objective value of RSP(w) is maximized when $w = w_N^{RSP\star}$, and define $\overline{\boldsymbol{x}}_N^{RSP\star} \in [0,1]^N$ as any vector that satisfies

$$(\overline{\boldsymbol{x}}_N^{RSP\star}, w_N^{RSP\star}) \in \operatorname*{arg\,max}_{\substack{w\in\mathbb{R} \\ \overline{\boldsymbol{x}} \in \mathcal{F}_N(w)}} \hat{\pi}_N(\overline{\boldsymbol{x}}, w). \quad \text{(EC.21)}$$

We need two auxiliary results. Lemma EC.11 is proved at the end of this section.

LEMMA EC.11. *Under Assumptions 1–3,*

$$\lim_{N\to\infty} \frac{1}{N} \hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}, w_N^{RSP\star}) = \lim_{N\to\infty} \frac{1}{N} \hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}) = \pi^\star.$$

For $w \in \mathbb{R}$, define

$$\pi(\phi, w) = \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w-u))\phi(u,r) f_\mu(u,r) du dr.$$

This is in contrast to the objective value $\pi(\phi)$ to (RLOP) that we have been using, which satisfies $\pi(\phi) = \pi(\phi, w(\phi))$. The following lemma, whose proof is detailed in Section EC.7.2, is analogous to Lemma EC.8.

LEMMA EC.12. *Given a feasible solution $\phi : \mathbb{R}^2 \to [0,1]$ to (RLOP) such that $\pi(\phi) > 0$, and any $\epsilon > 0$, there exists an F_μ-continuity set $A \in \mathcal{B}(\mathbb{R}^2)$ such that $\mathbb{1}_A$ is an interior solution to (RLOP); i.e.,*

$$\int_A f_\mu(u,r) du dr < \eta.$$

Furthermore,

$$\int_A (1 - F_\varepsilon(w(\phi) - u)) f_\mu(u,r) du dr + \frac{\beta}{\gamma} \cdot \int_{\mathbb{R}} (1 - F_\xi(w(\phi) - v)) f_\nu(v) dv < \beta,$$

and $\pi(\mathbb{1}_A, w^{++}(\phi)) \geq \pi(\phi) - \epsilon$, where $w^{++}(\phi)$ is defined by

$$w^{++}(\phi) = \sup\left\{ w \geq w(\phi) : \int_{\mathbb{R}} r(1 - F_\varepsilon(w-u))\phi(u,r) f_\mu(u,r) du dr = \pi(\phi) \right\}.$$

Proof of Proposition 3. Recall that

$$\boldsymbol{x}_N \in \operatorname*{arg\,max}_{\boldsymbol{x} \in \{0,1\}^N \cap \mathcal{F}_N(w_N^{RSP\star})} \hat{\pi}_N(\boldsymbol{x}, w_N^{RSP\star})$$

is the optimal solution to SP($w_N^{RSP\star}$). Theorem 2 with $\epsilon = 0$ implies that

$$\limsup_{N\to\infty} \frac{1}{N}(\hat{\pi}_N(\boldsymbol{x}_N^{SP\star}) - \hat{\pi}_N(\boldsymbol{x}_N)) = 0 \text{ if and only if } \limsup_{N\to\infty} \frac{1}{N}(\pi_N(\boldsymbol{x}_N^{OP\star}) - \pi_N(\boldsymbol{x}_N)) = 0,$$

and since both $(1/N)\pi_N(\boldsymbol{x}_N^{OP\star}) \to \pi^\star$ and $(1/N)\hat{\pi}_N(\boldsymbol{x}_N^{SP\star}) \to \pi^\star$ by Proposition 2, to prove that $(1/N)\pi(\boldsymbol{x}_N) \to \pi^\star$, it suffices to show that $(1/N)\hat{\pi}_N(\boldsymbol{x}_N) \to \pi^\star$. Since

$$0 \leq \limsup_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N) \leq \limsup_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N^{SP\star}) = \pi^\star,$$

Proposition 3 follows trivially if $\pi^\star = 0$. So going forward we assume that $\pi^\star > 0$.

We now show that for any subsequence $\{N'\} \subset \{N\}$, there exists another subsequence $\{N''\} \subset \{N'\}$ such that $(1/N)\hat{\pi}_N(\boldsymbol{x}_{N''}) \to \pi^\star$, which implies the result. Fix $\{N''\} \subset \{N'\}$ and recall $\overline{\boldsymbol{x}}_N^{RSP\star}$ defined in (EC.21). By Lemmas EC.5 and EC.6, we may choose $\{N''\} \subset \{N'\}$ such that $\mu_{\overline{\boldsymbol{x}}_{N''}^{RSP\star}} \Rightarrow \phi\mu$ for some $\phi : \mathbb{R}^2 \to [0,1]$. We now argue that $(1/N'')\hat{\pi}(\boldsymbol{x}_{N''}) \to \pi^\star$. Going forward we write N in place of N''.

Since $\limsup_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{x}_N)\le \pi^\star$ due to Lemmas EC.2 and EC.4, we need only to show that $\liminf_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{x}_N)\ge \pi^\star$. Note that

$$\pi(\phi)=\lim_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star})=\pi^\star, \tag{EC.22}$$

where the first equality follows from $\mu_{\overline{\boldsymbol{x}}_N^{RSP\star}}\Rightarrow \phi\mu$ and Lemma EC.7, and the second from Lemma EC.11. Furthermore, we claim that for any $\epsilon>0$, there exists $\{\boldsymbol{y}_N\in\{0,1\}^N\}_{N=1}^\infty$ such that $\boldsymbol{y}_N$ is feasible to SP$(w_N^{RSP\star})$ for large-enough N and that $\liminf_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{y}_N,w_N^{RSP\star})\ge\pi(\phi)-\epsilon$. Given this claim, which we verify later, we now show that $\liminf_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{x}_N)\ge\pi^\star$.

Since $\boldsymbol{x}_N$ is a feasible solution to SP$(w_N^{RSP\star})$, it follows that

$$\sum_{i=1}^{N}\big(1-F_\varepsilon(w_N^{RSP\star}-u_i)\big)x_{N,i}+\sum_{j=1}^{N_0}\big(1-F_\xi(w_N^{RSP\star}-v_j)\big)\le B,$$

which, when combined with the definition of $w_{(B+1)}(\boldsymbol{x})$ in (2), implies that $w_N^{RSP\star}\ge w_{(B+1)}(\boldsymbol{x}_N)$. Therefore, for any $\epsilon>0$, we have

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N)&=\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N,w_{(B+1)}(\boldsymbol{x}_N))\ge\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N,w_N^{RSP\star})\\&\ge\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{y}_N,w_N^{RSP\star})\ge\pi(\phi)-\epsilon=\pi^\star-\epsilon.\end{aligned}$$

The first inequality follows because $\hat{\pi}_N(\boldsymbol{x}_N,w)=\sum_{i=1}^N r_i x_{N,i}(1-F_\varepsilon(w-u_i))$ is nonincreasing in w, the second inequality is because $\boldsymbol{x}_N$ is optimal to SP$(w_N^{RSP\star})$ and $\boldsymbol{y}_N$ is feasible to SP$(w_N^{RSP\star})$ for large-enough N, the third inequality is due to the claim we made, and the last equality follows from (EC.22). Since $\epsilon>0$ is arbitrary, we conclude that $\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{x}_N)\ge\pi^\star$.

We now prove our claim that for any $\epsilon>0$, there exists $\{\boldsymbol{y}_N\in\{0,1\}^N\}_{N=1}^\infty$ such that $\boldsymbol{y}_N$ is feasible to SP$(w_N^{RSP\star})$ for large-enough N and $\liminf_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{y}_N,w_N^{RSP\star})\ge\pi(\phi)-\epsilon$. By Lemma EC.12, we can find an F_μ-continuity set $A\in\mathcal{B}(\mathbb{R}^2)$ such that

$$\int_A(1-F_\varepsilon(w(\phi)-u))f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi)-v))f_\nu(v)dv<\beta, \tag{EC.23}$$

$$\int_A f_\mu(u,r)dudr<\eta, \tag{EC.24}$$

$$\pi(\mathbb{1}_A,w^{++}(\phi))\ge\pi(\phi)-\epsilon=\pi^\star-\epsilon, \tag{EC.25}$$

where the last equality follows from (EC.22). Defining

$$\boldsymbol{y}_N=\big(\mathbb{1}_A(u_1,r_1),\dots,\mathbb{1}_A(u_N,r_N)\big),$$

we first show that $\boldsymbol{y}_N$ is feasible to SP$(w_N^{RSP\star})$ for large-enough N. Since $\overline{\boldsymbol{x}}_N^{RSP\star}\in\mathcal{F}_N(w_N^{RSP\star})$, we have

$$\sum_{i=1}^{N}\big(1-F_\varepsilon(w_N^{RSP\star}-u_i)\big)\overline{x}_{N,i}^{RSP\star}+\sum_{j=1}^{N_0}\big(1-F_\xi(w_N^{RSP\star}-v_j)\big)\le B.$$

It follows that

$$\liminf_{N\to\infty} w_N^{RSP\star} \geq \liminf_{N\to\infty} w_{(B+1)}(\overline{\boldsymbol{x}}_N^{RSP\star}) \geq w(\phi),$$

where the first inequality is due to the definition of $w_{(B+1)}(\overline{\boldsymbol{x}})$ in (EC.5), and the second follows from Lemma EC.7 and the fact that $\mu_{\overline{\boldsymbol{x}}_N^{RSP\star}} \Rightarrow \phi\mu$. As a result, for any $\delta > 0$, $w_N^{RSP\star} \geq w(\phi) - \delta$ for large-enough N. With the monotonicity of $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$, we see that for any $\delta > 0$,

$$\begin{aligned}
&\limsup_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(1-F_\varepsilon(w_N^{RSP\star}-u_i))y_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0}(1-F_\xi(w_N^{RSP\star}-v_j)) - \frac{B}{N}\right] \\
\leq\ &\limsup_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(1-F_\varepsilon(w(\phi)-\delta-u_i))y_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0}(1-F_\xi(w(\phi)-\delta-v_j)) - \frac{B}{N}\right],
\end{aligned}$$

where $y_{N,i}$ is the ith element of $\boldsymbol{y}_N$. Since A is a F_μ-continuity set, Lemma EC.1 yields $\mu_{\boldsymbol{y}_N} \Rightarrow \mathbb{1}_A\mu$. Together with $\nu_{N_0} \Rightarrow \nu$ by (EC.6), the right-hand side equals

$$\int_A (1-F_\varepsilon(w(\phi)-\delta-u))f_\mu(u,r)dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi)-\delta-v))f_\nu(v)dv - \beta.$$

Taking $\delta \to 0$, the dominated convergence theorem implies that

$$\begin{aligned}
&\limsup_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(1-F_\varepsilon(w_N^{RSP\star}-u_i))y_{N,i} + \frac{1}{N}\sum_{j=1}^{N_0}(1-F_\xi(w_N^{RSP\star}-v_j)) - \frac{B}{N}\right] \\
\leq\ &\int_A (1-F_\varepsilon(w(\phi)-u))f_\mu(u,r)dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi)-v))f_\nu(v)dv - \beta < 0,
\end{aligned} \tag{EC.26}$$

where the strict inequality follows from (EC.23). Similarly, by (EC.24) and the fact that $\mu_{\boldsymbol{y}_N} \Rightarrow \mathbb{1}_A\mu$, we have

$$\limsup_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N} y_{N,i} - \eta\right] = \int_A f_\mu(u,r)dudr - \eta < 0. \tag{EC.27}$$

Combining (EC.26) and (EC.27) implies that $\boldsymbol{y}_N$ is feasible to SP$(w_N^{RSP\star})$ for large-enough N. Finally, we show that $\liminf_{N\to\infty}(1/N)\hat{\pi}_N(\boldsymbol{y}_N, w_N^{RSP\star}) \geq \pi^\star - \epsilon$. By (EC.25), it is sufficient to show that

$$\liminf_{N\to\infty}\frac{1}{N}\hat{\pi}_N(\boldsymbol{y}_N, w_N^{RSP\star}) \geq \pi(\mathbb{1}_A, w^{++}(\phi)).$$

We first prove that $\limsup_{N\to\infty} w_N^{RSP\star} \leq w^{++}(\phi)$ by contradiction. Assume the opposite is true. Then there exists a subsequence $\{N'\} \subset \{N\}$ such that $w_{N'}^{RSP\star} > w^{++}(\phi) + \delta$ for some fixed $\delta > 0$ and any N. Since $\pi(\phi) = \pi^\star$ from (EC.22), then

$$\begin{aligned}
\pi(\phi) = \pi^\star &= \lim_{N\to\infty}\frac{1}{N'}\hat{\pi}_{N'}(\overline{\boldsymbol{x}}_{N'}^{RSP\star}, w_{N'}^{RSP\star}) \\
&= \lim_{N\to\infty}\frac{1}{N'}\sum_{i=1}^{N'} r_i(1-F_\varepsilon(w_{N'}^{RSP\star}-u_i))\overline{x}_{N',i}^{RSP\star} \\
&\leq \limsup_{N\to\infty}\frac{1}{N'}\sum_{i=1}^{N'} r_i(1-F_\varepsilon(w^{++}(\phi)+\delta-u_i))\overline{x}_{N',i}^{RSP\star},
\end{aligned}$$

where the second equality is due to Lemma EC.11. Since $\mu_{\overline{\boldsymbol{x}}_{N'}^{RSP\star}} \Rightarrow \phi\mu$ and $r \le r_{\max}$, the right-hand side equals

$$\int_{\mathbb{R}^2} r(1-F_\varepsilon(w^{++}(\phi)+\delta-u))\phi(u,r)f_\mu(u,r)dudr < \pi(\phi),$$

where the strict inequality is due to the definition of $w^{++}(\phi)$ in Lemma EC.12. Thus we have reached a contradiction. Now since $\limsup_{N\to\infty} w_N^{RSP\star} \le w^{++}(\phi)$, we know that for any $\delta > 0$, $w_N^{RSP\star} \le w^{++}(\phi)+\delta$ for large-enough N. Therefore

