

Electronic companion to “Optimizing Input Data Collection for Ranking and Selection”

EC.1. Proofs and Discussions on the Theory in Section 3

EC.1.1. Discussion on Assumption 4 and Proof of Proposition 1

We first show that the strong consistency of the MAP to θ_0 can be guaranteed under Assumption 4 as long as $m_\ell \rightarrow \infty$ for all $1 \leq \ell \leq L$, which is weaker than Condition 1 since it does not require the convergence of ratio $c_\ell m_\ell / T$.

LEMMA EC.1. *If Assumption 4 holds and $\lim_{T \rightarrow \infty} m_\ell = \infty$ for all $1 \leq \ell \leq L$, then $\hat{\theta}_T \xrightarrow{a.s.} \theta_0$.*

Proof. Observe that $\theta_0^\ell = \arg \min_{\theta^\ell} D^{\text{KL}}(\theta_0^\ell || \theta^\ell) = \arg \min_{\theta^\ell} E_{Z^\ell \sim f_{\theta_0^\ell}} \left[-\log \left(\frac{f_{\theta^\ell}(Z^\ell)}{f_{\theta_0^\ell}(Z^\ell)} \right) \right]$ and $\hat{\theta}_T^\ell = \arg \min_{\theta^\ell} \frac{1}{m_\ell} \left\{ -\log \pi_0^\ell(\theta^\ell) + \sum_{i=1}^{m_\ell} -\log f_{\theta^\ell}(Z_i^\ell) \right\} = \arg \min_{\theta^\ell} \frac{1}{m_\ell} \left\{ -\log \pi_0^\ell(\theta^\ell) + \sum_{i=1}^{m_\ell} -\log \left(f_{\theta^\ell}(Z_i^\ell) / f_{\theta_0^\ell}(Z_i^\ell) \right) \right\}$. Considering the two optimization problems' objective functions, $\lim_{m_\ell \rightarrow \infty} \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} -\log \left(\frac{f_{\theta^\ell}(Z_i^\ell)}{f_{\theta_0^\ell}(Z_i^\ell)} \right) = E_{Z^\ell \sim f_{\theta_0^\ell}} \left[-\log \left(\frac{f_{\theta^\ell}(Z^\ell)}{f_{\theta_0^\ell}(Z^\ell)} \right) \right]$ almost surely uniformly over $\theta^\ell \in \Theta^\ell$ from Assumption 4(c). By Proposition 3 in Shapiro (2003), the convergence of the optimal solution then follows, i.e., $\hat{\theta}_T^\ell \xrightarrow{a.s.} \theta_0^\ell$. Q.E.D.

Some sufficient conditions for GC class have been studied in the empirical process literature. We briefly mention three conditions. One is provided in Theorem 19.4 in van der Vaart (2000) by exploiting a bracketing number of a function class. From Examples 19.7 and 19.8 in van der Vaart (2000), we can derive sufficient conditions for Assumption 4(c) by assuming Lipschitz continuity and existence of an integrable envelope function, respectively. The detailed theory is beyond the scope of this paper; see van der Vaart (2000) and van der Vaart and Wellner (1996) for further reading.

Proposition 1 states a sufficient condition that implies Assumption 4(c). Below, we prove it for the case of $L = 1$, which can be extended to $L > 1$ easily.

Proof of Proposition 1 ($L = 1$). From the properties of exponential family: $E[S(\mathbf{Z})] = \nabla A(\theta_0)$, $\text{Cov}[S(\mathbf{Z})] = \nabla^2 A(\theta_0)$, and the KL divergence is $D^{\text{KL}}(\theta_0 || \theta) = A(\theta) - A(\theta_0) - \langle \nabla A(\theta_0), \theta - \theta_0 \rangle$. Therefore,

$$\begin{aligned} \sup_{\theta \in \Theta} \left| \frac{1}{m} \sum_{i=1}^m \left(-\log \frac{f_\theta}{f_{\theta_0}} \right) (\mathbf{Z}_i) - D^{\text{KL}}(\theta_0 || \theta) \right| &= \sup_{\theta \in \Theta} \left| \left\langle \theta_0 - \theta, \frac{1}{m} \sum_{i=1}^m (S(\mathbf{Z}_i) - \nabla A(\theta_0)) \right\rangle \right| \\ &\leq 2C \frac{1}{m} \sum_{i=1}^m \|S(\mathbf{Z}_i) - \nabla A(\theta_0)\|_1 =: F \end{aligned}$$

where $C = \max_{\theta \in \Theta} \|\theta\|_\infty$, which is bounded from compactness of Θ , and the inequality follows from Hölder's inequality. From Example 19.8 in van der Vaart (2000), if F is integrable, then $\{\log(f_\theta/f_{\theta_0})\}_{\theta \in \Theta}$ is P_{θ_0} -

Glivenko-Cantelli. Hence, it suffices to show that F is integrable. By the second-order differentiability of $A(\theta)$, F is indeed integrable since

$$\mathbb{E}[F] = 2CE[\|S(\mathbf{Z}) - \nabla A(\theta_0)\|_1] \leq 2C \sum_{j=1}^{\dim(S)} \sqrt{\text{Var}(S_j(\mathbf{Z}))} = 2C \sum_{j=1}^{\dim(S)} \sqrt{\frac{\partial^2}{\partial \theta_j^2} A(\theta_0)} < \infty,$$

where S_j represents the j -th element of S . For the first inequality we use $\mathbb{E}[X]^2 \leq \mathbb{E}[X^2]$. Q.E.D.

EC.1.2. Proof of Theorem 1

Let us first show Part (b). Because $P_{\pi_m}(\Theta) = 1$, observe that

$$\begin{aligned} -\frac{1}{T} \log P_{\pi_m}(\Theta_{i_0}^c) &= -\frac{1}{T} \log P_{\pi_m}(\Theta_{i_0}^c) + \frac{1}{T} \log P_{\pi_m}(\Theta) \\ &= -\frac{1}{T} \log \int_{\Theta_{i_0}^c} \frac{\pi_m(\theta)}{\pi_m(\theta_0)} d\theta + \frac{1}{T} \log \int_{\Theta} \frac{\pi_m(\theta)}{\pi_m(\theta_0)} d\theta. \end{aligned} \quad (\text{EC.1})$$

To bound the integral, we first show the following uniform convergence of the log-likelihood ratio, $\log(\pi_m(\theta)/\pi_m(\theta_0))$:

$$\varepsilon_T := \sup_{\theta \in \Theta} \left| \frac{1}{T} \log \left(\frac{\pi_m(\theta)}{\pi_m(\theta_0)} \right) + \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \text{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) \right| \rightarrow 0 \text{ P}_{\theta_0}\text{-a.s.} \quad (\text{EC.2})$$

as $m_\ell \rightarrow \infty$ for all $1 \leq \ell \leq L$. From the definition of π_m , we have

$$\frac{1}{T} \log \left(\frac{\pi_m(\theta)}{\pi_m(\theta_0)} \right) + \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \text{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) = \frac{1}{T} \sum_{\ell=1}^L \log \left(\frac{\pi_0^\ell(\theta^\ell)}{\pi_0^\ell(\theta_0^\ell)} \right) + \sum_{\ell=1}^L \frac{m_\ell}{T} \left(\frac{1}{m_\ell} \sum_{i=1}^{m_\ell} \log \frac{f_{\theta^\ell}^\ell(Z_i^\ell)}{f_{\theta_0^\ell}^\ell(Z_i^\ell)} + \text{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) \right) + o(1).$$

The $o(1)$ term is uniform in θ by Condition 1 and Assumption 4. For each ℓ , combining Assumption 4 (c) with $\text{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) := -\mathbb{E}_{Z^\ell \sim f_{\theta_0^\ell}^\ell} \left[\log \left(f_{\theta^\ell}^\ell(Z^\ell) / f_{\theta_0^\ell}^\ell(Z^\ell) \right) \right]$, each term in the second summation converges to zero uniformly over $\theta^\ell \in \Theta^\ell$ almost surely. Further, the first term also converges to zero uniformly by the boundedness of π_0^ℓ . Thus, $\varepsilon_T \rightarrow 0$ almost surely. For simplicity, we denote $I(\theta) := \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \text{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell)$ for $\theta = (\theta^1, \dots, \theta^L)$ from now on.

From (EC.2), we have the following two-sided bound

$$\exp(-TI(\theta) - T\varepsilon_T) \leq \frac{\pi_m(\theta)}{\pi_m(\theta_0)} \leq \exp(-TI(\theta) + T\varepsilon_T).$$

By integrating all sides over $\theta \in \Theta_{i_0}^c$, we get

$$\exp(-T\varepsilon_T) \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta \leq \int_{\Theta_{i_0}^c} \frac{\pi_m(\theta)}{\pi_m(\theta_0)} d\theta \leq \exp(T\varepsilon_T) \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta.$$

After taking logarithms on all sides, dividing by T , and letting $T \rightarrow \infty$, we obtain

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{T} \log \int_{\Theta_{i_0}^c} \frac{\pi_m(\theta)}{\pi_m(\theta_0)} d\theta - \frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta \right\} = 0, \text{ P}_{\theta_0}\text{-a.s.}, \quad (\text{EC.3})$$

thanks to (EC.2). Our next goal is to approximate the second term in (EC.3). Let θ_1 be a minimizer of $I(\theta)$ in the closure of $\Theta_{i_0}^c$. Then, we have

$$\frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta + I(\theta_1) \leq \frac{1}{T} \log \left(\exp(-TI(\theta_1)) \int_{\Theta_{i_0}^c} d\theta \right) + I(\theta_1) = \frac{1}{T} \log \text{Vol}(\Theta_{i_0}^c)$$

where the volume $\text{Vol}(\tilde{\Theta}) := \int_{\tilde{\Theta}} d\theta$. Taking the limsup with respect to T on both sides of the inequality, we have

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta + I(\theta_1) \leq 0.$$

Equivalently,

$$\liminf_{T \rightarrow \infty} -\frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta \geq \inf_{\theta \in \text{int}(\Theta_{i_0}^c)} I(\theta) \text{ P}_{\theta_0}\text{-a.s.} \quad (\text{EC.4})$$

Next, we aim to show

$$\limsup_{T \rightarrow \infty} -\frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta \leq \inf_{\theta \in \text{int}(\Theta_{i_0}^c)} I(\theta) \text{ P}_{\theta_0}\text{-a.s.} \quad (\text{EC.5})$$

When $\text{int}(\Theta_{i_0}^c) = \emptyset$, (EC.5) is straightforward since the infimum over the empty set is ∞ by convention. Suppose $\text{int}(\Theta_{i_0}^c) \neq \emptyset$. Since a Cartesian product of compact sets is compact, Assumption 4(a) implies that $\Theta = \prod_{\ell=1}^L \Theta^\ell$ is also compact. Assumptions 4(a) and (d) imply uniform continuity of $I(\theta)$ for $\theta \in \Theta$. In other words, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|\theta - \nu\|_\infty \leq 2\delta$ implies $|I(\theta) - I(\nu)| \leq \epsilon$. For such $\delta > 0$, compactness of Θ implies that there exists finite open cover $\{o_1, o_2, \dots, o_M\}$, where o_j is a δ -radius open ball in Θ such that $\Theta \subset \cup_{j=1}^M o_j$. Let us remove all balls that do not intersect with $\text{int}(\Theta_{i_0}^c)$. Abusing the notation, let us say $\text{int}(\Theta_{i_0}^c) \subset \cup_{j=1}^M o_j$. Let θ_2 be a minimizer of $\inf_{\theta \in \text{int}(\Theta_{i_0}^c)} I(\theta)$. Then, $\theta_2 \in \text{cl}(\text{int}(\Theta_{i_0}^c))$ and θ_2 must be contained in at least one of $\{\text{cl}(o_j), 1 \leq j \leq M\}$. Without loss of generality, suppose $\theta_2 \in \text{cl}(o_1)$. Furthermore, $\text{Vol}(o_1 \cap \text{int}(\Theta_{i_0}^c)) > 0$ since $o_1 \cap \text{int}(\Theta_{i_0}^c)$ is a nonempty open set. Consequently,

$$\begin{aligned} \frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta + I(\theta_2) &\geq \frac{1}{T} \log \int_{\text{int}(\Theta_{i_0}^c) \cap o_1} \exp(-TI(\theta)) d\theta + I(\theta_2) \\ &\geq \frac{1}{T} \log \int_{\text{int}(\Theta_{i_0}^c) \cap o_1} \exp(-TI(\theta_2) - T\epsilon) d\theta + I(\theta_2) \\ &= \frac{1}{T} \log \text{Vol}(\text{int}(\Theta_{i_0}^c) \cap o_1) - \epsilon \end{aligned}$$

where the second inequality holds by the uniform continuity. Hence, we can see that

$$\liminf_{T \rightarrow \infty} \left\{ \frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta + I(\theta_2) \right\} \geq -\epsilon,$$

a.s., for any $\epsilon > 0$, which implies (EC.5).

By Assumption 3, we have $\theta_0 \in \text{int}(\Theta)$. It implies that $\inf_{\theta \in \text{int}(\Theta)} I(\theta) = \inf_{\theta \in \Theta} I(\theta) = 0$. Therefore, we obtain $\lim_{T \rightarrow \infty} -\frac{1}{T} \log \int_{\Theta} \exp(-TI(\theta)) d\theta = 0$, P_{θ_0} -a.s. Finally, we have

$$\begin{aligned} \liminf_{T \rightarrow \infty} -\frac{1}{T} \log P_{\pi_m}(\Theta_{i_0}^c) &\stackrel{\text{(EC.1)}}{=} \liminf_{T \rightarrow \infty} \left\{ -\frac{1}{T} \log \int_{\Theta_{i_0}^c} \frac{\pi_m(\theta)}{\pi_m(\theta_0)} d\theta + \frac{1}{T} \log \int_{\Theta} \frac{\pi_m(\theta)}{\pi_m(\theta_0)} d\theta \right\} \\ &\stackrel{\text{(EC.3)}}{=} \liminf_{T \rightarrow \infty} \left\{ -\frac{1}{T} \log \int_{\Theta_{i_0}^c} \exp(-TI(\theta)) d\theta + \frac{1}{T} \log \int_{\Theta} \exp(-TI(\theta)) d\theta \right\} \\ &\stackrel{\text{(EC.4)}}{\geq} \inf_{\theta \in \Theta_{i_0}^c} I(\theta) \end{aligned}$$

as desired. To show $\limsup_{T \rightarrow \infty} -\frac{1}{T} \log P_{\pi_m}(\Theta_{i_0}^c) \leq \inf_{\theta \in \text{int}(\Theta_{i_0}^c)} I(\theta)$, a similar argument can be applied to (EC.5). Consequently, $\lim_{T \rightarrow \infty} -\frac{1}{T} \log P_{\pi_m}(\Theta_{i_0}^c) = \inf_{\theta \in \text{int}(\Theta_{i_0}^c)} I(\theta)$ for some $I(\theta) > 0$ from Condition 1.

We proceed to show Part (a). From Part (b), $P_{\pi_m}(\Theta_{i_0}) \rightarrow 1$ and $P_{\pi_m}(\Theta_j) \rightarrow 0$ for $j \neq i_0$. Let us define $\mathcal{E} \subset \Omega^{\mathbb{N}}$ such that $\lim_{T \rightarrow \infty} P_{\pi_m}(\Theta_i) = 1\{i = i_0\}$ for all sample paths of input data that belong in $\mathcal{E}$. Then, for each sample path in $\mathcal{E}$, there exists T_1 such that for all $T \geq T_1$, $P_{\pi_m}(\Theta_{i_0}) > 1/2$ and $P_{\pi_m}(\Theta_i) < 1/2$. This implies $i^*(\pi_m) = i_0$ for all $T \geq T_1$. Since $P_{\theta_0}^{\mathbb{N}}(\mathcal{E}) = 1$, the proof is complete. Q.E.D.

REMARK EC.1. For the case when the input model belongs to an exponential family, Proposition 5 of Russo (2020) also derives a posterior large deviation rate. This result coincides with our Theorem 1 when $L = 1$, however, Theorem 1 is more general as it does not restrict the distribution family.

EC.1.3. Extension to Good Selection

Suppose that a decision-maker is content with selecting any “good” solution in $\mathcal{I}_{\delta} = \{i : \eta_i(\theta_0) - \delta < \min_{j \neq i} \eta_j(\theta_0)\}$ for some user-chosen $\delta > 0$. These are the solutions whose performance under θ_0 is within δ from an optimum. By definition, any optimum under θ_0 is in $\mathcal{I}_{\delta}$. Ignoring the stochastic error in simulation for the moment, the good selection event occurs when $i^*(\pi_m) \in \mathcal{I}_{\delta}$. In the following, we show that our framework can be modified to guarantee good selection almost surely as $T \rightarrow \infty$.

