Online Appendix for "Separation of non-ergodic uniform convergence rates for regularized learning in games"

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Appendix A: Complete Table of Uniform and Universal Convergence Rates

We present Table EC.1, which is the complete version of Table 1 for uniform and universal non-ergodic convergence rates of OGDA and OMWU in two-player zero-sum games. We note that the linear universal last-iterate rate of $C_1 \exp(-\frac{T}{C_2})$ only implies a $O(T^{-1}C_2 \ln C_1)$ random-iterate convergence rate since the first $C_2 \ln C_1$ iterates have constant duality gap and will be sampled each with probability $\frac{1}{T}$ in the calculation of expected duality gap. By definition, any bound on the last-iterate and random-iterate convergence rate implies the same rate for best-iterate convergence rates. The exact definitions of the different notions of convergence rates are given in Definition 1 and Definition 2.

Convergence	OGDA (uniform)	OMWU (uniform)	OMWU (universal)
Last iterate	$O(T^{-1/2})$	$\Omega(1)$	$C_1 \exp(-\frac{T}{C_2})^{\dagger}$
Random iterate		$\Omega\left(\frac{1}{\log T}\right)$	$O\left(\min\left\{T^{-\frac{1}{4}}\delta^{-\frac{1}{2}}, T^{-1}C_2 \ln C_1\right\}\right)$
Best iterate		$O(T^{-\frac{1}{6}})$	$O\left(\min\left\{T^{-\frac{1}{6}}, T^{-\frac{1}{4}}\delta^{-\frac{1}{2}}, C_1 \exp(-\frac{T}{C_2})\right\}\right)$ ‡

Table EC.1 *Uniform* and *universal* convergence rates of OGDA and OMWU in zero-sum games with a fully mixed equilibrium. Here $\delta > 0$ is the minimum probability in the Nash equilibrium. *Universal* convergence rates allow for dependence on δ while *uniform* convergence rates are δ-independent (the constants only depend on the dimensions of the problems and the norm of the payoff matrix). Our new results are highlighted in gray. †: $C_1, C_2 = \Omega(\exp(\frac{1}{\delta}))$. ‡: This upper bound only holds for 2×2 games.

Appendix B: Proofs for Section 2

B.1. Proof of Lemma 1

Proof Let $e_1 < e_2$. Denote $x_1 = F_{\eta,R}(e_1)$ and $x_2 = F_{\eta,R}(e_2)$. By definition, we have $e_2(x_2 - x_1) \leq \frac{1}{\eta}(R(x_1) - R(x_2)) \leq e_1(x_2 - x_1)$. Since $e_1 < e_2$, we have $x_2 \leq x_1$. □

B.2. Proof of Proposition 1

Proof By definition, $F_{\eta,R}(\frac{E}{\eta}) = \arg\min_{x\in[0,1]} \left\{x \cdot \frac{E}{\eta} + \frac{1}{\eta}R(x)\right\} = \arg\min_{x\in[0,1]} \{x \cdot E + R(x)\} = F_{1,R}(E)$. The second claim on the Lipschitzness follows directly. □

Appendix C: Proofs for Section 3

C.1. Proof of Lemma 2

Proof Assume that $x[1] \ge \frac{1}{1+\delta}, y[1] \ge \frac{1}{2} + \epsilon$. We have $\mathrm{DualityGap}(x,y) = \left(\frac{1}{2}+\delta\right)x[1] - (1 - y[1]) \ge \frac{1}{2}\frac{1+2\delta}{1+\delta} - \frac{1}{2} + \epsilon \ge \epsilon$. □

C.2. The Case of Non-increasing Step Size

We focus on the following update rule with non-increasing step sizes $\{\eta_t\}$: for each $t \ge 1$, x^t is updated as follows (the update rule for y^t is symmetric):

$$x^t = \arg\min_{x \in \Delta^{d_1}} \left\{ \left\langle x, \sum_{k=1}^{t-1} \eta_k \ell_x^k + \eta_t m_x^t \right\rangle + R(x) \right\}. \tag{EC.1}$$

REMARK EC.1. The update rule (EC.1) covers the following classes of algorithms:

1. OFTRL with constant step size for any regularizer;
2. OOMD with constant/non-increasing step sizes for any Legendre regularizer.

However, we note that (EC.1) does not cover the case of OFTRL with decreasing step sizes, where at each period t, the same step size η_t is used for all the previous losses, i.e., where the update is

$$x^t = \arg\min_{x \in \Delta^{d_1}} \left\{ \left\langle x, \eta_t \left(\sum_{k=1}^{t-1} \ell_x^k + m_x^t \right) \right\rangle + R(x) \right\}. \tag{EC.2}$$

Recall that we focus on 2×2 games, and we assume that the predictor m^t is a convex combination of 0 and the loss vectors $\{\ell^{t-1}, \ell^t\}$. Recall that we also define $g_x^t := m_x^t[1] - m_x^t[2] \in [-1,1]$ and $g_y^t := m_y^t[1] - m_y^t[2] \in [-1,1]$, which is analog to the definition $e_x^t = \ell_x^t[1] - \ell_x^t[2]$ and $e_y^t = \ell_y^t[1] - \ell_y^t[2]$. Using this notation and $F_{1,R}$, the update rule in (EC.1) can be written as

$$x^t[1] = F_{1,R}\left(\sum_{k=1}^{t-1} \eta_k e_x^k + \eta_t g_x^t \right), x^t[2] = 1 - x^t[1],$$

and the update rule for y^t is symmetric.

We consider non-increasing step sizes $\{\eta_t\}$ that satisfy the following assumption.

ASSUMPTION EC.1. *The step sizes $\{\eta_t\}$ satisfy*

1. $\eta_t \ge \eta_{t+1} > 0$ *for all* $t \ge 1$.
2. $\sum_{t=1}^{\infty} \eta_t = +\infty$.

C.3. Proof of Theorem 3

Before we proceed to the proof, we prove a useful proposition on the predictors g_x^t and g_y^t.

PROPOSITION EC.1. *$g_x^t \in [-\frac{1}{2}, \frac{1}{2}+\delta]$ and $g_y^t \in [-\delta, 1]$ for all $t \geq 0$.*

Proof Since m^t is a convex combination of 0 and the loss vectors $\{\ell^{t-1}, \ell^t\}$, the value of g_x^t is a convex combination of 0 and $\{e^{t-1}, e^t\}$. In particular, we have $g_x^t \in [-\frac{1}{2}, \frac{1}{2}+\delta]$ and $g_y^t \in [-\delta, 1]$ for all t by item 2 in Proposition 2. □

We present a full technical version of Theorem 3 with non-increasing step sizes.

THEOREM EC.1. *Assume the regularizer R satisfies Assumption 1 and Assumption 2. For any $\delta \in (0, \hat{\delta})$, where $\hat{\delta} = \min\{\frac{1}{100}, \frac{c_1}{6}, \frac{c_1^2}{300}, \delta'\}$ is a constant depending only on the constants c_1 and δ' defined in Assumption 2, let $\{x^t, y^t\}$ be the iterates produced by the dynamics (EC.1) on A_δ (defined in (2)) with any non-increasing step sizes $\{\eta_t\}$ that satisfies Assumption EC.1 and $\eta_1 \leq \frac{1}{4L}$. Then the following holds:*

1. *Define T_s the smallest iteration when $x^{T_s}[1] \geq \frac{3}{4}$ and T_1 the smallest iteration when $x^{T_1}[1] \geq \frac{1}{1+\delta}$. Then $y^t[1] \leq \frac{1}{2} - c_1, \forall t \in [T_s, T_1 - 1]$.*
2. *Define $T_\ell > 0$ the smallest integer such that $\sum_{k=T_1}^{T_1+T_\ell} \eta_k \geq \frac{3\eta_1}{2c_1}$. Define $T_h > 0$ the smallest integer such that $\sum_{k=T_1}^{T_1+T_h} \eta_k \geq \frac{c_1}{3L\delta}$. Define $T_2 > T_1$ the first iteration when $y^{T_2}[1] \geq \frac{1}{2(1+\delta)}$. Then we have $T_2 \geq T_h \geq \frac{c_1}{3\eta_1 L\delta} - 1 \geq \frac{c_1}{\delta}$. Moreover*
$$x^t[1] \geq \frac{1}{1+\delta}, \quad \forall t \in [T_1 + T_\ell, T_1 + T_h]$$
$$x^t[1] \geq \frac{1+c_3}{1+c_3+\delta}, \quad \forall t \in [T_1 + T_h, T_2]$$
3. *Define $T_m > 0$ the smallest integer such that $\sum_{k=T_2}^{T_2+T_m-1} \eta_k \in [\frac{c_1^2}{20L\delta}, \frac{c_1^2}{20L\delta} + \frac{1}{4L}]$. Then we have $T_m \geq \frac{c_1^2}{5\delta}$ and $\mathrm{DualityGap}(x^t, y^t) \geq c_2, \forall t \in [T_2 + \lceil \frac{T_m}{2} \rceil, T_2 + T_m]$.*

Proof Since $\delta \leq \hat{\delta} = \min\{\frac{1}{100}, \frac{c_1}{6}, \frac{c_1^2}{300}, \delta'\}$, Assumption 2 holds.

Stage I: Recall that OFTRL is initialized with $x^1[1] = y^1[1] = \frac{1}{2}$. We define two time steps for Stage I's analysis:

- $T_s > 1$: the smallest iteration such that $x^{T_s}[1] \geq \frac{3}{4}$
- $T_1 > T_s$: the smallest iteration such that $x^{T_1}[1] \geq \frac{1}{1+\delta}$.

We note that T_s and T_1 exist and postpone the proof of this fact to the end of Stage I. By definition of T_s and the fact that $F_{1,R}(0) = \frac{1}{2}$ and $F_{1,R}$ is L-Lipschitz, we have

$$F_{1,R}\left(\sum_{k=1}^{T_s-1} \eta_k e_x^k + \eta_{T_s} g_x^{T_s}\right) \geq \frac{3}{4} \implies \sum_{k=1}^{T_s-1} \eta_k e_x^k + \eta_{T_s} g_x^{T_s} \leq -\frac{1}{4L}.$$

Since $e_x^t \ge -\frac{1}{2}$ and $g_x^t \ge -\frac{1}{2}$, the above further implies $\sum_{k=1}^{T_s} \eta_k \ge \frac{1}{2L}$. Since the step sizes are non-increasing and $\eta_1 \le \frac{1}{4L}$, we have $\sum_{k=1}^{T_s-1} \eta_k \ge \frac{1}{2L} - \frac{1}{4L} = \frac{1}{4L}$. We define $T_s' \ge 2$ the first iteration such that $\sum_{k=1}^{T_s'-1} \eta_k \ge \frac{1}{4L}$. Note that $T_s' \le T_s$.

Since $x^t[1] < \frac{3}{4}$ for all $1 \le t \le T_s' - 1$, we know that $e_y^t = \ell_y^t[1] - \ell_y^t[2] = 1 - (1+\delta)x^t[1] > \frac{1-3\delta}{4} \ge \frac{6}{25}$ (as $\delta \le \frac{1}{100}$) for all $1 \le t \le T_s - 1$. Moreover, for all $1 \le t \le T_1 - 1$, we know that $e_y^t \ge 0$ as $x^t[1] \le \frac{1}{1+\delta}$. We thus have for any $t \in [T_s', T_1]$, $y^t[1]$ is upper bounded by

$$F_{1,R}\left(\sum_{k=1}^{t-1} \eta_k e_y^k + \eta_t g_y^t\right) \le F_{1,R}\left(\frac{6}{25}\sum_{k=1}^{T_s'-1} \eta_k - \eta_t \delta\right) \le F_{1,R}\left(\frac{3}{50L} - \frac{1}{400L}\right) \le F_{1,R}\left(\frac{1}{20L}\right) = \frac{1}{2} - c_1, \tag{EC.3}$$

where the first inequality follows from $e_y^k \ge \frac{6}{25}$ for $k \in [1, T_s' - 1]$, $e_y^k \ge 0$ for $k \in [1, T_1 - 1]$, and $g_y^t \ge -\delta \ge -\frac{1}{100}$ by Proposition EC.1 and definition of δ; the second inequality follows from $\sum_{k=1}^{T_s'-1} \eta_k \ge \frac{1}{4L}$; the last equality holds by Assumption 2.

This completes the proof of Stage I as well as item 1 in Theorem EC.1, where $x^{T_1}[1] \ge \frac{1}{1+\delta}$ and $y^{T_1}[1] \le \frac{1}{2} - c_1$. Before we proceed to the next stage, we prove the existence of T_s and T_1.

Existence of T_s and T_1. It suffices to prove that T_1 exists, as it implies the existence of T_s. Assume for the sake of contradiction that T_1 does not exist, *i.e.*, $x^t[1] < \frac{1}{1+\delta}$ for all $t \ge 1$. By the same analysis as for Stage I, we get $y^t[1] \le \frac{1}{2} - c_1$ for all $t \ge T_s'$. This implies that $e_x^t = -\frac{1}{2} + (1+\delta)y^t[1] \le \frac{\delta}{2} - c_1 \le -\frac{c_1}{2}$ for all $t \ge T_s'$. Then $\sum_{k=1}^{t} \eta_k e_x^t + \eta_t g_x^t \to -\infty$ as $t \to +\infty$. As a consequence, $x^t[1] = F_{1,R}(\sum_{k=1}^{t} \eta_k e_x^t + \eta_t g_x^t) \to 1$ as $t \to +\infty$ by item 2 in Assumption 1. But this contradicts with the assumption that $x^t[1] < \frac{1}{1+\delta}$ for all $t \ge 1$. This completes the proof.