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\boldsymbol{y}_N, w_N^{RSP\star}) &= \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^N r_i(1-F_\varepsilon(w_N^{RSP\star}-u_i))y_{N,i} \\
&\ge \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^N r_i(1-F_\varepsilon(w^{++}(\phi)+\delta-u_i))y_{N,i} \\
&= \int_A r(1-F_\varepsilon(w^{++}(\phi)+\delta-u))f_\mu(u,r)dudr.
\end{aligned}$$

where the last equality is due to $\mu_{\boldsymbol{y}_N} \Rightarrow \mathbb{1}_A\mu$, and $r \le r_{\max}$. Taking $\delta \to 0$ and applying the dominated convergence theorem (which we can apply because $r \le r_{\max}$), the limit of the right-hand side equals $\pi(\mathbb{1}_A, w^{++}(\phi))$. $\square$

Proof of Lemma EC.11. Recall that $\boldsymbol{x}_N^{SP\star}$ is the optimal solution to (SP). By the definition of $(\overline{\boldsymbol{x}}_N^{RSP\star}, w_N^{RSP\star})$ in (EC.21), we have

$$\hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}, w_N^{RSP\star}) \ge \max_{\substack{w\in\mathbb{R} \\ \boldsymbol{x}\in\{0,1\}^N \cap \mathcal{F}_N(w)}} \hat{\pi}_N(\boldsymbol{x}, w) = \hat{\pi}(\boldsymbol{x}_N^{SP\star}, w_{(B+1)}(\boldsymbol{x}_N^{SP\star})) = \hat{\pi}_N(\boldsymbol{x}_N^{SP\star}).$$

Furthermore, recalling the definition of $w_{(B+1)}(\overline{\boldsymbol{x}})$ in (9), it follows from $\overline{\boldsymbol{x}}_N^{RSP\star} \in \mathcal{F}_N(w_N^{RSP\star})$ that $w_{(B+1)}(\overline{\boldsymbol{x}}_N^{RSP\star}) \le w_N^{RSP\star}$. Since $\hat{\pi}(\overline{\boldsymbol{x}}, w)$ is nonincreasing in w, we have

$$\hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}, w_N^{RSP\star}) \le \hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}, w_{(B+1)}(\overline{\boldsymbol{x}}_N^{RSP\star})) = \hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}),$$

implying that

$$\hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}) \ge \hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}, w_N^{RSP\star}) \ge \hat{\pi}_N(\boldsymbol{x}_N^{SP\star}).$$

Since $\hat{\pi}_N(\boldsymbol{x}_N^{SP\star}) \to \pi^\star$ by Proposition 2, it suffices to prove that $\limsup_{N\to\infty} \frac{1}{N}\hat{\pi}_N(\overline{\boldsymbol{x}}_N^{RSP\star}) \le \pi^\star$. By Lemmas EC.5 and EC.6, any subsequence $\{N'\} \subset \{N\}$ has a further subsequence $\{N''\} \subset \{N'\}$ such that $\mu_{\overline{\boldsymbol{x}}_{N''}^{RSP\star}} \Rightarrow \phi\mu$ for some $\phi : \mathbb{R}^2 \to [0,1]$, and therefore

$$\int_{\mathbb{R}^2} \phi(u,r)f_\mu(u,r)dudr = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \overline{x}_{N,i}^{RSP\star} \le \eta,$$

meaning that ϕ is feasible to (RLOP), where the inequality is because $\overline{\boldsymbol{x}}_N^{RSP\star} \in \mathcal{F}_N(w_N^{RSP\star})$. It then follows from Lemma EC.7 that $\lim_{N\to\infty}(1/N'')\hat{\pi}_{N''}(\overline{\boldsymbol{x}}_{N''}^{RSP\star}) = \pi(\phi) \le \pi^\star$. $\square$

EC.6. Proof of Proposition 5.

We require an auxiliary lemmas, which is proved at the end of this section after we prove Proposition 5.

Lemma EC.13. *Consider the relaxed knapsack-type problem*

$$\max_{\boldsymbol{x}\in[0,1]^N} \sum_{i=1}^{N} p_i x_i \quad s.t. \quad \sum_{i}^{N} q_i x_i \le b, \quad \sum_{i=1}^{N} x_i \le s.$$

where $p_i, q_i > 0$ *for* $i = 1, 2, ..., N$*. Then there exists an optimal solution* $\boldsymbol{x}^\star$ *such that* $\big|\{i \in [N] : 0 < x_i^\star < 1\}\big| \le 2$.

Proof of Proposition 5 Throughout the proof we omit the subscript N for convenience. We first prove (18). Note that

$$\begin{aligned} 0 &\le \pi(\boldsymbol{x}^{OP\star}) - \pi(\boldsymbol{x}^{SP\star}) \\ &= \pi(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{OP\star}) + \hat{\pi}(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{SP\star}) + \hat{\pi}(\boldsymbol{x}^{SP\star}) - \pi(\boldsymbol{x}^{SP\star}), \end{aligned} \tag{EC.28}$$

where the inequality follows from the optimality of $x^{OP\star}$. The second difference, $\hat{\pi}(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{SP\star})$, is nonpositive by the optimality of $\hat{\boldsymbol{x}}^{SP\star}$. The result then follows once we show that

$$|\pi(\boldsymbol{x}) - \hat{\pi}(\boldsymbol{x})| \le r_{\max}\sqrt{B},$$

for any assortment $\boldsymbol{x}$. Recall that $W(\boldsymbol{x}) = \big\{U_i : x_i = 1, i \in [N]\big\} \cup \big\{V_j : j \in [N_0])\big\}$, let $S(\boldsymbol{x}) = \{i \in [N] : x_i = 1\}$, and define the random quantities

$$\mathcal{P}(\boldsymbol{x}) = \{i \in S(\boldsymbol{x}) \cup \{N+1, ..., N+N_0\} : W_i > W_{(B+1)}(\boldsymbol{x})\} \quad \text{and} \quad \Pi(\boldsymbol{x}) = \sum_{i=1}^{N} r_i \mathbb{1}\{i \in \mathcal{P}(\boldsymbol{x})\},$$

$$\hat{\mathcal{P}}(\boldsymbol{x}) = \{i \in S(\boldsymbol{x}) \cup \{N+1, ..., N+N_0\} : W_i > w_{(B+1)}(\boldsymbol{x})\} \quad \text{and} \quad \hat{\Pi}(\boldsymbol{x}) = \sum_{i=1}^{N} r_i \mathbb{1}\{i \in \hat{\mathcal{P}}(\boldsymbol{x})\}.$$

It follows from (OP) and (SP) that $\pi(\boldsymbol{x}) = \mathbb{E}(\Pi(\boldsymbol{x}))$ and $\hat{\pi}(\boldsymbol{x}) = \mathbb{E}(\hat{\Pi}(\boldsymbol{x}))$, so to bound $\pi(\boldsymbol{x}) - \hat{\pi}(\boldsymbol{x})$, we analyze $\Pi(\boldsymbol{x}) - \hat{\Pi}(\boldsymbol{x})$. Letting $A\Delta B = (A\backslash B) \cup (B\backslash A)$ be the symmetric difference of two sets A and B, it follows that

$$|\Pi(\boldsymbol{x}) - \hat{\Pi}(\boldsymbol{x})| = \sum_{i=1}^{N} r_i \mathbb{1}\{i \in \mathcal{P}(\boldsymbol{x})\Delta\hat{\mathcal{P}}(\boldsymbol{x})\} \le r_{\max} \cdot \big|\mathcal{P}(\boldsymbol{x})\Delta\hat{\mathcal{P}}(\boldsymbol{x})\big|, \tag{EC.29}$$

where $|\cdot|$ denotes both the absolute value and cardinality of a set. Note that by definition, since $\mathcal{P}(\boldsymbol{x})$ contains all the i such that $W_i > W_{(B+1)}(\boldsymbol{x})$, and $\hat{\mathcal{P}}(\boldsymbol{x})$ contains all the i such that $W_i > w_{(B+1)}(\boldsymbol{x})$, we must have $\mathcal{P}(\boldsymbol{x}) \supset \hat{\mathcal{P}}(\boldsymbol{x})$ if $w_{(B+1)}(\boldsymbol{x}) \ge W_{(B+1)}(\boldsymbol{x})$, or $\mathcal{P}(\boldsymbol{x}) \subset \hat{\mathcal{P}}(\boldsymbol{x})$ otherwise. Therefore

$$\big|\mathcal{P}(\boldsymbol{x})\Delta\hat{\mathcal{P}}(\boldsymbol{x})\big| = \big||\hat{\mathcal{P}}(\boldsymbol{x})| - |\mathcal{P}(\boldsymbol{x})|\big|.$$

Note that $|\mathcal{P}(\boldsymbol{x})| = \min(B, \|\boldsymbol{x}\|_1 + N_0)$, because $W_{(B+1)}(\boldsymbol{x})$ is the $(B+1)$st order statistic. In addition, we have

$$|\hat{\mathcal{P}}(\boldsymbol{x})| = \sum_{i=1}^{N} \mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\} x_i + \sum_{j=1}^{N_0} \mathbb{1}\{W_{j+N} > w_{(B+1)}(\boldsymbol{x})\},$$

and by the definition of $w_{(B+1)}(\boldsymbol{x})$ in (2),

$$\mathbb{E}|\hat{\mathcal{P}}(\boldsymbol{x})| = \sum_{i=1}^{N} \mathbb{P}(W_i > w_{(B+1)}(\boldsymbol{x})) x_i + \sum_{j=1}^{N_0} \mathbb{P}(W_{j+N} > w_{(B+1)}(\boldsymbol{x})) = \min(B, \|\boldsymbol{x}\|_1 + N_0).$$

Combined with (EC.29) this yields

$$|\pi(\boldsymbol{x}) - \hat{\pi}(\boldsymbol{x})| = \mathbb{E}\big(\Pi(\boldsymbol{x}) - \hat{\Pi}(\boldsymbol{x})\big) \le r_{\max} \mathbb{E}\big||\hat{\mathcal{P}}(\boldsymbol{x})| - |\mathcal{P}(\boldsymbol{x})|\big| = r_{\max} \mathbb{E}\big||\hat{\mathcal{P}}(\boldsymbol{x})| - \mathbb{E}|\hat{\mathcal{P}}(\boldsymbol{x})|\big|. \tag{EC.30}$$

Since $\mathbb{E}|X - \mathbb{E}X| \le \sqrt{\mathbb{E}|X - \mathbb{E}X|^2} = \sqrt{\mathrm{Var}(X)}$, it follows that the right-hand side is bounded by

$$\begin{aligned} r_{\max} \cdot \sqrt{\mathrm{Var}(|\hat{\mathcal{P}}(\boldsymbol{x})|)} &= r_{\max} \cdot \sqrt{\sum_{i=1}^{N} \mathrm{Var}(\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\}) x_i + \sum_{j=1}^{N_0} \mathrm{Var}(\mathbb{1}\{W_{j+N} > w_{(B+1)}(\boldsymbol{x})\})} \\ &\le r_{\max} \cdot \sqrt{\sum_{i=1}^{N} \mathbb{P}(W_i > w_{(B+1)}(\boldsymbol{x})) x_i + \sum_{j=1}^{N_0} \mathbb{P}(W_{j+N} > w_{(B+1)}(\boldsymbol{x}))} \\ &= r_{\max} \cdot \sqrt{\min(B, \|\boldsymbol{x}\|_1 + N_0)} \le r_{\max} \cdot \sqrt{B}. \end{aligned}$$

where the last inequality is because $\mathrm{Var}(\mathbb{1}(A)) = \mathbb{P}(A)(1 - \mathbb{P}(A)) \le \mathbb{P}(A)$ for any measurable set A, and the last equality is from the definition of $w_{(B+1)}(\boldsymbol{x})$ in (2).