For some $i \in \mathcal{I}_{\delta}$, we shift $\eta_i(\theta_b)$ by subtracting δ for all θ_b , i.e., $\tilde{\eta}_i(\theta_b) = \eta_i(\theta_b) - \delta$, and $\tilde{\eta}_j(\theta_b) = \eta_j(\theta_b)$ for all $j \neq i$. This can be done by simply subtracting δ from all of the simulation outputs observed for i . Under this modification, Solution i becomes the unique optimum. Let $\tilde{i}(\pi_m)$ denote the MPB for the modified problem. Then, following the same steps as in the proof of Theorem 1, we can (a) show that $\tilde{i}(\pi_m)$ converges to i almost surely; and (b) derive the limit of $-\frac{1}{T} \log(1 - P_{\pi_m}(\tilde{\Theta}_{\tilde{i}(\pi_m)}))$, where $\tilde{\Theta}_{\tilde{i}(\pi_m)} = \{\theta_b : \tilde{\eta}_{\tilde{i}(\pi_m)}(\theta_b) \leq \min_{i \neq \tilde{i}(\pi_m)} \tilde{\eta}_i(\theta_b)\}$. From the latter, one can formulate a program to optimize the input data collection ratios as in (2) to most efficiently find Solution i . Because we consider the case when $\mathcal{I}_{\delta}$ contains multiple solutions, we may target the solution producing the largest convergence rate in (b) instead of selecting an arbitrary solution in $\mathcal{I}_{\delta}$. Of course, with simulation error, $\mathcal{I}_{\delta}$ is estimated with error, which can be analyzed similarly as in our development in Section 4 although we do not pursue further in this work.

EC.2. Proof of Statements in Section 4

EC.2.1. Proof of Theorem 2

Strong consistency of i_T^* . As shown in Theorem 1, we have $\lim_{T \rightarrow \infty} \pi_m(\theta_0) = 1$ almost surely. Hence, i_T^* converges to the estimated conditional optimum at θ_0 almost surely. Thanks to $N_{i,T}(\theta_b) \rightarrow \infty$ for all (i, θ_b) in Condition 2, we can apply the strong law of large numbers, and this yields the estimated conditional optimum at θ_0 converges to $i_0 = \arg \min_{1 \leq i \leq k} \eta_i(\theta_0)$ almost surely.

The convergence rate of $E[P_{\pi_m}(\Theta_{i_T^*,T})1\{i_T^* = i^*(\pi_m)\}|\pi_m]$. For each $\theta_b \in \Theta$, define

$$\tilde{G}_i(\theta_b) := \lim_{T \rightarrow \infty} -\frac{1}{T} \log P(i_T^b = i).$$

Namely, $\tilde{G}_i(\theta_b)$ is the exponential decay rate of the probability such that $i(\neq i^b)$ is falsely selected as i^b . The expression for $\tilde{G}_i(\theta_b)$ is derived under Assumption 6 by Kim et al. (2025) to analyze the convergence rate of $E[1\{i_T^* = i^*(\pi_m)\}|\pi_m]$ for fixed π_m , i.e., no additional input data sampling is allowed. However, in our problem, as π_m becomes more concentrated with additional input data, $\tilde{G}_i(\theta_b)$ can eventually be eliminated in the rate analysis. Hence, we do not present an analytical expression of $\tilde{G}_i(\theta_b)$.

Let us first establish the following:

$$\begin{aligned} & \lim_{T \rightarrow \infty} -\frac{1}{T} \log(1 - E[P_{\pi_m}(\Theta_{i_T^*,T})1\{i_T^* = i^*(\pi_m)\}|\pi_m]) \\ &= \min \left\{ \min_{i \neq i_0} G_i(\theta_0), \min_{\tilde{\Theta} \in \mathcal{A}} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) + \sum_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} \tilde{G}_{i_0}(\theta_b) \right) \right\}, \end{aligned} \quad (\text{EC.6})$$

where $\mathcal{A} := \{\tilde{\Theta} \subseteq \Theta | \theta_0 \in \tilde{\Theta}\}$ is the collection of all subsets of Θ that include θ_0 . The proof will proceed by first showing some intermediate results in the following steps that will be needed to complete the proof.

Step 1. For sufficiently large T such that $i^*(\pi_m) = i_0$ and $\pi_m(\theta_0) > 1/2$, we have $E[P_{\pi_m}(\Theta_{i_0,T}^c)1\{i_T^* = i_0\}|\pi_m] = \sum_{\tilde{\Theta} \in \mathcal{A}} P_{\pi_m}(\tilde{\Theta}^c)P(\Theta_{i_0,T} = \tilde{\Theta})$.

Proof of Step 1. Suppose T is sufficiently large as stated. Let $\mathcal{B}$ be the power set of $\{\theta_1, \dots, \theta_B\}$. Observe that

$$\begin{aligned} E[P_{\pi_m}(\Theta_{i_0,T}^c)1\{i_T^* = i_0\}|\pi_m] &= \sum_{\tilde{\Theta} \in \mathcal{B}} E[P_{\pi_m}(\tilde{\Theta}^c)1\{i_T^* = i_0, \Theta_{i_0,T} = \tilde{\Theta}\}|\pi_m] \\ &= \sum_{\tilde{\Theta} \in \mathcal{B}} P_{\pi_m}(\tilde{\Theta}^c)P(i_T^* = i_0, \Theta_{i_0,T} = \tilde{\Theta}). \end{aligned}$$

When $\pi_m(\theta_0) > 1/2$, $\theta_0 \in \Theta_{i_0,T}$ if and only if $i_T^* = i_0$. Thus, if $\theta_0 \notin \tilde{\Theta}$, then $P\{i_T^* = i_0, \Theta_{i_0,T} = \tilde{\Theta}\} = 0$. Moreover, if $\theta_0 \in \tilde{\Theta}$, then $1\{i_T^* = i_0, \Theta_{i_0,T} = \tilde{\Theta}\} = 1\{\Theta_{i_0,T} = \tilde{\Theta}\}$. Consequently,

$$\sum_{\tilde{\Theta} \in \mathcal{B}} P_{\pi_m}(\tilde{\Theta}^c)P(i_T^* = i_0, \Theta_{i_0,T} = \tilde{\Theta}) = \sum_{\tilde{\Theta} \in \mathcal{A}} P_{\pi_m}(\tilde{\Theta}^c)P(i_T^* = i_0, \Theta_{i_0,T} = \tilde{\Theta}) = \sum_{\tilde{\Theta} \in \mathcal{A}} P_{\pi_m}(\tilde{\Theta}^c)P(\Theta_{i_0,T} = \tilde{\Theta})$$

as desired. $\square$

Step 2. For each $\tilde{\Theta} \in \mathcal{A}$, $\lim_{T \rightarrow \infty} -\frac{1}{T} \log P(\Theta_{i_0,T} = \tilde{\Theta}) = \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) + \sum_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} \tilde{G}_{i_0}(\theta_b)$.

Proof of Step 2. Notice that event $\{\tilde{\Theta} = \Theta_{i_0, T}\}$ can be written as

$$\{\Theta_{i_0, T} = \tilde{\Theta}\} = \left(\cap_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \{i_T^b \neq i_0\}\right) \cap \left(\cap_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} \{i_T^b = i_0\}\right) \cap \left(\cap_{\theta_b \in \tilde{\Theta} \cap \Theta_{i_0}} \{i_T^b = i_0\}\right) \cap \left(\cap_{\theta_b \notin \tilde{\Theta} \cup \Theta_{i_0}} \{i_T^b \neq i_0\}\right). \quad (\text{EC.7})$$

By independence across b , we have

$$\lim_{T \rightarrow \infty} -\frac{1}{T} \log P(\Theta_{i_0, T} = \tilde{\Theta}) = \lim_{T \rightarrow \infty} \left\{ \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} -\frac{1}{T} \log P(i_T^b \neq i_0) + \sum_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} -\frac{1}{T} \log P(i_T^b = i_0) \right\},$$

where the convergence rates of $\cap_{\theta_b \in \tilde{\Theta} \cap \Theta_{i_0}} \{i_T^b = i_0\}$ and $\cap_{\theta_b \notin \tilde{\Theta} \cup \Theta_{i_0}} \{i_T^b \neq i_0\}$ in (EC.7) are negligible since $\lim_{T \rightarrow \infty} P(i_T^b = i_0) = 1$ if $\theta_b \in \tilde{\Theta} \cap \Theta_{i_0}$ and $\lim_{T \rightarrow \infty} P(i_T^b \neq i_0) = 1$ if $\theta_b \notin \Theta_{i_0}$. Combining this with $\lim_{T \rightarrow \infty} -\frac{1}{T} \log P(i_T^b \neq i_0) = \min_{i \neq i_0} G_i(\theta_b)$ and $\lim_{T \rightarrow \infty} -\frac{1}{T} \log P(i_T^b = i_0) = \tilde{G}_{i_0}(\theta_b)$ completes the proof.

□

Thanks to Step 1, for all sufficiently large T such that $i^*(\pi_m) = i_0$ and $\pi_m(\theta_0) > 1/2$, we have

$$\begin{aligned} 1 - E[P_{\pi_m}(\Theta_{i_T^*, T}) 1\{i_T^* = i^*(\pi_m)\} | \pi_m] &= P(i_T^* \neq i_0) + E[P_{\pi_m}(\Theta_{i_0, T}^c) 1\{i_T^* = i_0\} | \pi_m] \\ &= P(i_T^* \neq i_0) + \sum_{\tilde{\Theta} \in \mathcal{A}} E[P_{\pi_m}(\tilde{\Theta}^c) 1\{i_T^* = i_0, \Theta_{i_0, T} = \tilde{\Theta}\} | \pi_m] \\ &= P(i_T^* \neq i_0) + \sum_{\tilde{\Theta} \in \mathcal{A}} P_{\pi_m}(\tilde{\Theta}^c) P(\Theta_{i_0, T} = \tilde{\Theta}). \end{aligned}$$

Then, from Lemma EC.4,

$$\begin{aligned} &\lim_{T \rightarrow \infty} -\frac{1}{T} \log(1 - E[P_{\pi_m}(\Theta_{i_T^*, T}) 1\{i_T^* = i^*(\pi_m)\} | \pi_m]) \\ &= \min \left\{ \lim_{T \rightarrow \infty} -\frac{1}{T} \log P(i_T^* \neq i_0), \min_{\tilde{\Theta} \in \mathcal{A}} \left(\lim_{T \rightarrow \infty} -\frac{1}{T} \log P_{\pi_m}(\tilde{\Theta}^c) + \lim_{T \rightarrow \infty} -\frac{1}{T} \log P(\Theta_{i_0, T} = \tilde{\Theta}) \right) \right\}. \end{aligned}$$

Because $\pi_m(\theta_0) > 1/2$, $\{i_T^* \neq i_0\}$ is equivalent to $\{i_T^b \neq i_0\}$ at $\theta_b = \theta_0$. Therefore, $\lim_{T \rightarrow \infty} -\frac{1}{T} \log P(i_T^* \neq i_0) = \min_{i \neq i_0} G_i(\theta_0)$, where $\min_{i \neq i_0} G_i(\theta_0)$ is the exponential convergence rate of the PFS at θ_0 . Combining Step 2 with Theorem 1, we can further observe

$$\begin{aligned} &\lim_{T \rightarrow \infty} -\frac{1}{T} \log P_{\pi_m}(\tilde{\Theta}^c) + \lim_{T \rightarrow \infty} -\frac{1}{T} \log P(\Theta_{i_0, T} = \tilde{\Theta}) \\ &= \min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) + \sum_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} \tilde{G}_{i_0}(\theta_b). \end{aligned} \quad (\text{EC.8})$$

Putting all results together (EC.6) follows.

To show the main result, we further simplify (EC.8) next. Pick $\tilde{\Theta} \in \mathcal{A}$ such that $\tilde{\Theta} \setminus \Theta_{i_0} \neq \emptyset$. Define $\bar{\Theta} = \tilde{\Theta} \cap \Theta_{i_0}$. Since $\theta_0 \in \bar{\Theta}$, we have $\bar{\Theta} \in \mathcal{A}$, $\bar{\Theta} \setminus \Theta_{i_0} = \emptyset$, $\Theta_{i_0} \setminus \bar{\Theta} = \Theta_{i_0} \setminus \tilde{\Theta}$, and $\tilde{\Theta}^c \subseteq \bar{\Theta}^c$. Hence,

$$\begin{aligned} &\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) + \sum_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} \tilde{G}_{i_0}(\theta_b) \\ &\leq \min_{\theta \in \bar{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \bar{\Theta}} \min_{i \neq i_0} G_i(\theta_b) + \sum_{\theta_b \in \bar{\Theta} \setminus \Theta_{i_0}} \tilde{G}_{i_0}(\theta_b). \end{aligned} \quad (\text{EC.9})$$

Define $\mathcal{A}_1 = \{\tilde{\Theta} \in \mathcal{A} | \tilde{\Theta} \subseteq \Theta_{i_0}\}$ as a subcollection of $\mathcal{A}$. From (EC.9), it follows that the minimum function over $\tilde{\Theta} \in \mathcal{A}$ in (EC.6) is attained at $\tilde{\Theta}$ such that $\tilde{\Theta} \subseteq \Theta_{i_0}$, i.e., $\tilde{\Theta} \in \mathcal{A}_1$. Namely,

$$\begin{aligned} & \min_{\tilde{\Theta} \in \mathcal{A}} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) + \sum_{\theta_b \in \tilde{\Theta} \setminus \Theta_{i_0}} \tilde{G}_{i_0}(\theta_b) \right) \\ &= \min_{\tilde{\Theta} \in \mathcal{A}_1} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right) \end{aligned}$$

In the following, we simplify (EC.6) by showing that we can substitute $\mathcal{A}_1$ with $\tilde{\mathcal{A}} := \{\tilde{\Theta} \in \mathcal{A}_1 | |\Theta_{i_0} \setminus \tilde{\Theta}| \leq 1\}$. The proof is divided into Steps 3 and 4.

Step 3. We have

$$\begin{aligned} & \min_{\tilde{\Theta} \in \mathcal{A}_1} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right) \\ &= \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\}. \end{aligned} \quad (\text{EC.10})$$

Proof of Step 3. We will first show that

$$\min_{\tilde{\Theta} \in \mathcal{A}_1} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right) = \min_{\tilde{\Theta} \in \tilde{\mathcal{A}}} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right). \quad (\text{EC.11})$$

Let us pick $\tilde{\Theta} \in \mathcal{A}_1 \setminus \tilde{\mathcal{A}}$, i.e., $|\Theta_{i_0} \setminus \tilde{\Theta}| \geq 2$. For fixed $\beta, \bar{\theta} := \arg \min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell)$ belongs to either (a) $\Theta_{i_0}^c$ or (b) $\Theta_{i_0} \setminus \tilde{\Theta}$. If Case (a) holds, then pick arbitrary $\tilde{\theta} \in \Theta_{i_0} \setminus \tilde{\Theta}$, and define $\bar{\Theta} = \Theta_{i_0} \setminus \{\tilde{\theta}\}$. If Case (b) is satisfied, then let $\bar{\Theta} = \Theta_{i_0} \setminus \{\tilde{\theta}\}$. Then, for both cases, we have $\tilde{\Theta} \subseteq \bar{\Theta}$, which implies

$$\begin{aligned} \min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) &\leq \min_{\theta \in \bar{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell), \\ \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) &\leq \sum_{\theta_b \in \Theta_{i_0} \setminus \bar{\Theta}} \min_{i \neq i_0} G_i(\theta_b). \end{aligned} \quad (\text{EC.12})$$

Because $\bar{\theta} \in \bar{\Theta}^c \subseteq \tilde{\Theta}^c$, the equality holds for the first inequality in (EC.12). This leads us to the following inequality.

$$\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \leq \min_{\theta \in \bar{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \bar{\Theta}} \min_{i \neq i_0} G_i(\theta_b). \quad (\text{EC.13})$$

Combining (EC.13) with $\tilde{\Theta} \in \mathcal{A}_1$ yields that the minimum over $\tilde{\mathcal{A}}$ in (EC.11) is always attained at some $\bar{\Theta} \in \mathcal{A}_1$, which proves (EC.11).

Now it suffices to show that

$$\begin{aligned} & \min_{\tilde{\Theta} \in \tilde{\mathcal{A}}} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right) \\ &= \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\}. \end{aligned} \quad (\text{EC.14})$$

Let $\mathcal{B}_i := \{\tilde{\Theta} \in \tilde{\mathcal{A}} \mid |\Theta_{i_0} \setminus \tilde{\Theta}| = i\}$ for $i = 0, 1$. Then, we have $\tilde{\mathcal{A}} = \mathcal{B}_0 \cup \mathcal{B}_1$. Because $\mathcal{B}_0 = \{\Theta_{i_0}\}$, we have

$$\min_{\tilde{\Theta} \in \mathcal{B}_0} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right) = \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell). \quad (\text{EC.15})$$

Since each $\tilde{\Theta} \in \mathcal{B}_1$ can be written as $\tilde{\Theta} = \Theta_{i_0} \setminus \{\theta_b\}$ for each $\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}$,

$$\begin{aligned} & \min_{\tilde{\Theta} \in \mathcal{B}_1} \left(\min_{\theta \in \tilde{\Theta}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \sum_{\theta_b \in \Theta_{i_0} \setminus \tilde{\Theta}} \min_{i \neq i_0} G_i(\theta_b) \right) \\ &= \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right). \end{aligned} \quad (\text{EC.16})$$

Combining (EC.15) with (EC.16) yields (EC.14), which completes the proof. $\square$

The following step further simplifies (EC.10).