Stage II: We have $y^{T_1}[1] \le \frac{1}{2} - c_1$. We define the following two time steps:

- $T_1 + T_h$: the first iteration after T_1 such that $\sum_{k=T_1}^{T_1+T_h} \eta_k \ge \frac{c_1}{3L\delta}$.
- T_2: the first iteration after T_1 such that $y^{T_2}[1] \ge \frac{1}{2(1+\delta)}$.

The existence of T_h follows from Assumption EC.1. We prove the existence of T_2 in Claim EC.1.

We have $\sum_{k=T_1}^{T_1+T_h} \eta_k \in [\frac{c_1}{3L\delta}, \frac{c_1}{2L\delta}]$ as $\eta_t \le \frac{1}{4L} \le \frac{c_1}{6\delta L}$. For any $t = [T_1, T_1 + T_h - 1]$, we have

$$\begin{aligned} y^t[1] &= F_{1,R}\left(\sum_{k=1}^{T_1-1} \eta_k e_y^k + \sum_{k=T_1}^{t-1} \eta_k e_y^k + \eta_t g_y^t\right) \le F_{1,R}\left(\sum_{k=1}^{T_1-1} \eta_k e_y^k\right) + L \cdot \max\left\{0, -\sum_{k=T_1}^{t-1} \eta_k e_y^k - \eta_t g_y^t\right\} \\ &\le \frac{1}{2} - c_1 + \delta L \cdot \sum_{k=T_1}^{t} \eta_k \le \frac{1}{2} - c_1 + \delta L \cdot \sum_{k=T_1}^{T_1+T_h} \eta_k \le \frac{1}{2} - \frac{c_1}{2}, \end{aligned}$$

where the first inequality holds since $F_{1,R}$ is non-increasing and L-Lipschitz; the second inequality holds since $F_{1,R}(\sum_{k=1}^{T_1-1}\eta_k e_y^k) \le \frac{1}{2} - c_1$ (by (EC.3)) and $e_y^k, g_y^k \ge -\delta$; the last inequality holds since $\sum_{k=T_1}^{T_1+T_h}\eta_k \in [\frac{c_1}{3L\delta}, \frac{c_1}{2L\delta}]$.

For all $t \in [T_s', T_1+T_h-1]$, since $y^t[1] \le \frac{1}{2} - \frac{c_1}{2}$, we have

$$e_x^t = \ell_x^t[1] - \ell_x^t[2] = -\frac{1}{2} + (1+\delta)y^t[1] \le \frac{-1+(1+\delta)(1-c_1)}{2} \le \frac{\delta - c_1}{2} \le -\frac{c_1}{3},$$

where in the last inequality we use $\delta \le \frac{c_1}{6}$. For all $t \in [T_s', T_2-1]$, we have $y^t[1] \le \frac{1}{2(1+\delta)}$, which implies $e_x^t \le 0$.

We first show that $x^t[1] \ge \frac{1}{1+\delta}$ for $t \in [T_1+T_\ell, T_1+T_h]$, where $T_\ell > 0$ is the smallest integer such that $\sum_{k=T_1}^{T_1+T_\ell}\eta_k \ge \frac{3\eta_1}{2c_1}$. For $t \in [T_1+T_\ell, T_1+T_h]$, we have

$$\begin{aligned}
x^t[1] = F_{1,R}\left(\sum_{k=1}^{t}\eta_k e_x^k + \eta_t g_x^t\right) &= F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} + \sum_{k=T_1}^{t-1}\eta_k e_x^k + \eta_t g_x^t - \eta_{T_1} g_x^{T_1}\right)\\
&\ge F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{c_1}{3}\cdot\sum_{k=T_1}^{T_1+T_\ell}\eta_k + \frac{1}{2}\cdot\eta_{T_1}\right)\\
&\ge F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{1}{2}\cdot\eta_1 + \frac{1}{2}\cdot\eta_{T_1}\right)\\
&\ge F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1}\right) \ge \frac{1}{1+\delta},
\end{aligned}$$

where in the first inequality we use $e_x^k \le -\frac{c_1}{3}$ for $k \in [T_1, T_1+T_h-1]$, $-\frac{1}{2} \le g_x^k \le 0$ for all $k \in [T_1, T_2-1]$; in the second inequality we use $\sum_{k=T_1}^{T_1+T_\ell}\eta_k \ge \frac{3\eta_1}{2c_1}$; in the third inequality we use $\eta_1 \ge \eta_T$.

Now we consider $x^t[1]$ for $t \in [T_1+T_h, T_2]$. By a similar analysis as above, we have

$$\begin{aligned}
x^t[1] &= F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} + \sum_{k=T_1}^{t-1}\eta_k e_x^k + \eta_t g_x^t - \eta_{T_1} g_x^{T_1}\right)\\
&\ge F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} + \sum_{k=T_1}^{T_1+T_h-1}\eta_k e_x^k + \eta_t g_x^t - \eta_{T_1} g_x^{T_1}\right)\\
&\ge F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{c_1}{3}\cdot\sum_{k=T_1}^{T_1+T_h-1}\eta_k + 2\eta_{T_1}\right)\\
&\ge F_{1,R}\left(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{c_1^2}{9L\delta} + \frac{1}{2L}\right).
\end{aligned}$$

Moreover, since $F_{1,R}(\sum_{k=1}^{T_1-1}\eta_k e_x^k + \eta_{T_1} g_x^{T_1}) = x^{T_1}[1] \ge \frac{1}{1+\delta}$ and $-\frac{c_1^2}{9L\delta} + \frac{1}{2L} \le -\frac{c_1^2}{30L\delta}$, we have $x^t[1] \ge \frac{1+c_3}{1+c_3+\delta}$ for all $t \in [T_1+T_h, T_2]$ by item 1 in Assumption 2.

CLAIM EC.1. *T_2 exists.*

Proof Assume for the sake of contradiction that T_2 does not exist, *i.e.*, $y^t[1] < \frac{1}{2(1+\delta)}$ for all $t \geq T_1$ (since we know $y^t[1] \leq \frac{1}{2} - \frac{c_1}{2}$ for all $t \in [T_1, T_1 + T_h - 1]$). Then by the analysis of Stage II, we have $x^t[1] \geq \frac{1+c_3}{1+c_3+\delta}$ for all $t \geq T_1$. This implies $e_y^t \leq -\frac{c_3\delta}{2}$ for all $t \geq T_1$. As a result, we have $\sum_{k=1}^{t-1} \eta_k e_y^k + \eta_t g_y^t \to -\infty$ as $t \to \infty$. By item 2 in Assumption 1, we get $y^t[1] = F_{1,R}(\sum_{k=1}^{t-1} \eta_k e_y^k + \eta_t g_y^t) \geq \frac{1}{2}$ as $t \to \infty$. But this contradicts with the assumption that $y^t[1] < \frac{1}{2(1+\delta)}$ for all $t \geq T_1$. □

Stage III: We define $T_m > 0$ the smallest integer such that $\sum_{k=T_2}^{T_2+T_m-1} \eta_k \in [\frac{c_1^2}{20L\delta}, \frac{c_1^2}{20L\delta} + \frac{1}{4L}]$. The existence of T_m is due to Assumption EC.1 and $\eta_1 \leq \frac{1}{4L}$. Since η_k is non-increasing, we have $T_m \geq \frac{c_1^2}{5\delta}$.

We define $T_3 = T_2 + T_m$. For any $t \in [T_2, T_3]$, we have

$$
\begin{aligned}
x^t[1] &= F_{1,R}\left(\sum_{k=1}^{T_2-1} \eta_k e_x^k + \eta_{T_2} g_x^{T_2} + \sum_{k=T_2}^{t-1} \eta_k e_x^k + \eta_t g_x^t - \eta_{T_2} g_x^{T_2}\right) \\
&\geq F_{1,R}\left(\sum_{k=1}^{T_1-1} \eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{c_1^2}{9L\delta} + \frac{1}{2L} + \sum_{k=T_2}^{t-1} \eta_k e_x^k + \eta_t g_x^t - \eta_{T_2} g_x^{T_2}\right) \quad \text{(by above inequality)} \\
&\geq F_{1,R}\left(\sum_{k=1}^{T_1-1} \eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{c_1^2}{9L\delta} + \frac{1}{2L} + \sum_{k=T_2}^{T_2+T_m-1} \eta_k + 2 \cdot \frac{1}{4L}\right) \\
&\qquad\qquad (t \leq T_2 + T_m,\ e_x^k, g_x^k \leq 1, \eta_1 \leq \tfrac{1}{4L}) \\
&\geq F_{1,R}\left(\sum_{k=1}^{T_1-1} \eta_k e_x^k + \eta_{T_1} g_x^{T_1} - \frac{c_1^2}{30L\delta}\right) \qquad (\textstyle\sum_{k=T_2}^{T_2+T_m-1} \eta_k \leq \frac{c_1^2}{20L\delta} + \frac{1}{4L},\ \delta < \frac{c_1^2}{300}) \\
&\geq \frac{1+c_3}{1+c_3+\delta},
\end{aligned}
$$

where the last inequality holds since $F_{1,R}(\sum_{k=1}^{T_1-1} \eta_k e_x^k + \eta_{T_1} g_x^{T_1}) = x^{T_1}[1] \geq \frac{1}{1+\delta}$ and item 1 in Assumption 2.

This implies for all $t \in [T_2, T_3]$, $e_y^t = 1 - (1+\delta)x^t[1] = -\frac{c_3\delta}{1+c_3+\delta} \leq -\frac{c_3\delta}{2}$ as $c_3 \leq \frac{1}{2}$. Moreover, we know that $g_y^t \geq -\delta$ for any t. Then for any $t \in [T_2 + \lceil \frac{T_m}{2} \rceil, T_3]$

$$
\begin{aligned}
y^t[1] = F_{1,R}\left(\sum_{k=1}^{t-1} \eta_k e_y^k + \eta_t g_y^t\right) &= F_{\eta,R}\left(\sum_{k=1}^{T_2-1} \eta_k e_y^k + \eta_{T_2} g_y^{T_2} + \sum_{k=T_2}^{t-1} \eta_k e_y^k + \eta_t g_y^t - \eta_{T_2} g_y^{T_2}\right) \\
&\geq F_{1,R}\left(\sum_{k=1}^{T_2-1} \eta_k e_y^k + \eta_{T_2} g_y^{T_2} - \frac{c_3\delta}{2} \cdot \sum_{k=T_2}^{T_2+\lceil \frac{T_m}{2} \rceil - 1} \eta_k + \delta \eta_{T_2}\right) \\
&\geq F_{1,R}\left(\sum_{k=1}^{T_2-1} \eta_k e_y^k + \eta_{T_2} g_y^{T_2} - \frac{c_3\delta}{2} \cdot \frac{c_1^2}{40L\delta} + \delta \cdot \frac{1}{4L}\right)
\end{aligned}
$$

$$= F_{1,R}\left(\sum_{k=1}^{T_2-1} \eta_k e_y^k + \eta_{T_2} g_y^{T_2} - \frac{c_3 c_1^2}{80L\delta} + \frac{\delta}{4L}\right) \geq F_{1,R}\left(\sum_{k=1}^{T_2-1} \eta_k e_y^k + \eta_{T_2} g_y^{T_2} - \frac{c_3 c_1^2}{120L\delta} + \frac{\delta}{4L}\right) \geq \frac{1}{2} + c_2,$$

where in the first inequality we use $e_y^t \leq -\frac{c_3\delta}{2}$ for all $t \in [T_2, T_3]$ and $g_y^t \geq -\delta$ for all t; in the second inequality we use $\sum_{k=T_2}^{T_2+\lceil\frac{T_m}{2}\rceil-1} \eta_k \geq \frac{1}{2}\sum_{k=1}^{T_2+T_m-1} \eta_k \geq \frac{c_1^2}{40L\delta}$; in the last inequality we use item 2 in Assumption 2.

Now we have prove that for $t \in [T_2 + \lceil\frac{T_m}{2}\rceil, T_3 = T_2 + T_m]$, $x^t[1] \geq \frac{1}{1+\delta}$ and $y^t[1] \geq \frac{1}{2} + c_2$. By Lemma 2, the duality gap of (x^t, y^t) is at least c_2 for all $t \in [T_2 + \lceil\frac{T_m}{2}\rceil, T_3 = T_2 + T_m]$. This completes the proof.

C.4. Proof of Theorem 4

Proof Assume for the sake of contradiction that there is a function f that satisfies both conditions. Then for any $A \in [0,1]^{2\times 2}$, we have the OFTRL learning dynamics over A satisfies

1. $\mathrm{DualityGap}(x^T, y^T) \leq f(2,2,T)$ for all T.
2. $\lim_{T\to\infty} f(2,2,T) \to 0$.

Since $\lim_{T\to\infty} f(2,2,T) \to 0$, we know there exists $T_0 > 0$ such that for any $t \geq T_0$, $\mathrm{DualityGap}(x^t, y^t) \leq f(2,2,t) < c_2$. Now let $\delta \leq \min\{\hat{\delta}, \frac{c_1}{3\eta L T_0}\}$ where $\hat{\delta}$ is defined in Theorem 3. Then by Theorem 3, we know there exists an iteration $t \geq \frac{c_1}{3\eta L\delta} \geq T_0$ such that $\mathrm{DualityGap}(x^t, y^t) \geq c_2$. This contradiction completes the proof. □

C.5. Verifying Assumption 2 for Different Regularizers

In this section, we present proofs that all the commonly used regularizers satisfy Assumption 2. We first present a useful lemma on the Lipschitzness of $F_{1,R}$. Then we verify Assumption 2 holds for the negative entropy (Lemma EC.2), the squared Euclidean norm (Lemma EC.3), the log regularizer (Lemma EC.4), and the Tsallis entropy (Lemma EC.5).