Having proved (18), we now prove (19). Recall that $w^{RSP\star}$ is the threshold that maximizes $\mathrm{RSP}(w)$; see the discussion around (RSP). By Lemma EC.13, we can find

$$\overline{\boldsymbol{y}} \in \operatorname*{arg\,max}_{\overline{\boldsymbol{x}} \in \mathcal{F}(w^{RSP\star})} \hat{\pi}(\overline{\boldsymbol{x}}, w^{RSP\star}). \tag{EC.31}$$

such that $|i \in [N] : 0 < \overline{y}_i < 1| \le 2$, where $\mathcal{F}(\cdot)$ is defined in (EC.20). Define $\boldsymbol{z} \in \{0,1\}^N$ by $z_i := \lfloor \overline{y}_i \rfloor$, $i \in [N]$. Since (RSP) is a relaxation of (SP) and $\overline{\boldsymbol{y}}$ is an optimal solution to the fixed-threshold relaxation at $w^{RSP\star}$, we have $\hat{\pi}(\boldsymbol{x}^{SP\star}) \le \hat{\pi}(\overline{\boldsymbol{y}}, w^{RSP\star})$. Moreover, since $\boldsymbol{z} \in \mathcal{F}(w^{RSP\star})$, the optimality of $\boldsymbol{x}^{RSP\star}$ in the binary fixed-threshold problem implies $\hat{\pi}(\boldsymbol{z}, w^{RSP\star}) \le \hat{\pi}(\boldsymbol{x}^{RSP\star}, w^{RSP\star})$. Finally, since $\boldsymbol{x}^{RSP\star} \in \mathcal{F}(w^{RSP\star})$, we have $w_{(B+1)}(\boldsymbol{x}^{RSP\star}) \le w^{RSP\star}$, and hence, by the monotonicity of $\hat{\pi}(\boldsymbol{x}, w)$ in w, $\hat{\pi}(\boldsymbol{x}^{RSP\star}, w^{RSP\star}) \le \hat{\pi}(\boldsymbol{x}^{RSP\star})$. Therefore, since $\boldsymbol{x}^{RSP\star}$ is feasible for (SP), $\boldsymbol{z} \le \overline{\boldsymbol{y}}$, and $\|\boldsymbol{z} - \overline{\boldsymbol{y}}\|_1 \le 2$, we obtain

$$0 \le \hat{\pi}(\boldsymbol{x}^{SP\star}) - \hat{\pi}(\boldsymbol{x}^{RSP\star}) \le \hat{\pi}(\overline{\boldsymbol{y}}, w^{RSP\star}) - \hat{\pi}(\boldsymbol{x}^{RSP\star}, w^{RSP\star}) \le \hat{\pi}(\overline{\boldsymbol{y}}, w^{RSP\star}) - \hat{\pi}(\boldsymbol{z}, w^{RSP\star}) \le 2 r_{\max}.$$

Proceeding as in (EC.28), we have

$$
\begin{aligned}
0 \leq \pi(\boldsymbol{x}^{OP\star}) - \pi(\boldsymbol{x}^{RSP\star}) = {} & \pi(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{OP\star}) + \hat{\pi}(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{RSP\star}) + \hat{\pi}(\boldsymbol{x}^{RSP\star}) - \pi(\boldsymbol{x}^{RSP\star}) \\
& \leq 2r_{\max}\sqrt{B} + \hat{\pi}(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{RSP\star}) \\
& = 2r_{\max}\sqrt{B} + \hat{\pi}(\boldsymbol{x}^{OP\star}) - \hat{\pi}(\boldsymbol{x}^{SP\star}) + \hat{\pi}(\boldsymbol{x}^{SP\star}) - \hat{\pi}(\boldsymbol{x}^{RSP\star}) \leq 2r_{\max}\sqrt{B} + 2r_{\max}.
\end{aligned}
$$

□

To conclude the section, we now prove Lemma EC.13.

Proof of Lemma EC.13. Let $P = \{\, x \in [0,1]^N : \sum_{i=1}^N q_i x_i \leq b,\ \sum_{i=1}^N x_i \leq s\}$. Since the objective function $\sum_{i=1}^N p_i x_i$ is linear, the fundamental theorem of linear programming ensures the existence of an optimal solution $x^\star$ at some vertex of P. Furthermore, there are precisely N linearly independent constraints active at $x^\star$; see, for instance, Theorem 3.4 of Vanderbei (2015). In our case, P is defined by $2N+2$ inequalities: $0 \leq x_i \leq 1$ for $i = 1, \ldots, N$, $\sum_{i=1}^N q_i x_i \leq b$, and $\sum_{i=1}^N x_i \leq s$. At a vertex, exactly N of these must be tight and independent. We conclude that at least $N-2$ of the coordinates of $\boldsymbol{x}$ are binary, since at most two of the N binding constraints are due to $\sum_{i=1}^N q_i x_i = b$ and $\sum_{i=1}^N x_i = s$.

□

EC.7. Proofs of the Auxiliary Lemmas.

EC.7.1. Proof of Lemma EC.1.

We require an auxiliary lemma that can be proved using the Portmanteau theorem (Theorem 2.1 in Billingsley (2013)) by working with the normalized measures $\lambda_n/\lambda_n(S)$; we omit the detailed proof.

LEMMA EC.14 **(Portmanteau Theorem for Nonnegative Measures)**. *Let $\{\lambda_n\}_{n=1}^\infty$, λ be finite nonnegative Borel measures on a metric space S. Then $\lambda_n \Rightarrow \lambda$ if and only if $\lim_{n\to\infty} \lambda_n(A) = \lambda(A)$ for any λ-continuity set A.*

Given Lemma EC.14, we are now ready to prove Lemma EC.1.

Proof of Lemma EC.1. Fix a λ-continuity set A. If $\lambda(A) = 0$, then for any bounded and continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$,

$$
\left| \int_{\mathbb{R}^2} \varphi(x) d(\mathbb{1}_A \lambda_n)(x) \right| = \left| \int_A \varphi(x) d\lambda_n(x) \right| \leq ||\varphi||_\infty |\lambda_n(A)| \to \lambda(A) = 0 \quad \text{as } n \to \infty,
$$

where the convergence holds due to Lemma EC.14, implying that $\mathbb{1}_A \lambda_n \Rightarrow \mathbb{1}_A \lambda$. Now suppose that $\lambda(A) > 0$, implying the existence of some N such that $\lambda_n(A) > 0$ for all $n > N$. Assuming $n > N$, we define the probability measures

$$
\overline{\lambda}_n(B) = \frac{\lambda_n(A \cap B)}{\lambda_n(A)} \quad \text{and} \quad \overline{\lambda}(B) = \frac{\lambda(A \cap B)}{\lambda(A)}, \quad B \in \mathcal{B}(S).
$$

It suffices to prove that $\overline{\lambda}_n \Rightarrow \overline{\lambda}$, because then, for any bounded and continuous function $\varphi : S \to \mathbb{R}$,

$$\int_{\mathbb{R}^2} \varphi(x) d(\mathbb{1}_A \lambda_n)(x) = \lambda_n(A) \int_{\mathbb{R}^2} \varphi(x) d\overline{\lambda}_n(x) \to \lambda(A) \int_{\mathbb{R}^2} \varphi(x) d\overline{\lambda}(x) = \int_{\mathbb{R}^2} \varphi(x) d(\mathbb{1}_A \lambda)(x).$$

Let $\mathcal{U} \subset \mathcal{B}(S)$ denote the collection of all λ-continuity sets. Since $\partial(A \cap B) \subset \partial A \cup \partial B$, it follows that $(A \cap B) \in \mathcal{U}$ for any $B \in \mathcal{U}$, which further implies that $\lim_{n\to\infty} \overline{\lambda}_n(B) = \overline{\lambda}(B)$ for all $B \in \mathcal{U}$. It is straightforward to show that $\mathcal{U}$ is a convergence-determining class by showing that it satisfies the conditions of Corollary 1 to Theorem 2.2 of Billingsley (2013), implying $\overline{\lambda}_n \Rightarrow \overline{\lambda}$. $\square$

EC.7.2. Proof of Lemmas EC.8 and EC.12.

We first state and prove two auxiliary lemmas.

Lemma EC.15. *Let $\{f_i : \mathbb{R}^2 \to \mathbb{R}\}_{i=1}^d$ be integrable and measurable functions and let $\phi : \mathbb{R}^2 \to [0,1]$. Then for any $\epsilon > 0$, there exists a finite collection of disjoint open rectangles in $\mathbb{R}^2$ such that their union A satisfies*

$$\left| \int_{\mathbb{R}^2} f_i(x)\phi(x)\mathrm{d}x - \int_A f_i(x)\mathrm{d}x \right| < \epsilon \quad i = 1, 2, ..., d$$

Proof of Lemma EC.15. We first note that Theorem 2.4 in Chapter 2 of Stein and Shakarchi (2009) says that for any $\epsilon > 0$, there exist step functions $g_1(x), ..., g_d(x)$ such that

$$\int_{\mathbb{R}^2} |f_i(x) - g_i(x)| dx < \epsilon/2, \quad i = 1, 2, ..., d,$$

where we recall that a function $g(x)$ is a step function if $g(x) = \sum_{j=1}^n w^{(j)} \mathbb{1}\{x \in (a^{(j)}, b^{(j)})\}$ for some $n \geq 0$ and rectangles $\{(a^{(j)}, b^{(j)}) \subset \mathbb{R}^2\}_{j=1}^n$. We now construct a set $A \subset \mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} g_i(x)\phi(x)\mathrm{d}x = \int_A g_i(x)\mathrm{d}x, \quad i = 1, \ldots, d,$$

which implies that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} f_i(x)\phi(x)\mathrm{d}x - \int_A f_i(x)\mathrm{d}x \right| \\
=&\left| \left(\int_{\mathbb{R}^2} f_i(x)\phi(x)\mathrm{d}x - \int_{\mathbb{R}^2} g_i(x)\phi(x)\mathrm{d}x \right) - \left(\int_A f_i(x)\mathrm{d}x - \int_A g_i(x)\mathrm{d}x \right) \right| \\
\leq& 2\int_{\mathbb{R}^2} |f_i(x) - g_i(x)|\mathrm{d}x < \epsilon.
\end{aligned}$$

To construct such a set A, we note that although each $g_i(x)$ can be represented by n_i rectangles, by decomposing rectangles into unions of smaller rectangles, we can find a common value n and weights $\{w_i^{(j)}\}_{j=1}^n$ such that $g_i(x) = \sum_{j=1}^n w_i^{(j)} \mathbb{1}\{x \in (a^{(j)}, b^{(j)})\}$ for all $x \in \mathbb{R}^2$ except on a set of Lebesgue-measure zero. Defining

$$A_j = \left(a_1^{(j)}, a_2^{(j)} + \frac{1}{b_2^{(j)} - b_1^{(j)}} \int_{(a^{(j)}, b^{(j)})} \phi(x)\, dx \right) \times \left(b_1^{(j)}, b_2^{(j)} \right)$$

and setting $A = \cup_{j=1}^n A_j$, we conclude that for any $i = 1, \ldots, d$,

$$\int_{\mathbb{R}^2} g_i(x)\phi(x)\mathrm{d}x = \sum_{j=1}^n w_i^{(j)} \int_{(a^{(j)},b^{(j)})} \phi(x)\,dx = \sum_{j=1}^n w_i^{(j)} \int_{A_j} 1dx = \int_A g_i(x)\,dx.$$

□

For the next lemma, we recall that μ is the measure on $\mathbb{R}^2$ that has CDF F_μ, and that given a measurable function $\phi : \mathbb{R}^2 \to [0,1]$, the measure $\phi\mu$ is defined in (EC.7). Finally, we recall $w(\phi)$ and $w^+(\phi)$ defined in (EC.1) and (EC.2).

LEMMA EC.16. *If $\{\phi_n : \mathbb{R}^2 \to [0,1]\}_{n=1}^\infty$ is such that $\phi_n\mu \Rightarrow \phi\mu$, then $\lim_{n\to\infty} \pi(\phi_n) = \pi(\phi)$ and*

$$w(\phi) \le \liminf_{n\to\infty} w(\phi_n) \le \limsup_{n\to\infty} w^+(\phi_n) \le w^+(\phi). \tag{EC.32}$$

Furthermore, if $\phi_n(x) \to \phi(x)$ μ-a.s., then $\phi_n\mu \Rightarrow \phi\mu$.

Proof of Lemma EC.16. Assuming for now that the inequalities in display (EC.32) holds, we observe that for any $\epsilon > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \pi(\phi_n) &= \limsup_{n\to\infty} \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi_n) - u))\phi_n(u,r) f_\mu(u,r)dudr \\
&\le \limsup_{n\to\infty} \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - \epsilon - u))\phi_n(u,r) f_\mu(u,r)dudr \\
&= \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - \epsilon - u))\phi(u,r) f_\mu(u,r)dudr
\end{aligned}$$

where the first inequality follows from the monotonicity of $F_\varepsilon(\cdot)$ and the last equality follows from $\phi_n\mu \Rightarrow \phi\mu$ and the fact that $r \le r_{\max}$ by Assumption 3. Similarly,

$$\begin{aligned}
\liminf_{n\to\infty} \pi(\phi_n) &= \liminf_{n\to\infty} \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi_n) - u))\phi_n(u,r) f_\mu(u,r)dudr \\
&\ge \liminf_{n\to\infty} \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^+(\phi) + \epsilon - u))\phi_n(u,r) f_\mu(u,r)dudr \\
&= \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^+(\phi) + \epsilon - u))\phi(u,r) f_\mu(u,r)dudr
\end{aligned}$$

Since ϵ is arbitrary, we conclude that

$$\limsup_{n\to\infty} \pi(\phi_n) \le \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - u))\phi(u,r) f_\mu(u,r)dudr = \pi(\phi).$$

and by (EC.4),

$$\begin{aligned}
\liminf_{n\to\infty} \pi(\phi_n) &\ge \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^+(\phi) - u))\phi(u,r) f_\mu(u,r)dudr \\
&= \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - u))\phi(u,r) f_\mu(u,r)dudr = \pi(\phi),
\end{aligned}$$

implying that $\lim_{n\to\infty}\pi(\phi_n)=\pi(\phi)$. We now prove (EC.32) by contradiction. We first show $w(\phi)\le \liminf_{n\to\infty} w(\phi_n)$. If $w(\phi)=-\infty$ then this holds trivially, so we assume that $w(\phi)>-\infty$. First we note from (EC.3) that

$$0=\int_{\mathbb{R}^2}(1-F_\varepsilon(w(\phi_n)-u))\phi_n(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi_n)-v))f_\nu(v)dv-\beta, \tag{EC.33}$$

$$0=\int_{\mathbb{R}^2}(1-F_\varepsilon(w^+(\phi_n)-u))\phi_n(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w^+(\phi_n)-v))f_\nu(v)dv-\beta. \tag{EC.34}$$

Suppose that $\liminf_{n\to\infty} w(\phi_n)<w(\phi)$. Then there exists $\epsilon>0$ such that $w(\phi_n)\le w(\phi)-\epsilon$ for large-enough n. Taking $n\to\infty$ on both sides of (EC.33) and using $\phi_n\mu\Rightarrow\phi\mu$ leads to the contradiction

$$\begin{aligned}0&\ge \lim_{n\to\infty}\int_{\mathbb{R}^2}(1-F_\varepsilon(w(\phi)-\epsilon-u))\phi_n(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi)-\epsilon-v))f_\nu(v)dv-\beta\\&=\int_{\mathbb{R}^2}(1-F_\varepsilon(w(\phi)-\epsilon-u))\phi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi)-\epsilon-v))f_\nu(v)dv-\beta>0,\end{aligned}$$

where the final strict inequality follows from (EC.3).