Step 4. We have

$$\begin{aligned} & \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\} \\ &= \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta_b^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\}. \end{aligned}$$

Proof of Step 4. As the first inner minimization function does not depend on θ_b , we can switch the order of the minimization functions as

$$\begin{aligned} & \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\} \\ &= \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right\}. \end{aligned} \quad (\text{EC.17})$$

Similarly, we have

$$\begin{aligned} & \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta_b^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\} \\ &= \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta_b^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right\}. \end{aligned} \quad (\text{EC.18})$$

Combining (EC.17) with (EC.18), it suffices to show that for each $\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}$,

$$\begin{aligned} & \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right\} \\ &= \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell), \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta_b^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right\}. \end{aligned} \quad (\text{EC.19})$$

Let $\tilde{\theta} = \arg \min_{\theta \in (\Theta_{i_0} \setminus \{\theta_b\})^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell)$. There are two possibilities a) $\tilde{\theta} \in \Theta_{i_0}^c$ or b) $\tilde{\theta} = \theta_b$. In Case a), we have $\sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta_b^\ell) \geq \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell)$. Therefore, both sides of (EC.19) attain the minimum, $\min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathcal{D}^{\text{KL}}(\theta_0^\ell \parallel \theta^\ell)$, and the equality holds. For Case b), (EC.19) holds straightforwardly. This concludes the proof.

To summarize, the LDR of the posterior preference can be characterized by

$$\begin{aligned} & \lim_{T \rightarrow \infty} -\frac{1}{T} \log(1 - \mathbb{E}[\mathbb{P}_{\pi_m}(\Theta_{i_T^*, T}) \mathbb{1}\{i_T^* = i^*(\pi_m)\} | \pi_m]) \\ &= \min \left\{ \min_{i \neq i_0} G_i(\theta_0), \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbb{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell), \min_{\theta_b \in \Theta_{i_0} \setminus \{\theta_0\}} \left(\sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbb{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\} \\ &= \min \left\{ \min_{\theta \in \Theta_{i_0}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbb{D}^{\text{KL}}(\theta_0^\ell || \theta^\ell), \min_{\theta_b \in \Theta_{i_0}} \left(\sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbb{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) + \min_{i \neq i_0} G_i(\theta_b) \right) \right\}. \end{aligned}$$

The last equality holds from noticing $\mathbb{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) = 0$ when $\theta_b = \theta_0$. Q.E.D.

EC.3. Proof of Statements in Sections 5 and 6

EC.3.1. Proof of Theorem 3

Before we present the proof of Theorem 3, we first present an auxiliary result that involves the following program defined for some fixed $\epsilon > 0$:

$$\begin{aligned} V(\epsilon) = & \max_{\alpha, \beta} \min_{\theta_b \in \Theta} \left\{ \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbb{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) + \alpha(\theta_b) G^*(\theta_b) \mathbb{1}\{\theta_b \in \Theta_{i_0}\} \right\} \\ & \text{subject to } \mathbf{1}_B^\top \alpha + \mathbf{1}_L^\top \beta = 1, \alpha \geq \epsilon \mathbf{1}_B, \beta \geq \epsilon \mathbf{1}_L \end{aligned} \quad (\text{EC.20})$$

That is, all input and simulation sampling fractions are at least ϵ . Program (9) in Algorithm 1 is a plug-in version of (EC.20) when $\epsilon = \epsilon_s$. The optimal objective function value of (8) corresponds to $V(0)$. Let the solution to Program (EC.20) be $(\alpha^*(\epsilon), \beta^*(\epsilon))$. The following lemma shows that if Algorithm 1 chooses $\epsilon_s = \epsilon$ at all $s \geq 1$, then the sampling ratios converge to $(\alpha^*(\epsilon), \beta^*(\epsilon))$ as $T \rightarrow \infty$.

LEMMA EC.2. *Suppose Assumptions 1–5 hold and that the algorithm used in Step 9 has strong consistency of the sampling ratios to the rate-optimal static ratios in Glynn and Juneja (2004). Further, suppose Algorithm 1 chooses $\epsilon_s = \epsilon$ at all $s \geq 1$ and $(\alpha^*(\epsilon), \beta^*(\epsilon))$ is unique. Let α^s and β^s respectively represent the cumulative simulation and input data sampling ratios after the s -th batch allocation in OSAR. Then, we have $\lim_{s \rightarrow \infty} \alpha^s = \alpha^*(\epsilon)$ and $\lim_{s \rightarrow \infty} \beta^s = \beta^*(\epsilon)$ almost surely.*

Proof. We first define $N_i^s(\theta_b)$ and $m_{\ell,s}$ to be the cumulative numbers of replications made at (i, θ_b) and data collected from the ℓ -th input, respectively, after the s -th batch allocation. As $T \rightarrow \infty$, we have $s \rightarrow \infty$.

From Steps 8–9 of Algorithm 1, we have $\hat{n}_{s_1}(\theta_b) \sim \text{Bin}(\Delta, \hat{\alpha}_{s_1}(\theta_b))$ and $\hat{m}_{\ell, s_1} \sim \text{Bin}(\lceil \Delta / c_\ell \rceil, \Delta \hat{\beta}_{\ell, s_1} / \{c_\ell \lceil \Delta / c_\ell \rceil\})$ for each $1 \leq s_1 \leq s$. We will proceed by the following steps.

Step 1. With probability one, $\lim_{s \rightarrow \infty} \sum_{i=1}^k N_i^s(\theta_b) = \infty$ and $\lim_{s \rightarrow \infty} m_{\ell,s} = \infty$ for all $1 \leq b \leq B$ and $1 \leq \ell \leq L$. Moreover, $\lim_{T \rightarrow \infty} N_{i,T}(\theta_b) = \infty$ for all $1 \leq i \leq k$, $1 \leq b \leq B$.

Proof of Step 1. Since $\hat{\alpha}_{s_1}(\theta_b) \geq \epsilon$ and $\hat{\beta}_{\ell, s_1} \geq \epsilon$ for all $1 \leq b \leq B$ and $1 \leq \ell \leq L$, we have $\lim_{t \rightarrow \infty} \sum_{s_1=1}^s \hat{\alpha}_{s_1}(\theta_b) = \infty$ and $\lim_{s \rightarrow \infty} \sum_{s_1=1}^s \hat{\beta}_{\ell, s_1} = \infty$. From Lemma EC.5, one can show that $\lim_{s \rightarrow \infty} \sum_{i=1}^k N_i^s(\theta_b) = \lim_{s \rightarrow \infty} \sum_{s_1=1}^s \hat{n}_{s_1}(\theta_b) = \infty$ and $\lim_{s \rightarrow \infty} m_{\ell,s} = \lim_{s \rightarrow \infty} \sum_{s_1=1}^s \hat{m}_{\ell, s_1} = \infty$, which means that we run infinitely many simulation replications at all $\theta_b \in \Theta$. Consequently, $\lim_{T \rightarrow \infty} N_{i,T}(\theta_b) = \infty$ because the R&S subroutine of Algorithm 1 simulates all solutions at each θ_b infinitely often w. p. 1. $\square$

Combining Step 1 with that the subroutine algorithm achieves the rate-optimal static allocation yields that $\lim_{T \rightarrow \infty} G_{i,T}(\theta_b) = G^*(\theta_b)$ holds almost surely for all $1 \leq b \leq B$.

Step 2. $\lim_{s \rightarrow \infty} \hat{D}^{\text{KL}}(\hat{\theta}^\ell || \theta_b^\ell) = D^{\text{KL}}(\theta_0^\ell || \theta_b^\ell)$ almost surely.

Proof of Step 2. Since $\lim_{s \rightarrow \infty} m_{\ell,s} = \infty$ holds almost surely for all $1 \leq \ell \leq L$, $\lim_{s \rightarrow \infty} \hat{\theta}^\ell = \theta_0^\ell$ a.s. from Lemma EC.1. Since Θ is finite, for all sufficiently large s , $\hat{\theta}^\ell = \theta_0^\ell$. Hence, $\hat{D}^{\text{KL}}(\hat{\theta}^\ell || \theta_b^\ell) = \hat{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell)$ holds for all sufficiently large s . Combining this with Glivenko-Cantelli property in Assumption 4(c), we have $\lim_{s \rightarrow \infty} \hat{D}^{\text{KL}}(\hat{\theta}^\ell || \theta_b^\ell) = D^{\text{KL}}(\theta_0^\ell || \theta_b^\ell)$ a.s., as desired. $\square$

Step 3. For $\hat{\alpha}^s$ and $\hat{\beta}^s$ in (9), we have $\lim_{s \rightarrow \infty} \hat{\alpha}^s = \alpha^*$ and $\lim_{s \rightarrow \infty} \hat{\beta}^s = \beta^*$ almost surely.

Proof of Step 3. Recall that Algorithm 1 solves the plug-in Program (9) with coefficients $G_{i,T}(\theta_b)$ and $\hat{D}^{\text{KL}}(\hat{\theta}^\ell || \theta_b^\ell)$ at each s to obtain $\hat{\alpha}^s$ and $\hat{\beta}^s$. Combining Step 2 with the strong consistency of $G_{i,T}(\theta_b)$ shows that each plug-in coefficient in (9) converges to the corresponding quantity in (EC.20) almost surely. Furthermore, $i_T^* = i_0$ and $\Theta_{i_T^*,T} = \Theta_{i_0}$ for all sufficiently large T w. p. 1.

Next, it remains to show that the optimal solution of (9) converges to that of (EC.20). The objective function of (9) is continuous and concave. Thus, from Proposition 24 and Theorem 25 in Shapiro (2003), the convergence of the optimal solution follows. $\square$

Step 4. We have $\lim_{s \rightarrow \infty} \alpha^s = \alpha^*$ and $\lim_{s \rightarrow \infty} \beta^s = \beta^*$.

Proof of Step 4. For ease of exposition, we assume $m_0 = 0$ and $n_0 = 0$; the extension to nonzero m_0 and n_0 case is straightforward. Then, for all $1 \leq b \leq B$ and $1 \leq \ell \leq L$,

$$\alpha_s(\theta_b) = (s\Delta)^{-1} \sum_{s_1=1}^s \hat{n}_{s_1}(\theta_b) \text{ and } \beta_{\ell,s} = (s\Delta)^{-1} \sum_{s_1=1}^s c_\ell \hat{m}_{\ell,s_1}.$$

From Lemma EC.5 in Section EC.8, we can show that

$$\frac{\sum_{s_1=1}^s \hat{n}_{s_1}(\theta_b)}{\Delta \sum_{s_1=1}^s \hat{\alpha}_{s_1}(\theta_b)} \rightarrow 1 \text{ and } \frac{\sum_{s_1=1}^s c_\ell \hat{m}_{\ell,s_1}}{\Delta \sum_{s_1=1}^s \hat{\beta}_{\ell,s_1}} \rightarrow 1. \quad (\text{EC.21})$$

From the convergence of the Cesaro's mean, we can verify that for all $1 \leq b \leq B$ and $1 \leq \ell \leq L$,

$$s^{-1} \sum_{s_1=1}^s \hat{\alpha}_{s_1}(\theta_b) \rightarrow \alpha^*(\theta_b) \text{ and } s^{-1} \sum_{s_1=1}^s \hat{\beta}_{\ell,s_1} \rightarrow \beta_\ell^*. \quad (\text{EC.22})$$

Combining (EC.21) with (EC.22) completes the proof. **Q.E.D.**

The proof of Theorem 3 is a straightforward extension of Lemma EC.2.

Proof of Theorem 3. Since $\sum_{s=1}^\infty \epsilon_s = \infty$ holds, we have $\lim_{s \rightarrow \infty} \sum_{s_1=1}^s \hat{\alpha}_{s_1}(\theta_b) = \infty$ and $\lim_{s \rightarrow \infty} \sum_{s_1=1}^s \hat{\beta}_{\ell,s_1} = \infty$ for all $1 \leq \ell \leq L$ and $1 \leq b \leq B$. Hence, Step 1 of the proof of Lemma EC.2 holds. Further, one can easily confirm that Steps 2–4 can be shown without any modification. **Q.E.D.**

Lastly, for interested readers, we characterize the optimality gap of $(\alpha^*(\epsilon), \beta^*(\epsilon))$ for Program (8).

PROPOSITION EC.1. For $\epsilon \in (0, \frac{1}{B+L})$, we have $0 \leq 1 - \frac{V(\epsilon)}{V(0)} \leq \epsilon(B+L)$.

Proof. Program (EC.20) can be rewritten as

$$\begin{aligned}
 V(\epsilon) = \max \quad & C \\
 \text{subject to} \quad & C \leq \alpha(\theta_b)G^*(\theta_b) + \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell), \forall \theta_b \in \Theta_{i_0}, \\
 & C \leq \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell), \forall \theta_b \in \Theta_{i_0}^c, \\
 & \mathbf{1}_B^\top \alpha + \mathbf{1}_L^\top \beta = 1, \alpha(\theta_b) \geq \epsilon, \beta_\ell \geq \epsilon, \forall 1 \leq b \leq B, 1 \leq \ell \leq L.
 \end{aligned} \tag{EC.23}$$

Since (EC.23) has a smaller constraint on α and β , we have $V(0) \geq V(\epsilon)$, which proves the first inequality. For the second inequality, let us define $\alpha^\epsilon(\theta_b) = \frac{\alpha(\theta_b) - \epsilon}{1 - \epsilon(B+L)}$ and $\beta_\ell^\epsilon = \frac{\beta_\ell - \epsilon}{1 - \epsilon(B+L)}$ for $1 \leq \ell \leq L$. Then, after some algebras, Program (EC.23) can be written as

$$\begin{aligned}
 V(\epsilon) = \max \quad & C \\
 \text{s.t.} \quad & \frac{C - \epsilon \left(G^*(\theta_b) + \sum_{\ell=1}^L \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) / c_\ell \right)}{1 - \epsilon(B+L)} \leq \alpha^\epsilon(\theta_b)G^*(\theta_b) + \sum_{\ell=1}^L \frac{\beta_\ell^\epsilon}{c_\ell} \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell), \forall \theta_b \in \Theta_{i_0}, \\
 & \frac{C - \epsilon \left(\sum_{\ell=1}^L \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) / c_\ell \right)}{1 - \epsilon(B+L)} \leq \sum_{\ell=1}^L \frac{\beta_\ell^\epsilon}{c_\ell} \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell), \forall \theta_b \in \Theta_{i_0}^c, \\
 & \mathbf{1}_B^\top \alpha^\epsilon(\theta_b) + \mathbf{1}_L^\top \beta^\epsilon = 1, \alpha^\epsilon(\theta_b) \geq 0, \beta_\ell^\epsilon \geq 0, \forall 1 \leq b \leq B, 1 \leq \ell \leq L.
 \end{aligned}$$

Let $C_0 = \min \left\{ \min_{\theta_b \in \Theta_{i_0}} \left(G^*(\theta_b) + \sum_{\ell=1}^L \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) / c_\ell \right), \min_{\theta_b \in \Theta_{i_0}^c} \sum_{\ell=1}^L \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell) / c_\ell \right\}$. Then, one can see that $V(\epsilon)$ is bounded from below by $\bar{V}(\epsilon)$, which is defined as

$$\begin{aligned}
 \bar{V}(\epsilon) = \max \quad & C \\
 \text{s.t.} \quad & \frac{C - \epsilon C_0}{1 - \epsilon(B+L)} \leq \alpha^\epsilon(\theta_b)G^*(\theta_b) + \sum_{\ell=1}^L \frac{\beta_\ell^\epsilon}{c_\ell} \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell), \forall \theta_b \in \Theta_{i_0}, \\
 & \frac{C - \epsilon C_0}{1 - \epsilon(B+L)} \leq \sum_{\ell=1}^L \frac{\beta_\ell^\epsilon}{c_\ell} \mathbf{D}^{\text{KL}}(\theta_0^\ell || \theta_b^\ell), \forall \theta_b \in \Theta_{i_0}^c, \\
 & \mathbf{1}_B^\top \alpha^\epsilon(\theta_b) + \mathbf{1}_L^\top \beta^\epsilon = 1, \alpha^\epsilon(\theta_b) \geq 0, \beta_\ell^\epsilon \geq 0, \forall 1 \leq b \leq B, 1 \leq \ell \leq L.
 \end{aligned}$$

From the definition of $V(0)$, we have $\bar{V}(\epsilon) = (1 - \epsilon(B+L))V(0) + \epsilon C_0$. Consequently, we obtain that

$$V(0) - V(\epsilon) \leq V(0) - \bar{V}(\epsilon) = \epsilon((B+L)V(0) - C_0) \leq \epsilon(B+L)V(0).$$

Dividing $V(0)$ on both sides, the second inequality follows. Q.E.D.

Since B and L are known in advance, Proposition EC.1 lets us control the relative optimality gap with arbitrary precision by choosing ϵ .