LEMMA EC.1. *If the regularizer R is 1-strongly convex, then $F_{1,R}$ is $\frac{1}{2}$-Lipschitz.*

Proof Notice that $R(x) + R(1-x)$ is 2-strongly convex. Thus by standard analysis (see e.g., (Luo 2022, Lemma 4)) we know $F_{1,R}$ is $\frac{1}{2}$-Lipschitz. □

By Lemma EC.1, we can choose $L = \frac{1}{2}$ for any 1-strongly convex regularizer in Assumption 1. We summarize the results in the table below. We recall that for each regularizer R, the constant c_1 is defined as $\frac{1}{2} - F_{1,R}(\frac{1}{20L}) > 0$.

Regularizer	$R(x)$ for $x \in \Delta^d$	L	c_2	c_3	δ'
Negative entropy	$\sum_{i=1}^d x[i]\log x[i]$	$\frac{1}{2}$	$F_{1,R}(-\frac{c_1^2}{480L})$	$\frac{1}{2}$	$\frac{c_1^2}{480L}$
Squared Euclidean norm	$\frac{1}{2}\sum_{i=1}^d x[i]^2$	$\frac{1}{2}$	$\frac{c_1^2}{960L}$	$\frac{1}{2}$	$\frac{c_1^2}{480L}$
Log regularizer	$-\sum_{i=1}^d \log(x[i])$	$\frac{1}{2}$	$\sqrt{\frac{1}{4}+(\frac{c_3c_1^2}{240L})^2}-\frac{c_3c_1^2}{240L}$	$\frac{c_1^2}{60L}$	$\frac{c_3c_1^2}{2160L}$
Tsallis entropy	$\frac{1-\sum_{i=1}^d(x[i])^\beta}{1-\beta}, \beta \in (0,1)$	$\frac{1}{2\beta}$	$F_{1,R}(-\frac{c_3c_1^2}{240L}-\frac{1}{2})$	$\frac{1}{2}$	†

Table EC.2 Constants for Assumption 2 for commonly used regularizers.

$\dagger = \min\{(c_1^2(1-\beta)/(120L\beta c_3^{1-\beta}))^{1/\beta}, \frac{c_3c_1^2}{120}, \frac{1-\beta}{8\beta}\cdot\frac{c_3c_1^2}{480L}\}$.

C.5.1. Negative Entropy

Lemma EC.2 (**Assumption 2 holds for the entropy regularizer**). *Consider the negative entropy regularizer R defined as $R(x) = x\log x + (1-x)\log(1-x)$. Then $F_{1,R}$ is $L = \frac{1}{2}$-Lipschitz. Moreover, $c_1 = \frac{1}{2} - F_{1,R}(\frac{1}{20L})$ is a universal constant and Assumption 2 holds with $\delta' = \frac{c_1^2}{480L}$, $c_2 = F_{1,R}(-\frac{c_1^2}{480L}) - \frac{1}{2}$, and $c_3 = \frac{1}{2}$.*

Proof It is easy to verify that $F_{1,R}(x)$ has a closed-form representation $F_{1,R}(E) = \frac{1}{1+\exp(E)}$. Thus $L = \frac{1}{2}$ and $c_1 = \frac{1}{2} - F_{1,R}(\frac{1}{20L})$ is a universal constant. We also choose $c_3 = \frac{1}{2}$.

If $F_{1,R}(E) \ge \frac{1}{1+\delta}$, then we have $E \le -\log(1/\delta)$. We note that

$$\exp\left(-\frac{c_1^2}{30L\delta}\right) \le \frac{1}{1+c_3} \Rightarrow \frac{1}{1+\exp\left(-\frac{c_1^2}{30L\delta}-\log(1/\delta)\right)} \ge \frac{1+c_3}{1+c_3+\delta}.$$

Thus $\delta \le \delta_1 = \frac{c_1^2}{30L\log(1+c_3))} = \frac{c_1^2}{30\log(\frac{3}{2}))L}$ suffices for item 1 in Assumption 2.

If $F_{1,R}(E) \ge \frac{1}{2(1+\delta)} = \frac{1}{1+1+2\delta}$, we have $E \le \log(1+2\delta)$. Note that since $\log(1+2y) \le 2y$ for $y > 0$, we have

$$\delta \le \frac{c_3c_1^2}{480L} \Rightarrow -\frac{c_3c_1^2}{120L} + \log(1+2\delta) < -\frac{c_3c_1^2}{240L} \Rightarrow F_{1,R}\left(-\frac{c_3c_1^2}{120L}+E\right) > F_{1,R}\left(-\frac{c_3c_1^2}{240L}\right).$$

Thus item 2 in Assumption 2 holds for any $\delta \le \delta_2 = \frac{c_3c_1^2}{480L} = \frac{c_1^2}{960L}$ and $c_2 = F_{1,R}(-\frac{c_3c_1^2}{240L}) - \frac{1}{2} = F_{1,R}(-\frac{c_1^2}{480L}) - \frac{1}{2}$. Combining the above, we know Assumption 2 holds for the negative entropy regularizer with $\delta' = \frac{c_1^2}{960L}$ and $c_2 = F_{1,R}(-\frac{c_1^2}{480L}) - \frac{1}{2}$. □

C.5.2. Squared Euclidean Norm Regularizer

Lemma EC.3 (**Assumption 2 holds for the Euclidean regularizer**). *Consider the Euclidean regularizer R defined as $R(x) = \frac{1}{2}(x^2 + (1-x)^2)$. We have $L = \frac{1}{2}$ and $c_1 = \frac{1}{20}$. We also have Assumption 2 holds with $\delta' = \frac{c_1^2}{480L}$, $c_2 = \frac{c_1^2}{960L}$, and $c_3 = \frac{1}{2}$.*

Proof It is easy to verify that $F_{1,R}(x)$ has a closed-form representation

$$F_{1,R}(x) = \begin{cases} 1 & \text{if } x \le -1 \\ \frac{1-x}{2} & \text{if } x \in (-1,1) \\ 0 & \text{if } x \ge 1 \end{cases}$$

Thus $F_{1,R}$ is L-Lipschitz with $L = \frac{1}{2}$. Moreover, $c_1 = \frac{1}{2} - F_{1,R}(\frac{1}{20L}) = \frac{1}{20}$. We choose $c_3 = \frac{1}{2}$.

Fix any E such that $F_{1,R}(E) \ge \frac{1}{1+\delta}$. We have $E \le -\frac{1-\delta}{1+\delta} < 0$. We note that for any $\delta \le \frac{c_1^2}{30L} = \frac{c_1^2}{15}$, $F_{1,R}(-\frac{c_1^2}{30L\delta} + E) \ge F_{1,R}(-1) = 1$. Thus $\delta \le \delta_1 = \frac{c_1^2}{30L}$ suffices for item 1 in Assumption 2.

Fix any E such that $F_{1,R}(E) \ge \frac{1}{2(1+\delta)} = \frac{1}{2(1+\delta)}$. We have $E \le \frac{\delta}{1+\delta} \le \delta$. The for any $\delta \le \frac{c_3 c_1^2}{240L}$, we have $F_{1,R}(-\frac{c_3 c_1^2}{120L} + E) \ge F_{1,R}(-\frac{c_3 c_1^2}{240L}) = \frac{1}{2} + \frac{c_3 c_1^2}{480L}$. Thus item 2 in Assumption 2 holds for any $\delta \le \delta_2 = \frac{c_1^2}{480L}$ and $c_2 = \frac{c_1^2}{960L}$. Combining the above, we know Assumption 2 holds for the negative entropy regularizer with $\delta' = \min\{\delta_1, \delta_2\} = \frac{c_1^2}{480L}$ and $c_2 = \frac{c_1^2}{960L}$. □

C.5.3. Log Barrier

LEMMA EC.4 (**Assumption 2 holds for the log barrier**). *Consider the log barrier regularizer R defined as $R(x) = -\log(x) - \log(1-x)$. Then Assumption 2 holds with the following choices of constants: $c_1 = \sqrt{\frac{1}{4} + 400L^2} - 20L > 0, c_3 = \frac{c_1^2}{60L}, c_2 = \sqrt{\frac{1}{4} + (\frac{c_3 c_1^2}{240L})^2} - \frac{c_3 c_1^2}{240L} > 0, \delta' = \frac{c_3 c_1^2}{2160L}$.*

Proof By setting the gradient of $x \cdot E + R(x)$ to 0, we get a closed-form expression of $F_{1,R}$:

$$F_{1,R}(E) = \begin{cases} \frac{1}{2} + \frac{1}{E} - \sqrt{\frac{1}{4} + \frac{1}{E^2}} & \text{if } E > 0 \\ \frac{1}{2} & \text{if } E = 0 \\ \frac{1}{2} + \frac{1}{E} + \sqrt{\frac{1}{4} + \frac{1}{E^2}} & \text{if } E < 0. \end{cases}$$

For $x \in (0,1)$, the $F_{1,R}$ function admits an inverse function defined as $F_{1,R}^{-1}(x) = \frac{2x-1}{x^2-x}$. Thus we know $E_0 := F_{1,R}^{-1}(\frac{1}{1+\delta}) = -\frac{1-\delta^2}{\delta}$ satisfies $F_{1,R}(E_0) = \frac{1}{1+\delta}$. Moreover, we can calculate

$$F_{1,R}^{-1}\left(\frac{1+c_3}{1+c_3+\delta}\right) = -\frac{(1+c_3)^2 - \delta^2}{(1+c_3)\delta} = -\frac{1+c_3}{\delta} + \frac{\delta}{1+c_3} = E_0 - \frac{c_3}{\delta} - \frac{c_3\delta}{1+c_3}.$$

Thus we can choose $c_3 = \frac{c_1^2}{60L}$ so that, since $\delta < 1/2$ and $c_3 > 0$, $E_0 - \frac{c_1^2}{30L\delta} = E_0 - \frac{c_3}{\delta} - \frac{c_3}{\delta} \le E_0 - \frac{c_3}{\delta} - \frac{c_3\delta}{1+c_3}$. Thus we have $F_{1,R}(E_0 - \frac{c_1^2}{30L\delta}) \ge F_{1,R}(E_0 - \frac{c_3}{\delta} - \frac{c_3\delta}{1+c_3}) \ge \frac{1+c_3}{1+c_3+\delta}$.

We calculate $E_1 := F_{1,R}^{-1}(\frac{1}{2(1+\delta)}) = \frac{4(\delta+\delta^2)}{1+2\delta} \le 8\delta$. Then we can choose $\delta \le \delta' := \frac{c_3 c_1^2}{2160L}$ so that $F_{1,R}(-\frac{c_3 c_1^2}{120L} + \frac{\delta}{4L} + E_1) \ge F_{1,R}(-\frac{c_3 c_1^2}{120L} + 9\delta) \ge F_{1,R}(-\frac{c_3 c_1^2}{240L}) = \frac{1}{2} + c_2$, where $c_2 = \sqrt{\frac{1}{4} + (\frac{c_3 c_1^2}{240L})^2} - \frac{c_3 c_1^2}{240L} > 0$ by the closed-form expression of $F_{1,R}$. □

C.5.4. Negative Tsallis Entropy For $x \in [0,1]$, the negative Tsallis entropy is a family of regularizers parameterized by $\beta \in (0,1)$:

$$R(x) = \frac{1-x^\beta}{1-\beta}. \tag{EC.4}$$

The corresponding $F_{1,R}$ is $F_{1,R}(E) = \arg\min_{x\in(0,1)} \left\{ x \cdot E + \frac{1-x^\beta}{1-\beta} + \frac{1-(1-x)^\beta}{1-\beta} \right\}$. For $x \in (0,1)$, we note that $F_{1,R}$ has an inverse function $F_{1,R}^{-1}(x) = \frac{\beta}{1-\beta}(x^{\beta-1} - (1-x)^{\beta-1})$.

LEMMA EC.5 (**Assumption 2 holds for Tsallis entropy**). *Consider Tsallis entropy parameterized by $\beta \in (0,1)$. Then $L = \frac{1}{2\beta}$ and Assumption 2 holds with the following choices of constants: $c_1 = \frac{1}{2} - F_{1,R}(\frac{1}{20L}) > 0, c_3 = \frac{1}{2}, c_2 = F_{1,R}(-\frac{c_3 c_1^2}{240L}) - \frac{1}{2} > 0, \delta' = \min\{(\frac{c_1^2(1-\beta)}{120L\beta c_3^{1-\beta}})^{\frac{1}{\beta}}, \frac{c_3 c_1^2}{120}, \frac{1-\beta}{8\beta}\cdot\frac{c_3 c_1^2}{480L}\}$.*

Proof We choose $c_3 = \frac{1}{2}$. We have $c_1 = \frac{1}{2} - F_{1,R}(\frac{1}{20L})$ is a constant.