Similarly, we show that $\limsup_{n\to\infty} w^+(\phi_n)\le w^+(\phi)$. Consider first the case that $w^+(\phi)>-\infty$. If $\limsup_{n\to\infty} w^+(\phi_n)>w^+(\phi)$, then $w^+(\phi_n)\ge w^+(\phi)+\epsilon$ for some $\epsilon>0$ and large-enough n. It follows from (EC.34), from (EC.3) and the convergence $\phi_n\mu\Rightarrow\phi\mu$ that

$$\begin{aligned}0&\le \lim_{n\to\infty}\int_{\mathbb{R}^2}(1-F_\varepsilon(w^+(\phi)+\epsilon-u))\phi_{n'}(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w^+(\phi)+\epsilon-v))f_\nu(v)dv-\beta\\&=\int_{\mathbb{R}^2}(1-F_\varepsilon(w^+(\phi)+\epsilon-u))\phi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w^+(\phi)+\epsilon-v))f_\nu(v)dv-\beta<0,\end{aligned}$$

which is a contradiction. Now we consider the case $w^+(\phi)=-\infty$. Suppose that there exists $M>0$ and a subsequence n' such that $\limsup_{n\to\infty} w^+(\phi_{n'})>-M$. We have, from (EC.34), (EC.3) and the convergence $\phi_n\mu\Rightarrow\phi\mu$, that

$$\begin{aligned}0&\le \lim_{n\to\infty}\int_{\mathbb{R}^2}(1-F_\varepsilon(-M-u))\phi_{n'}(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(-M-v))f_\nu(v)dv-\beta\\&=\int_{\mathbb{R}^2}(1-F_\varepsilon(-M-u))\phi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(-M-v))f_\nu(v)dv-\beta<0,\end{aligned}$$

which is a contradiction. Finally, we show that $\phi_n(x)\to\phi(x)$ μ-a.s. implies $\phi_n\mu\Rightarrow\phi\mu$. Indeed, for any bounded and continuous function $h:\mathbb{R}^2\to\mathbb{R}$, the dominated convergence theorem together with $|h(u,r)\phi_n(u,r)|\le|h(u,r)|$ implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}h(u,r)\phi_n(u,r)f_\mu(u,r)dudr=\int_{\mathbb{R}^2}h(u,r)\phi(u,r)f_\mu(u,r)dudr.$$

□

Proof of Lemma EC.8. Assume without loss of generality that $\int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)dudr > 0$, because otherwise $A=\varnothing$ satisfies all the conditions with $\epsilon=0$. We first consider the case when $w(\phi)>-\infty$. Fix a feasible solution $\phi:\mathbb{R}^2\to[0,1]$ and $\epsilon>0$. We make two claims. First, there exists an interior solution $\varphi(x)\in[0,1]$ such that $|\pi(\varphi)-\pi(\phi)|\le\epsilon/2$ and $w(\phi)-\epsilon/2\le w(\varphi)\le w^+(\varphi)\le w^+(\phi)+\epsilon/2$. Second, there exists a measurable F_μ-continuity set A such that $\mathbb{1}_A(x)$ is an interior solution, that $|\pi(\varphi)-\pi(A)|\le\epsilon/2$, and that $w(\varphi)-\epsilon/2\le w(A)\le w^+(A)\le w^+(\varphi)+\epsilon/2$. The two claims imply Lemma EC.8 because $|\pi(\phi)-\pi(A)|\le|\pi(\phi)-\pi(\varphi)|+|\pi(\varphi)-\pi(A)|\le\epsilon$ and $w(\phi)-\epsilon\le w(\varphi)-\frac{\epsilon}{2}\le w(A)\le w^+(A)\le w^+(\varphi)+\frac{\epsilon}{2}\le w^+(\phi)+\epsilon$. We now prove both claims. Note that $\alpha\phi(x)$ is an interior solution for any constant $\alpha\in(0,1)$ because $\alpha\int_{\mathbb{R}^2}\phi(x)\mu(dx)<\alpha\eta\le\eta$. Since $\alpha\phi(x)\to\phi(x)$ point-wise as $\alpha\to1$, Lemma EC.16 implies that there exists some α_ϵ close to one such that $\alpha_\epsilon\phi(x)$ satisfies the first claim. Fixing $\varphi(x)=\alpha_\epsilon\phi(x)$, we now prove the second claim. First, we note by (EC.4) that

$$\begin{aligned}\pi(\varphi)&=\int_{\mathbb{R}^2}r(1-F_\varepsilon(w(\varphi)-u))\varphi(u,r)f_\mu(u,r)dudr\\&=\int_{\mathbb{R}^2}r(1-F_\varepsilon(w^+(\varphi)-u))\varphi(u,r)f_\mu(u,r)dudr.\end{aligned}$$

Since $r\le r_{\max}$, it follows by the dominated convergence theorem that there exists a small-enough $\delta>0$ such that $0<\delta<\epsilon/2$ and

$$\begin{aligned}\pi(\varphi)-\frac{\epsilon}{4}&<\int_{\mathbb{R}^2}r(1-F_\varepsilon(w^+(\varphi)+\delta-u))\varphi(u,r)f_\mu(u,r)dudr\\&\le\int_{\mathbb{R}^2}r(1-F_\varepsilon(w(\varphi)-\delta-u))\varphi(u,r)f_\mu(u,r)dudr<\pi(\varphi)+\frac{\epsilon}{4},\end{aligned}\tag{EC.35}$$

where the second inequality is due to (EC.3). We fix this δ and observe that by (EC.3),

$$\begin{aligned}&\int_{\mathbb{R}^2}(1-F_\varepsilon(w^+(\varphi)+\delta-u))\varphi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w^+(\varphi)+\delta-v))f_\nu(v)dv<\beta,\\&\int_{\mathbb{R}^2}(1-F_\varepsilon(w(\varphi)-\delta-u))\varphi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\varphi)-\delta-v))f_\nu(v)dv>\beta.\end{aligned}\tag{EC.36}$$

Fix $\epsilon_1=\eta-\int_{\mathbb{R}^2}\varphi(u,r)f_\mu(u,r)dudr$ and note that $\epsilon_1>0$ because $\varphi(x)$ is an interior solution. Lemma EC.15 implies that there exists a set A, composed of finite unions of open rectangles, such that

$$\left|\int_A f_\mu(u,r)dudr-\int_{\mathbb{R}^2}\varphi(u,r)f_\mu(u,r)dudr\right|<\epsilon_1.\tag{EC.37}$$

Since $F_\mu(u,r)$ has a density and A is a union of open rectangles, then A must be an F_μ-continuity set. Furthermore, (EC.37) implies that

$$\int_A f_\mu(u,r)dudr<\int_{\mathbb{R}^2}\varphi(u,r)f_\mu(u,r)dudr+\epsilon_1<\eta,$$

meaning that $\mathbb{1}_A(x)$ is an interior solution. We now show that A can be modified to keep its previous properties while also satisfying $w(\varphi) - \delta \le w(A) \le w^+(A) \le w^+(\varphi) + \delta$. Namely, we redefine A by applying Lemma EC.15 to find an $F_\mu-$continuity set satisfying (EC.37) as well as

$$\left| \int_A (1 - F_\varepsilon(w^+(\varphi) + \delta - u)) f_\mu(u,r) du dr - \int_{\mathbb{R}^2} (1 - F_\varepsilon(w^+(\varphi) + \delta - u)) \varphi(u,r) f_\mu(u,r) du dr \right| < \epsilon_2, \tag{EC.38}$$

$$\left| \int_A (1 - F_\varepsilon(w(\varphi) - \delta - u)) f_\mu(u,r) du dr - \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\varphi) - \delta - u)) \varphi(u,r) f_\mu(u,r) du dr \right| < \epsilon_3, \tag{EC.39}$$

$$\left| \int_A r(1 - F_\varepsilon(w^+(\varphi) + \delta - u)) f_\mu(u,r) du dr - \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^+(\varphi) + \delta - u)) \varphi(u,r) f_\mu(u,r) du dr \right| < \epsilon/4, \tag{EC.40}$$

$$\left| \int_A r(1 - F_\varepsilon(w(\varphi) - \delta - u)) f_\mu(u,r) du dr - \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\varphi) - \delta - u)) \varphi(u,r) f_\mu(u,r) du dr \right| < \epsilon/4, \tag{EC.41}$$

where we define

$$\begin{aligned}
\epsilon_2 &= \beta - \int_{\mathbb{R}^2} (1 - F_\varepsilon(w^+(\varphi) + \delta - u)) \varphi(u,r) f_\mu(u,r) du dr - \frac{\beta}{\gamma} \int_{\mathbb{R}} (1 - F_\xi(w^+(\varphi) + \delta - v)) f_\nu(v) dv \\
\epsilon_3 &= \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\varphi) - \delta - u)) \varphi(u,r) f_\mu(u,r) du dr + \frac{\beta}{\gamma} \int_{\mathbb{R}} (1 - F_\xi(w(\varphi) - \delta - v)) f_\nu(v) dv - \beta,
\end{aligned}$$

and observe that both $\epsilon_2 > 0$ and $\epsilon_3 > 0$ by (EC.36). As a consequence,

$$\begin{aligned}
&\int_A (1 - F_\varepsilon(w^+(\varphi) + \delta - u)) f_\mu(u,r) du dr + \frac{\beta}{\gamma} \int_{\mathbb{R}} (1 - F_\xi(w^+(\varphi) + \delta - v)) f_\nu(v) dv \\
&< \int_{\mathbb{R}^2} (1 - F_\varepsilon(w^+(\varphi) + \delta - u)) \varphi(u,r) f_\mu(u,r) du dr + \frac{\beta}{\gamma} \int_{\mathbb{R}} (1 - F_\xi(w^+(\varphi) + \delta - v)) f_\nu(v) dv + \epsilon_2 = \beta
\end{aligned}$$

and

$$\begin{aligned}
&\int_A (1 - F_\varepsilon(w(\varphi) - \delta - u)) f_\mu(u,r) du dr + \frac{\beta}{\gamma} \int_{\mathbb{R}} (1 - F_\xi(w(\varphi) - \delta - v)) f_\nu(v) dv \\
&> \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\varphi) - \delta - u)) \varphi(u,r) f_\mu(u,r) du dr + \frac{\beta}{\gamma} \int_{\mathbb{R}} (1 - F_\xi(w(\varphi) - \delta - v)) f_\nu(v) dv - \epsilon_3 = \beta.
\end{aligned}$$

It follows from the characterization of $w(\phi)$ and $w^+(\phi)$ in (EC.3), and the monotonicity of F_ε and F_ξ that $w(\varphi) - \delta \le w(A) \le w^+(A) \le w^+(\varphi) + \delta$. Finally,

$$\begin{aligned}
\pi(A) = \int_A r(1 - F_\varepsilon(w(A) - u)) f_\mu(u,r) du dr &= \int_A r(1 - F_\varepsilon(w^+(A) - u)) f_\mu(u,r) du dr \\
&\ge \int_A r(1 - F_\varepsilon(w^+(\varphi) + \delta - u)) f_\mu(u,r) du dr \\
&\ge \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^+(\varphi) + \delta - u)) \varphi(u,r) f_\mu(u,r) du dr - \frac{\epsilon}{4} > \pi(\varphi) - \frac{\epsilon}{2},
\end{aligned}$$

where the second equality is due to (EC.4), the first inequality is by monotonicity of $F_\varepsilon(\cdot)$, the second inequality is due to (EC.40), and the final inequality is due to (EC.35). Similarly,

$$\begin{aligned}\pi(A) &= \int_A r(1-F_\varepsilon(w(A)-u))f_\mu(u,r)dudr \le \int_A r(1-F_\varepsilon(w(\varphi)-\delta-u))f_\mu(u,r)dudr \\ &\le \int_{\mathbb{R}^2} r(1-F_\varepsilon(w(\varphi)-\delta-u))\varphi(u,r)f_\mu(u,r)dudr + \frac{\epsilon}{4} < \pi(\varphi)+\frac{\epsilon}{2},\end{aligned}$$

and we conclude that $|\pi(\varphi)-\pi(A)|<\epsilon/2$.