EC.3.2. Proof of Theorem 4

We first state a result auxiliary to Theorem 4.

LEMMA EC.3. *In addition to the assumptions made in Lemma EC.2, suppose Assumption 6 holds. Let α^s and β^s respectively represent the cumulative simulation and input data sampling ratios after the s -th batch allocation in OSAR⁺. Then, (α^s, β^s) converges to $(\alpha^*(\epsilon), \beta^*(\epsilon))$ almost surely as $T \rightarrow \infty$.*

This lemma can be proved in a similar manner as that of Lemma EC.2. Steps 1, 2, and 4 can proceed without modification. To see Step 3, it suffices to show the convergence of $G_{i,T}(\theta_b)$ to $G^*(\theta_b)$ in Algorithm 2. From the closed-form expression of $\hat{\mu}_{i,T}(\theta_b)$ in (12), one can confirm $\lim_{T \rightarrow \infty} \hat{\mu}_{i,T} = (\eta_i(\theta_b))_{1 \leq b \leq B}$ a.s. The rest follows from Step 3 in the proof of Lemma EC.2. Then, we can extend this result to show Theorem 4 by applying the same argument made in the proof of Theorem 3.

EC.3.3. Proof of Proposition 2

The optimal solution to (15) can be written as $g_{i,T}(x) = \sum_{b=1}^{B+1} c_b K_i(\theta_b, x)$. Then, we have

$$[(g_{i,T}(\theta_b))_{1 \leq b \leq B}^T (g_{i,T}(\hat{\theta}_{i,r}))_{1 \leq r \leq N_{i,T}(\hat{\theta})}^T]^T = \tilde{\mathbf{K}}_{i,T}^T \mathbf{c}_{i,T},$$

where $\mathbf{c}_{i,T} = (c_b)_{1 \leq b \leq B+1}$ is a $(B+1)$ -dimensional column vector. Then, (15) can be rewritten as

$$\mathbf{c}_{i,T} = \arg \min_{\mathbf{c}} \left\{ \frac{1}{2} (\mathbf{x} - \tilde{\mathbf{K}}_{i,T}^T \mathbf{c})^T \tilde{\Sigma}_{i,T}^{-1} (\mathbf{x} - \tilde{\mathbf{K}}_{i,T}^T \mathbf{c}) + \frac{\kappa}{2} \mathbf{c}^T \tilde{\mathbf{K}}_{i,T} \mathbf{c} \right\}. \quad (\text{EC.24})$$

Here $\mathbf{x} = [\boldsymbol{\mu}_{i,T}^T \mathbf{Y}_i^T]^T - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\theta})}$. After some algebra, (EC.24) is reformulated as

$$\mathbf{c}_{i,T} = \arg \min_{\mathbf{c}} \left\{ \frac{1}{2} \mathbf{c}^T (\tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^T + \kappa \tilde{\mathbf{K}}_{i,T}) \mathbf{c} - (\tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{x})^T \mathbf{c} \right\}. \quad (\text{EC.25})$$

Let $\mathbf{A}_{i,T} = \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^T + \kappa \tilde{\mathbf{K}}_{i,T}$. Since $\mathbf{A}_{i,T}$ is symmetric and positive semidefinite (psd), its eigendecomposition can be written as

$$\mathbf{A}_{i,T} = \mathbf{V} \mathbf{D} \mathbf{V}^T \quad (\text{EC.26})$$

for orthogonal matrix $\mathbf{V}$ and $\mathbf{D} = \text{diag}(\{\sigma_b\}_{1 \leq b \leq B+1})$, where $\{\sigma_b\}_{1 \leq b \leq B+1}$ is the set of the eigenvalues of $\mathbf{A}_{i,T}$. From the positive semi-definiteness of $\mathbf{A}_{i,T}$, we have $\sigma_b \geq 0$ for all $1 \leq b \leq B+1$. Without loss of generality, assume $\sigma_b > 0$ for all $1 \leq b \leq R$ and $\sigma_b = 0$ for all $R+1 \leq b \leq B+1$ for some nonnegative integer R . In this case, (EC.26) can be regarded as the singular value decomposition of $\mathbf{A}_{i,T}$, and the pseudoinverse of $\mathbf{A}_{i,T}$, $\mathbf{A}_{i,T}^\dagger$, is given as $\mathbf{A}_{i,T}^\dagger = \mathbf{V} \mathbf{D}^\dagger \mathbf{V}^T$, where $\mathbf{D}^\dagger = \text{diag}(\{\sigma_1^{-1}, \dots, \sigma_R^{-1}, 0, \dots, 0\})$

Letting $\mathbf{d} = \mathbf{V}^T \mathbf{c}$, (EC.25) can be written as

$$\mathbf{c}_{i,T} = \mathbf{V} \mathbf{d}^* \text{ where } \mathbf{d}^* = \arg \min_{\mathbf{d}} \left\{ \frac{1}{2} \mathbf{d}^T \mathbf{D} \mathbf{d} - (\mathbf{V}^T \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{x})^T \mathbf{d} \right\}.$$

Let the b th elements of $\mathbf{d}$ and $\mathbf{V}^T \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{x}$ be d_b and z_b , respectively, for $b = 1 \leq b \leq B+1$. We show that $z_b = 0$ for all $b \geq R+1$ for which it suffices to have $\tilde{\mathbf{K}}_{i,T}^T \mathbf{V} \mathbf{e}_b = \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1}$ for all $b \geq R+1$. To see this, observe that $\mathbf{e}_b^T \mathbf{V}^T \mathbf{A}_{i,T} \mathbf{V} \mathbf{e}_b = \mathbf{e}_b^T \mathbf{D} \mathbf{e}_b = 0$ holds for all $b \geq R+1$. Thus, $0 = \mathbf{e}_b^T \mathbf{V}^T \mathbf{A}_{i,T} \mathbf{V} \mathbf{e}_b =$

$(\mathbf{V}\mathbf{e}_b)^\top (\tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^\top + \kappa \tilde{\mathbf{K}}_{i,T}) \mathbf{V}\mathbf{e}_b \geq (\tilde{\mathbf{K}}_{i,T}^\top \mathbf{V}\mathbf{e}_b)^\top \tilde{\Sigma}_{i,T}^{-1} (\tilde{\mathbf{K}}_{i,T}^\top \mathbf{V}\mathbf{e}_b) \geq 0$, where the first inequality follows because $\tilde{\mathbf{K}}_{i,T}$ is a psd matrix. Therefore, $\tilde{\mathbf{K}}_{i,T}^\top \mathbf{V}\mathbf{e}_b = \mathbf{0}_{(B+N_{i,T}(\theta)) \times 1}$ as desired, which implies

$$\mathbf{d}^* = \arg \min_{d_b, 1 \leq b \leq R} \left\{ \sum_{b=1}^R \left(\frac{\sigma_b}{2} d_b^2 - z_b d_b \right) \right\}$$

From the first-order condition, the optimal solution is attained at $d_b = z_b / \sigma_b$. Consequently, $\mathbf{d}^* = \mathbf{D}^\dagger \mathbf{V}^\top \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{x}$ and $\mathbf{c}_{i,T} = \mathbf{V} \mathbf{D}^\dagger \mathbf{V}^\top \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{x} = \mathbf{A}_{i,T}^\dagger \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{x}$, as desired.

EC.3.4. Proof of Theorem 5

In this proof, we introduce notation $\Theta_{\text{a.s.}}$ to compare two positive random sequences $\{a_T\}$ and $\{b_T\}$. Specifically, $b_T = \Theta_{\text{a.s.}}(a_T)$ means that $0 < \liminf_{T \rightarrow \infty} b_T / a_T \leq \limsup_{T \rightarrow \infty} b_T / a_T < \infty$ holds almost surely.

We proceed in the following steps.

Step 1. We have $N_{i,T}(\theta_b) \xrightarrow{\text{a.s.}} \infty$ almost surely and $N_{i,T}(\theta_b) = \Theta_{\text{a.s.}}(T)$, for all $1 \leq i \leq k$ and $1 \leq b \leq B$.

Proof of Step 1. Proceeding as in Step 1 of the proof of Theorem 3, we have $\lim_{T \rightarrow \infty} N_{i,T}(\theta_b) = \infty$ for all $1 \leq i \leq k, 1 \leq b \leq B$.

To show $N_{i,T}(\theta_b) = \Theta_{\text{a.s.}}(T)$, first note that $N_{i,T}(\theta_b) \leq T$ as the number of simulation replications at (i, θ_b) cannot exceed the budget, T , which implies that $\limsup_{T \rightarrow \infty} \frac{N_{i,T}(\theta_b)}{T} < \infty$ for all $1 \leq i \leq k$. Since $\alpha(\theta_b) \geq \epsilon$ for all $1 \leq b \leq B$, we have $\epsilon \leq \liminf_{T \rightarrow \infty} \frac{\sum_{i=1}^k N_{i,T}(\theta_b)}{T}$. As the R&S subroutine in Algorithm 3 achieves the asymptotically optimal allocation ratios in Glynn and Juneja (2004), the following must be satisfied at each θ_b :

$$\lim_{T \rightarrow \infty} T^{-2} \left\{ \frac{N_{i^b,T}^2(\theta_b)}{\lambda_{i^b}^2(\theta_b)} - \sum_{i \neq i^b} \frac{N_{i,T}^2(\theta_b)}{\lambda_i^2(\theta_b)} \right\} = 0 \text{ and } \lim_{T \rightarrow \infty} \frac{G_{i,T}(\theta_b)}{G_{j,T}(\theta_b)} = 1 \text{ for all } i \neq j, i \neq i^b, j \neq i^b. \quad (\text{EC.27})$$

Suppose $\liminf_{T \rightarrow \infty} \frac{N_{i^b,T}(\theta_b)}{T} = 0$. Then, there exists subsequence $\{T_\ell\}_{\ell \geq 1}$ such that $\lim_{\ell \rightarrow \infty} \frac{N_{i^b,T_\ell}(\theta_b)}{T_\ell} = 0$. Since $\sum_{i=1}^k N_{i,T_\ell}(\theta_b) / T_\ell \geq \epsilon / 2$ holds for all sufficiently large ℓ , we have $\sum_{i \neq i^b} N_{i,T_\ell}(\theta_b) / T_\ell \geq \epsilon / 4$. By the pigeonhole principle, there exists a fixed j_1 such that $N_{j_1,T_\ell}(\theta_b) / T_\ell > \epsilon / (4(k-1))$ infinitely often, i.e., $\limsup_{\ell \rightarrow \infty} N_{j_1,T_\ell}(\theta_b) / T_\ell \geq \epsilon / (4(k-1))$. We can therefore choose a further subsequence $\{S_\ell\}_{\ell \geq 1} \subseteq \{T_\ell\}_{\ell \geq 1}$ such that $\lim_{\ell \rightarrow \infty} N_{j_1,S_\ell}(\theta_b) / S_\ell \geq \epsilon / (4(k-1)) > 0$. Consequently, we obtain

$$\begin{aligned} \liminf_{\ell \rightarrow \infty} \left\{ S_\ell^{-2} \left\{ \frac{N_{i^b,S_\ell}^2(\theta_b)}{\lambda_{i^b}^2(\theta_b)} - \sum_{i \neq i^b} \frac{N_{i,S_\ell}^2(\theta_b)}{\lambda_i^2(\theta_b)} \right\} \right\} &\leq \liminf_{\ell \rightarrow \infty} \left\{ S_\ell^{-2} \left\{ \frac{N_{i^b,S_\ell}^2(\theta_b)}{\lambda_{i^b}^2(\theta_b)} - \frac{N_{j_1,S_\ell}^2(\theta_b)}{\lambda_{j_1}^2(\theta_b)} \right\} \right\} \\ &\leq \liminf_{\ell \rightarrow \infty} \frac{N_{i^b,S_\ell}^2(\theta_b)}{S_\ell^2 \lambda_{i^b}^2(\theta_b)} - \liminf_{\ell \rightarrow \infty} \frac{N_{j_1,S_\ell}^2(\theta_b)}{S_\ell^2 \lambda_{j_1}^2(\theta_b)} \\ &= - \liminf_{\ell \rightarrow \infty} \frac{N_{j_1,S_\ell}^2(\theta_b)}{S_\ell^2 \lambda_{j_1}^2(\theta_b)} < 0 \end{aligned}$$

contradicting the first condition of (EC.27). The second inequality follows from $\liminf_{T \rightarrow \infty} a_T + \liminf_{T \rightarrow \infty} b_T \leq \liminf_{T \rightarrow \infty} (a_T + b_T)$ for any two real sequences, $\{a_T\}$ and $\{b_T\}$. This proves $N_{i^b,T}(\theta_b) = \Theta_{\text{a.s.}}(T)$.

Next, we show that there exists $i \neq i^b$ such that $\liminf_{T \rightarrow \infty} \frac{N_{i,T}(\theta_b)}{T} > 0$ by contradiction.

Suppose otherwise. Since $N_{i^b,T}(\theta_b) = \Theta_{\text{a.s.}}(T)$, we have $\liminf_{T \rightarrow \infty} \frac{N_{i,T}(\theta_b)}{N_{i^b,T}(\theta_b)} = 0$ for all $i \neq i^b$. We first establish that $\limsup_{T \rightarrow \infty} N_{j,T}(\theta_b)/N_{i,T}(\theta_b) < \infty$ for all $i \neq j, j \neq i^b, i \neq i^b$. By means of contradiction, assume that there exist $i_1 \neq i_2 \neq i^b$ and subsequence $\{T_\ell\}_{\ell \geq 1}$ satisfying $\lim_{\ell \rightarrow \infty} N_{i_2,T_\ell}(\theta_b)/N_{i_1,T_\ell}(\theta_b) = \infty$. Therefore,

$$\limsup_{\ell \rightarrow \infty} \frac{G_{i_1,T_\ell}(\theta_b)}{G_{i_2,T_\ell}(\theta_b)} \leq \frac{\limsup_{\ell \rightarrow \infty} \frac{(\mu_{i_1,T_\ell}(\theta_b) - \mu_{i^b,T_\ell}(\theta_b))^2}{2\lambda_{i_1}^2(\theta_b)}}{\liminf_{\ell \rightarrow \infty} \frac{(\mu_{i_2,T_\ell}(\theta_b) - \mu_{i^b,T_\ell}(\theta_b))^2}{2(\lambda_{i_2}^2(\theta_b) + \lambda_{i^b}^2(\theta_b)N_{i_2,T_\ell}(\theta_b)/N_{i^b,T_\ell}(\theta_b))}} \times \limsup_{\ell \rightarrow \infty} \frac{N_{i_1,T_\ell}(\theta_b)}{N_{i_2,T_\ell}(\theta_b)} = 0.$$

Because every convergent subsequence of a convergent sequence converges to the same limit as the original sequence, this implies that $\lim_{T \rightarrow \infty} \frac{G_{i,T}(\theta_b)}{G_{i_2,T}(\theta_b)} = 0$, which contradicts the second condition in (EC.27).

This establishes that $\limsup_{T \rightarrow \infty} N_{j,T}(\theta_b)/N_{i,T}(\theta_b) < \infty$ for all $i \neq j, j \neq i^b, i \neq i^b$. Then, for a fixed $i_3 \neq i^b$, there exists a subsequence $\{T_\ell\}_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \rightarrow \infty} N_{i_3,T_\ell}(\theta_b)/N_{i^b,T_\ell}(\theta_b) = 0$. For all i such that $i \notin \{i_3, i^b\}$, there exists $C_i \in (0, \infty)$ satisfying $N_{i,T}(\theta_b)/N_{i_3,T}(\theta_b) \leq C_i$ for all sufficiently large T as $\limsup_{T \rightarrow \infty} N_{i,T}(\theta_b)/N_{i_3,T}(\theta_b) < \infty$. From this observation, we have $0 \leq \lim_{\ell \rightarrow \infty} N_{i,T_\ell}(\theta_b)/N_{i^b,T_\ell}(\theta_b) \leq C_i \lim_{\ell \rightarrow \infty} N_{i_3,T_\ell}(\theta_b)/N_{i^b,T_\ell}(\theta_b) = 0$, which implies $\lim_{\ell \rightarrow \infty} N_{i,T_\ell}(\theta_b)/N_{i^b,T_\ell}(\theta_b) = 0$. Consequently, we have $\lim_{\ell \rightarrow \infty} \frac{N_{j,T_\ell}(\theta_b)}{N_{i^b,T_\ell}(\theta_b)} = 0$ for all $j \neq i^b$. Therefore, the equality below follows:

$$\liminf_{\ell \rightarrow \infty} \left\{ T_\ell^{-2} \left\{ \frac{N_{i^b,T_\ell}^2(\theta_b)}{\lambda_{i^b}^2(\theta_b)} - \sum_{i \neq i^b} \frac{N_{i,T_\ell}^2(\theta_b)}{\lambda_i^2(\theta_b)} \right\} \right\} = \frac{1}{\lambda_{i^b}^2(\theta_b)} \liminf_{\ell \rightarrow \infty} \frac{N_{i^b,T_\ell}^2(\theta_b)}{T_\ell^2} > 0. \quad (\text{EC.28})$$

However, the first condition of (EC.27) stipulates that $\liminf_{\ell \rightarrow \infty} T_\ell^{-2} \left(\frac{N_{i^b,T_\ell}^2(\theta_b)}{\lambda_{i^b}^2(\theta_b)} - \sum_{i \neq i^b} \frac{N_{i,T_\ell}^2(\theta_b)}{\lambda_i^2(\theta_b)} \right) = 0$, which makes (EC.28) a contradiction.