We note that $E_0 := F_{1,R}^{-1}(\frac{1}{1+\delta}) = \frac{\beta}{1-\beta}((1+\delta)^{1-\beta} - (\frac{1+\delta}{\delta})^{1-\beta})$ satisfies $F_{1,R}(E_0) = \frac{1}{1+\delta}$. Similarly, we calculate

$$\begin{aligned}
E_1 &:= F_{1,R}^{-1}(\frac{1+c_3}{1+c_3+\delta}) \\
&= \frac{\beta}{1-\beta}\left(\left(\frac{1+c_3+\delta}{1+c_3}\right)^{1-\beta} - \left(\frac{1+c_3+\delta}{\delta}\right)^{1-\beta}\right) \\
&\ge \frac{\beta}{1-\beta}\left((1+\delta)^{1-\beta} - 2 - \left(\frac{1+c_3+\delta}{\delta}\right)^{1-\beta}\right) \\
&\ge \frac{\beta}{1-\beta}\left((1+\delta)^{1-\beta} - \left(\frac{1+\delta}{\delta}\right)^{1-\beta} - \left(\frac{c_3}{\delta}\right)^{1-\beta} - 2\right) \\
&= E_0 - \frac{\beta}{1-\beta}\left(\left(\frac{c_3}{\delta}\right)^{1-\beta} + 2\right)
\end{aligned}$$

where in the first inequality we use the fact that $(1+\delta)^{1-\beta} \le 2$ since $\delta \le 1$; the second inequality we use the inequality $(x+y)^{1-\beta} \le x^{1-\beta} + y^{1-\beta}$. We note that

$$\delta \le \delta_1 := \left(\frac{c_1^2(1-\beta)}{120L\beta c_3^{1-\beta}}\right)^{\frac{1}{\beta}} \Rightarrow -\frac{c_1^2}{30L\delta} \le -\frac{\beta}{1-\beta}\left(\left(\frac{c_3}{\delta}\right)^{1-\beta} + 2\right). \tag{EC.5}$$

Thus for any $\delta \le \delta_1$, we have for any E such that $F_{1,R}(E) \ge \frac{1}{1+\delta}$,

$$-\frac{c_1^2}{30L\delta} + E \le -\frac{c_1^2}{30L\delta} + E_0 \le E_0 - \frac{\beta}{1-\beta}\left(\left(\frac{c_3}{\delta}\right)^{1-\beta} + 2\right) \le E_1.$$

The above implies $F_{1,R}(-\frac{c_1^2}{30L\delta} + E) \ge \frac{1+c_3}{1+c_3+\delta}$ and the first item in Assumption 2 is satisfied.

We define E_2

$$E_2 := F_{1,R}^{-1}\left(\frac{1}{2(1+\delta)}\right) = \frac{\beta}{1-\beta}\left((2+2\delta)^{1-\beta} - \left(\frac{2+2\delta}{1+2\delta}\right)^{1-\beta}\right)$$
$$= \frac{\beta}{1-\beta}(2+2\delta)^{1-\beta}\cdot\left(1-\left(\frac{1}{1+2\delta}\right)^{1-\beta}\right)$$
$$\le \frac{4\beta}{1-\beta}\cdot\left(1-\left(1-\frac{2\delta}{1+2\delta}\right)^{1-\beta}\right)$$
$$\le \frac{4\beta}{1-\beta}\cdot\frac{2\delta}{1+\delta} = \frac{8\beta\delta}{(1-\beta)(1+\delta)}$$

where in the first inequality we use $(2+2\delta)^{1-\beta} \le 4$ since $0 \le \delta \le 1$ and $\beta \in (0,1)$; in the second inequality we use the basic inequality $(1-x)^r \le 1-x$ for $r, x \in (0,1)$. We define

$$\delta_2 := \min\{\frac{c_3 c_1^2}{120}, \frac{1-\beta}{8\beta}\cdot\frac{c_3 c_1^2}{480L}\}$$

Then for any $\delta \le \delta_2$ and E such that $F_{1,R}[E] \ge \frac{1}{2(1+\delta)}$, we have

$$-\frac{c_3c_1^2}{120L} + \frac{\delta}{4L} + E \le -\frac{c_3c_1^2}{120L} + \frac{\delta}{4L} + E_2 \le -\frac{c_3c_1^2}{120L} + \frac{c_3c_1^2}{480L} + \frac{8\beta\delta}{(1-\beta)(1+\delta)}$$
$$\le -\frac{c_3c_1^2}{120L} + \frac{c_3c_1^2}{480L} + +\frac{c_3c_1^2}{480L} = -\frac{c_3c_1^2}{240L}.$$

Thus we know $F_{1,R}(-\frac{c_3c_1^2}{120L} + \frac{\delta}{4L} + E) \ge F_{1,R}(-\frac{c_3c_1^2}{240L})$ and item 2 in Assumption 2 is satisfied by $c_2 = F_{1,R}(-\frac{c_3c_1^2}{240L}) - \frac{1}{2} > 0$. Combining the above, we can choose $\hat{\delta} = \min\{\delta_1, \delta_2\}$ so that both items in Assumption 2 hold for $\delta \le \hat{\delta}$. □

Appendix D: Proofs for Section 4

In this section, we present our lower bounds for random-iterate convergence rates of OFTRL learning dynamics. The main theorem in this section is Theorem EC.2 (the technical version of Theorem 5) that shows there exists $T = O(\frac{f_R(\delta)}{\delta})$ such that all the iterates $t \in [T, T+\Omega(\frac{1}{\delta})]$ have a duality gap lower bounded by a constant. Here $f_R(\delta) := -F_{1,R}^{-1}(\frac{1}{1+\delta})$ is a quantity that depends on the regularizer R. We provide upper bounds of $f_R(\delta)$ in Lemma EC.6. At the end, we combine Theorem 5 and Lemma EC.6 to prove Theorem 6.

D.1. Bounds on $f_R(\delta)$ for Different Regularizers

LEMMA EC.6. *Let $f_R(\delta) := -F_{1,R}^{-1}(\frac{1}{1+\delta})$. If $E \ge -f_R(\delta)$, then $F_{1,R}(E) \le \frac{1}{1+\delta}$. Moreover, we have the following upper bounds on $f_R(\delta)$ for $\delta \in (0, \frac{1}{2})$ and regularizer R:*

1. *Negative entropy:* $f_R(\delta) \le \log(\frac{1}{\delta})$;
2. *Squared Euclidean norm:* $f_R(\delta) \le 1$.
3. *Log barrier:* $f_R(\delta) \le \frac{1}{\delta}$.
4. *Negative Tsallis entropy* ($\beta \in (0,1)$)*:* $f_R(\delta) \le \frac{2\beta}{1-\beta}(\frac{1}{\delta})^{1-\beta}$.

Proof Since $f_R(\delta) = -F_{1,R}^{-1}(\frac{1}{1+\delta})$, we have

$$F_{1,R}(-f_R(\delta)) = F_{1,R}\left(F_{1,R}^{-1}\left(\frac{1}{1+\delta}\right)\right) = \frac{1}{1+\delta}.$$

Since $F_{1,R}$ is a non-increasing function (Lemma 1), we know if $E \ge -f_R(\delta)$, then $F_{1,R}(E) \le \frac{1}{1+\delta}$. This finishes the proof of the first claim.

We then prove upper bounds for $f_R(\delta)$ for different regularizer R. Our proof strategy is to derive the closed-form expression of $F_{1,R}$ (by setting the gradient of Equation (1) to be 0) and that of its inverse function $F_{1,R}^{-1}$ and directly bound $f_R(\delta)$.

Negative Entropy: $R(x) = x\log x + (1-x)\log(1-x)$. For negative entropy, $F_{1,R}$ admits the following closed-form expression:

$$F_{1,R}(E) = \frac{1}{1+\exp(E)} \Rightarrow F_{1,R}^{-1}\left(\frac{1}{1+\delta}\right) = \log\delta$$

Hence, $f_R(\delta) = -F_{1,R}^{-1}(\frac{1}{1+\delta}) = \log\frac{1}{\delta}$.

Squared Euclidean Norm: $R(x) = \frac{1}{2}(x^2 + (1-x)^2)$. In this case, we can verify that $F_{1,R}^{-1}(x)$ admits a closed-form for $x \in (0,1)$:

$$F_{1,R}(E) = \frac{1-E}{2}, \forall E \in (-1,1) \Rightarrow F_{1,R}^{-1}(x) = 1-2x, \forall x \in (0,1).$$

Hence, $f_R(\delta) = -F_{1,R}^{-1}(\frac{1}{1+\delta}) = \frac{1-\delta}{1+\delta} \le 1$.

Log Barrier: $R(x) = -\log x - \log(1-x)$. In this case, we can verify that $F_{1,R}^{-1}(x)$ admits a closed-form for $x \in (0,1)$:

$$F_{1,R}^{-1}(x) = \frac{2x-1}{x^2-x}, \forall x \in (0,1).$$

Hence, we have

$$f_R(\delta) = -F_{1,R}^{-1}\left(\frac{1}{1+\delta}\right) = -\frac{2(1+\delta)-(1+\delta)^2}{1-(1+\delta)} = \frac{1-\delta^2}{\delta} \le \frac{1}{\delta}.$$

Negative Tsallis Entropy: $R(x) = \frac{1}{1-\beta}(1 - x^\beta + 1 - (1-x)^\beta)$. In this case, we can verify that $F_{1,R}^{-1}(x)$ admits a closed-form for $x \in (0,1)$:

$$F_{1,R}^{-1}(x) = \frac{\beta}{1-\beta}\left(x^{\beta-1} - (1-x)^{\beta-1}\right), \forall x \in (0,1).$$

Hence, we have

$$f_R(\delta) = -F_{1,R}^{-1}\left(\frac{1}{1+\delta}\right) = \frac{\beta}{1-\beta}\left(\left(\frac{\delta}{1+\delta}\right)^{\beta-1} - \left(\frac{1}{1+\delta}\right)^{\beta-1}\right) \le \frac{\beta}{1-\beta}\left(\frac{1+\delta}{\delta}\right)^{1-\beta} \le \frac{2\beta}{1-\beta}\left(\frac{1}{\delta}\right)^{1-\beta}.$$

This completes the proof. □

D.2. Proof of Theorem 5

We prove the following theorem that implies Theorem 5.

THEOREM EC.2. *Assume the regularizer R satisfies Assumption 1 and Assumption 2. For any $\delta \in (0, \hat{\delta})$ where $\hat{\delta} = \min\{\frac{1}{100}, \frac{c_1}{6}, \frac{c_1^2}{300}, \delta'\}$ is a constant that depending only on the constants c_1 and δ' defined in Assumption 2, the OFTRL dynamics on $\mathcal{A}_\delta$ (defined in (2)) with any non-increasing step size $\{\eta_t\}$ satisfying Assumption EC.1 satisfies* $\text{DualityGap}(x^t, y^t) \ge c_2$ *for all $t \in [T_2 + \lceil \frac{T_m}{2} \rceil, T_2 + T_m]$, where $c_2 > 0$ is a constant defined in Assumption 2. Moreover, we have*

1. $\sum_{k=1}^{T_2+T_m} \eta_k \le \frac{13+6Lf_R(\delta)}{c_1 c_3 L\delta}$.
2. $\sum_{k=T_2}^{T_2+T_m-1} \eta_k \ge \frac{c_1^2}{20L\delta}$.

REMARK EC.2. When $\eta_t = \eta \le \frac{1}{4L}$ are fixed, there exists $T = T_2 + \lceil \frac{T_m}{2} \rceil \le \frac{13+6Lf_R(\delta)}{c_1 c_3 \eta L\delta}$ such that $\text{DualityGap}(x^t, y^t) \ge c_2$ for all $t \in [T, T + \frac{c_1^2}{40\eta L\delta}]$. This implies Theorem 5.

Proof of Theorem EC.2 Recall that $\eta_1 \le \frac{1}{4L}$ and the step sizes are non-increasing. We will use notations and results presented in Theorem EC.1. We restate some definitions and key facts:

- T_s' is the smallest iteration when $\sum_{k=1}^{T_s'-1} \eta_k \ge \frac{1}{4L}$, we have $\sum_{k=1}^{T_s'} \eta_k \in [\frac{1}{4L}, \frac{3}{4L}]$.
- T_1 is the smallest iteration when $x^{T_1}[1] \ge \frac{1}{1+\delta}$.
- $T_\ell > 0$ the smallest integer such that $\sum_{k=T_1}^{T_1+T_\ell} \eta_k \ge \frac{3\eta_1}{2c_1}$. Thus $\sum_{k=T_1}^{T_1+T_\ell} \eta_k \in [\frac{3\eta_1}{2c_1}, \frac{2\eta_1}{c_1}]$.
- $T_h > 0$ is the smallest integer such that $\sum_{k=T_1}^{T_1+T_h} \eta_k \in [\frac{c_1}{3L\delta} \frac{c_1}{2L\delta}]$
- $T_2 > T_1$ is the smallest iteration when $y^{T_2}[1] \ge \frac{1}{2(1+\delta)}$;
- We have $e_x^t \le -\frac{c_1}{3}$ for all $t \in [T_s', T_1 + T_h - 1]$.
- $T_m > 0$ is the smallest integer such that $\sum_{k=T_2}^{T_2+T_m-1} \eta_k \in [\frac{c_1^2}{20L\delta}, \frac{c_1^2}{20L\delta} + \frac{1}{4L}]$. We have

$$\text{DualityGap}(x^t, y^t) \ge c_2, \quad \forall t \in \left[T_2 + \lceil \frac{T_m}{2} \rceil, T_2 + T_m\right]$$

Notice that the interval $[T_2+\lceil\frac{T_m}{2}\rceil, T_2+T_m]$ has length at least $\frac{T_m}{3}$. Moreover T_m satisfies $\sum_{k=T_2}^{T_2+T_m-1} \geq \frac{c_1^2}{20L\delta} = \Omega(\frac{1}{\delta})$.

In the following, we prove that $\sum_{k=1}^{T_2}\eta_k = O(\frac{f_R(\delta)}{\delta})$. We proceed with two steps. We first upper bound T_1, then we use the obtained bound to further bound T_2.