We now consider the case when $w(\phi)=-\infty$. As before, we can find $\varphi=\alpha_\epsilon\phi$ for some $0<\alpha_\epsilon<1$ such that

$$\begin{aligned}&|\pi(\phi)-\pi(\varphi)|<\epsilon/2 \\ &\int_{\mathbb{R}^2}\varphi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}} f_\nu(v)dv < \int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}} f_\nu(v)dv \le \beta \\ &\int_{\mathbb{R}^2}\varphi(u,r)f_\mu(u,r)dudr = \alpha_\epsilon\int_{\mathbb{R}^2}\varphi(u,r)f_\mu(u,r)dudr \le \alpha\eta<\eta,\end{aligned}$$

and, as before, we apply Lemma EC.15 to find a F_μ-continuity set A such that

$$|\pi(\varphi)-\pi(A)|<\epsilon/2, \quad \int_A f_\mu(u,r)dudr+\frac{\beta}{\gamma}\int_{\mathbb{R}} f_\nu(v)dv<\beta, \quad \int_A f_\mu(u,r)dudr<\eta.$$

In turn, these six inequalities imply that $|\pi(\phi)-\pi(A)|\le|\pi(\phi)-\pi(\varphi)|+|\pi(\varphi)-\pi(A)|\le\epsilon$, that $w(A)=w^+(A)=-\infty$, and that A is an interior solution. Note that since $w(\phi)=-\infty$, it must be true that $w(\phi)-\epsilon\le w(A)\le w^+(A)\le w^+(\phi)+\epsilon$, which concludes the proof. □

Proof of Lemma EC.12. We first consider the case $w(\phi)=-\infty$. In this case,

$$\int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)\,du\,dr+\frac{\beta}{\gamma}\le\beta.$$

Moreover, since $\pi(\phi)>0$, we have $\int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)\,du\,dr>0$. Choose $\alpha\in(0,1)$ sufficiently close to one such that $\alpha\pi(\phi)\ge\pi(\phi)-\epsilon$ and $\alpha\int_{\mathbb{R}^2}\phi(u,r)f_\mu(u,r)\,du\,dr+\frac{\beta}{\gamma}<\beta$. By the same continuity-set approximation argument used in Lemma EC.8, there exists an F_μ-continuity set $A\in\mathcal{B}(\mathbb{R}^2)$ such that

$$\int_A f_\mu(u,r)\,du\,dr+\frac{\beta}{\gamma}<\beta \qquad \text{and} \qquad \int_A r f_\mu(u,r)\,du\,dr\ge\pi(\phi)-\epsilon.$$

It follows immediately that $\mathbb{1}_A$ is an interior feasible solution to (RLOP). Since $w(\phi)=-\infty$, we also have $1-F_\varepsilon(w(\phi)-u)=1$ and $1-F_\xi(w(\phi)-v)=1$. Therefore,

$$\int_A(1-F_\varepsilon(w(\phi)-u))\,f_\mu(u,r)\,du\,dr+\frac{\beta}{\gamma}\int_{\mathbb{R}}(1-F_\xi(w(\phi)-v))\,f_\nu(v)\,dv=\int_A f_\mu(u,r)\,du\,dr+\frac{\beta}{\gamma}<\beta.$$

Finally, under the standing assumption that F_ε is strictly increasing on $\mathbb{R}$, the equality $\int_{\mathbb{R}^2} r\,(1-F_\varepsilon(w-u))\,\phi(u,r)f_\mu(u,r)\,du\,dr=\pi(\phi)$ can hold only at $w=-\infty$. Hence $w^{++}(\phi)=-\infty$, and $\pi(\mathbb{1}_A, w^{++}(\phi)) = \pi(\mathbb{1}_A, -\infty) = \int_A r f_\mu(u,r)\, du\, dr \geq \pi(\phi) - \epsilon$. This proves the lemma when $w(\phi) = -\infty$.

It remains to consider the case $w(\phi) > -\infty$. Since $\phi(x)$ is a feasible solution to (RLOP), it follows that $\int_{\mathbb{R}^2} \alpha\phi(u,r) f_\mu(u,r) dudr < \eta$ for any $\alpha \in (0,1)$, and also that

$$\begin{aligned}\beta &= \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - u))\phi(u,r) f_\mu(u,r) dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi) - v)) f_\nu(v) dv \\ &> \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - u))\alpha\phi(u,r) f_\mu(u,r) dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi) - v)) f_\nu(v) dv,\end{aligned}$$

where the equality is due to (EC.1) and the inequality follows from the assumption that

$$\pi(\phi) = \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w(\phi) - u))\phi(u,r) f_\mu(u,r) dudr > 0.$$

Furthermore, we can choose α_ϵ close enough to one such that $\pi(\alpha_\epsilon\phi, w^{++}(\phi)) > \pi(\phi) - \frac{\epsilon}{2}$, because

$$\begin{aligned}\lim_{\alpha\to 1} \pi(\alpha\phi, w^{++}(\phi)) &= \lim_{\alpha\to 1} \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^{++}(\phi) - u))\alpha\phi(u,r) f_\mu(u,r) dudr \\ &= \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^{++}(\phi) - u))\phi(u,r) f_\mu(u,r) dudr = \pi(\phi),\end{aligned}$$

where the last equality follows from the definition of $w^{++}(\phi)$. Fixing $\varphi(x) = \alpha_\epsilon\phi(x)$, we now prove the existence of a set A satisfying the conditions of Lemma EC.12. By Lemma EC.15, there is a set A composed of a union of finitely many open rectangles (meaning that A is an F_μ-continuity set), such that

$$\begin{aligned}&\left| \int_A f_\mu(u,r) dudr - \int_{\mathbb{R}^2} \varphi(u,r) f_\mu(u,r) dudr \right| < \epsilon_1 \\ &\left| \int_A (1 - F_\varepsilon(w(\phi) - u)) f_\mu(u,r) dudr - \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - u))\varphi(u,r) f_\mu(u,r) dudr \right| < \epsilon_2 \\ &\left| \int_A r(1 - F_\varepsilon(w^{++}(\phi) - u)) f_\mu(u,r) dudr - \int_{\mathbb{R}^2} r(1 - F_\varepsilon(w^{++}(\phi) - u))\varphi(u,r) f_\mu(u,r) dudr \right| < \frac{\epsilon}{2}.\end{aligned}$$

where

$$\begin{aligned}&\epsilon_1 = \eta - \int_{\mathbb{R}^2} \varphi(u,r) f_\mu(u,r) dudr > 0 \\ &\epsilon_2 = \beta - \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - u))\varphi(u,r) f_\mu(u,r) dudr - \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi) - v)) f_\nu(v) dv > 0.\end{aligned}$$

It follows that

$$\int_A f_\mu(u,r) dudr < \int_{\mathbb{R}^2} \varphi(u,r) f_\mu(u,r) dudr + \epsilon_1 = \eta$$

and that

$$\begin{aligned}&\int_A (1 - F_\varepsilon(w(\phi) - u)) f_\mu(u,r) dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi) - v)) f_\nu(v) dv \\ &\qquad < \int_{\mathbb{R}^2} (1 - F_\varepsilon(w(\phi) - u))\varphi(u,r) f_\mu(u,r) dudr + \frac{\beta}{\gamma}\int_{\mathbb{R}} (1 - F_\xi(w(\phi) - v)) f_\nu(v) dv + \epsilon_2 = \beta,\end{aligned}$$

and, by noting that $\int_A r(1-F_\varepsilon(w^{++}(\phi)-u))f_\mu(u,r)dudr = \pi(\mathbb{1}_A, w^{++}(\phi))$, we arrive at

$$\pi(\mathbb{1}_A, w^{++}(\phi)) \geq \pi(\varphi, w^{++}(\phi)) - \frac{\epsilon}{2} \geq \pi(\phi) - \epsilon.$$

□

EC.8. MILP Formulations of the Surrogate Problem

Recall that (SP′) is defined by

$$\begin{aligned}
\max_{w\in\mathbb{R},\ \boldsymbol{x}\in\{0,1\}^N} \quad & \hat{\pi}_N(\boldsymbol{x}, w) = \sum_{i=1}^N x_i r_i \big(1 - F_\varepsilon(w-u_i)\big) \\
\text{s.t.} \quad & \sum_{i=1}^N x_i \big(1 - F_\varepsilon(w-u_i)\big) + \sum_{j=1}^{N_0} \big(1 - F_\xi(w-v_j)\big) \leq B, \\
& ||\boldsymbol{x}||_1 \leq \eta N.
\end{aligned}$$

In this section we consider two special cases for $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$ where this optimization problem can be written as a MILP. We first specify the finite interval over which the threshold variable w is optimized in the MILP formulations. Let $u_{\min} = \min_{i\in[N]} u_i$. In the nontrivial case $B < N + N_0$, define

$$w_H = \inf\left\{ w \in \mathbb{R} : \sum_{i=1}^N \big(1 - F_\varepsilon(w-u_i)\big) + \sum_{j=1}^{N_0} \big(1 - F_\xi(w-v_j)\big) \leq B \right\}.$$

By monotonicity, w_H is an upper bound on the optimal threshold. For the lower bound, if $B \leq N_0$, define

$$w_L = \sup\left\{ w \in \mathbb{R} : 1 - F_\varepsilon(w - u_{\min}) + \sum_{j=1}^{N_0} \big(1 - F_\xi(w-v_j)\big) \geq B \right\}.$$

In this case, w_L is a valid lower bound on the optimal threshold. If $B > N_0$, such a finite lower bound need not exist. In this case, for a prescribed absolute tolerance $\epsilon > 0$, choose any finite w_L such that $\sum_{i=1}^N r_i F_\varepsilon(w_L - u_i) \leq \epsilon$. Such a w_L exists because $F_\varepsilon(w_L - u_i) \to 0$ as $w_L \to -\infty$ for every i. Optimizing over $w \in [w_L, w_H]$ then gives an ϵ_{milp}-optimal solution to (SP′). Indeed, if the unrestricted optimal threshold for some assortment lies below w_L, replacing it by w_L decreases the objective by at most

$$\sum_{i=1}^N x_i r_i F_\varepsilon(w_L - u_i) \leq \sum_{i=1}^N r_i F_\varepsilon(w_L - u_i) \leq \epsilon.$$

Therefore, in the MILP formulations below, we impose the explicit bound $w_L \leq w \leq w_H$.

EC.8.1. Uniform Distribution

Assume that $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$ follow the uniform distribution over $[-d,d]$, namely,

$$F_\varepsilon(t)=F_\xi(t)=\frac{1}{2d}(t+d)\mathbb{1}(-d\le t<d)+\mathbb{1}(t\ge d),\quad t\in\mathbb{R}.$$

With $\boldsymbol{x},w,u_i$, and v_j being as in (SP′), we define the binary variables $\underline{y}_i=\mathbb{1}(w-u_i<-d)$, $y_i=\mathbb{1}(-d\le w-u_i<d)$, and $\overline{y}_i=\mathbb{1}(w-u_i\ge d)$ for $i=1,\dots,N$, and similarly $\underline{z}_j=\mathbb{1}(w-v_j<-d)$, $z_j=\mathbb{1}(-d\le w-v_j<d)$, and $\overline{z}_j=\mathbb{1}(w-v_j\ge d)$ for $j=1,\dots,N_0$. We further define $q_i=x_i\cdot y_i$, $\underline{q}_i=x_i\cdot\underline{y}_i$, and $s_i=w\cdot q_i$ for $i=1,\dots,N$, as well as $t_j=w\cdot z_j$ for $j=1,\dots,N_0$. resulting in the MILP formulation

$$\begin{aligned}
\max_{\substack{w\in[w_L,w_H],\boldsymbol{z}\in\{0,1\}^{N_0}\\ \boldsymbol{x},\boldsymbol{y},\overline{\boldsymbol{y}},\underline{\boldsymbol{y}},\boldsymbol{q},\underline{\boldsymbol{q}}\in\{0,1\}^N\\ \boldsymbol{s}\in\mathbb{R}^N,\boldsymbol{t}\in\mathbb{R}^{N_0}}}\quad & \sum_{i=1}^N\left[\frac{r_i(d+u_i)}{2d}q_i-\frac{r_i}{2d}s_i+r_i\underline{q}_i\right]\\
\text{s.t.}\quad & \sum_{i=1}^N\left[\frac{d+u_i}{2d}q_i-\frac{1}{2d}s_i+\underline{q}_i\right]+\sum_{j=1}^{N_0}\left[\frac{d+v_j}{2d}z_j-\frac{1}{2d}t_j+\underline{z}_j\right]\le B,\\
& \sum_{i=1}^N x_i\le\eta N,\\
& \underline{y}_i\le 1+\tfrac{-d-w+u_i}{M},\ \overline{y}_i\le 1+\tfrac{w-u_i-d}{M},\ y_i\le 1+\tfrac{d-w+u_i}{M},\ y_i\le 1+\tfrac{d+w-u_i}{M},\ \underline{y}_i+y_i+\overline{y}_i=1,\\
& \underline{z}_j\le 1+\tfrac{-d-w+v_j}{M},\ \overline{z}_j\le 1+\tfrac{w-v_j-d}{M},\ z_j\le 1+\tfrac{d-w+v_j}{M},\ z_j\le 1+\tfrac{d+w-v_j}{M},\ \underline{z}_j+z_j+\overline{z}_j=1,\\
& q_i\le x_i,\ q_i\le y_i,\ q_i\ge x_i+y_i-1,\quad \underline{q}_i\le x_i,\ \underline{q}_i\le\underline{y}_i,\ \underline{q}_i\ge x_i+\underline{y}_i-1, i=1,\dots,N,\\
& -Mq_i\le s_i\le Mq_i,\ w-M(1-q_i)\le s_i\le w+M(1-q_i), i=1,\dots,N,\\
& -Mz_j\le t_j\le Mz_j,\ w-M(1-z_j)\le t_j\le w+M(1-z_j), j=1,\dots,N_0.
\end{aligned}$$

for any constant M satisfying $M>d+\max\{|w_L|,|w_H|\}+\max_{1\le i\le N}|u_i|+\max_{1\le j\le N_0}|v_j|$.