Hence, this implies that there exists i such that $\liminf_{T \rightarrow \infty} N_{i,T}(\theta_b)/T > 0$; let us denote it by j_2 .

Finally, we show that $N_j(\theta_b) = \Theta_{\text{a.s.}}(T)$ for all j . Suppose that there exists $j_3 \notin \{i^b, j_2\}$ such that $\liminf_{T \rightarrow \infty} \frac{N_{j_3,T}(\theta_b)}{T} = 0$. Observe that

$$G_{j_3,T}(\theta_b) = \frac{(\mu_{i^b,T}(\theta_b) - \mu_{j_3,T}(\theta_b))^2}{2 \left(\frac{\lambda_{i^b}^2(\theta_b)}{N_{i^b,T}(\theta_b)/T} + \frac{\lambda_{j_3}^2(\theta_b)}{N_{j_3,T}(\theta_b)/T} \right)} \leq \frac{(\mu_{i^b,T}(\theta_b) - \mu_{j_3,T}(\theta_b))^2}{2\lambda_{j_3}^2(\theta_b)} \frac{N_{j_3,T}(\theta_b)}{T},$$

which implies $\liminf_{T \rightarrow \infty} G_{j_3,T}(\theta_b) = 0$. Because $N_{j_2,T}(\theta_b) = \Theta_{\text{a.s.}}(T)$ and $N_{i^b,T}(\theta_b) = \Theta_{\text{a.s.}}(T)$, $G_{j_2,T}(\theta_b) = \Theta_{\text{a.s.}}(1)$. Accordingly, we have $\liminf_{T \rightarrow \infty} \frac{G_{j_3,T}(\theta_b)}{G_{j_2,T}(\theta_b)} = 0$, which contradicts the second condition of (EC.27). This concludes the proof. $\square$

Step 2. We have $\lim_{T \rightarrow \infty} \bar{\mathbf{K}}_{i,T} = \mathbf{K}_{i,B+1}$. For all sufficiently large T , $\bar{\mathbf{K}}_{i,T}$ is positive definite.

Proof of Step 2. Observe that $\mathbf{K}_{i,B+1}$ and $\bar{\mathbf{K}}_{i,T}$ can be viewed as block matrices by writing them as

$$\mathbf{K}_{i,B+1} = \begin{bmatrix} \mathbf{K}_{i,B} & \mathbf{u} \\ \mathbf{u}^\top & K_i(\theta_0, \theta_0) \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{K}}_{i,T} = \begin{bmatrix} \mathbf{K}_{i,B} & \mathbf{u}_T \\ \mathbf{u}_T^\top & K_i(\hat{\theta}_T, \hat{\theta}_T) \end{bmatrix}$$

where $\mathbf{K}_{i,B} = (K_i(\boldsymbol{\theta}_b, \boldsymbol{\theta}_{b'}))_{1 \leq b, b' \leq B}$ is a Gram matrix of K_i at Θ^+ , $\mathbf{u} = (K_i(\boldsymbol{\theta}_b, \boldsymbol{\theta}_0))_{1 \leq b \leq B}$, and $\mathbf{u}_T = (K_i(\boldsymbol{\theta}_b, \hat{\boldsymbol{\theta}}_T))_{1 \leq b \leq B}$. Thanks to the continuity of K_i and the strong consistency of $\hat{\boldsymbol{\theta}}_T$, $\lim_{T \rightarrow \infty} \bar{\mathbf{K}}_{i,T} = \mathbf{K}_{i,B+1}$.

Further, the definition of RKHS implies that $\mathbf{K}_{i,B}$ and $\bar{\mathbf{K}}_{i,T}$ are positive semidefinite. Thus, to show the positive definiteness of $\bar{\mathbf{K}}_{i,T}$, it suffices to show $\det(\bar{\mathbf{K}}_{i,T}) > 0$ for all sufficiently large T . By Lemma EC.6, we have $\det(\mathbf{K}_{i,B+1}) = \det(\mathbf{K}_{i,B})(K_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) - \mathbf{u}^\top \mathbf{K}_{i,B}^{-1} \mathbf{u})$ and $\det(\bar{\mathbf{K}}_{i,T}) = \det(\mathbf{K}_{i,B})(K_i(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\theta}}_T) - \mathbf{u}_T^\top \mathbf{K}_{i,B}^{-1} \mathbf{u}_T)$. The positive definiteness of $\mathbf{K}_{i,B+1}$ yields $\det(\mathbf{K}_{i,B}) > 0$ and $K_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) - \mathbf{u}^\top \mathbf{K}_{i,B}^{-1} \mathbf{u} > 0$. From the continuity of K_i and the strong consistency of $\hat{\boldsymbol{\theta}}_T$, we have $\lim_{T \rightarrow \infty} K_i(\hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\theta}}_T) - \mathbf{u}_T^\top \mathbf{K}_{i,B}^{-1} \mathbf{u}_T = K_i(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) - \mathbf{u}^\top \mathbf{K}_{i,B}^{-1} \mathbf{u} > 0$. Hence, for all sufficiently large T , $\det(\bar{\mathbf{K}}_{i,T}) > 0$, as required. $\square$

From Step 2, we can replace the Moore-Penrose inverse in (16) with the exact inverse for all sufficiently large T . To show the strong consistency of $(g_{i,T}(\boldsymbol{\theta}_b))_{1 \leq b \leq B+1}$, it is enough to verify

$$\lim_{T \rightarrow \infty} (\bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T}^\top + \kappa \bar{\mathbf{K}}_{i,T})^{-1} \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \left([\boldsymbol{\mu}_{i,T}^\top \mathbf{Y}_i^\top]^\top - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\boldsymbol{\theta}})} \right) = \mathbf{K}_{i,B+1}^{-1} (\mathbf{y}_i - \gamma_i \mathbf{1}_{B+1}),$$

where $\mathbf{y}_i := (\eta_i(\boldsymbol{\theta}_b))_{1 \leq b \leq B+1}$. We rewrite the left-hand side sequence as

$$\begin{aligned} & (\bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T}^\top + \kappa \bar{\mathbf{K}}_{i,T})^{-1} \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \left([\boldsymbol{\mu}_{i,T}^\top \mathbf{Y}_i^\top]^\top - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\boldsymbol{\theta}})} \right) \\ &= (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T}^\top + \kappa \bar{\Sigma}_{i,T})^{-1} \bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \left([\boldsymbol{\mu}_{i,T}^\top \mathbf{Y}_i^\top]^\top - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\boldsymbol{\theta}})} \right) = \mathbf{A}_T \mathbf{b}_T, \end{aligned}$$

where $\bar{\Sigma}_{i,T}$, $\mathbf{A}_T$, and $\mathbf{b}_T$ are defined as

$$\begin{aligned} \bar{\Sigma}_{i,T} &:= \text{diag} \left(\left\{ \lambda_i^2(\boldsymbol{\theta}_b) / N_{i,T}(\boldsymbol{\theta}_b) \right\}_{1 \leq b \leq B+1} \right), \\ \mathbf{A}_T &:= (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T}^\top + \kappa \bar{\Sigma}_{i,T})^{-1}, \\ \mathbf{b}_T &:= \bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \left([\boldsymbol{\mu}_{i,T}^\top \mathbf{Y}_i^\top]^\top - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\boldsymbol{\theta}})} \right). \end{aligned}$$

We will analyze the limits of $\mathbf{A}_T$ and $\mathbf{b}_T$, respectively, in Steps 3 and 4.

Step 3. We have $\lim_{T \rightarrow \infty} \mathbf{b}_T = \mathbf{y}_i - \gamma_i \mathbf{1}_{B+1}$ almost surely.

Proof of Step 3. Observe that the ℓ -th element of $\mathbf{b}_T$ is calculated as

$$\begin{aligned} (\mathbf{b}_T)_\ell &= \sum_{b_1=1}^{B+1} (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1})_{\ell, b_1} \left\{ \bar{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \left([\boldsymbol{\mu}_{i,T}^\top \mathbf{Y}_i^\top]^\top - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\boldsymbol{\theta}})} \right) \right\}_{b_1} \\ &= \sum_{b_1=1}^{B+1} \frac{\lambda_i^2(\boldsymbol{\theta}_\ell)}{N_{i,T}(\boldsymbol{\theta}_\ell)} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell, b_1} \left\{ \sum_{b_2=1}^B \frac{K_i(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}) N_{i,T}(\boldsymbol{\theta}_{b_2})}{\lambda_i^2(\boldsymbol{\theta}_{b_2})} (\mu_{i,T}(\boldsymbol{\theta}_{b_2}) - \gamma_i) + \sum_{r=1}^{N_{i,T}(\hat{\boldsymbol{\theta}})} \frac{K_i(\boldsymbol{\theta}_{b_1}, \hat{\boldsymbol{\theta}}_{i,r})}{\lambda_i^2(\hat{\boldsymbol{\theta}}_{i,r})} (Y_i(\hat{\boldsymbol{\theta}}_{i,r}) - \gamma_i) \right\} \\ &= \underbrace{\sum_{b_2=1}^B \sum_{b_1=1}^{B+1} \frac{\lambda_i^2(\boldsymbol{\theta}_\ell)}{N_{i,T}(\boldsymbol{\theta}_\ell)} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell, b_1} \frac{K_i(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}) N_{i,T}(\boldsymbol{\theta}_{b_2})}{\lambda_i^2(\boldsymbol{\theta}_{b_2})} (\mu_{i,T}(\boldsymbol{\theta}_{b_2}) - \gamma_i)}_{:=\text{(I)}} \\ &\quad + \underbrace{\sum_{r=1}^{N_{i,T}(\hat{\boldsymbol{\theta}})} \sum_{b_1=1}^{B+1} \frac{\lambda_i^2(\boldsymbol{\theta}_\ell)}{N_{i,T}(\boldsymbol{\theta}_\ell)} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell, b_1} \frac{K_i(\boldsymbol{\theta}_{b_1}, \hat{\boldsymbol{\theta}}_{i,r})}{\lambda_i^2(\hat{\boldsymbol{\theta}}_{i,r})} (Y_i(\hat{\boldsymbol{\theta}}_{i,r}) - \gamma_i)}_{:=\text{(II)}}. \end{aligned}$$

We can rewrite (I) as

$$(I) = \sum_{b_2=1}^B \frac{\lambda_i^2(\theta_\ell) \lambda_i^{-2}(\theta_{b_2})}{N_{i,T}(\theta_\ell) N_{i,T}^{-1}(\theta_{b_2})} (\mu_{i,T}(\theta_{b_2}) - \gamma_i) \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \theta_{b_2}) \right) \quad (\text{EC.29})$$

For all sufficiently large T , we have $\bar{\mathbf{K}}_{i,T}^{-1} \bar{\mathbf{K}}_{i,T} = I$, so $\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \theta_{b_2}) = \mathbf{1}\{\ell = b_2\}, b_2 \leq B$. Accordingly, the limit of (EC.29) is given as

$$\lim_{T \rightarrow \infty} (I) = \sum_{b_2=1}^B (\eta_i(\theta_\ell) - \gamma_i) \mathbf{1}\{b_2 = \ell\} = (\eta_i(\theta_\ell) - \gamma_i) \mathbf{1}\{\ell \leq B\}. \quad (\text{EC.30})$$

Now, let us show the limit of (II), which can be rewritten as

$$(II) = \frac{\lambda_i^2(\theta_\ell) N_{i,T}(\hat{\theta})}{N_{i,T}(\theta_\ell)} \frac{1}{N_{i,T}(\hat{\theta})} \sum_{r=1}^{N_{i,T}(\hat{\theta})} \frac{Y_i(\hat{\theta}_{i,r}) - \gamma_i}{\lambda_i^2(\hat{\theta}_{i,r})} \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) \right)$$

Let $X_T = \frac{1}{N_{i,T}(\hat{\theta})} \sum_{r=1}^{N_{i,T}(\hat{\theta})} \frac{Y_i(\hat{\theta}_{i,r}) - \gamma_i}{\lambda_i^2(\hat{\theta}_{i,r})} \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) \right)$. Since each simulation error is independent, Assumption 6 gives that the standardized simulation errors, $\epsilon_{i,r} := (Y_i(\hat{\theta}_{i,r}) - \eta_i(\hat{\theta}_{i,r})) / \lambda_i(\hat{\theta}_{i,r})$, are i.i.d. standard normal random variables for $r = 1, \dots, N_{i,T}$. The strong law of large numbers implies that

$$\lim_{T \rightarrow \infty} N_{i,T}^{-1}(\hat{\theta}) \sum_{r=1}^{N_{i,T}(\hat{\theta})} \epsilon_{i,r} = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} N_{i,T}^{-1}(\hat{\theta}) \sum_{r=1}^{N_{i,T}(\hat{\theta})} |\epsilon_{i,r}| = \sqrt{2/\pi} \quad \text{a.s.} \quad (\text{EC.31})$$

Combining Step 2 with the continuity of K_i , $\hat{\theta}_T \rightarrow \theta_0$, and $\hat{\theta}_{i,r} \rightarrow \theta_0$, we obtain

$$\lim_{T,r \rightarrow \infty} \sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) = \mathbf{1}\{\ell = B+1\} \quad \text{a.s.}$$

The continuity of $\eta_i(\theta)$ and $\lambda_i(\theta)$, together with $\lambda_i(\theta_0) > 0$, then gives

$$\lim_{T,r \rightarrow \infty} \frac{\eta_i(\hat{\theta}_{i,r}) - \gamma_i}{\lambda_i^2(\hat{\theta}_{i,r})} \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) \right) = \frac{\eta_i(\theta_0) - \gamma_i}{\lambda_i^2(\theta_0)} \mathbf{1}\{\ell = B+1\}, \quad (\text{EC.32})$$

and

$$\lim_{T,r \rightarrow \infty} \lambda_i^{-1}(\hat{\theta}_{i,r}) \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) \right) = \lambda_i^{-1}(\theta_0) \mathbf{1}\{\ell = B+1\}. \quad (\text{EC.33})$$

One can decompose X_T as follows:

$$\begin{aligned} X_T = & N_{i,T}^{-1}(\hat{\theta}) \sum_{r=1}^{N_{i,T}(\hat{\theta})} \frac{\eta_i(\hat{\theta}_{i,r}) - \gamma_i}{\lambda_i^2(\hat{\theta}_{i,r})} \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) \right) + \frac{\mathbf{1}\{\ell = B+1\}}{\lambda_i(\theta_0)} N_{i,T}^{-1}(\hat{\theta}) \sum_{r=1}^{N_{i,T}(\hat{\theta})} \epsilon_{i,r} \\ & + N_{i,T}^{-1}(\hat{\theta}) \sum_{r=1}^{N_{i,T}(\hat{\theta})} \epsilon_{i,r} \times \left[\lambda_i^{-1}(\hat{\theta}_{i,r}) \sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) - \lambda_i^{-1}(\theta_0) \mathbf{1}\{\ell = B+1\} \right]. \end{aligned} \quad (\text{EC.34})$$

From the Cesàro mean property, the first term converges to $\frac{\eta_i(\theta_0) - \gamma_i}{\lambda_i^2(\theta_0)} \mathbf{1}\{\ell = B+1\}$. The second term converges to zero by the strong law established above. We next show that the third term converges to zero. For any $\delta > 0$, there exists R_δ such that $\left| \lambda_i^{-1}(\hat{\theta}_{i,r}) \left(\sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) \right) - \lambda_i^{-1}(\theta_0) \mathbf{1}\{\ell = B+1\} \right| \leq \delta$ for all sufficiently large T and $r \geq R_\delta$ a.s. Then, the absolute value of the third term is bounded above by

$$\frac{1}{N_{i,T}(\hat{\theta})} \sum_{r=1}^{R_\delta-1} |\epsilon_{i,r}| \times \left| \frac{1}{\lambda_i(\hat{\theta}_{i,r})} \sum_{b_1=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{\ell,b_1} K_i(\theta_{b_1}, \hat{\theta}_{i,r}) - \frac{\mathbf{1}\{\ell = B+1\}}{\lambda_i(\theta_0)} \right| + \frac{\delta}{N_{i,T}(\hat{\theta})} \sum_{r=R_\delta}^{N_{i,T}(\hat{\theta})} |\epsilon_{i,r}|.$$

The first term converges to zero as $N_{i,T}(\hat{\theta}) \rightarrow \infty$. The second term converges to $\delta\sqrt{2/\pi}$ by (EC.31). Thus, the limit superior of the absolute value of the third term in (EC.34) is at most $\delta\sqrt{2/\pi}$. Letting $\delta \rightarrow 0^+$ shows that the third term also converges to zero. Consequently, we have $\lim_{T \rightarrow \infty} X_T = \frac{\eta_i(\theta_0) - \gamma_i}{\lambda_i^2(\theta_0)} 1\{\ell = B+1\}$ a.s. For $\ell \leq B$, Step 1 stipulates that $N_{i,T}(\hat{\theta})/N_{i,T}(\theta_\ell) = O(1)$, whereas this ratio equals one if $\ell = B+1$. Combining these observations with the continuity of λ_i , we obtain

$$\lim_{T \rightarrow \infty} (\text{II}) = (\eta_i(\theta_0) - \gamma_i) 1\{\ell = B+1\}. \quad (\text{EC.35})$$

From (EC.30) and (EC.35), we conclude that $\lim_{T \rightarrow \infty} (\mathbf{b}_T)_\ell = \eta_i(\theta_\ell) - \gamma_i$, as desired. $\square$

Step 4. $\lim_{T \rightarrow \infty} \mathbf{A}_T = \mathbf{K}_{i,B+1}^{-1}$ holds almost surely.