Bounding T_1. By item 1 of Theorem EC.1, we have $y^t[1] \leq \frac{1}{2} - c_1$ for all $t \in [T_s', T_1-1]$. Recall that $e_y^t = 1-(1+\delta)x^t[1]$, this implies $e_x^t \leq -\frac{c_1}{2}$ for all $t \in [T_s', T_1-1]$. Since T_1 is the *first* iteration that $x^{T_1}[1] \geq \frac{1}{1+\delta}$, it follows that

$$\frac{1}{1+\delta} \geq x^{T_1-1}[1] = F_{1,R}\left(\sum_{k=1}^{T_1-2}\eta_k e_x^k + \eta_{T_1-1}g_x^{T_1-1}\right) \geq F_{1,R}\left(\sum_{k=1}^{T_s'}\eta_k - \frac{c_1}{2}\cdot\sum_{k=T_s'+1}^{T_1-2}\eta_k\right)$$

By definition of $f_R(\delta)$ and monotonicity of $F_{1,R}$, we deduce

$$\sum_{k=1}^{T_s'}\eta_k - \frac{c_1}{2}\cdot\sum_{k=T_s'+1}^{T_1-2}\eta_k \geq -f_R(\delta) \Rightarrow \sum_{k=T_s'+1}^{T_1-2}\eta_k \leq \frac{2f_R(\delta)}{c_1} + \frac{3}{2c_1L},$$

where we use $\sum_{k=1}^{T_s'}\eta_k \leq \frac{3}{4L}$. Applying this fact again with $\eta_{T_1-1} \leq \frac{1}{4L}$ and $c_1 \leq \frac{1}{2}$ gives $\sum_{k=1}^{T_1-1}\eta_k \leq \frac{2f_R(\delta)}{c_1} + \frac{2}{c_1L}$.

Bounding T_2. We first give several bounds on e_y^t.

1. For all t, we have $e_y^t \leq 1$ by Proposition 2.
2. For $t \in [T_1+T_\ell, T_1+T_h]$, we have $e_y^t \leq 0$. This is because we have $x^t[1] \geq \frac{1}{1+\delta}$ for $t \in [T_1+T_\ell, T_1+T_h]$ by item 2 of Theorem EC.1.
3. For $t \in [T_1+T_h, T_2]$, we have $x^t[1] \geq \frac{1+c_3}{1+c_3+\delta}$ by item 2 of Theorem EC.1, which implies $e_y^t = 1-(1+\delta)x^t[1] = -\frac{c_3\delta}{1+c_3+\delta} \leq -\frac{c_3\delta}{2}$ as $1+c_3+\delta \leq 2$. Moreover, $g_y^{T_2-1} \leq 0$ as it is convex combination of $\{0, e_y^{T_2-2}, e_y^{T_2-1}\}$.

Combining the above gives

$$\begin{aligned}
\frac{1}{2} \geq y^{T_2-1}[1] &= F_{1,R}\left(\sum_{t=1}^{T_2-2}\eta_t e_y^t + \eta_{T_2-1}g_y^{T_2-1}\right)\\
&= F_{1,R}\left(\sum_{t=1}^{T_1-1}\eta_t e_y^t + \sum_{t=T_1}^{T_1+T_\ell}\eta_t e_y^t + \sum_{t=T_1+T_\ell+1}^{T_1+T_h-1}\eta_t e_y^t + \sum_{t=T_1+T_h}^{T_2-2}\eta_t e_y^t + \eta_{T_2-1}g_y^{T_2-1}\right)\\
&\geq F_{1,R}\left(\sum_{t=1}^{T_1-1}\eta_t + \sum_{t=T_1}^{T_1+T_\ell}\eta_t - \frac{c_3\delta}{2}\cdot\sum_{t=T_1+T_h}^{T_2-2}\eta_t\right).
\end{aligned}$$

By monotonicity of $F_{1,R}$ and $F_{1,R}(0) = \frac{1}{2}$ (Assumption 1), we have

$$\sum_{t=1}^{T_1-1}\eta_t + \sum_{t=T_1}^{T_1+T_\ell}\eta_t - \frac{c_3\delta}{2}\cdot\sum_{t=T_1+T_h}^{T_2-2}\eta_t \geq 0 \Rightarrow \sum_{t=T_1+T_h}^{T_2-2}\eta_t \leq \frac{2}{c_3\delta}\cdot\left(\frac{2f_R(\delta)}{c_1} + \frac{2}{c_1L} + \frac{2\eta_1}{c_1}\right) \leq \frac{8+4Lf_R(\delta)}{c_1c_3L\delta}.$$

Combining the bounds on step size again we get

$$\sum_{t=1}^{T_2} \eta_t \le \sum_{t=1}^{T_1-1} \eta_t + \sum_{t=T_1}^{T_1+T_h-1} \eta_t + \sum_{t=T_1+T_h}^{T_2-2} \eta_t + 2\eta_1 \le \frac{12+6Lf_R(\delta)}{c_1c_3L\delta}.$$

Recall that $\mathrm{DualityGap}(x^t, y^t) \ge c_2$ for all $t \in [T_2 + \lceil \frac{T_m}{2} \rceil, T_2 + T_m]$. We conclude that there is one iteration $T := T_2 + \lceil \frac{T_m}{2} \rceil$ such that the following iterations $[T, T + \lfloor \frac{T_m}{2} \rfloor]$ all have duality gap larger than the constant $c_2 > 0$. The length of the interval is $\lfloor \frac{T_m}{2} \rfloor + 1 \ge \frac{T_m}{2}$, which satisfies $\sum_{k=T_2}^{T_2+T_m-1} \eta_k \ge \frac{c_1^2}{20L\delta}$. Moreover, we can bound $\sum_{k=1}^{T+\lfloor \frac{T_m}{2} \rfloor} \eta_k = \sum_{k=1}^{T_2+T_m} \eta_k \le \frac{12+6Lf_R(\delta)}{c_1c_3L\delta} + \frac{c_1^2}{20L\delta} + \frac{1}{4L} \le \frac{13+6Lf_R(\delta)}{c_1c_3L\delta}$. □

D.3. Proof of Theorem 6

We only present the proof for the negative entropy regularizer since the proof for the other regularizers follows in the same way.

For OFTRL with negative entropy regularizer, by Theorem 5 and Lemma EC.6, we have $\mathrm{DualityGap}(x^t, y^t) \ge c_2$ for all $t \in [T_2 + \lceil \frac{T_m}{2} \rceil, T_2 + T_m]$ with $\sum_{k=1}^{T_2+T_m} \eta_k = O(\frac{\log(\frac{1}{\delta})}{\delta})$ and $\sum_{k=T_2}^{T_2+T_m-1} \eta_k = \Omega(\frac{1}{\delta})$. Since the step sizes are non-increasing, we have $\frac{T_m}{T_2+T_m} = \Omega(\frac{1}{\log(\frac{1}{\delta})})$. Thus we have

$$\frac{1}{T_2+T_m} \sum_{t=1}^{T_2+T_m} \mathrm{DualityGap}(x^t, y^t) \ge \frac{\sum_{t=T_2+\lceil \frac{T_m}{2} \rceil}^{T_2+T_m} \mathrm{DualityGap}(x^t, y^t)}{T_2+T_m}$$
$$\ge \frac{c_2}{2} \cdot \frac{T_m}{T_2+T_m} = \Omega(\frac{1}{\log(\frac{1}{\delta})}) = \Omega(\frac{1}{\log(T_2+T_m)}),$$

where the last equality holds since $\log(T_2+T_m) = \Omega(\log \frac{1}{\delta})$. We note that $T_2 + T_m$ can be arbitrarily large as $\delta \to 0$. Therefore, the uniform random-iterate convergence rate of OFTRL with negative entropy regularizer is at most $\Omega(\frac{1}{\log T})$.

Appendix E: Extension to Higher Dimensions

In this section, we show that the pathological behaviors highlighted in Section 3 and Section 4 may also happen beyond the hard matrix game (2). In particular, we show that there exists matrix games of size $2n \times 2n$ where the OFTRL iterates of the players behave similarly as the OFTRL iterates over the 2×2 game A_δ.

We start by showing an equivalence result for a single player (say, the first player). We assume that a decision maker is using OFTRL with a 1-strongly convex (w.r.t. the ℓ_2 norm) and separable regularizer $R(x) = R_1(x_1) + R_2(x_2)$ to choose decisions. At a given time time t, they see a loss $\ell^t \in [0, 1]^2$.

Now consider the following $2n$-dimensional decision problem: The player uses OFTRL using the regularizer $\hat{R}(\hat{x}) = \sum_{i=1}^{n} R_1(\hat{x}_i) + \sum_{i=n+1}^{2n} R_2(\hat{x}_i)$, *i.e.*, they use R_1 on the first half of actions, and R_2 on the second half. This is again a 1-strongly convex regularizer (w.r.t. the ℓ_2 norm). Suppose the decision maker sees the rescaled and *duplicated* version of the losses $\ell^1, \dots, \ell^T$ from the 2-dimensional case: $\hat{\ell}_i^t = \frac{1}{n^\alpha}\ell_1^t$ if $i \le n$, and $\hat{\ell}_i^t = \frac{1}{n^\alpha}\ell_2^t$ if $i > n$. The parameter α will be chosen later based on the regularizer.

Now we wish to show that by choosing α in the right way, we get that the decisions for the 2-dimensional and $2n$-dimensional OFTRL algorithms are equivalent. Let $x^1, \dots, x^T$ be the 2-dimensional OFTRL decisions, and let $\hat{x}^1, \dots, \hat{x}^T$ be the $2n$-dimensional OFTRL decisions. Then, we want to show that $\sum_{i=1}^{n} \hat{x}_i^t = x^t[1]$ and $\sum_{i=n+1}^{2n} \hat{x}_i^t = x^t[2]$ for all t.

LEMMA EC.7. *Let the losses $\hat{\ell}^1, \dots, \hat{\ell}^T$ satisfy the duplication procedure given in the preceding paragraph. Then for any time t, we have $\hat{x}_1^t = \cdots = \hat{x}_n^t$ and $\hat{x}_{n+1}^t = \cdots = \hat{x}_{2n}^t$.*

Proof Suppose not and let $\hat{x}^t$ be the corresponding solution. Then the optimal solution is such that $\hat{x}_i^t \neq \hat{x}_k^t$ for some i, k both less than n, or both greater than n. But then, by symmetry, we have that there is more than one optimal solution to the OFTRL optimization problem at time t: the objective is exactly the same if we create a new solution where we swap the values of $\hat{x}_i^t$ and $\hat{x}_k^t$. This is a contradiction due to strong convexity. $\square$

From Lemma EC.7, we have that the OFTRL decision problem in $2n$ dimensions can equivalently be written as a 2-dimensional decision problem: Since the first n entries must be the same, we can simply optimize over that one shared value, say $x^t[1]$, which we use for all n entries, and similarly we use $x^t[2]$ for the second half of the entries. Let $\mathrm{Dupl} : \Delta^2 \to \Delta^{2n}$ be a function that maps the two-dimensional solution into the corresponding duplicated $2n$-dimensional solution. The equivalent 2-dimensional problem is then:

$$\begin{aligned}
\hat{x}^t &= \mathrm{Dupl}\left[\operatorname*{arg\,min}_{x \in \frac{1}{n}\cdot\Delta^2} \left\{ \frac{n}{n^\alpha}\left\langle x, \sum_{\tau=1}^{t-1} \ell^\tau + m^t \right\rangle + \frac{n}{\eta} R_1(x[1]) + \frac{n}{\eta} R_2(x[2]) \right\}\right] \\
&= \mathrm{Dupl}\left[\frac{1}{n} \cdot \operatorname*{arg\,min}_{x \in \Delta^2} \left\{ \frac{n}{n^\alpha}\left\langle \frac{1}{n}x, \sum_{\tau=1}^{t-1} \ell^\tau + m^t \right\rangle + \frac{n}{\eta} R(x/n) \right\}\right] \\
&= \mathrm{Dupl}\left[\frac{1}{n} \cdot \operatorname*{arg\,min}_{x \in \Delta^2} \left\{ \left\langle x, \sum_{\tau=1}^{t-1} \ell^\tau + m^t \right\rangle + \frac{n^{\alpha+1}}{\eta} R(x/n) \right\}\right].
\end{aligned}$$

The next theorem shows that we can choose α for different regularizers and construct $2n \times 2n$ loss matrices whose learning dynamics are equivalent to the learning dynamics in 2×2 games given in the preceding sections.

THEOREM EC.3. *For any loss matrix $A \in [0,1]^{2\times 2}$, there exists a loss matrix $\hat{A} \in [0, n^{-\alpha}]^{2n\times 2n}$ such that for the Euclidean ($\alpha = 1$), entropy ($\alpha = 0$), Tsallis ($\beta \in (0,1)$ and $\alpha = -1+\beta$), and log ($\alpha = -1$) regularizers, the resulting OFTRL learning dynamics are equivalent in the two games.*

Proof Recall the equivalent 2-dimensional problem:

$$\begin{aligned}\hat{x}^t &= \mathrm{Dupl}\left[\operatorname*{arg\,min}_{x \in \frac{1}{n}\cdot\Delta^2}\left\{\frac{n}{n^\alpha}\left\langle x, \sum_{\tau=1}^{t-1}\ell^\tau + m^t\right\rangle + \frac{n}{\eta}R_1(x[1]) + \frac{n}{\eta}R_2(x[2])\right\}\right] \\ &= \mathrm{Dupl}\left[\frac{1}{n}\cdot\operatorname*{arg\,min}_{x\in\Delta^2}\left\{\frac{n}{n^\alpha}\left\langle \frac{1}{n}x, \sum_{\tau=1}^{t-1}\ell^\tau + m^t\right\rangle + \frac{n}{\eta}R(x/n)\right\}\right] \\ &= \mathrm{Dupl}\left[\frac{1}{n}\cdot\operatorname*{arg\,min}_{x\in\Delta^2}\left\{\left\langle x, \sum_{\tau=1}^{t-1}\ell^\tau + m^t\right\rangle + \frac{n^{\alpha+1}}{\eta}R(x/n)\right\}\right].\end{aligned}$$

Euclidean regularizer: this regularizer is homogeneous of degree two. Choosing $\alpha = 1$, the inner minimization problem is exactly the same as the one solved by OFTRL in two dimensions.