EC.8.2. Negative Exponential Distribution

Assume that $F_\varepsilon(\cdot)$ and $F_\xi(\cdot)$ follow the negative exponential distribution with rate λ. Namely,

$$F_\varepsilon(t)=F_\xi(t)=1-(1-e^{\lambda t})\mathbb{1}(t\le 0),\quad t\in\mathbb{R}.$$

With $\boldsymbol{x},w,u_i$, and v_j being as in (SP′), we define the variable $p=e^{\lambda w}\ge 0$, the constants $\bar{u}_i=e^{\lambda u_i}, i=1,2,...,N$, and $\bar{v}_j=e^{\lambda v_j}, j=1,2,...,N_0$, as well as the variables $y_i=\mathbb{1}(w\le u_i)=\mathbb{1}(p\le\bar{u}_i), i=1,2,...,N$, and $z_j=\mathbb{1}(w\le v_j)=\mathbb{1}(p\le\bar{v}_j), j=1,2,...,N_0$. We further define the variables $q_i=x_i\cdot y_i$ and $s_i=p\cdot q_i$ for $i=1,2,...,N$, and $t_j=p\cdot z_j$ for $j=1,2,...,N_0$. we arrive at the MILP formulation

$$\max_{\substack{p\in[e^{\lambda w_L},e^{\lambda w_H}],\boldsymbol{z}\in\{0,1\}^{N_0}\\ \boldsymbol{x},\boldsymbol{y},\boldsymbol{q}\in\{0,1\}^N\\ \boldsymbol{s}\in\mathbb{R}^N,\boldsymbol{t}\in\mathbb{R}^{N_0}}}\quad \sum_{i=1}^N r_iq_i-\sum_{i=1}^N\frac{r_i}{\bar{u}_i}s_i$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{i=1}^{N}\left(q_i-\frac{s_i}{\bar{u}_i}\right)+\sum_{j=1}^{N_0}\left(z_j-\frac{t_j}{\bar{v}_j}\right)\le B, \quad \sum_{i=1}^{N} x_i \le \eta N,\\
& y_i \le 1+(\bar{u}_i-p)/M, \quad y_i \ge (\bar{u}_i-p)/M, \quad i=1,\dots,N,\\
& z_j \le 1+(\bar{v}_j-p)/M, \quad z_j \ge (\bar{v}_j-p)/M, \quad j=1,\dots,N_0,\\
& q_i \le x_i,\, q_i\le y_i,\, q_i \ge x_i+y_i-1, \quad i=1,\dots,N,\\
& -Mq_i \le s_i \le Mq_i, \quad p-M(1-q_i)\le s_i \le p+M(1-q_i), \quad i=1,\dots,N,\\
& -Mz_j \le t_j \le Mz_j, \quad p-M(1-z_j)\le t_j \le p+M(1-z_j), \quad j=1,\dots,N_0.
\end{aligned}
$$

for any constant M satisfying $M > e^{\lambda w_H} + \max_{1\le i\le N} e^{\lambda u_i} + \max_{1\le j\le N_0} e^{\lambda v_j}$.

EC.9. Proof of Proposition 6.

The following technical lemma is needed. It is proved at the end of this section.

LEMMA EC.17. *Assume that* $(X,Y)\sim\mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$ *follows a 2-dimensional normal distribution. Then for any* $a,b\in\mathbb{R}$,

$$
\left|Cov\big(\mathbb{1}\{X<a\},\mathbb{1}\{Y<b\}\big)\right| \le \frac{|\rho|}{4}.
$$

Proof of Proposition 6. Proceeding as in (EC.28) in the proof of Proposition 5, it is sufficient to show that

$$
|\pi_N(\boldsymbol{x})-\hat{\pi}_N(\boldsymbol{x})| \le r_{\max}\sqrt{B+\frac{1}{2}\sum_{1\le i<j\le N+N_0}|\rho_{ij}|} \tag{EC.42}
$$

for any assortment $\boldsymbol{x}\in\{0,1\}^N$. To show (EC.42), by following the same procedure of the proof of Proposition 5, we can show that $|\pi_N(\boldsymbol{x})-\hat{\pi}_N(\boldsymbol{x})|\le r_{\max}\cdot\sqrt{\mathrm{Var}(|\hat{\mathcal{P}}(\boldsymbol{x})|)}$. where we recall that $\hat{\mathcal{P}}(\boldsymbol{x})$ is defined by

$$
\hat{\mathcal{P}}(\boldsymbol{x}) = \{i\in S(\boldsymbol{x})\cup\{N+1,...,N+N_0\}: W_i > w_{(B+1)}(\boldsymbol{x})\},
$$

and we have

$$
|\hat{\mathcal{P}}(\boldsymbol{x})| = \sum_{i=1}^{N}\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\}x_i + \sum_{j=1}^{N_0}\mathbb{1}\{W_{j+N} > w_{(B+1)}(\boldsymbol{x})\}.
$$

Note that the random components are not independent in this case. We then have

$$
\begin{aligned}
\mathrm{Var}\left(|\hat{\mathcal{P}}(\boldsymbol{x})|\right) &= \sum_{i=1}^{N}\mathrm{Var}\left(\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\}\right)x_i + \sum_{j=1}^{N_0}\mathrm{Var}\left(\mathbb{1}\{W_{j+N} > w_{(B+1)}(\boldsymbol{x})\}\right)\\
&\quad + 2\sum_{1\le i<j\le N+N_0}\mathrm{Cov}\left(\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\},\mathbb{1}\{W_j > w_{(B+1)}(\boldsymbol{x})\}\right)\\
&\qquad\qquad \cdot\mathbb{1}\{i>N \text{ or } x_i=1\}\,\mathbb{1}\{j>N \text{ or } x_j=1\}\\
&\le B + 2\sum_{1\le i<j\le N+N_0}\left|\mathrm{Cov}\left(\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\},\mathbb{1}\{W_j > w_{(B+1)}(\boldsymbol{x})\}\right)\right|.
\end{aligned}
$$

Note that

$$\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\} = \begin{cases} \mathbb{1}\{-\varepsilon_i < u_i - w_{(B+1)}(\boldsymbol{x})\} & 1 \le i \le N \\ \mathbb{1}\{-\xi_{i-N} < v_{i-N} - w_{(B+1)}(\boldsymbol{x})\} & N < i \le N + N_0 \end{cases}$$

Lemma EC.17 then implies that for $1 \le i < j \le N + N_0$,

$$\left|\mathrm{Cov}\big(\mathbb{1}\{W_i > w_{(B+1)}(\boldsymbol{x})\}, \mathbb{1}\{W_j > w_{(B+1)}(\boldsymbol{x})\}\big)\right| \le \frac{|\rho_{ij}|}{4}.$$

We conclude that

$$|\pi_N(\boldsymbol{x}) - \hat{\pi}_N(\boldsymbol{x})| \le r_{\max} \cdot \sqrt{B + 2 \cdot \sum_{1 \le i < j \le N + N_0} \frac{|\rho_{ij}|}{4}} = r_{\max} \cdot \sqrt{B + \frac{1}{2} \cdot \sum_{1 \le i < j \le N + N_0} |\rho_{ij}|}$$

$\square$

Proof of Lemma EC.17. Let

$$\Phi(a,b;t) := \frac{1}{2\pi\sqrt{1-t^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp\left(-\frac{x^2 - 2txy + y^2}{2(1-t^2)}\right) \mathrm{d}x\mathrm{d}y$$

be the C.D.F. function of $(X_t, Y_t) \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & t\\ t & 1\end{bmatrix}\right)$ for $t \in [-1,1]$. We have

$$\begin{aligned}\left|\mathrm{Cov}\big(\mathbb{1}\{X < a\}, \mathbb{1}\{Y < b\}\big)\right| &= \left|\mathbb{P}(X < a, Y < b) - \mathbb{P}(X < a)\mathbb{P}(Y < b)\right| \\ &= \left|\Phi(a,b;\rho) - \Phi(a,b;0)\right| = \left|\int_0^{\rho} \frac{\partial \Phi(a,b;t)}{\partial t} \mathrm{d}t\right|\end{aligned}$$

From (Plackett 1954) we have $\frac{\partial \Phi(a,b;t)}{\partial t} = \frac{1}{2\pi\sqrt{1-t^2}} \exp\left(-\frac{a^2 - 2tab + b^2}{2(1-t^2)}\right)$, which is positive for all $t \in [-1,1]$. Therefore

$$\begin{aligned}\left|\mathrm{Cov}\big[\mathbb{1}(X > a), \mathbb{1}(Y > b)\big]\right| &= \left|\int_0^{\rho} \frac{\partial \Phi(a,b;t)}{\partial t} \mathrm{d}t\right| = \left|\int_0^{\rho} \frac{1}{2\pi\sqrt{1-t^2}} \exp\left(-\frac{a^2 - 2tab + b^2}{2(1-t^2)}\right) \mathrm{d}t\right| \\ &\le \left|\int_0^{\rho} \frac{1}{2\pi\sqrt{1-t^2}} \mathrm{d}t\right| = \int_0^{|\rho|} \frac{1}{2\pi\sqrt{1-t^2}} \mathrm{d}t = \frac{1}{2\pi} \arcsin(|\rho|) \le \frac{|\rho|}{4}\end{aligned}$$

where the last inequality is due to $\arcsin(x) \le \pi x/2$ for $x \in [0,1]$. $\square$

EC.10. Identifiability: proof of Theorem 1 and Proposition 1.

EC.10.1. Proof of Theorem 1.

We first introduce the concept of a Tchebycheff system, followed by a few auxiliary lemmas, all of which are proved at the end of this section after the proof of Theorem 1.

Definition EC.1 (Tchebycheff system (T-system)). Let $f_0, \ldots, f_n : [a,b] \to \mathbb{R}$ where $[a,b]$ is a finite interval. We say that $\{f_0, \ldots, f_n\}$ is a Tchebycheff system on $[a,b]$ if for every choice of distinct nodes $a \le t_0 < \cdots < t_n \le b$, the determinant of the matrix $(f_j(t_k))_{j,k=0}^{n}$ is nonzero.

LEMMA EC.18. *Let $w_1 < \cdots < w_M$ be real numbers and let $F(x) = \int_{-\infty}^{x} f(s)\,ds$ with $f \in PF_\infty$ such that $\int_{\mathbb{R}} |f(s)|ds < \infty$, and define*

$$\phi_m(t) = F(w_m - t) - F(w_1 - t) = \int_{w_1}^{w_m} f(s-t)\,ds, \quad m = 2, \ldots, M.$$

Then $\{\phi_m\}_{m=2}^{M}$ is a T-system on every finite interval $[a,b]$.

We abuse notation and use $w_{\boldsymbol{\theta}}(\boldsymbol{x})$ to denote the SP threshold introduced in (2). Since we only consider assortments such that $B < \|\boldsymbol{x}\|_1 + N_0$, the threshold $w_{\boldsymbol{\theta}}(\boldsymbol{x})$ satisfies

$$\sum_{i=1}^{N} x_i \left(1 - F_\varepsilon(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - u_i)\right) + \sum_{j=1}^{N_0} \left(1 - F_\xi(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - v_j)\right) = B. \tag{EC.43}$$

Now given a collection of assortments $\mathcal{X}$, let $M_{\boldsymbol{\theta}}(\mathcal{X}) = |\{w_{\boldsymbol{\theta}}(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}\}|$ be the number of distinct thresholds generated by $\mathcal{X}$. It follows that $M_{\boldsymbol{\theta}}(\mathcal{X}) \le |\mathcal{X}|$.

LEMMA EC.19. *Assume that the distribution F_ε admit strictly positive density F_ε. Given N_0 and B, the equality $M_{\boldsymbol{\theta}}(\mathcal{X}) = |\mathcal{X}|$ holds for all $(\mathbf{u}, \mathbf{v})$ outside of a set of Lebesgue measure zero.*

Proof of Theorem 1. We show that the conclusion of the theorem holds if $M_{\boldsymbol{\theta}}(\mathcal{X}) \ge 2N_0 + 2$. The result then follows by Lemma EC.19.

Fix $\boldsymbol{\theta}$, $\boldsymbol{\theta}'$, and $\mathcal{X}$ such that $M_{\boldsymbol{\theta}}(\mathcal{X}) \ge 2N_0 + 2$. Since the SP-choice model is invariant under constant shift of the utilities, we assume without loss of generality that $u_1 = u_1' = 0$. Suppose that the choice probabilities satisfy $P_{\boldsymbol{\theta}'}(i \mid \boldsymbol{x}) = P_{\boldsymbol{\theta}}(i \mid \boldsymbol{x})$, for all $i \in [N]$ and $\boldsymbol{x} \in \mathcal{X}$. We show, in order, that $\mathbf{u} = \mathbf{u}'$ and $w_{\boldsymbol{\theta}}(\boldsymbol{x}) = w_{\boldsymbol{\theta}'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$, that $(N_0, B) = (N_0', B')$, and that $\mathbf{v}$, $\mathbf{v}'$ are equal when sorted in ascending order. The result then follows. Since $P_{\boldsymbol{\theta}'}(i \mid \boldsymbol{x}) = P_{\boldsymbol{\theta}}(i \mid \boldsymbol{x})$, it follows by definition that $F_\varepsilon\big(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - u_i\big) = F_\varepsilon\big(w_{\boldsymbol{\theta}'}(\boldsymbol{x}) - u_i'\big), \quad i \in [N], \boldsymbol{x} \in \mathcal{X}$. Since $B < \|\boldsymbol{x}\|_1 + N_0$, both $w_{\boldsymbol{\theta}}(\boldsymbol{x})$ and $w_{\boldsymbol{\theta}'}(\boldsymbol{x})$ are finite for all $x \in \mathcal{X}$, and our assumption that F_ε is strictly increasing implies that for all i such that $x_i = 1$,

$$w_{\boldsymbol{\theta}}(\boldsymbol{x}) - u_i = w_{\boldsymbol{\theta}'}(\boldsymbol{x}) - u_i' \quad \text{or, equivalently,} \quad w_{\boldsymbol{\theta}'}(\boldsymbol{x}) - w_{\boldsymbol{\theta}}(\boldsymbol{x}) = \Delta_i(\boldsymbol{x}),$$

where $\Delta_i(\boldsymbol{x}) = u_i - u_i'$. Note that $\Delta_i(\boldsymbol{x})$ depends on $\boldsymbol{x}$ only via $x_i = 1$.