Proof of Step 4. Since $\lim_{T \rightarrow \infty} \bar{\Sigma}_{i,T} = \mathbf{0}_{B+1,B+1}$, it suffices to verify $\lim_{T \rightarrow \infty} \bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^\top = \mathbf{K}_{i,B+1}$. For $1 \leq b_1, b_2 \leq B+1$,

$$\begin{aligned} (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^\top)_{b_1, b_2} &= \sum_{\ell=1}^{B+1} (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} (\tilde{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^\top)_{\ell, b_2} \\ &= \sum_{\ell=1}^{B+1} (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} \sum_{\ell_1=1}^{B+N_{i,T}(\hat{\theta})} (\tilde{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1})_{\ell, \ell_1} (\tilde{\mathbf{K}}_{i,T}^\top)_{\ell_1, b_2} \\ &= \underbrace{\sum_{\ell=1}^{B+1} (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} \sum_{\ell_1=1}^B (\tilde{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1})_{\ell, \ell_1} (\tilde{\mathbf{K}}_{i,T}^\top)_{\ell_1, b_2}}_{:= (\text{III})} \\ &\quad + \underbrace{\sum_{\ell=1}^{B+1} (\bar{\Sigma}_{i,T} \bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} \sum_{\ell_1=1}^{N_{i,T}(\hat{\theta})} (\tilde{\mathbf{K}}_{i,T} \bar{\Sigma}_{i,T}^{-1})_{\ell, B+\ell_1} (\tilde{\mathbf{K}}_{i,T}^\top)_{B+\ell_1, b_2}}_{:= (\text{IV})}. \end{aligned}$$

Let us see (III) first. One can derive that

$$\begin{aligned} (\text{III}) &= \sum_{\ell=1}^{B+1} \sum_{\ell_1=1}^B \frac{\lambda_i^2(\theta_{b_1})}{N_{i,T}(\theta_{b_1})} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} K_i(\theta_\ell, \theta_{\ell_1}) \frac{N_{i,T}(\theta_{\ell_1})}{\lambda_i^2(\theta_{\ell_1})} K_i(\theta_{\ell_1}, \theta_{b_2}) \\ &= \sum_{\ell_1=1}^B \frac{\lambda_i^2(\theta_{b_1}) \lambda_i^{-2}(\theta_{\ell_1})}{N_{i,T}(\theta_{b_1}) N_{i,T}^{-1}(\theta_{\ell_1})} K_i(\theta_{\ell_1}, \theta_{b_2}) \left(\sum_{\ell=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} K_i(\theta_\ell, \theta_{\ell_1}) \right). \end{aligned}$$

From Step 2 again, we have $\sum_{\ell=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} K_i(\theta_\ell, \theta_{\ell_1}) = 1\{b_1 = \ell_1\}$ if $\ell_1 \leq B$. From this result, the limit of (III) is given as

$$\lim_{T \rightarrow \infty} (\text{III}) = \sum_{\ell_1=1}^B K_i(\theta_{\ell_1}, \theta_{b_2}) 1\{b_1 = \ell_1\} = K_i(\theta_{b_1}, \theta_{b_2}) 1\{b_1 \leq B\}.$$

Further, (IV) can be calculated as

$$\begin{aligned} (\text{IV}) &= \sum_{\ell_1=1}^{N_{i,T}(\hat{\theta})} \sum_{\ell=1}^{B+1} \frac{\lambda_i^2(\theta_{b_1})}{N_{i,T}(\theta_{b_1})} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} \frac{K_i(\theta_\ell, \hat{\theta}_{i, \ell_1})}{\lambda_i^2(\hat{\theta}_{i, \ell_1})} K_i(\theta_{b_2}, \hat{\theta}_{i, \ell_1}) \\ &= \frac{N_{i,T}(\hat{\theta})}{N_{i,T}(\theta_{b_1})} \frac{1}{N_{i,T}(\hat{\theta})} \sum_{\ell_1=1}^{N_{i,T}(\hat{\theta})} \frac{\lambda_i^2(\theta_{b_1})}{\lambda_i^2(\hat{\theta}_{i, \ell_1})} K_i(\theta_{b_2}, \hat{\theta}_{i, \ell_1}) \left(\sum_{\ell=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} K_i(\theta_\ell, \hat{\theta}_{i, \ell_1}) \right). \end{aligned}$$

From $\lim_{T, \ell_1 \rightarrow \infty} \sum_{\ell=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} K_i(\theta_\ell, \hat{\theta}_{i, \ell_1}) = 1\{b_1 = B+1\}$ and the continuity of λ_i^2 , we can see that

$$\lim_{T, \ell_1 \rightarrow \infty} \frac{\lambda_i^2(\theta_{b_1})}{\lambda_i^2(\hat{\theta}_{i, \ell_1})} K_i(\theta_{b_2}, \hat{\theta}_{i, \ell_1}) \left(\sum_{\ell=1}^{B+1} (\bar{\mathbf{K}}_{i,T}^{-1})_{b_1, \ell} K_i(\theta_\ell, \hat{\theta}_{i, \ell_1}) \right) = \frac{\lambda_i^2(\theta_{b_1})}{\lambda_i^2(\theta_0)} K_i(\theta_{b_2}, \theta_0) 1\{b_1 = B+1\}.$$

Applying the Cesàro mean property, one can derive the limit of (IV) as

$$\lim_{T \rightarrow \infty} (\text{IV}) = K_i(\boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_0) 1\{b_1 = B + 1\}.$$

We finally have $\lim_{T \rightarrow \infty} (\text{III}) + (\text{IV}) = (\mathbf{K}_{i, B+1})_{b_1, b_2}$, which completes the proof. $\square$ Q.E.D.

REMARK EC.2. We show that if BOLD is adopted as the R&S subroutine in OSAR⁺⁺, then the condition in Theorem 5, $N_{i,T}(\hat{\boldsymbol{\theta}}) \xrightarrow{a.s.} \infty$ for all $1 \leq i \leq k$, holds.

Proof. For each sample path, define the set of solutions simulated infinitely many times at the sequence of MAPs as $\mathcal{I} = \{i : N_{i,T}(\hat{\boldsymbol{\theta}}) \rightarrow \infty\}$. Since $\liminf_{T \rightarrow \infty} \sum_{i=1}^k N_{i,T}(\hat{\boldsymbol{\theta}})/T \geq \epsilon$, we have $\mathcal{I} \neq \emptyset$. Our goal is to show $\mathcal{I} = \{1, 2, \dots, k\}$. Suppose otherwise, i.e., $\mathcal{I}^c \neq \emptyset$. Then, there exists T_0 such that $(\ell, \hat{\boldsymbol{\theta}})$ is not chosen to simulate for any $T \geq T_0$ and any $\ell \in \mathcal{I}^c$. For each $i \in \mathcal{I}$, $\lim_{T \rightarrow \infty} \hat{\mu}_{i,T}(\hat{\boldsymbol{\theta}}) = \eta_i(\boldsymbol{\theta}_0)$. For $i \in \mathcal{I}^c$, combining (16) with Steps 1–2, and that i is simulated finitely many times, yields $\hat{\mu}_{i,T}(\hat{\boldsymbol{\theta}}) = L_i, \forall T \geq T_0$, for sample-path dependent constant L_i .

Let b_T denote the estimated best at $\hat{\boldsymbol{\theta}}$ after spending budget $T \geq T_0$. If $b_T \in \mathcal{I}^c$, then $N_{b_T,T}^2(\hat{\boldsymbol{\theta}})/\lambda_{b_T}^2(\hat{\boldsymbol{\theta}})$ is bounded, whereas $\sum_{i \neq b_T} N_{i,T}^2(\hat{\boldsymbol{\theta}})/\lambda_i^2(\hat{\boldsymbol{\theta}})$ diverges as there exists $i \in \mathcal{I}$ such that $i \neq b_T$. Then, the global-balance condition allocates simulation runs to $(b_T, \hat{\boldsymbol{\theta}})$, contradicting the definition of T_0 . Hence, $b_T \in \mathcal{I}$ eventually. By the consistency of $\hat{\mu}_{i,T}(\hat{\boldsymbol{\theta}})$ on $\mathcal{I}$, b_T converges to $j_0 := \arg \min_{i \in \mathcal{I}} \eta_i(\boldsymbol{\theta}_0)$. Because $j_0 \in \mathcal{I}$, the pairwise-balance condition of BOLD will be invoked infinitely often. Otherwise, only j_0 would eventually be simulated, making $N_{j_0,T}^2(\hat{\boldsymbol{\theta}})/\lambda_{j_0}^2(\hat{\boldsymbol{\theta}})$ diverge and $\sup_T \sum_{i \neq j_0} N_{i,T}^2(\hat{\boldsymbol{\theta}})/\lambda_i^2(\hat{\boldsymbol{\theta}}) < \infty$, which contradicts the global-balance condition.

Whenever we test the pairwise balance condition of BOLD, it selects $i \neq j_0$ minimizing

$$\Gamma_{i,T} := \left(\sum_{i=1}^k N_{i,T}(\hat{\boldsymbol{\theta}}) \right) \times G_{i,T}(\hat{\boldsymbol{\theta}}) = \frac{(\hat{\mu}_{i,T}(\hat{\boldsymbol{\theta}}) - \hat{\mu}_{j_0,T}(\hat{\boldsymbol{\theta}}))^2}{2 \left(\lambda_{j_0}^2(\hat{\boldsymbol{\theta}})/N_{j_0,T}(\hat{\boldsymbol{\theta}}) + \lambda_i^2(\hat{\boldsymbol{\theta}})/N_{i,T}(\hat{\boldsymbol{\theta}}) \right)}, \quad \forall i \neq j_0, \quad (\text{EC.36})$$

for all sufficiently large T . For $i \in \mathcal{I} \setminus \{j_0\}$, the numerator tends to a strictly positive constant and the denominator tends to zero, so $\Gamma_{i,T} \rightarrow \infty$ whereas we have $\Gamma_{i_1,T} = O(1)$ for some $i_1 \in \mathcal{I}^c$. Consequently, for all sufficiently large T , BOLD will select a member of $\mathcal{I}^c$ to simulate. This contradicts the definition of T_0 , after which no member of $\mathcal{I}^c$ should be simulated. Thus, $\mathcal{I}^c = \emptyset$ and we have $\mathcal{I} = \{1, 2, \dots, k\}$. $\square$

EC.4. Supporting Results for Sections 7 and 8

EC.4.1. Extension to the Case under Model Misspecification

To handle a potential model misspecification, we first make the following assumption, which is a modification of Assumption 4.

ASSUMPTION EC.1. For each $1 \leq \ell \leq L$, $\mathcal{F}_{\Xi^\ell}$ and $\mathcal{F}_\ell$ are σ -algebras of prior π_0^ℓ and $Z^\ell \sim g^\ell$, respectively, and the following holds: Assumptions 4(a) and 4(b) hold and

- (c) The class of functions, $\{\log(f_{\theta^\ell}^\ell/g^\ell) : \theta^\ell \in \Xi^\ell\}$, is P_{g^ℓ} -Glivenko-Cantelli;
- (d) $D^\ell(\theta_*^\ell, \theta^\ell) := D^{\text{KL}}(g^\ell \| f_{\theta^\ell}^\ell) - D^{\text{KL}}(g^\ell \| f_{\theta_*^\ell}^\ell)$, is continuous in both θ^ℓ and θ_*^ℓ , $D^\ell(\theta_*^\ell, \theta^\ell) = 0 \iff \theta_*^\ell = \theta^\ell$, and $0 < \inf_{\theta^\ell \in \Xi^\ell} \pi_0^\ell(\theta^\ell) \leq \sup_{\theta^\ell \in \Xi^\ell} \pi_0^\ell(\theta^\ell) < \infty$.

Theorem EC.1 below derives an extended version of Theorem 1 under model misspecification.

THEOREM EC.1 (Extension of Theorem 1 under model misspecification). Suppose Condition 1, Assumptions 1–3, EC.1, and 5 hold. Then, with probability 1, (a) $i^*(\pi_m)$ converges to i_* as $T \rightarrow \infty$; and (b) $\lim_{T \rightarrow \infty} -\frac{1}{T} \log(1 - P_{\pi_m}(\Theta_{i^*}(\pi_m))) = \inf_{\theta \in \Theta_{i_*}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^\ell(\theta_*^\ell, \theta^\ell)$.

Proof. For all $1 \leq \ell \leq L$, $D^\ell(\theta_*^\ell, \theta^\ell) = E_{Z^\ell \sim g^\ell} \left[-\log \frac{f_{\theta^\ell}^\ell(Z^\ell)}{f_{\theta_*^\ell}^\ell(Z^\ell)} \right]$. Assumptions EC.1(c) and (d) yield

$$\begin{aligned} & \sup_{\theta \in \Theta} \left| \frac{1}{T} \log \left(\frac{\pi_m(\theta)}{\pi_m(\theta_*)} \right) + \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^\ell(\theta_*^\ell, \theta^\ell) \right| \\ &= \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{\ell=1}^L \log \left(\frac{\pi_0^\ell(\theta^\ell)}{\pi_0^\ell(\theta_*^\ell)} \right) + \sum_{\ell=1}^L \frac{m_\ell}{T} \left(\frac{1}{m_\ell} \sum_{i=1}^{m_\ell} \log \frac{f_{\theta^\ell}^\ell(Z_i^\ell)}{f_{\theta_*^\ell}^\ell(Z_i^\ell)} \right) + \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^\ell(\theta_*^\ell, \theta^\ell) \right| \\ &\leq \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{\ell=1}^L \log \left(\frac{\pi_0^\ell(\theta^\ell)}{\pi_0^\ell(\theta_*^\ell)} \right) \right| + \sum_{\ell=1}^L \frac{m_\ell}{T} \sup_{\theta^\ell \in \Theta^\ell} \left| \left(\frac{1}{m_\ell} \sum_{i=1}^{m_\ell} \log \frac{f_{\theta^\ell}^\ell(Z_i^\ell)}{f_{\theta_*^\ell}^\ell(Z_i^\ell)} + D^\ell(\theta_*^\ell, \theta^\ell) \right) \right| + o(1) \\ &\rightarrow 0 \text{ } P_g\text{-a.s.,} \end{aligned}$$

where uniform $o(1)$ term follows from Condition 1, Assumptions 4 and EC.1. Combining this result with the continuity of $I(\theta) = \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} D^\ell(\theta_*^\ell, \theta^\ell)$ from Assumption EC.1(d), we can proceed in the same way as in the proof of Theorem 1 to obtain the result. Q.E.D.

Now, we are ready to show Theorem 6; its proof proceeds similarly as that of Theorem 2.

Proof of Theorem 6. One can easily validate the statement by replacing i_0 and $D^{\text{KL}}(\theta_0^\ell \| \theta_b^\ell)$ with i_* and $D^\ell(\theta_*^\ell, \theta_b^\ell)$, respectively. All other parts of the proof of Theorem 2 remain valid without modification. Q.E.D.

EC.4.2. An Extended Discussion of Section 8.1.4

We first present the proof of Proposition 3.