Entropy regularizer: we set $\alpha = 0$ to get equivalence:

$$nR(x/n) = \sum_{i=1}^{2} x[i]\log(x[i]/n) = \sum_{i=1}^{2} x[i]\log x[i] - \sum_{i=1}^{2} x[i]\log n = \sum_{i=1}^{2} x[i]\log x[i] - \log n.$$

Now we have equivalence because the last term is a constant that does not affect the arg min.

Log regularizer: we set $\alpha = -1$ to get equivalence, using similar logic as for entropy:

$$R(x/n) = \sum_{i=1}^{2} -\log(x[i]/n) = 2\log n + \sum_{i=1}^{2} -\log x[i].$$

Tsallis entropy regularizer: we set $\alpha = -1+\beta$ to get equivalence, using similar logic as for entropy:

$$n^\beta R(x/n) = n^\beta \cdot \frac{1 - \sum_{i=1}^{2}(\frac{x[i]}{n})^\beta}{1-\beta} = \frac{n^\beta - 1}{1-\beta} + \frac{1 - \sum_{i=1}^{2} x[i]^\beta}{1-\beta}.$$

□

Using the reduction in Theorem EC.3, we can extend our lower bound results for the last-iterate convergence (Theorem 3) and the random-iterate convergence (Theorem 6) to higher dimensions.

Appendix F: Proofs for Section 5

F.1. Existing Results on Linear Last-Iterate Convergence

The next two lemmas are adapted from Wei et al. (2021).

LEMMA EC.8 **(Adapted from Lemma 1 of Wei et al. (2021)).** *Consider OWMU for a zero-sum game $A \in [0,1]^{d_1\times d_2}$ with $\eta \le \frac{1}{8}$. Let z^* be a Nash equilibrium. Define $\Theta^t = \mathrm{KL}(z^*, \widehat{z}^t) + \frac{1}{16}\mathrm{KL}(\widehat{z}^t, z^{t-1})$ and $\zeta^t = \mathrm{KL}(\widehat{z}^{t+1}, z^t) + \mathrm{KL}(z^t, \widehat{z}^t)$. Then for any $t \ge 2$, we have*

1. *For any z, $\eta\langle F(z^t), z^t - z\rangle \le \Theta^t - \Theta^{t+1} - \frac{15}{16}\zeta^t$.*
2. *$\Theta^{t+1} \le \Theta^t - \frac{15}{16}\zeta^t$.*
3. *$\sum_{t=2}^{\infty}\|z^{t+1}-z^t\|_1^2 = O(\log(d_1 d_2))$.*

LEMMA EC.9 **(Adapted from Lemma 19 of Wei et al. (2021)).** *Let matrix game $A \in [0,1]^{d_1\times d_2}$ has a fully mixed Nash equilibrium with minimum probability $\delta > 0$. Let $\{z^t\}_{t\ge 1}$ be the iterates produced by OMWU with uniform initialization and step size $\eta \le \frac{1}{8}$, then*

$$\min_{t\ge 1, i\in[d_1+d_2]}\{z^t[i], \widehat{z}^t[i]\} \ge \Omega\left(\frac{1}{d_1 d_2}\exp\left(-\frac{1}{\delta}\right)\right).$$

F.2. Global Phase: Missing Proofs in Section 5.2

F.2.1. Proof of Proposition 3

Proof Recall that $\ell_x^t = Ay^t$ and $\ell_y^t = -A^\top x^t$. By definition, we have

$$\begin{aligned}\sum_{t=1}^{T}\mathrm{DualityGap}(x^t, y^t) &= \sum_{t=1}^{T}\left(\max_{y\in\Delta^{d_2}}(x^t)^\top Ay - \min_{x\in\Delta^{d_1}} x^\top Ay^t\right)\\ &= \sum_{t=1}^{T}\left(\max_{y\in\Delta^{d_2}}(x^t)^\top Ay - (x^t)^\top Ay^t\right) + \sum_{t=1}^{T}\left((x^t)^\top Ay^t - \min_{x\in\Delta^{d_1}} x^\top Ay^t\right)\\ &= \mathcal{R}_y^T(\{y_\star^t\}) + \mathcal{R}_x^T(\{x_\star^t\}).\end{aligned}$$

□

F.2.2. Proof of Lemma 3

Proof By Lemma EC.8, for any interval $\mathcal{I} \in [s,e] \in [2,\infty]$[4], we have

$$\mathcal{R}_x^{\mathcal{I}} + \mathcal{R}_y^{\mathcal{I}} = \max_z \sum_{t=s}^{e}\left\langle F(z^t), z^t - z\right\rangle \le \frac{1}{\eta}\sum_{t=s}^{e}\Theta^t - \Theta^{t+1} - \zeta^t \le \frac{1}{\eta}\Theta^s.$$

For any $t \ge 2$, we can bound Θ^t by definition and the probability lower bound in Lemma EC.9:

$$\Theta^t \le \mathrm{KL}(z^*, \widehat{z}^t) + \mathrm{KL}(\widehat{z}^t, z^{t-1}) \le \sum_{i\in[d_1+d_2]} \log\frac{1}{\widehat{z}^t[i]} + \log\frac{1}{z^{t-1}[i]} \le O\left((d_1+d_2)\left(\log(d_1 d_2) + \frac{1}{\delta}\right)\right).$$

Combining the above two inequalities completes the proof. □

[4] Here we ignore the first iteration which contributes at most $O(1)$ regret.

F.2.3. Proof of Lemma 4

Proof Let $\mathcal{I} = [s, e]$. Then by definition, we have

$$\begin{aligned}
V^{\mathcal{I}} &= \sum_{t=s+1}^{e} \max_{z} |\langle F(z^t) - F(z^{t-1}), z\rangle| \\
&\le 2 \sum_{t=s+1}^{e} \left\|F(z^t) - F(z^{t-1})\right\|_\infty && \text{(Cauchy-Schwarz and } \|z\|_1 \le 2) \\
&\le 2\sqrt{|\mathcal{I}| \sum_{t=s+1}^{e} \left\|F(z^t) - F(z^{t-1})\right\|_\infty^2} && \text{(Cauchy-Schwarz)} \\
&\le 2\sqrt{|\mathcal{I}| \sum_{t=s+1}^{e} \left\|z^t - z^{t-1}\right\|_1^2} && (F \text{ is 1-Lipschitz since } A \in [0,1]^{d_1\times d_2}) \\
&\le O\left(\sqrt{|\mathcal{I}| \log(d_1 d_2)}\right). && \text{(Lemma EC.8 item 3)}
\end{aligned}$$

□

F.2.4. Proof of Theorem 8

Proof Let $\mathcal{I}_1 = [s_1, e_1], \ldots, \mathcal{I}_M = [s_M, e_M]$ be any partition of the T rounds. Recall that $z_\star^t \in \arg\min_z \langle F(z^t), z\rangle$. Then, the social dynamic regret on $\mathcal{I}_m$ is

$$\sum_{t\in\mathcal{I}_m} \left\langle F(z^t), z^t - z_\star^t \right\rangle = \sum_{t\in\mathcal{I}_m} \left\langle F(z^t), z^t - z_\star^{s_m} \right\rangle + \sum_{t\in\mathcal{I}_m} \left\langle F(z^t), z_\star^{s_m} - z_\star^t \right\rangle \le \mathcal{R}_z^{\mathcal{I}_m} + 2|\mathcal{I}_m| V^{\mathcal{I}_m},$$

where the last step is by the definition of interval regret and the fact that

$$\begin{aligned}
\left\langle F(z^t), z_\star^{s_m} - z_\star^t \right\rangle \le \left\langle F(z^t) - F(z^{s_m}), z_\star^{s_m} \right\rangle + \left\langle F(z^{s_m}) - F(z^t), z_\star^t \right\rangle &= \sum_{k=s_m+1}^{t} \left\langle F(z^k) - F(z^{k-1}), z_\star^{s_m} \right\rangle \\
&\quad + \sum_{k=s_m+1}^{t} \left\langle F(z^{k-1}) - F(z^k), z_\star^t \right\rangle \\
&\le 2V^{\mathcal{I}_m}.
\end{aligned}$$

where the first inequality is by optimality of $z_\star^{s_m}$. Therefore,

$$\begin{aligned}
\sum_{t=1}^{T} \left\langle F(z^t), z^t - z_\star^t \right\rangle &\le \sum_{m=1}^{M} (\mathcal{R}_z^{\mathcal{I}_m} + 2|\mathcal{I}_m| V^{\mathcal{I}_m}) \\
&\le O\left((d_1 + d_2)\log(d_1 d_2) \cdot \frac{M}{\eta\delta} + \max_{m\in[1,M]} |\mathcal{I}_m| \cdot V^{[1,T]}\right) && \text{(Lemma 3)} \\
&\le O\left((d_1 + d_2)\log(d_1 d_2) \cdot \left(\frac{M}{\eta\delta} + \max_{m\in[1,M]} |\mathcal{I}_m| \sqrt{T}\right)\right) && \text{(Lemma 4)}
\end{aligned}$$

We choose $M = \max\{1, T^{\frac{3}{4}}\delta^{\frac{1}{2}}\}$ and make sure each interval has length $O(\frac{T}{M})$, then we have

$$\frac{M}{\eta\delta} + \max_{m\in[1,M]} |\mathcal{I}_m|\sqrt{T} = O\left(\frac{M}{\eta\delta} + \frac{T^{\frac{3}{2}}}{M}\right) = O\left(\frac{T^{\frac{3}{4}}\delta^{-\frac{1}{2}}}{\eta}\right).$$

We note that the above holds for any T. This completes the proof. □

F.3. Initial Phase: Missing Proofs in Section 5.3

F.3.1. Proof of Proposition 4

Proof Fix any $\eta \in (0, \eta']$. Since $A = b_1 \cdot \mathbf{1} + b_2 A_{\delta_x,\delta_y}$, we know that the following two dynamics produce exactly the same trajectory $\{x^t, y^t\}$:

1. OMWU with step size η on A;
2. OMWU with step size $b_2\eta$ on A_{δ_x,δ_y}.

The reason is (1) the update rule of OMWU concerns only the relative loss between actions so the $b_1\mathbf{1}$ component does not matter; (2) $\eta(b_2 A_{\delta_x,\delta_y}) = (b_2\eta)A_{\delta_x,\delta_y}$. Therefore, if we assume that $\{x^t, y^t\}$ evaluated on A_{δ_x,δ_y} has a rate $O(\frac{1}{b_2\eta} f(T))$, then the same sequence $\{x^t, y^t\}$ evaluated on $A = b_1\mathbf{1} + b_2 A_{\delta_x,\delta_y}$ has a convergence rate of $O(b_2 \cdot \frac{1}{b_2\eta} f(T)) = O(\frac{1}{\eta} f(T))$. □

F.3.2. Proof of Theorem 9 By Assumption 3, Lemma 5, and Proposition 4, we only need to consider the following class of matrices without loss of generality:

$$A_{\delta_x,\delta_y} = \begin{bmatrix} \frac{1-\delta_y}{1-\delta_x} & \frac{1-\delta_x-\delta_y}{1-\delta_x} \\ 0 & 1 \end{bmatrix}, \frac{1}{100} \le \delta_x \le \delta_y \le 1 - \delta_x \tag{EC.6}$$

For simplicity, in the proof, we omit the subscript and denote by A the matrix A_{δ_x,δ_y}. We first summarize properties of A that can be verified by simple algebra.

Proposition EC.2. *Given Assumption 3, we have the following:*

- *The loss vectors of A are*

$$\ell_x = Ay = \left[1 - \frac{\delta_y}{1-\delta_x} + \frac{\delta_x}{1-\delta_x} y[1], 1 - y[1]\right], \ell_y = -A^\top x = -\left[\frac{1-\delta_y}{1-\delta_x} x[1], 1 - \frac{\delta_y}{1-\delta_x} x[1]\right]$$

- *The loss vectors of A satisfy:*

$$e_x := \ell_x[1] - \ell_x[2] = \frac{y[1]-\delta_y}{1-\delta_x} = \frac{1-\delta_y-y[2]}{1-\delta_x},$$
$$e_y := \ell_y[1] - \ell_y[2] = \frac{x[2]-\delta_x}{1-\delta_x} = \frac{1-\delta_x-x[1]}{1-\delta_x}.$$

Moreover,

$$e_x \in \left[-\frac{\delta_y}{1-\delta_x}, 1\right] \subseteq [-1,1], e_y \in \left[-\frac{\delta_x}{1-\delta_x}, 1\right] \subseteq [-2\delta_x, 1].$$

We consider two cases: (1) $\delta_y \ge \frac{1}{2}$; (2) $\delta_y < \frac{1}{2}$. We prove Theorem 9 for each case in the following.

Case 1: $\delta_y \geq \frac{1}{2}$.