It follows that $\Delta_i(\boldsymbol{x}) = \Delta_{i'}(\boldsymbol{x})$ if two items i, i' co-occur in some offered $\boldsymbol{x}$ (i.e., $x_i = x_{i'} = 1$). Since we assumed that the undirected co-occurrence graph of $\mathcal{X}$ is connected, at least one assortment (call it $\boldsymbol{x}^{(1)}$) must offer item $i = 1$, and, therefore, $\Delta_1(\boldsymbol{x}^{(1)}) = 0$. Furthermore, since there is a path in the co-occurrence graph from any item j to item $i = 1$, it follows that $u_j - u_j' = u_1 - u_1' = 0$ for all $j \in [N]$, and, therefore, $w_{\boldsymbol{\theta}}(\boldsymbol{x}) = w_{\boldsymbol{\theta}'}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathcal{X}$.

We now argue that $(N_0, B) = (N_0', B')$ and that $\mathbf{v}$, $\mathbf{v}'$ are equal when sorted in ascending order. Let $g(x) = 1 - F_\xi(x)$. Since $\mathbf{u} = \mathbf{u}'$ and $w_{\boldsymbol{\theta}}(\boldsymbol{x}) = w_{\boldsymbol{\theta}'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{X}$, it follows that

$$\sum_{i=1}^{N_0} x_i\big(1 - F_\varepsilon(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - u_i)\big) + \sum_{j=1}^{N_0} g(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - v_j) = B,$$
$$\sum_{i=1}^{N_0} x_i\big(1 - F_\varepsilon(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - u_i)\big) + \sum_{k=1}^{N_0'} g(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - v_k') = B'. \quad \boldsymbol{x} \in \mathcal{X}.$$

Subtracting the two equations yields

$$\sum_{j=1}^{N_0} g(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - v_j) - \sum_{k=1}^{N_0'} g(w_{\boldsymbol{\theta}}(\boldsymbol{x}) - v_k') = B - B', \quad x \in \mathcal{X}. \tag{EC.44}$$

Letting $w_1 < \cdots < w_{M_{\boldsymbol{\theta}}(\mathcal{X})}$ be the distinct thresholds in $\{w_{\boldsymbol{\theta}}(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}\}$, sorted in increasing order, it follows that

$$\sum_{j=1}^{N_0}\big(g(w_m - v_j) - g(w_1 - v_j)\big) = \sum_{k=1}^{N_0'}\big(g(w_m - v_k') - g(w_1 - v_k')\big), \quad m = 2, \ldots, M_{\boldsymbol{\theta}}(\mathcal{X}).$$

For $m = 2, \ldots, M_{\boldsymbol{\theta}}(\mathcal{X})$ we define $\phi_m(x) = g(w_1 - x) - g(w_m - x) = F(w_m - x) - F(w_1 - x)$, so that

$$\sum_{j=1}^{N_0} \phi_m(v_j) = \sum_{k=1}^{N_0'} \phi_m(v_k') \qquad m = 2, \ldots, M_{\boldsymbol{\theta}}(\mathcal{X}). \tag{EC.45}$$

By Lemma EC.18, we know that $\{\phi_m(\cdot)\}_{m=2}^{M_{\boldsymbol{\theta}}(\mathcal{X})}$ is a Tchebycheff system on any finite interval $[a, b]$. Fix any finite interval $[a, b]$ that contains all points in $\{v_1, \ldots, v_{N_0}\} \cup \{v_1', \ldots, v_{N_0'}'\}$. Since $M_{\boldsymbol{\theta}}(\mathcal{X}) \geq 2N_0 + 2$, we have $N_0 < \frac{M_{\boldsymbol{\theta}}(\mathcal{X}) - 1}{2}$, the principal representation Theorem of T-systems (Karlin and Studden 1966, Chapter 2, Theorem 2.1) implies the uniqueness of the representation in (EC.45). In particular, it implies that $N_0 = N_0'$ and that $\mathbf{v}$ and $\mathbf{v}'$ are equal when sorted in ascending order. It follows from (EC.44) that $B = B'$. $\square$

We now prove Lemma EC.18 and Lemma EC.19.

Proof of Lemma EC.18 Given $\{a_{ij} \in \mathbb{R} : 1 \leq i, j \leq M\}$, we write $(a_{ij})_{i,j=1}^M$ to denote the matrix whose (i, j)th entry is a_{ij}. We need to show that for any $a \leq t_2 < \cdots < t_M \leq b$, $\det\big(\phi_m(t_j)\big)_{j,m=2}^M > 0$.

Since $\phi_m(t_j) = \int_{\mathbb{R}} \mathbf{1}_{\{w_1 < s \leq w_m\}} f(s - t_j)\, ds$, the Andréief identity (Andréief 1883) yields

$$\det\big(\phi_m(t_j)\big)_{j,m=2}^M = \frac{1}{(M-1)!} \int_{\mathbb{R}^{M-1}} \det\big(f(s_{k-1} - t_j)\big)_{j,k=2}^M \det\big(\mathbf{1}_{\{w_1 < s_{k-1} \leq w_m\}}\big)_{k,m=2}^M ds. \tag{EC.46}$$

Furthermore since for any permutation $\{\sigma(1), \ldots, \sigma(M-1)\}$ of $\{1, \ldots, M-1\}$,

$$\det\big(f(s_{\sigma(k-1)} - t_j)\big)_{j,k=2}^M = \operatorname{sgn}(\sigma) \det\big(f(s_{k-1} - t_j)\big)_{j,k=2}^M,$$
$$\det\big(\mathbf{1}_{\{w_1 < s_{\sigma(k-1)} \leq w_m\}}\big)_{k,m=2}^M = \operatorname{sgn}(\sigma) \det\big(\mathbf{1}_{\{w_1 < s_{k-1} \leq w_m\}}\big)_{k,m=2}^M,$$

where $\mathrm{sgn}(\sigma) = 1$ if the permutation consists of an even number of pairwise swaps of indices, and $\mathrm{sgn}(\sigma) = -1$ if the number of swaps is odd. Therefore, the integrand in (EC.46) is invariant under any permutation of s. Defining $\Lambda = \{s \in \mathbb{R}^{M-1} : s_1 < \cdots < s_{M-1}\}$ and noting that there are $(M-1)!$ possible permutations of s, it follows that the right-hand side of (EC.46) equals

$$\frac{1}{(M-1)!} \cdot (M-1)! \int_\Lambda (\mathrm{sgn}(\sigma))^2 \det\left(f(s_{k-1} - t_j)\right)_{j,k=2}^M \det\left(\mathbf{1}_{\{w_1 < s_{k-1} \le w_m\}}\right)_{k,m=2}^M ds$$
$$= \int_\Lambda \det\left(f(s_{k-1} - t_j)\right)_{j,k=2}^M \det\left(\mathbf{1}_{\{w_1 < s_{k-1} \le w_m\}}\right)_{k,m=2}^M.$$

To conclude, we argue that the integrand is positive on some set with positive Lebesgue measure, and nonnegative elsewhere on Λ. Since f is PF_∞, Definition 3 yields

$$\det\left(f(s_{k-1} - t_j)\right)_{j,k=2}^M > 0 \qquad \text{for all } s \in \Lambda.$$

Next, we let $A(s) = \left(\mathbf{1}_{\{w_1 < s_{k-1} \le w_m\}}\right)_{k,m=2}^M$ and define the interlacing region

$$R = \{s \in \mathbb{R}^{M-1} : w_1 < s_1 \le w_2 < s_2 \le w_3 < \cdots < s_{M-1} \le w_M\} \subset \Lambda.$$

For any row k, the entries of $A(s)$ satisfy $A_{k,m}(s) = 0$ for $m < k$ and $A_{k,k}(s) = 1$, implying that $A(s)$ is upper triangular with ones on the diagonal and, therefore, $\det A(s) = 1$ on R. Since R has positive Lebesgue measure, it remains to show that $\det A(s) = 0$ for $s \notin R$.

The structure of $A(s)$ is that each row consists of leading zeros followed by ones. Moreover, because $s_1 < \cdots < s_{M-1}$ and $w_2 < \cdots < w_M$, the number of leading zeros is nondecreasing in the row index. Therefore, $\det A(s) \neq 0$ if and only if $A(s)$ is upper triangular with ones on the diagonal, which holds precisely when $w_{k-1} < s_{k-1} \le w_k$ for each $2 \le k \le M$ or, equivalently, $s \in R$. $\square$

Proof of Lemma EC.19. Denote by λ^d the d-dimensional Lebesgue measure on $\mathbb{R}^d$. By the definition of $M_{\boldsymbol{\theta}}(\mathcal{X})$, it is sufficient to show that for λ^{N+N_0}-a.e. $(\mathbf{u}, \mathbf{v})$ the values $w_{\boldsymbol{\theta}}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{X}$ are pairwise distinct. Fix any distinct $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$ and choose i with $x_i \neq y_i$. Write $\mathbf{u}_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$. Since F_ε has a positive density, by the characterization of $w_{\boldsymbol{\theta}}(\boldsymbol{x})$ in (EC.43), the map $u_i \mapsto w_{\boldsymbol{\theta}}(\boldsymbol{x})$ is strictly increasing if $x_i = 1$ and constant if $x_i = 0$. The same result applies to $\boldsymbol{y}$. Therefore the function $\Phi_{\boldsymbol{x},\boldsymbol{y}}(u_i, \mathbf{u}_{-i}, \mathbf{v}) = w_\theta(\boldsymbol{x}) - w_\theta(\boldsymbol{y})$ is strictly monotone in u_i (increasing if $x_i = 1, y_i = 0$, decreasing if $x_i = 0, y_i = 1$). Consequently, for each fixed $(\mathbf{u}_{-i}, \mathbf{v})$ the set $S_{\boldsymbol{x},\boldsymbol{y}}(\mathbf{u}_{-i}, \mathbf{v}) = \{\, u_i \in \mathbb{R} : \Phi_{\boldsymbol{x},\boldsymbol{y}}(u_i, \mathbf{u}_{-i}, \mathbf{v}) = 0 \,\}$ contains at most one point and thus has 1-dimensional Lebesgue measure 0. By Fubini's theorem,

$$\lambda^{N+N_0}\big(\{(\mathbf{u}, \mathbf{v}) : w_\theta(\boldsymbol{x}) = w_\theta(\boldsymbol{y})\}\big) = \int_{\mathbb{R}^{N+N_0-1}} \lambda^1\big(S_{\boldsymbol{x},\boldsymbol{y}}(\mathbf{u}_{-i}, \mathbf{v})\big)\, d\lambda^{N+N_0-1}(\mathbf{u}_{-i}, \mathbf{v}) = 0.$$

Since $\mathcal{X}$ is finite, the union over $\boldsymbol{x} \neq \boldsymbol{y}$ is still of Lebesgue measure zero. Hence for λ^{N+N_0}-a.e. $(\mathbf{u}, \mathbf{v})$ the values $w_{\boldsymbol{\theta}}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{X}$ are pairwise distinct. $\square$

EC.10.2. Proof of Proposition 1.

Proof of Proposition 1. Fix $\boldsymbol{\theta} = (N_0, B, \mathbf{u}, \mathbf{v})$, pick any $N_0' > B' \geq B$, and set $\mathbf{u}' = \mathbf{u}$. Our goal is to find some $\mathbf{v}' \in \mathbb{R}^{N_0'}$ such that $P_{\boldsymbol{\theta}'}(i \mid \boldsymbol{e}) = P_{\boldsymbol{\theta}}(i \mid \boldsymbol{e})$ for all $i \in [N]$. We will shortly show that we can always find $c \in \mathbb{R}$ such that

$$\sum_{i=1}^{N} \big(1 - F_\varepsilon(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - u_i)\big) + N_0'\big(1 - F_\xi(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - c)\big) = B', \tag{EC.47}$$

implying that $w_{\boldsymbol{\theta}'}(\boldsymbol{e}) = w_{\boldsymbol{\theta}}(\boldsymbol{e})$ if $\mathbf{v}' = (c, c, ..., c) \in \mathbb{R}^{N_0'}$ and, therefore,

$$P_{\boldsymbol{\theta}'}(i \mid \boldsymbol{e}) = 1 - F_\xi(w_{\boldsymbol{\theta}'}(\boldsymbol{e}) - u_i') = 1 - F_\xi(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - u_i) = P_{\boldsymbol{\theta}}(i \mid \boldsymbol{e}).$$

To verify the existence of such a c, we rearrange (EC.47) to get

$$1 - F_\xi(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - c) = \frac{1}{N_0'}\left(B' - \sum_{i=1}^{N}\big(1 - F_\varepsilon(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - u_i)\big)\right).$$

The right hand side is strictly less than one because $N_0' > B'$, and strictly greater than zero because

$$B' - \sum_{i=1}^{N}\big(1 - F_\varepsilon(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - u_i)\big) = B' - B + \sum_{j=1}^{N_0}\big(1 - F_\xi(w_{\boldsymbol{\theta}}(\boldsymbol{e}) - v_j)\big) > 0,$$

which implies the existence of such a c. $\square$

EC.11. Extension to Random Consideration Set (RCS) Setting

In this section we derive the surrogate problem for the random consideration set (RCS) setting, which is a class of consider-then-choose models where customers first form a consideration set before making a choice. The RCS framework studied in the literature (e.g., Manzini and Mariotti (2014), Gallego and Li (2024), Aouad et al. (2024)) assumes that each product in the offered assortment is considered independently based on a probabilistic process—akin to flipping coins. Formally, let $p(i|\boldsymbol{x})$ denote the probability that item i is considered given assortment $\boldsymbol{x} \in \{0,1\}^N$. The inclusion of products into the consideration set is assumed to be mutually independent and independent of the random utilities $\{U_i\}_{i\in[N]}$ and $\{V_j\}_{j\in[N_0]}$. The consideration set $C \subset S(\boldsymbol{x})$ is therefore drawn with probability

$$\prod_{i\in C} p(i|\boldsymbol{x}) \prod_{i\in S(\boldsymbol{x})\setminus C}\big(1 - p(i|\boldsymbol{x})\big).$$