Proof of Proposition 3. Observe that $E[\text{OC}_t] = \sum_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0)) P(i_t = i)$ can be bounded from above and below as follows:

$$P(i_t \neq i_0) \times \min_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0)) \leq E[\text{OC}_t] \leq P(i_t \neq i_0) \times \max_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0))$$

Equivalently,

$$\begin{aligned} \frac{\log P(i_t \neq i_0)}{t} + \frac{\log \min_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0))}{t} &\leq \frac{1}{t} \log E[\text{OC}_t] \\ &\leq \frac{\log P(i_t \neq i_0)}{t} + \frac{\log \max_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0))}{t} \end{aligned} \quad (\text{EC.37})$$

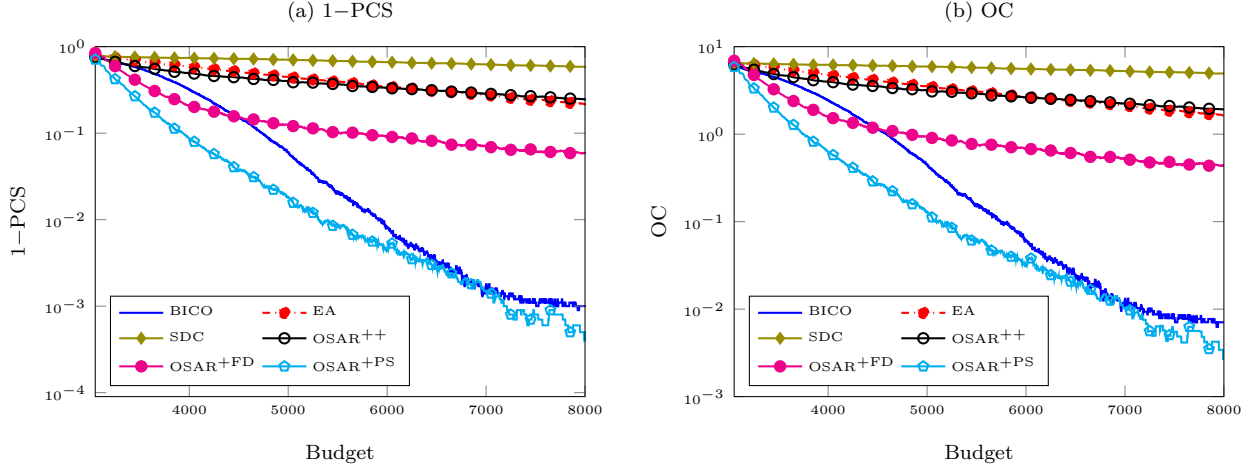


Figure EC.1 For the food contamination example, we display log-scaled 1-PCS and OC in panels (a) and (b), respectively. The results are obtained by averaging over 10,000 macroruns.

Since $\lim_{t \rightarrow \infty} \frac{\log \max_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0))}{t} = \lim_{t \rightarrow \infty} \frac{\log \min_{i \neq i_0} (\eta_i(\theta_0) - \eta_{i_0}(\theta_0))}{t} = 0$ holds as i_0 is unique, taking $t \rightarrow \infty$ on both sides of (EC.37) completes the proof. Q.E.D.

Figure EC.1 below is provided as a comparison of benchmark algorithms in terms of PFS and OC applied to the food contamination example in Section 8.2. Conclusions in Section 8.1.4 hold here, too—the algorithms perform very similarly in both PFS and OC.

EC.5. Efficient Updates of the Nyström KRR Model

In this subsection, we discuss an efficient method to update $\mathbf{c}_{i,T}$ discussed in Remark 4. Define $\mathbf{A}_{i,T} = \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \tilde{\mathbf{K}}_{i,T}^\top + \kappa \tilde{\mathbf{K}}_{i,T}$ and $\mathbf{b}_{i,T} = \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{y}_T$, where $\mathbf{y}_T = \left([\boldsymbol{\mu}_{i,T}^\top \mathbf{Y}_i^\top]^\top - \gamma_i \mathbf{1}_{B+N_{i,T}(\hat{\theta})} \right)$. From Proposition 2, Corollary EC.1 below shows that $\mathbf{c}_{i,T}$ becomes the optimal solution to the linear equation. Namely, we can learn $\mathbf{c}_{i,T}$ via the least squares method instead of computing the Moore-Penrose inverse of $\mathbf{A}_{i,T}$.

COROLLARY EC.1. *One can view $\mathbf{c}_{i,T}$ defined in (EC.24) as the least square solution to*

$$\min_{\mathbf{x}} \|\mathbf{A}_{i,T} \mathbf{x} - \mathbf{b}_{i,T}\|_2, \quad (\text{EC.38})$$

where $\mathbf{b}_{i,T} = \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{y}$.

Motivated by this, $\mathbf{c}_{i,T}$ can be computed by solving (EC.38). In this regard, we decompose $\mathbf{A}_{i,T}$ via the QR decomposition, $\mathbf{A}_{i,T} = \mathbf{Q}_{i,T} \mathbf{R}_{i,T}$, where $\mathbf{Q}_{i,T}$ is the orthogonal matrix (i.e., $\mathbf{Q}_{i,T}^\top = \mathbf{Q}_{i,T}^{-1}$) and $\mathbf{R}_{i,T}$ is the upper triangular matrix. Observe that

$$\begin{aligned} \|\mathbf{A}_{i,T} \mathbf{x} - \mathbf{b}_{i,T}\|_2^2 &= (\mathbf{A}_{i,T} \mathbf{x} - \mathbf{b}_{i,T})^\top (\mathbf{A}_{i,T} \mathbf{x} - \mathbf{b}_{i,T}) \\ &= (\mathbf{A}_{i,T} \mathbf{x} - \mathbf{b}_{i,T})^\top \mathbf{Q}_{i,T} \mathbf{Q}_{i,T}^\top (\mathbf{A}_{i,T} \mathbf{x} - \mathbf{b}_{i,T}) \\ &= \|\mathbf{R}_{i,T} \mathbf{x} - \mathbf{Q}_{i,T}^\top \mathbf{b}_{i,T}\|_2^2, \end{aligned}$$

where the second equality follows from $\mathbf{Q}_{i,T}\mathbf{Q}_{i,T}^\top = \mathbf{I}$. Hence, we have

$$\mathbf{c}_{i,T} = \arg \min_{\mathbf{x}} \|\mathbf{R}_{i,T}\mathbf{x} - \mathbf{Q}_{i,T}^\top \mathbf{b}_{i,T}\|_2. \quad (\text{EC.39})$$

We will exploit (EC.39) to evaluate $\mathbf{c}_{i,T}$ since it is easier to solve than (EC.38) as $\mathbf{R}_{i,T}$ is upper triangular.

Whenever we collect more simulation data at any (i, θ_b) , $\mathbf{A}_{i,T}$ and $\mathbf{b}_{i,T}$ are updated. Then, $\mathbf{Q}_{i,T}$ and $\mathbf{R}_{i,T}$ should be updated accordingly. Proposition EC.2 below summarizes the recursive formulae of $\mathbf{A}_{i,T}$ and $\mathbf{b}_{i,T}$, respectively. From Proposition EC.2, one can confirm that $\mathbf{A}_{i,T+1}$ can be written as $\mathbf{A}_{i,T+1} = \mathbf{A}_{i,T} + \mathbf{u}\mathbf{v}^\top$ for some $(B+1)$ -dimensional vectors, $\mathbf{u}$ and $\mathbf{v}$. Exploiting this, we can efficiently update $\mathbf{Q}_{i,T+1}$ and $\mathbf{R}_{i,T+1}$ in a sequential manner using the MATLAB built-in function, `qrupdate`. It is known that the computational complexity of this job is $O(B^2)$, which is cheaper than that of a new QR decomposition, $O(B^3)$ (Golub and Loan 1996).

PROPOSITION EC.2. *Assuming that Y_i^{new} is the first new observation after spending budget T , $\mathbf{A}_{i,T}$ and $\mathbf{b}_{i,T}$ are updated as $\mathbf{A}_{i,T+1} = \mathbf{A}_{i,T} + \lambda_i^{-2}(\theta_b)\mathbf{k}_{i,b}\mathbf{k}_{i,b}^\top$ and $\mathbf{b}_{i,T+1} = \mathbf{b}_{i,T} + \lambda_i^{-2}(\theta_b)(Y_i^{\text{new}} - \gamma_i)\mathbf{k}_{i,b}$, where $\mathbf{k}_{i,b} = \tilde{\mathbf{K}}_{i,T}\mathbf{e}_b$.*

Proof of Proposition EC.2. We consider two cases, (a) $\theta_b \in \Theta^+$ and (b) $\theta_b = \hat{\theta}$, separately. Notice that $\tilde{\mathbf{K}}_{i,T+1} = \tilde{\mathbf{K}}_{i,T}$ and $\tilde{\mathbf{K}}_{i,T+1} = \tilde{\mathbf{K}}_{i,T}$ hold if the MAP is not updated (Case (a)).

(a) Suppose $\theta_b \in \Theta^+$. Then,

$$\mathbf{A}_{i,T+1} = \tilde{\mathbf{K}}_{i,T+1}\tilde{\Sigma}_{i,T+1}^{-1}\tilde{\mathbf{K}}_{i,T+1}^\top + \kappa\tilde{\mathbf{K}}_{i,T+1} = \tilde{\mathbf{K}}_{i,T}\tilde{\Sigma}_{i,T}^{-1}\tilde{\mathbf{K}}_{i,T}^\top + \kappa\tilde{\mathbf{K}}_{i,T} + \lambda_i^{-2}(\theta_b)\mathbf{k}_{i,b}\mathbf{k}_{i,b}^\top = \mathbf{A}_{i,T} + \lambda_i^{-2}(\theta_b)\mathbf{k}_{i,b}\mathbf{k}_{i,b}^\top,$$

which proves the recursive formula for $\mathbf{A}_{i,T+1}$. Recall $\mathbf{y}_T = \left([\mu_{i,T}^\top \ \mathbf{Y}_i^\top]^\top - \gamma_i\mathbf{1}_{B+N_{i,T}(\hat{\theta})}\right)$. Let $\tilde{\mu} = \mu_{i,T+1}(\theta_b) - \mu_{i,T}(\theta_b)$. Then, we have

$$\begin{aligned} \mathbf{b}_{i,T+1} &= \tilde{\mathbf{K}}_{i,T+1}\tilde{\Sigma}_{i,T+1}^{-1}\mathbf{y}_{T+1} \\ &= \tilde{\mathbf{K}}_{i,T}(\tilde{\Sigma}_{i,T}^{-1} + \lambda_i^{-2}(\theta_b)\mathbf{e}_b\mathbf{e}_b^\top)(\mathbf{y}_T + \tilde{\mu}\mathbf{e}_b) \\ &= \tilde{\mathbf{K}}_{i,T}\left[\tilde{\Sigma}_{i,T}^{-1}\mathbf{y}_T + \left\{\lambda_i^{-2}(\theta_b)\mathbf{e}_b^\top\mathbf{y}_T + \lambda_i^{-2}(\theta_b)\tilde{\mu} + \tilde{\mu}\left(\frac{N_{i,T}(\theta_b)}{\lambda_i^2(\theta_b)}\right)\right\}\mathbf{e}_b\right] \\ &= \tilde{\mathbf{K}}_{i,T}\left[\tilde{\Sigma}_{i,T}^{-1}\mathbf{y}_T + \lambda_i^{-2}(\theta_b)\{\mu_{i,T}(\theta_b) - \gamma_i + (N_{i,T}(\theta_b) + 1)\tilde{\mu}\}\mathbf{e}_b\right] \\ &= \tilde{\mathbf{K}}_{i,T}\tilde{\Sigma}_{i,T}^{-1}\mathbf{y}_T + \lambda_i^{-2}(\theta_b)(Y_i^{\text{new}} - \gamma_i)\mathbf{k}_{i,b} = \mathbf{b}_{i,T} + \lambda_i^{-2}(\theta_b)(Y_i^{\text{new}} - \gamma_i)\mathbf{k}_{i,b}, \end{aligned}$$

where the fifth equality follows from $\mu_{i,T}(\theta_b) - \gamma_i + (N_{i,T}(\theta_b) + 1)\tilde{\mu} = Y_i^{\text{new}} - \gamma_i$.

(b) Since $\tilde{\mathbf{K}}_{i,T+1} = [\tilde{\mathbf{K}}_{i,T} \ \mathbf{k}_{i,B+1}]$ and $\tilde{\Sigma}_{i,T+1}^{-1} = \begin{bmatrix} \tilde{\Sigma}_{i,T}^{-1} & \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1} \\ \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1}^\top & \lambda_i^{-2}(\hat{\theta}_T) \end{bmatrix}$, we have

$$\begin{aligned} \mathbf{A}_{i,T+1} &= \tilde{\mathbf{K}}_{i,T+1}\tilde{\Sigma}_{i,T+1}^{-1}\tilde{\mathbf{K}}_{i,T+1}^\top + \kappa\tilde{\mathbf{K}}_{i,T+1} \\ &= [\tilde{\mathbf{K}}_{i,T} \ \mathbf{k}_{i,B+1}] \begin{bmatrix} \tilde{\Sigma}_{i,T}^{-1} & \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1} \\ \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1}^\top & \lambda_i^{-2}(\hat{\theta}_T) \end{bmatrix} [\tilde{\mathbf{K}}_{i,T} \ \mathbf{k}_{i,B+1}]^\top + \kappa\tilde{\mathbf{K}}_{i,T} \\ &= \tilde{\mathbf{K}}_{i,T}\tilde{\Sigma}_{i,T}^{-1}\tilde{\mathbf{K}}_{i,T}^\top + \lambda_i^{-2}(\hat{\theta}_T)\mathbf{k}_{i,B+1}\mathbf{k}_{i,B+1}^\top + \kappa\tilde{\mathbf{K}}_{i,T} = \mathbf{A}_{i,T} + \lambda_i^{-2}(\hat{\theta}_T)\mathbf{k}_{i,B+1}\mathbf{k}_{i,B+1}^\top. \end{aligned}$$

To analyze $\mathbf{b}_{i,T+1}$, we first observe $\mathbf{y}_{T+1} = [\mathbf{y}_T^\top Y_i^{\text{new}} - \gamma_i]^\top$ and

$$\begin{aligned}\tilde{\mathbf{K}}_{i,T+1} \tilde{\Sigma}_{i,T+1}^{-1} \mathbf{y}_{T+1} &= [\tilde{\mathbf{K}}_{i,T} \ \mathbf{k}_{i,B+1}] \left[\begin{array}{c|c} \tilde{\Sigma}_{i,T}^{-1} & \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1} \\ \hline \mathbf{0}_{(B+N_{i,T}(\hat{\theta})) \times 1}^\top & \lambda_i^{-2}(\hat{\theta}_T) \end{array} \right] \mathbf{y}_{T+1} \\ &= \tilde{\mathbf{K}}_{i,T} \tilde{\Sigma}_{i,T}^{-1} \mathbf{y}_T + \lambda_i^{-2}(\hat{\theta}_T) (Y_i^{\text{new}} - \gamma_i) \mathbf{k}_{i,B+1} \\ &= \mathbf{b}_{i,T} + \lambda_i^{-2}(\hat{\theta}) (Y_i^{\text{new}} - \gamma_i) \mathbf{k}_{i,B+1},\end{aligned}$$

which completes the proof. Q.E.D.

Corollary EC.2 below presents the revised formulae for $\mathbf{A}_{i,T}$ and $\mathbf{b}_{i,T}$ when the variances, $\lambda_i^2(\theta_b)$, are unknown.

COROLLARY EC.2. *Let $\lambda_{i,T}^2(\theta_b)$ be the sample variance at (i, θ_b) after spending sampling budget T . After observing Y_i^{new} , $\mathbf{A}_{i,T}$ and $\mathbf{b}_{i,T}$ are updated as follows:*

(a) *If $\theta_b \in \Theta^+$, we have*

$$\mathbf{A}_{i,T+1} = \mathbf{A}_{i,T} + \tilde{\lambda} \mathbf{k}_{i,b} \mathbf{k}_{i,b}^\top \text{ and } \mathbf{b}_{i,T+1} = \mathbf{b}_{i,T} + \tilde{\mu} \mathbf{k}_{i,b},$$

$$\text{where } \tilde{\lambda} = \left(\frac{N_{i,T}(\theta_b)+1}{\lambda_{i,T+1}^2(\theta_b)} - \frac{N_{i,T}(\theta_b)}{\lambda_{i,T}^2(\theta_b)} \right), \tilde{\mu} = \frac{N_{i,T}(\theta_b)+1}{\lambda_{i,T+1}^2(\theta_b)} (\mu_{i,T+1}(\theta_b) - \gamma_i) - \frac{N_{i,T}(\theta_b)}{\lambda_{i,T}^2(\theta_b)} (\mu_{i,T}(\theta_b) - \gamma_i), \text{ and } \mathbf{k}_{i,b} = \tilde{\mathbf{K}}_{i,T} \mathbf{e}_b.$$

(b) *If $\theta_b = \hat{\theta}$, we can see*

$$\mathbf{A}_{i,T+1} = \mathbf{A}_{i,T} + \lambda_{i,T}^{-2}(\hat{\theta}) \mathbf{k}_{i,B+1} \mathbf{k}_{i,B+1}^\top \text{ and } \mathbf{b}_{i,T+1} = \mathbf{b}_{i,T} + \lambda_{i,T}^{-2}(\hat{\theta}) (Y_i^{\text{new}} - \gamma_i) \mathbf{k}_{i,B+1},$$

$$\text{where } \mathbf{k}_{i,B+1} = (K_i(\hat{\theta}, \theta_b))_{1 \leq b \leq B+1}.$$

EC.6. Larger Disparities in Data Costs

In this section, we examine the behavior of OSAR when the input data collection cost is much higher/lower than the simulation cost. We test two extreme cases when $\mathbf{c} = (100, 100)$ —expensive input data—and $\mathbf{c} = (0.01, 0.01)$ —cheap input data—fixing the simulation cost at one. Figure EC.2 shows the numerical performances of OSAR, OSAR (gCEI), OSAR⁺, and BICO applied to the Baseline case in Section 8.1.1.