LEMMA EC.10. *Consider matrix A defined in Equation (EC.6) that satisfies $\delta_y \in [\frac{1}{2}, 1)$. Let $\{x^t, y^t\}_{t\geq 1}$ be the iterates produced by OMWU dynamics with uniform initialization and step size $\eta \leq 1$. Let T_1 be the first iteration such that $x^t[1] \geq 1-\delta_x$. Then*

1. $\mathrm{DualityGap}(x^t, y^t) \leq \frac{x^t[2]}{x^t[1]} \leq 9\exp(-\frac{\eta t}{42})$, *for all $t \in [1, T_1 - 1]$;*
2. $\mathrm{DualityGap}(x^{T_1}, y^{T_1}) \leq 2\delta_x$.
3. *For all $t \in [1, T_1]$, there exists a universal constant $C > 0$ such that*

$$\min_{k\in[1,t]} \mathrm{DualityGap}(x^k, y^k) \leq \frac{C}{\eta}\frac{1}{t}.$$

Proof The OMWU dynamics is initialized with uniform distributions (x^1, y^1) and step size $\eta \leq 1$. Denote by $T_0 = \lceil \frac{1}{\eta} \rceil + 1 \leq \frac{2}{\eta}$. Using the update rule, we have for all $t \in [1, T_0]$,

$$\frac{x^t[1]}{x^t[2]} = \frac{x^1[1]}{x^1[2]} \cdot \exp\left(-\eta E_x^{t-1} - \eta e_x^{t-1}\right) \leq e^{\eta t} \leq e^2,$$

where we use $x^1[1] = x^1[2] = \frac{1}{2}$ and $-e_x \leq 1$ by Proposition EC.2. This implies $x^t[1] \leq \frac{e^2}{e^2+1} < \frac{8}{9}$ for all $t \in [1, T_0]$. We define $T_1 > T_0$ the first iteration where $x^t[1] \geq 1 - \delta_x$.

We have the following two inequalities on $e_y^t = \ell_y^t[1] - \ell_y^t[2]$ by Proposition EC.2:

$$e_y^t = \frac{1-\delta_x - x^t[1]}{1-\delta_x} \geq 0, \forall t \in [1, T_1 - 1], \tag{EC.7}$$

$$e_y^t = \frac{1-\delta_x - x^t[1]}{1-\delta_x} \geq \frac{1}{10}, \forall t \in [1, T_0]. \tag{EC.8}$$

where in (EC.7) we use $x^t[1] < 1 - \delta_x$ for all $t \in [1, T_1 - 1]$; in (EC.8) we use $\delta_x \leq \frac{1}{100}$ and $x^t[1] \leq \frac{8}{9}$ for all $t \in [1, T_0]$.

For any $t \in [T_0, T_1]$, we have y^t satisfies

$$\begin{aligned}
\frac{y^t[1]}{y^t[2]} = \frac{y^1[1]}{y^1[2]} \cdot \exp\left(-\eta E_y^{T_0-1} - \sum_{k=T_0}^{t-1} e_y^t - e_y^{t-1}\right) &\leq \exp\left(-\eta E_y^{T_0-1}\right) && \text{(by (EC.7))}\\
&\leq \exp\left(-\frac{\eta(T_0-1)}{10}\right) && \text{(by (EC.8))}\\
&\leq \exp\left(-\frac{1}{10}\right) < \frac{10}{11}. && (T_0 - 1 \geq \tfrac{1}{\eta})
\end{aligned}$$

This implies $y^t[1] < \frac{10}{21}$ for all $t \in [T_0, T_1]$. Moreover, for all $t \in [T_0, T_1]$, we have e_x^t satisfies

$$e_x^t = \frac{y^t[1] - \delta_y}{1-\delta_x} \leq \frac{\frac{10}{21} - \frac{1}{2}}{1-\delta_x} \leq -\frac{1}{42},$$

where we use $y^t[1] \le \frac{10}{21}$ and $\delta_y \ge \frac{1}{2}$.

For all $t \in [T_0, T_1 - 1]$, using the fact that $e_x^t \in [-1, -\frac{1}{42}]$, we have x^t satisfies

$$\begin{aligned}
\frac{x^t[1]}{x^t[2]} &= \frac{x^1[1]}{x^1[2]} \cdot \exp\left(-2\eta e_x^{t-1} - \sum_{k=1}^{T_0-1} \eta e_x^k - \sum_{k=T_0}^{t-2} \eta e_x^k\right) \\
&\ge \exp\left(\frac{\eta(t-T_0)}{42} - \eta T_0\right) && (e_x^t \le -\tfrac{1}{42} \text{ for } t \in [T_0, T_1] \text{ and } e_x^t \in [-1/2, 1/2+\delta] \text{ for all } t) \\
&\ge \exp\left(\frac{\eta t}{42} - \frac{1}{21} - 2\right) && (T_0 \le \tfrac{2}{\eta}) \\
&\ge \frac{1}{9} \cdot \exp\left(\frac{\eta t}{42}\right). && (e^{--\frac{1}{21}-2} \ge \tfrac{1}{9})
\end{aligned}$$

Now we track the duality gap. Note that for $t \in [T_0, T_1 - 1]$, we have $x^t[1] \le 1 - \delta_x$ and $y^t[1] \le \frac{10}{21} \le \delta_y$. Therefore,

$$\mathrm{DualityGap}(x^t, y^t) = \frac{\delta_y}{1-\delta_x}(1 - x^t[1]) - \frac{\delta_x}{1-\delta_x} y^t[1] \le \frac{\delta_y}{1-\delta_x}(1 - x^t[1]) \le x^t[2] \le \frac{x^t[2]}{x^t[1]}.$$

where we use $0 < \delta_y \le 1 - \delta_x$ in the second inequality. Then we get for all $t \in [T_0, T_1 - 1]$,

$$\mathrm{DualityGap}(x^t, y^t) \le \frac{x^t[2]}{x^t[1]} \le 9 \exp\left(-\frac{\eta t}{42}\right).$$

Since $T_0 < \frac{2}{\eta}$, the above bounds on duality gap also holds for all $t \in [1, T_0]$. Thus we conclude that for all $t \in [1, T_1 - 1]$, $\mathrm{DualityGap}(x^t, y^t) \le 9 \cdot \exp(-\frac{\eta t}{42})$.

Moreover, since $x^{T_1}[1] \ge 1 - \delta_x$ and $y^{T_1}[1] < \frac{10}{21} \le \delta_y$, we have

$$\begin{aligned}
\mathrm{DualityGap}(x^{T_1}, y^{T_1}) &= \max_{i \in \{1,2\}} (A^\top x^t)[i] - \min_{i \in \{1,2\}} (Ay^t)[i] \\
&= \frac{1-\delta_y}{1-\delta_x} x^{T_1}[1] - 1 + \frac{\delta_y}{1-\delta_x} \\
&\le \frac{1}{1-\delta_x} - 1 \le 2\delta_x.
\end{aligned}$$

To summarize, we have shown the following:

- For $t \in [1, T_1 - 1]$, we have linear convergence rate: $\mathrm{DualityGap}(x^t, y^t) \le 9\exp(-\frac{\eta t}{42})$;
- At T_1, we have $\mathrm{DualityGap}(x^{T_1}, y^{T_1}) \le 2\delta_x$.

For all $t \in [1, T_1 - 1]$, we know there exists a universal constant C' such that $9\exp(-\frac{\eta t}{42}) \le \frac{C'}{\eta t}$. Thus we have $\min_{k \in [1,t]} \mathrm{DualityGap}(x^k, y^k) \le \frac{C}{\eta}\frac{1}{t}$ for all $t \in [1, T_1 - 1]$. For $t = T_1$, we have

$$\min_{k \in [1,T_1]} \mathrm{DualityGap}(x^k, y^k) \le \min_{k \in [1,T_1-1]} \mathrm{DualityGap}(x^k, y^k) \le \frac{C'}{\eta(T_1 - 1)} \le \frac{2C'}{\eta T_1}.$$

Let $C = 2C'$, we have for all $t \in [1, T_1]$, it holds that $\min_{k \in [1,t]} \mathrm{DualityGap}(x^k, y^k) \le \frac{C}{\eta}\frac{1}{t}$. ☐

Case 2: $\delta_y < \frac{1}{2}$.

LEMMA EC.11. *Consider matrix A defined in (EC.6) that satisfies Assumption 3 and $\delta_y \in (0, \frac{1}{2})$. Let $\{x^t, y^t\}_{t\geq 1}$ be the iterates produced by OMWU dynamics with uniform initialization and step size $\eta \leq \frac{1}{10}$. Denote T_1 the first iteration that $x^t[1] \geq 1-\delta_x$. Then*

1. $\mathrm{DualityGap}(x^{T_1}, y^{T_1}) \leq 2\delta_x$.
2. *For all $t \in [1, T_1]$, there exists a universal constant $C > 0$ such that*

$$\min_{k\in[1,t]} \mathrm{DualityGap}(x^k, y^k) \leq \frac{C}{\eta}\frac{\log^2 t}{t}.$$

Proof The proof of Lemma EC.11 is more involved than that of Lemma EC.10. We track the trajectory of OMWU dynamics in two phases:

- **Phase I:** $x^t[1]$ decreases and $y^t[1]$ decreases. At the end of phase I, $y^t[1] \leq \delta_y$ and the duality gap decreases to $O(\delta_y)$.
- **Phase II:** $x^t[1]$ increases and $y^t[1]$ continues to decrease. At the end of phase II, $x^t[1] \geq 1-\delta_x$ and the duality gap further decreases to $O(\delta_x)$.

Phase I: $x^t[1]$ and $y^t[1]$ both decreases. Recall that initially, $x^t[1] = y^t[1] = \frac{1}{2}$. We note that $e_x^t = \frac{y^t[1]-\delta_y}{1-\delta_x}$, thus as long as $y^t[1] \geq \delta_y$, we have $e_x^t \geq 0$. This implies $x^t[1] \leq x^1[1] = \frac{1}{2}$ until the first iteration when $y^t[1] < \delta_y$, which we denote as T_y. Now we know for all iterations $t \in [1, T_y]$, $x^t[1] \leq \frac{1}{2}$, this further implies $e_y^t = 1 - \frac{x^t[1]}{1-\delta_x} \geq \frac{1}{3}$ since $0 < \delta_x \leq \frac{1}{100}$. As a result, we have

$$\frac{y^t[1]}{y^t[2]} = \exp\left(-\eta E_y^{t-1} - \eta e_y^{t-1}\right) \leq \exp\left(-\frac{\eta t}{3}\right), \forall t \in [1, T_y].$$

Moreover, we know at time T_y, $y^{T_y}[1] \leq \delta_y$ for the first time. Consider the duality gap of $\{(x^t, y^t)\}_{t\in[1,T_y-1]}$: since $x^t[1] \leq 1-\delta_x$ and $y^t[1] \geq \delta_y$, we have $\forall t \in [1, T_y - 1]$,

$$\begin{aligned}\mathrm{DualityGap}(x^t[1], y^t[1]) &= \max_{i\in\{1,2\}} (A^\top x^t)[i] - \min_{i\in\{1,2\}} (Ay^t)[i] = y^t[1] - \frac{\delta_y}{1-\delta_x} x[1] \\ &\leq y^t[1] \leq \frac{y^t[1]}{y^t[2]} \leq \exp\left(-\frac{\eta t}{3}\right).\end{aligned}$$

Thus we have a linear convergence for $t \in [1, T_y - 1]$. For $t = T_y$, since $x^t[1] \leq 1-\delta_x$ and $y^t[1] \leq \delta_y$, the duality gap at (x^{T_y}, y^{T_y}) is

$$\begin{aligned}\mathrm{DualityGap}(x^{T_y}[1], y^{T_y}[1]) &= \max_{i\in\{1,2\}} (A^\top x^t)[i] - \min_{i\in\{1,2\}} (Ay^t)[i] \\ &= \frac{\delta_y}{1-\delta_x}(1 - x^{T_y}[1]) - \frac{\delta_x}{1-\delta_x} y^{T_y}[1] \leq 2\delta_y.\end{aligned}$$

This completes the analysis of Phase I.

Phase II: $y^t[1]$ continues to decrease but $x^t[1]$ increases. Now that at time T_y, $y^{T_y} \le \delta_y$, e_x^t could be negative since then and $x^t[1]$ might increase. However, we know that $e_y^t = 1 - \frac{x^t[1]}{1-\delta_x} \ge 0$ holds as long as $x^t[1] \le 1-\delta_x$. Thus $y^t[1]$ would continue to decrease until the first iteration $x^t[1] > 1-\delta_x$, which we denote as T_x. Now recall that $x^{T_y} \le \frac{1}{2}$, we argue $x^t \le \frac{3}{4}$ for all $t \le T_m := T_y + \lceil \frac{2}{\eta} \rceil$:

$$\frac{x^t[1]}{x^t[2]} = \exp\left(-\eta E_x^{T_y} - \eta \sum_{j=T_y+1}^{t-1} e_x^j - \eta e_x^{t-1}\right) \le \exp\left(\eta(t-T_y)\right) \le e^2, \forall t \in [T_y+1, T_m]$$

where we use $e_x^t \ge 0$ for all $t \in [1, T_y]$ and $e_x^t \ge -1$ for all t as well as $t - T_y \le \frac{2}{\eta}$. This implies $x^t[1] \le \frac{e^2}{e^2+1} \le \frac{8}{9}$ for all $t \in [1, T_m]$. It further implies $e_y^t = 1 - \frac{x^t[1]}{1-\delta_x} \ge \frac{1}{10}$ since $\delta_x \le \frac{1}{100}$.