Denote by $Z_i(\boldsymbol{x}) \sim \textsc{Bernoulli}(p(i|\boldsymbol{x}))$ the random variable representing whether an item $i \in S(\boldsymbol{x})$ belongs to the consideration set. We can then formulate the corresponding (OP) as follows:

$$\begin{aligned} \max_{\boldsymbol{x}\in\{0,1\}^N} \quad & \pi_N^{cs}(\boldsymbol{x}) = \sum_{i=1}^{N} x_i r_i \mathbb{P}\big(Z_i(\boldsymbol{x}) = 1, U_i > W_{(B+1)}^{cs}(\boldsymbol{x})\big), \\ & s.t. \ ||\boldsymbol{x}||_1 \leq \eta N, \end{aligned} \tag{OP}$$

where $W^{cs}_{(B+1)}(\boldsymbol{x})$ is the $(B+1)$th order statistic among the products in the consideration set and the outside options, which can be characterized by

$$W^{cs}_{(B+1)}(\boldsymbol{x}) = \inf\left\{W : \sum_{i=1}^{N} x_i \mathbb{1}\{Z_i(\boldsymbol{x}) = 1, U_i > W\} + \sum_{j=1}^{N_0} \mathbb{1}\{V_j > W\} \leq B\right\}.$$

To construct the corresponding surrogate problem, we approximate $W^{cs}_{(B+1)}(\boldsymbol{x})$ with a deterministic counterpart

$$w^{cs}_{(B+1)}(\boldsymbol{x}) = \inf\left\{w : \sum_{i=1}^{N} x_i p(i|\boldsymbol{x})\big(1 - F_\varepsilon(w - u_i)\big) + \sum_{j=1}^{N_0}\big(1 - F_\xi(w - v_j)\big) \leq B\right\}.$$

This leads to the following surrogate problem (SP)

$$\begin{aligned} \max_{\boldsymbol{x}\in\{0,1\}^N} \quad & \hat{\pi}^{cs}_N(\boldsymbol{x}) = \sum_{i=1}^{N} x_i r_i p(i|\boldsymbol{x})\big(1 - F_\varepsilon(w^{cs}_{(B+1)}(\boldsymbol{x}) - u_i)\big) \\ \text{s.t.} \quad & ||\boldsymbol{x}||_1 \leq \eta N. \end{aligned} \tag{SP}$$

In the RCS-MNL setting of Gallego and Li (2024), the consideration probability takes the form $p(i \mid \boldsymbol{x}) = \lambda_i x_i$, where λ_i is an item-specific consideration parameter. Under this specification, the surrogate problem retains the structure of the original (SP): for a fixed threshold w, both the objective and the threshold constraint are linear in $\boldsymbol{x}$. It can therefore be solved efficiently using the same line-search procedure discussed in Section 4.1.

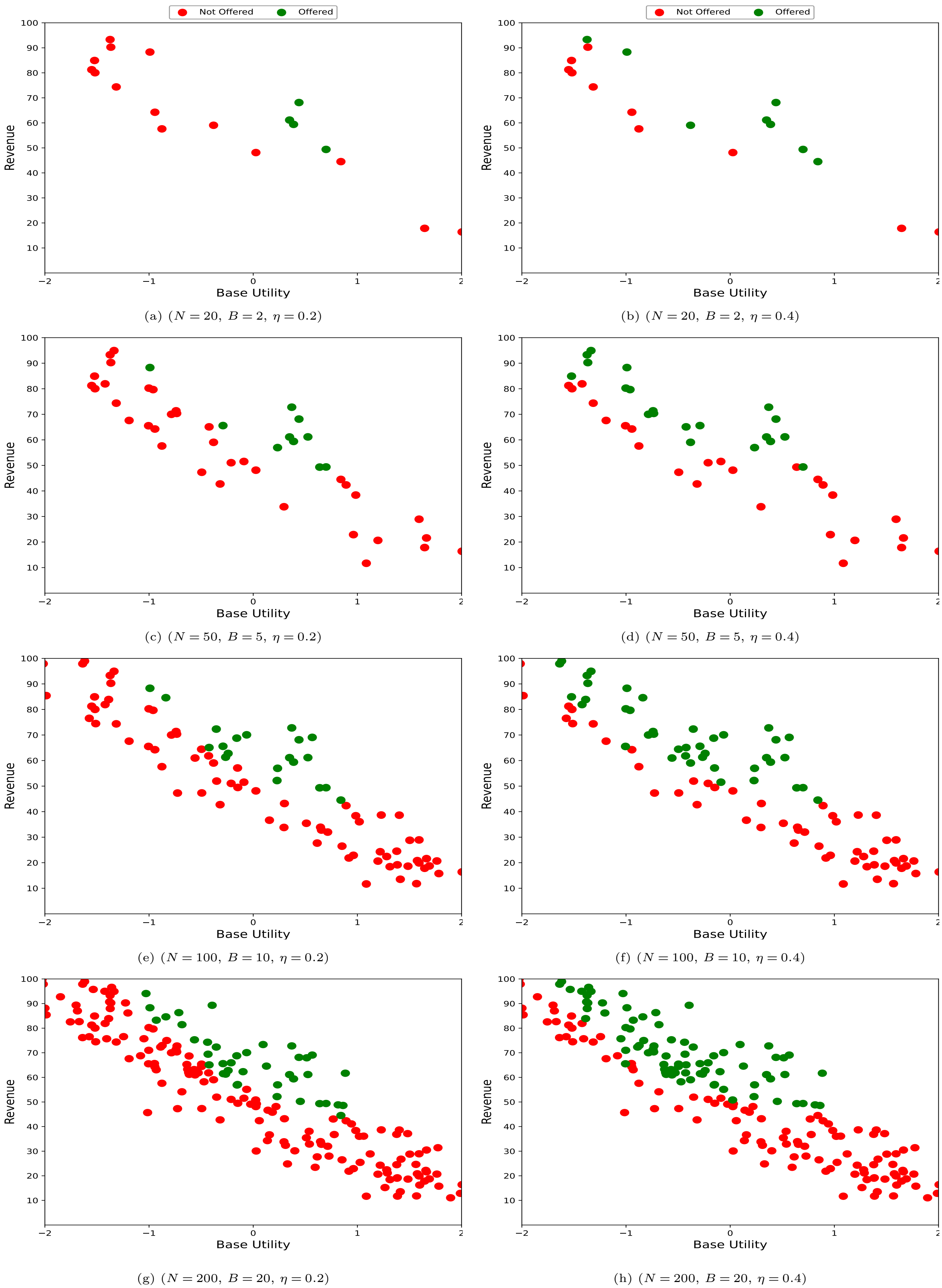
EC.12. Numerical Simulations: Revenue-Dependent Utilities

In Section 5, we considered a setting where the items' base utilities are independent of their respective revenues/prices. In this section, we repeat the same simulations assuming that the base utilities are decreasing functions of their respective revenues/prices.

In particular, we first generate $r_i \sim$ Uniform$(10, 100)$ for $1 \leq i \leq N$, and then set $u_i = -(r_i - 55)/25 + \epsilon_{u,i}$, where $\{\epsilon_{u,i} : 1 \leq i \leq N\}$ are i.i.d. following a Normal$(0, 0.5)$ distribution. Finally $\{v_j : 1 \leq j \leq N_0\}$ is generated, similar to $\{u_i : 1 \leq i \leq N\}$, by $v_j = -(r_{v,j} - 55)/25 + \epsilon_{v,j}$, where $\{r_{v,j} : j = 1, 2, ..., N_0\}$ are i.i.d. following a Uniform$(10, 100)$ distribution, and $\{\epsilon_{v,j} : 1 \leq j \leq N_0\}$ are i.i.d. following Normal$(0, 0.5)$ distribution. All elements of the set $\{r_i, \epsilon_{u,i} : 1 \leq i \leq N\} \cup \{r_{v,j}, \epsilon_{v,j} : 1 \leq j \leq N_0\}$ are mutually independent.

The scatter plot illustrating assortment selection is presented in Figure EC.1. Figure EC.2 displays the box plot comparing the relative performance of the SP-based solution to optimal assortments in a multi-purchase setting.

Figure EC.3 shows the relative performance of the RSP-based solution compared to the SP-based solution, while Figure EC.4 shows the performance of the SP-based solution under a single-purchase MNL setting.



(a) $(N = 20,\ B = 2,\ \eta = 0.2)$

(b) $(N = 20,\ B = 2,\ \eta = 0.4)$

(c) $(N = 50,\ B = 5,\ \eta = 0.2)$

(d) $(N = 50,\ B = 5,\ \eta = 0.4)$

(e) $(N = 100,\ B = 10,\ \eta = 0.2)$

(f) $(N = 100,\ B = 10,\ \eta = 0.4)$

(g) $(N = 200,\ B = 20,\ \eta = 0.2)$

(h) $(N = 200,\ B = 20,\ \eta = 0.4)$

Figure EC.1 Counterpart of Figure 1 with revenue-dependent utilities.

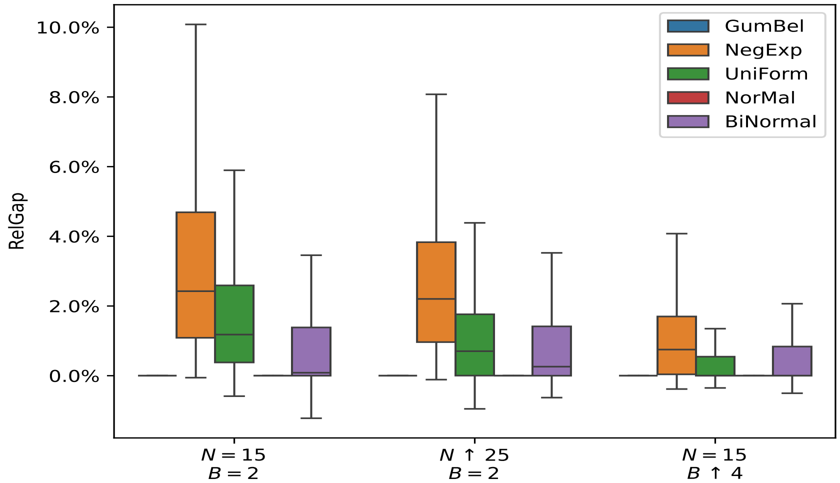
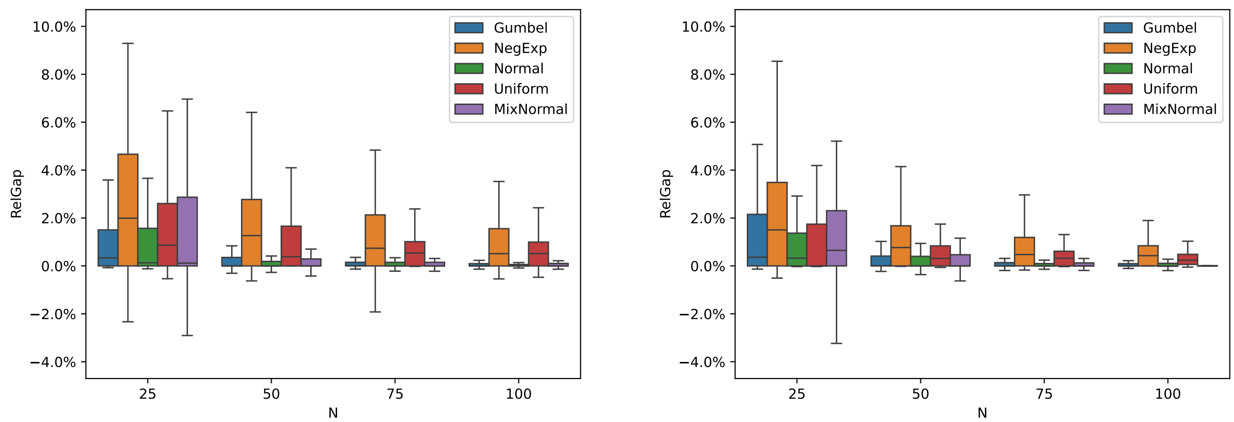
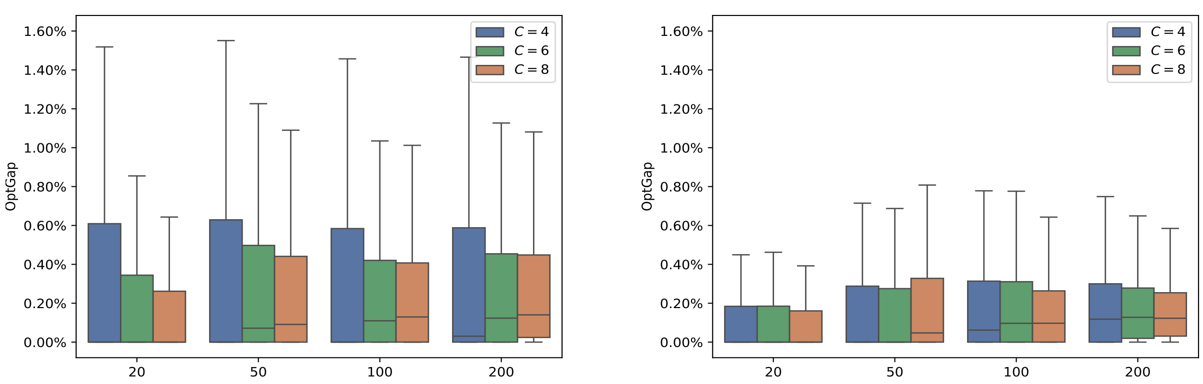


Figure EC.2 Counterpart of Figure 2 with revenue-dependent utilities.



(a) $B = 2$ (b) $B = 4$

Figure EC.3 Counterpart of Figure 3 with revenue-dependent utilities.



(a) Cardinality Constraint (b) Space Constraint

Figure EC.4 Counterpart of Figure 5 with revenue-dependent utilities.

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