In the expensive-input case (first row), BICO and OSAR⁺ slightly outperform OSAR and OSAR (gCEI) in the PCS convergence rate. Intuitively, when the input data collection is more expensive, both OSAR and BICO would run simulations more frequently relative to the input data. However, the input data sampling ratios, $\hat{\beta}_1$ and $\hat{\beta}_2$, in Figure EC.2(b), are still of the same order of magnitude as those in the Baseline case in Figure 4(b) for all algorithms. Although seemingly counterintuitive, this is because $\hat{\beta}_1$ and $\hat{\beta}_2$ are fractions of the *sampling budget* spent, which include the cost. Thus, considering that c_1 and c_2 are 100 times more expensive in this case than Baseline, the observed input data is much fewer than in Baseline. This coincides with the intuition that OSAR would simulate more as the input data becomes more expensive.

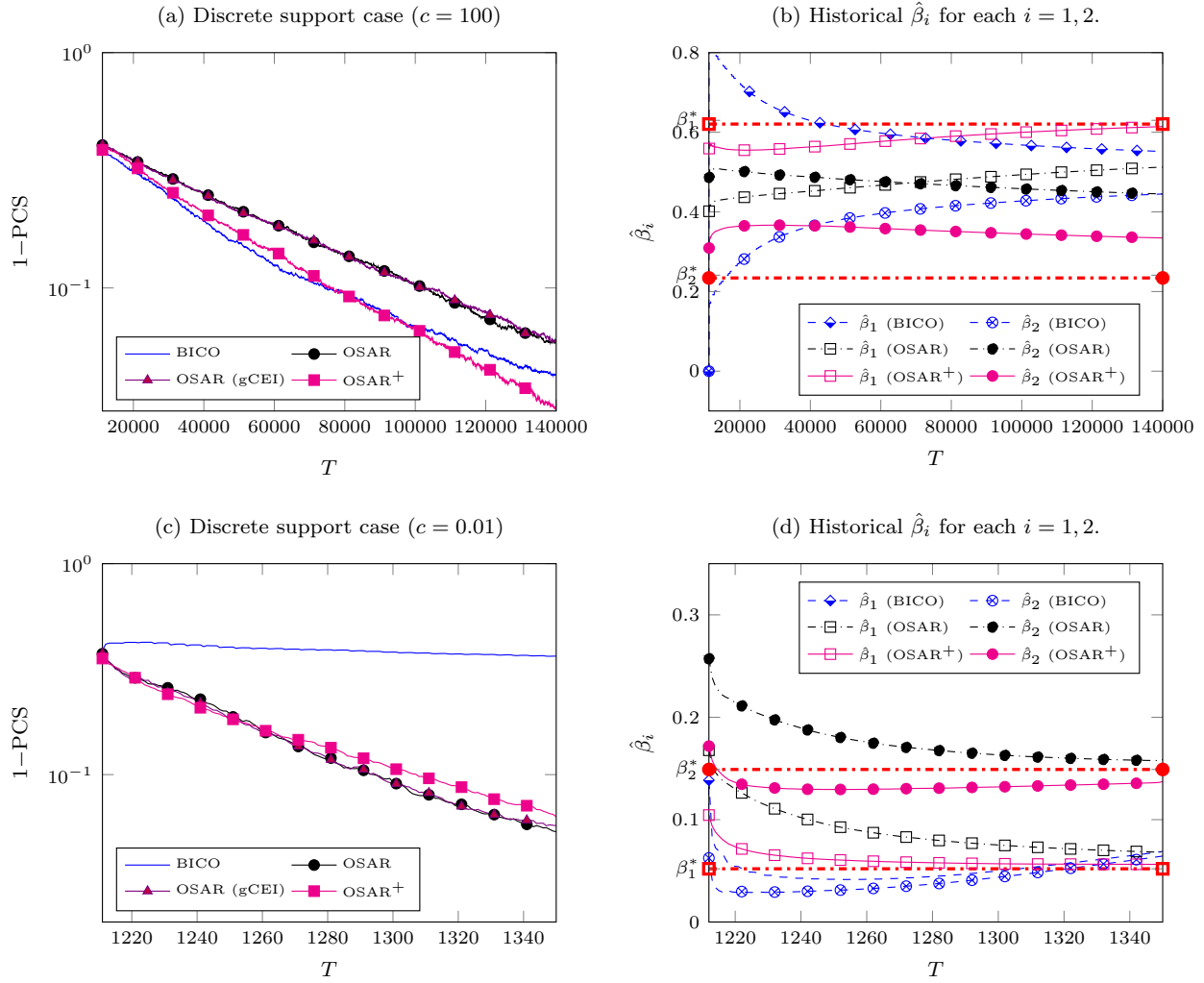


Figure EC.2 Each panel displays $1-PCS$ in log scale for discrete support described in Section 8 with $c = c_1 = c_2 = 100$ or $c = c_1 = c_2 = 0.01$, respectively. The results are obtained by averaging over 10,000 macroruns.

In the cheap-input case (second row), both OSAR and OSAR (gCEI) slightly outperform OSAR⁺ in later iterations while BICO performs significantly worse than the rest. In this case, the true parameter, θ_0 , can be learned accurately with a small fraction of the sampling budget and estimating the MPB reduces to solving an R&S problem accurately at θ_0 . Figure EC.2(d) shows that the optimal sampling fractions are smaller than those for the Baseline case in Figure 4(b). However, again, considering that c_1 and c_2 are 100 times cheaper than in Baseline, Figure EC.2(d) shows that all algorithms collect much more input data than in Baseline. This aligns with the intuition that OSAR would collect more input data as input data becomes cheaper.

Because OSAR and OSAR (gCEI) estimate the unknown means from sample averages of the simulations made at each solution-parameter pair, they are always unbiased. On the other hand, OSAR⁺ introduces bias by pooling the simulation outputs at other parameter values via KRR. Although this is typically more efficient for the case when simulation effort is spread out across different parameters, when input data is cheap,

the simulation effort is likely concentrated near θ_0 , which results in the slight underperformance observed for OSAR⁺. Lastly, BICO’s performance can be explained by that it spends a significantly smaller fraction of the budget on sampling from the second input data source compared to others (see Figure EC.2(d)). Consequently, the posterior probability does not concentrate sufficiently fast around θ_0 , which results in the slow convergence of the PCS.

EC.7. Implementation Details of BICO and SDC

This section discusses computational details of the Bayesian Information Collection and Optimization (BICO) algorithm suggested by Ungredda et al. (2022) and Separate Data Collection (SDC) for the examples in Section 8.

Unlike OSAR, BICO identifies the optimal solution by comparing the expected performance averaged over the joint posterior distribution of simulation error and input parameter. Another difference is that BICO allocates a single replication in each iteration for simulation, however, collects a batch of input data from a source, where the batch size is a user input. We adopt the batch size of 5 in all our experiments. To determine whether to collect more input data or simulation output, BICO computes their values of information (VoI). To compute the VoI of input data, they adopt Monte Carlo (MC) estimation, which requires two parameters, R_1 and R_2 ; the former is the MC sample size to compute the current expected performance of each solution, and the latter is the MC sample size to compute the improvement in each solution’s performance should we sample the batch input data from each input source. Again, these are suggested as user inputs and we adopt $R_1 = R_2 = 200$ in our implementations.

Algorithm 5 summarizes SDC. To determine how to split the sampling budget between input data collection and simulation, SDC follows the same logic as in Kim and Song (2022). Let LDR^{sim} be the optimal LDR of the probability of false selection of the MPB given a fixed set of input data, and $\text{LDR}^{\text{input}}$ be the optimal LDR of MPB’s posterior preference, respectively. Either rate is computed assuming that 100% of the sampling budget goes to serve either purpose. The authors propose balancing the two LDRs by maximizing $\max_{\gamma \geq 0} \min\{\text{LDR}^{\text{sim}}, \gamma \text{LDR}^{\text{input}}\} / (1 + \gamma)$, where γ is the ratio of the sampling budget spent on input data collection to that on simulation, and show that the maximum is attained if $\gamma = \text{LDR}^{\text{sim}} / \text{LDR}^{\text{input}} := \gamma^*$. Motivated by this, in each batch iteration, SDC aims to split Δ and allocate $\gamma^* / (1 + \gamma^*)$ to input data sampling and $1 / (1 + \gamma^*)$ to simulation sampling.

To learn unknown γ^* , Step 4 estimates $\text{LDR}^{\text{input}}$ and LDR^{sim} from the input models and simulation outputs by replacing unknown quantities with their corresponding sample estimates. In particular, computing LDR^{sim} requires the *balance weight*, $W_i(\theta_b)$, which quantifies the contribution of $\mathcal{G}_{i,t}(\theta_b)$ to the false-selection event of the MPB for each $1 \leq i \leq k, 1 \leq b \leq B$. We refer the readers to Kim et al. (2025) for the formal definition of $W_i(\theta_b)$. After calculating $\hat{\gamma}$ in Step 5, the remainder of SDC utilizes it to allocate Δ .

Algorithm 5 Separate Data Collection (SDC)

1: Run Steps 1–2 of Algorithm 1 (OSAR) and set $m_{\ell,1} = m_0$ for all $1 \leq \ell \leq L$; $s \leftarrow 1$.

2: **while** $t < T$ **do**

3: Run Steps 3–5 of Algorithm 2 (OSAR⁺).

4: Compute the LDRs from simulation and input model, $\widehat{\text{LDR}}^{\text{sim}}$ and $\widehat{\text{LDR}}^{\text{input}}$, defined as

$$\widehat{\text{LDR}}^{\text{sim}} := \min_{i \neq i_t^b, 1 \leq b \leq B} W_{i,t}(\theta_b) \mathcal{G}_{i,t}(\theta_b) \text{ and } \widehat{\text{LDR}}^{\text{input}} := \inf_{\theta \in \Theta_{i_t^*,t}^c} \sum_{\ell=1}^L \frac{m_{\ell,s}}{(\mathbf{c}^\top \mathbf{m})} \hat{\mathbf{D}}^{\text{KL}}(\hat{\theta}_t^\ell || \theta^\ell),$$

where $\mathcal{G}_{i,t}(\theta_b)$ is a rescaled version of $G_{i,t}(\theta_b)$ defined above (8) and $W_{i,t}(\theta_b)$ represents the balance weight estimated from $\{\hat{\mu}_{i,t}\}_{1 \leq i \leq k}$.

5: Estimate the splitting ratio $\hat{\gamma} = \widehat{\text{LDR}}^{\text{sim}} / \widehat{\text{LDR}}^{\text{input}}$.

6: Solve $\hat{\beta}_s := \arg \max_{\beta} \inf_{\theta \in \Theta_{i_t^*,t}^c} \sum_{\ell=1}^L \frac{\beta_\ell}{c_\ell} \hat{\mathbf{D}}^{\text{KL}}(\hat{\theta}_t^\ell || \theta^\ell)$ subject to $\beta \geq \epsilon_s \mathbf{1}_L$, $\mathbf{1}^\top \beta = 1$.

7: **[Input Data Sampling]** Draw $(\hat{m}_{\ell,s})_{1 \leq \ell \leq L} \sim \text{Bin}\left(\lceil \Delta / c_\ell \rceil, \frac{\hat{\beta}_{\ell,s} \hat{\gamma}}{(1+\hat{\gamma})} \times \frac{\Delta / c_\ell}{\lceil \Delta / c_\ell \rceil}\right)$ and collect $\hat{m}_{\ell,s}$ observations from the ℓ -th input data source.

8: **[Simulation Sampling]** Draw $N_s \sim \text{Bin}\left(\Delta, \frac{1}{\hat{\gamma}+1}\right)$ and run Algorithm 5 in Kim et al. (2025) with budget N_s .

9: Update $t \leftarrow t + \sum_{\ell=1}^L c_\ell \hat{m}_{\ell,s} + N_s$, $m_{\ell,s+1} \leftarrow m_{\ell,s} + \hat{m}_{\ell,s}$ for all $1 \leq \ell \leq L$, and $s \leftarrow s + 1$

10: **end while**

11: **return** i_T^* as the optimal solution.

EC.8. Auxiliary Results

This section contains auxiliary results invoked to show our main results and their proofs.

LEMMA EC.4. Suppose there exists sequence $\{a_{i,n}\}_{n=1}^\infty$ such that $a_{i,n} > 0$ and $\lim_{n \rightarrow \infty} -\frac{1}{n} \log a_{i,n} = \lambda_i \geq 0$ for each $i = 1, 2, \dots, K$. Then, we have $\lim_{n \rightarrow \infty} -\frac{1}{n} \log(\sum_{i=1}^K a_{i,n}) = \min_{1 \leq i \leq K} \lambda_i$.

Proof. Taking $-\frac{1}{n} \log(\cdot)$ operation on all sides of $\max_{1 \leq i \leq K} a_{i,n} \leq \sum_{i=1}^K a_{i,n} \leq K \max_{1 \leq i \leq K} a_{i,n}$, we have

$$-\frac{1}{n} \log K + \min_{1 \leq i \leq K} -\frac{1}{n} \log a_{i,n} \leq -\frac{1}{n} \log \left(\sum_{i=1}^K a_{i,n} \right) \leq \min_{1 \leq i \leq K} -\frac{1}{n} \log a_{i,n}. \quad (\text{EC.40})$$

which follows from that $-\log \max_{1 \leq i \leq K} a_{i,n} = \min_{1 \leq i \leq K} -\log a_{i,n}$. Letting $n \rightarrow \infty$ on both sides of (EC.40), we can obtain the desired result. Q.E.D.

LEMMA EC.5 (Modified version of Theorem 4.5.5 in Durrett (2019)). For each t , define $Y_t \sim \text{Bin}(\Delta, p_t)$, where $p_t \in \mathcal{F}_{t-1}$ and $\mathcal{F}_{t-1}$ is a filtration generated by the collected data until the $(t-1)$ -th iteration. Then, we have $\frac{\sum_{t=1}^T Y_t}{\Delta \sum_{t=1}^T p_t} \rightarrow 1$ a.s. conditional on $\{\sum_{t=1}^\infty p_t = \infty\}$.

Proof. Similar to the original proof, we construct a martingale $\{X_t\}_{t \geq 1}$ as $X_0 = 0$ and $X_t - X_{t-1} = \frac{Y_t}{\Delta} - p_t$.

This yields that

$$\sum_{t=1}^T Y_t / (\Delta \sum_{t=1}^T p_t) - 1 = X_T / \sum_{t=1}^T p_t.$$

The increasing process of X_t , say A_t , is defined as $A_t = \sum_{t_1=1}^t E[(X_{t_1} - X_{t_1-1})^2 | \mathcal{F}_{t_1-1}]$. Then, we have

$$A_t - A_{t-1} = E[(X_t - X_{t-1})^2 | \mathcal{F}_{t-1}] = \Delta^{-2} E[(Y_t - \Delta p_t)^2 | \mathcal{F}_{t-1}] = \Delta^{-1} p_t (1 - p_t) \leq p_t / \Delta.$$

Given $\{A_\infty < \infty\}$, Theorem 4.5.2 in Durrett (2019) shows that X_t converges to a constant. Hence, we have $\frac{X_T}{\sum_{t=1}^T p_t} \rightarrow 0$ conditional on $\{A_\infty < \infty, \sum_{t=1}^\infty p_t = \infty\}$. If $A_\infty = \infty$, we have $\{A_\infty = \infty\} \subseteq \{\sum_{t=1}^\infty p_t = \infty\}$. Applying Theorem 4.5.3 in Durrett (2019) with $f(t) = \max(t, 1)$, one can confirm that $X_T / f(A_T) \rightarrow 0$ conditional on $\{A_\infty = \infty\}$. Since $f(A_T) \leq \frac{\sum_{t=1}^T p_t}{\Delta}$, we finally have $X_T / \sum_{t=1}^T p_t \rightarrow 0$ conditional on $\{A_\infty = \infty\} = \{A_\infty = \infty, \sum_{t=1}^\infty p_t = \infty\}$. Combining two limit results verifies $\frac{X_T}{\sum_{t=1}^T p_t} \rightarrow 0$ given $\sum_{t=1}^\infty p_t = \infty$, as desired. Q.E.D.

LEMMA EC.6. Let A be a B -by- B invertible matrix. Define $A_1 = \left[\begin{array}{c|c} A & \mathbf{u} \\ \hline \mathbf{v}^\top & c \end{array} \right]$ for some B -dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, and constant c . Then, we have $\det(A_1) = (c - \mathbf{v}^\top A^{-1} \mathbf{u}) \det(A)$.

Proof. From a straightforward computation, one can confirm

$$A_1 = \left[\begin{array}{c|c} I_B & \mathbf{0}_B \\ \hline \mathbf{v}^\top A^{-1} & 1 \end{array} \right] \left[\begin{array}{c|c} A & \mathbf{0}_B \\ \hline \mathbf{0}_B^\top & c - \mathbf{v}^\top A^{-1} \mathbf{u} \end{array} \right] \left[\begin{array}{c|c} I_B & A^{-1} \mathbf{u} \\ \hline 0 & 1 \end{array} \right],$$

where I_B is a B -by- B identity matrix. Taking a determinant on both sides, we can obtain $\det(A_1) = (c - \mathbf{v}^\top A^{-1} \mathbf{u}) \det(A)$. Q.E.D.

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