Using $T_m - T_y \ge \lceil \frac{2}{\eta} \rceil$ and bounds on e_y^t, we have

$$\begin{aligned}
\forall t \in [T_m, T_x - 1], \quad \frac{y^t[1]}{y^t[2]} &= \exp\left(\eta E_y^{T_y-1} - \eta \sum_{k=T_y}^{t-1} e_y^k - \eta e_y^{t-1}\right) \\
&= \exp\left(-\eta E_y^{T_y-1} - \eta e_y^{T_y-1} - \eta \sum_{k=T_y}^{t-1} e_y^k - \eta e_y^{t-1} + \eta e_y^{T_y-1}\right) \\
&\le \frac{\delta_y}{1-\delta_y} \cdot \exp\left(-\eta \sum_{k=T_y}^{t-1} e_y^k - \eta e_y^{t-1} + \eta e_y^{T_y-1}\right) \\
&\le \frac{\delta_y}{1-\delta_y} \cdot \exp\left(-\eta \sum_{k=T_y}^{T_m-1} e_y^k + \eta e_y^{T_y-1}\right) \\
&\le \frac{\delta_y}{1-\delta_y} \cdot \exp\left(-\frac{\eta(T_m - T_y)}{10} + \eta\right) \\
&\le \frac{\delta_y}{1-\delta_y} \cdot \exp\left(-\frac{1}{10}\right) \le \frac{10}{11} \cdot \frac{\delta_y}{1-\delta_y}.
\end{aligned}$$

where: in the first inequality, we use $\exp(-\eta E_y^{T_y-1} - \eta e_y^{T_y-1}) = \frac{y^{T_y}[1]}{y^{T_y}[2]} \le \frac{\delta_y}{1-\delta_y}$; in the second inequality we use $e_y^t \ge 0$ for all $t < T_x$; in the third inequality, we use $e_y^t \ge \frac{1}{10}$ for all $t \in [1, T_m]$. In the last inequality, we use $T_m - T_y \ge \frac{2}{\eta}$ and $\eta \le \frac{1}{10}$. This implies

$$y^t[1] = \frac{1}{1 + y^t[2]/y^t[1]} \le \frac{1}{1 + \frac{11(1-\delta_y)}{10\delta_y}} = \frac{10\delta_y}{11 - \delta_y} \le \frac{20}{21}\delta_y, \forall t \in [T_m, T_x - 1]$$

where we use $\delta_y < \frac{1}{2}$ in the last inequality. Then $e_x^t = \frac{y^t[1] - \delta_y}{1-\delta_x} \le -\frac{\delta_y}{21}$ for all $t \in [T_m, T_x - 1]$.

Given that $e_x^t \le -\frac{\delta_y}{21}$ for all $t \in [T_m, T_x - 1]$, we know $x^t[1]$ increases during these iterations. In order to quantify the speed, we first provide some bounds on $T_m = T_y + \lceil \frac{2}{\eta} \rceil$. In the analysis in Phase

I, we know (1) $\frac{y^t[1]}{y^t[2]} \le \exp(-\frac{\eta t}{3}), \forall t \in [1, T_y]$ and (2) $y^t[1] \ge \delta_y$ for all $t \in [1, T_y - 1]$. Combining the two inequalities gives

$$\frac{\delta_y}{1-\delta_y} \le \exp\left(-\frac{\eta(T_y - 1)}{3}\right) \Rightarrow T_y \le \frac{3}{\eta}\log\frac{1}{\delta_y} + 1.$$

This implies $T_m - 1 = T_y - 1 + \lceil \frac{2}{\eta} \rceil \le \frac{4}{\eta}\log\frac{1}{\delta_y}$

Now we analyze x^t for $t \in [T_m, T_x]$:

$$\begin{aligned}
\frac{x^t[1]}{x^t[2]} = \exp\left(-\eta E_x^{t-1} - \eta e_x^{t-1}\right) &= \exp\left(-\eta E_x^{T_m-1} - \eta \sum_{k=T_m}^{t-1} \eta e_x^k - \eta e_x^{t-1}\right) \\
&\ge \exp\left(-\eta(T_m - 1) + \frac{\eta\delta_y(t - T_m + 1)}{21}\right) \\
&\ge \exp\left(-2\eta(T_m - 1) + \frac{\eta\delta_y t}{21}\right) \\
&\ge \delta_y^8 \cdot \exp\left(\frac{\eta\delta_y t}{21}\right).
\end{aligned}$$

where: in the first inequality we use $e_x^t \le 1$ for all t and $e_x^t \le -\frac{\delta_x}{21}$ for $t \in [T_m, T_x - 1]$; in the second inequality we use $\delta_y \le 1$; in the last inequality, we use $T_m - 1 \le \frac{4}{\eta}\log\frac{1}{\delta_y}$.

Now we analyze the duality gap. For $t \in [T_m, T_x - 1]$ we have

$$\text{DualityGap}(x^t, y^t) \le \frac{x^t[2]}{x^t[1]} \le \frac{1}{\delta_y^8} \cdot \exp\left(-\frac{\eta\delta_y t}{21}\right)$$

where the first inequality follows the same lines as for Phase I. For $t = T_x$, since $x^t[1] \ge 1 - \delta_x$ and $y^t[1] \le \delta_y$, we have

$$\begin{aligned}
\text{DualityGap}(x^{T_x}, y^{T_x}) &= \max_{i\in\{1,2\}} (A^\top x^t)[i] - \min_{i\in\{1,2\}} (Ay^t)[i] \\
&= \frac{1-\delta_y}{1-\delta_x} x^{T_x}[1] - 1 + \frac{\delta_y}{1-\delta_x} \le \frac{1}{1-\delta_x} - 1 \le 2\delta_x.
\end{aligned}$$

Combining Bounds. We combine the analysis in Phase I and II here. We claim the iterates $\{x^t, y^t\}_{t\in[1,T_x]}$ has the following convergence rates.

CLAIM EC.2. *There exists a universal constant C such that for any $t \in [1, T_x]$, the following best-iterate convergence holds:* $\min_{k\in[1,t]} \text{DualityGap}(x^k, y^k) \le \frac{C}{\eta} \cdot \frac{\log^2 t}{t}$.

Proof We prove the claim by considering the following cases.

1. $t \in [1, T_y]$. In this case, by analysis in Phase I, we have $\text{DualityGap}(x^t, y^t) \le \exp(-\frac{\eta t}{3})$. Then there exists a universal constant $C_1 > 0$ such that $\text{DualityGap}(x^t, y^t) \le \exp(-\frac{\eta t}{3}) \le \frac{C_1}{\eta t}$.

2. $t \in [T_y, T_m]$. We note that by analysis in Phase I, $\text{DualityGap}(x^{T_y}, y^{T_y}) \le 2\delta_y$; by analysis in Phase II, we have $T_m \le \frac{4}{\eta} \log \frac{1}{\delta_y}$. Thus for $t \in [T_y, T_m]$, we have

$$\min_{k\in[1,t]} \text{DualityGap}(x^k, y^k) \le \text{DualityGap}(x^{T_y}, y^{T_y}) \le 2\delta_y = \frac{8}{\eta}\frac{1}{\frac{4}{\eta}\frac{1}{\delta_y}} \le \frac{8}{\eta} \cdot \frac{1}{\frac{4}{\eta}\log\frac{1}{\delta_y}} \le \frac{8}{\eta}\frac{1}{T_m} \le \frac{8}{\eta} \cdot \frac{1}{t}.$$

where the third inequality uses $a \ge \log a, \forall a > 1$, the fourth inequality uses $T_m \le \frac{4}{\eta} \log \frac{1}{\delta_y}$, and the last inequality uses $t \le T_m$.

3. $t \in [T_m, T_x - 1]$. By analysis in Phase II and duality bound of (x^{T_y}, y^{T_y}), we have for any $t \in [T_m, T_x - 1]$, there exists a universal constant $C_2 = 8 \times 21$ such that

$$\min_{k\in[1,t]} \text{DualityGap}(x^k, y^k) \le \min\left\{2\delta_y, \frac{1}{\delta_y^8}\exp\left(-\frac{\eta\delta_y t}{21}\right)\right\} \le \frac{C_2}{\eta} \cdot \frac{\log^2 t}{t}.$$

The last inequality holds since (1) if $\delta_y \le \frac{C_2 \log^2 t}{\eta t}$, then the inequality holds; (2) if $\delta_y \ge \frac{C_2}{\eta}\frac{\log^2 t}{t}$, then we have

$$\begin{aligned}
&\delta_y \ge \frac{C_2}{\eta}\frac{\log^2 t}{t} \\
\Rightarrow &\frac{C_2}{\eta}\frac{\log\frac{1}{\delta_y}}{\delta_y} \le \frac{t}{\log^2 t}\log\frac{t}{\frac{C_2}{\eta}\log t} \le \frac{t}{\log t} && \left(\tfrac{C_2}{\eta} \ge 1\right) \\
\Leftrightarrow &8\log t + 8\log\frac{1}{\delta_y} \le \frac{\eta\delta_y t}{21} && (C_2 = 8\times 21) \\
\Rightarrow &\log t \le 8\log\delta_y + \frac{\eta\delta_y t}{21} \\
\Leftrightarrow &t \le \delta_y^8 \exp\left(\frac{\eta\delta_y t}{21}\right) \Leftrightarrow \frac{1}{\delta_y^8}\exp\left(-\frac{\eta\delta_y t}{21}\right) \le \frac{1}{t}.
\end{aligned}$$

4. $t = T_x$. We know $\text{DualityGap}(x^{T_x}, y^{T_x}) \le 2\delta_x$. Let $C' = \max\{C_1, C_2\}$. We can use the above bounds and conclude that

$$\min_{k\in[1,T_x]} \text{DualityGap}(x^k, y^k) \le \text{DualityGap}(x^{T_x-1}, y^{T_x-1}) \le \frac{C'}{\eta}\frac{\log^2(T_x-1)}{T_x-1} \le \frac{2C'}{\eta}\frac{\log^2(T_x)}{T_x}$$

Thus the claim holds with $C = 2C' = 2\max\{C_1, C_2\}$. □

Let $T_1 = T_x$, Lemma EC.11 then follows from Claim EC.2 and the fact that $\text{DualityGap}(x^{T_x}, y^{T_x}) \le 2\delta_x$. □

Combining Lemma EC.10 and Lemma EC.11 gives Theorem 9.

F.4. Combining Two-Phase Analysis: Proof of Theorem 7

Consider any matrix game $A \in [0,1]^{2\times 2}$ that has a fully-mixed Nash equilibrium with minimum probability $\delta > 0$. First, by Theorem 9, we know there exists an initial phase $[1, T_1]$ such that $O(\frac{\log^2 T}{T})$ best-iterate convergence rate holds and $\mathrm{DualityGap}(x^{T_1}, y^{T_1}) \le 2\delta$.

Thus we only need to consider iterations $t \ge T_1 + 1$. By Theorem 8, we know for all iteration t, the best-iterate convergence rate of $O(T^{-\frac{1}{4}}\delta^{-\frac{1}{2}})$ holds (note that $d_1 = d_2 = 2$ are constants). Then we have for all $T \ge T_1 + 1$ $\min_{t\in[1,T]} \mathrm{DualityGap}(x^t, y^t) \le \min\{2\delta, O(T^{-\frac{1}{4}}\delta^{-\frac{1}{2}})\} \le O(T^{-\frac{1}{6}})$. This completes the proof.

Appendix G: Numerical Experiments with Adaptive Stepsizes

In this section, we present our numerical results when OFTRL and OOMD are instantiated with the adaptive stepsizes of Hsieh et al. (2019) (given here for the x-player, the case of the y-player is symmetric):

$$\eta_t^x = 1/\sqrt{\epsilon + \sum_{k=1}^{t-1} \|\ell_x^k - \ell_x^{k-1}\|_2^2} \tag{EC.9}$$

for some constant $\epsilon > 0$. From Section 4.2 of Hsieh et al. (2021), we know that OFTRL with this choice of step sizes has last-iterate convergence and that $\{\eta_t^x\}$ and $\{\eta_t^y\}$ converge to strictly positive constants. We present our numerical experiments in Figure EC.1, where we choose $\epsilon = 0.1$. We still observe empirically the pathological behavior for the uniform last-iterate convergence rates of OFTRL with constant step sizes (Figure 1).

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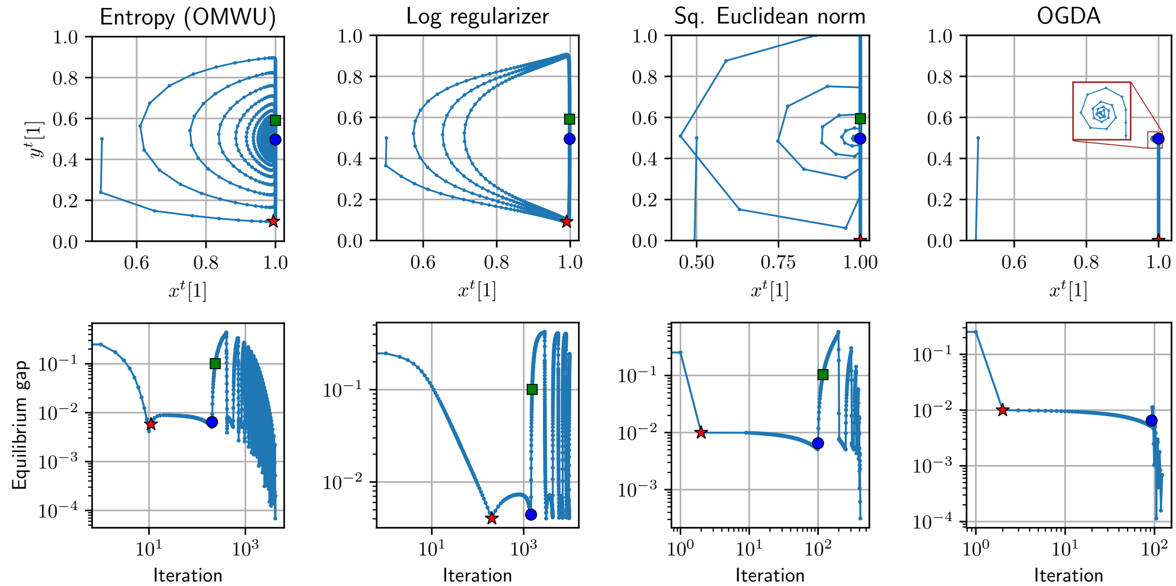

Figure EC.1 Dynamics (EC.2) with step sizes (EC.9), $\epsilon = 0.1$, $\delta = 0.01$.