Electronic Companion to
"Finite-Time Minimax Bounds and an Optimal Lyapunov Policy in Queueing Control"

Appendix

A. Summary of Main Notation

A table summarizing our notation is provided in Table EC.1.

Table EC.1 Summary of Main Notation

Symbol	Meaning
$Q(t)$	Queue length vector at time t
$A(t)$	Arrival vector at time t
$\lambda(t) = \mathbb{E}[A(t)]$	Arrival rate vector at time t
$\mathcal{D}_t$	Scheduling set at time t
$d(t) \in \mathcal{D}_t$	Schedule selected at time t
$S_d(t)$	Random departure vector associated with schedule d at time t
$D(t) = S_{d(t)}(t)$	Realized departure vector at time t
$\mathcal{F}_t$	History available before choosing $d(t)$
$\Phi = \{\phi_t\}_{t \in \mathbb{N}_0}$	Scheduling policy
ρ	Traffic intensity parameter
B	Capacity of the scheduling set
C_{a}, C_{d}	Arrival and departure variance parameters
$\mathcal{M}^\rho(B, C_{\mathrm{a}}, C_{\mathrm{d}})$	Model class
$\Gamma^\rho(B, C_{\mathrm{a}}, C_{\mathrm{d}}, T)$	Minimax expected total queue length at time T

B. Quadratic Objective for Queue Lengths

We now show that all the results established for the total queue length objective can be extended to the quadratic objective

$$\inf_{\Phi} \sup_{(\mathcal{D},A,S)\in\mathcal{M}^{\rho}(B,C_{\mathrm{a}},C_{\mathrm{d}})} \mathbb{E}^{\Phi}_{(A,S)}\left[\sum_{i=1}^{n} Q_i(T)^2\right].$$

For the lower bound, applying the Cauchy–Schwarz and Jensen's inequalities yields

$$\mathbb{E}^{\Phi}_{(A,S)}\left[\sum_{i=1}^{n} Q_i(T)^2\right] \geq \frac{1}{n}\mathbb{E}^{\Phi}_{(A,S)}\left[\sum_{i=1}^{n} Q_i(T)\right]^2 \geq \frac{1}{n}\left(\mathbb{E}^{\Phi}_{(A,S)}\left[\sum_{i=1}^{n} Q_i(T)\right]\right)^2.$$

Thus, the corresponding lower bound follows directly from the results for the total queue-length objective. For the upper bound, both the $\mathrm{LyapOpt}$ and $\mathrm{MaxWeight}$ policies are analyzed using a uniform Lyapunov method with the quadratic Lyapunov function $V(x) = \|x\|_2^2$. Hence, the corresponding results are directly obtained for $\sum_{i=1}^{n} Q_i(T)^2$.

Consequently, even under the quadratic objective, the $\mathrm{LyapOpt}$ policy achieves the finite-time lower bound up to a constant factor, provided that the conditions of Theorem 2 are satisfied, thereby confirming that the optimality of $\mathrm{LyapOpt}$ extends beyond the total queue-length objective.

C. On the Extension of the Pointwise Condition to Window-Based Conditions

The present results in Sections 3 and 4 are proved under the *pointwise constraint* $\lambda(t) \in \rho\Pi(\mathcal{D}_t)$ for every t. Extending the analysis to the weaker *window-based condition* used in prior work (e.g., Yang et al. (2023)) is not straightforward. For a window-length parameter $\tau > 1$, the window-based condition requires that for each t there exists an integer $\omega(t)$ with $1 \leq \omega(t) \leq \tau$ such that

$$\sum_{s=t}^{t+\omega(t)-1} \lambda(s) \in \rho \sum_{s=t}^{t+\omega(t)-1} \Pi(\mathcal{D}_s).$$

Under this alternative assumption the lower bounds remain valid, but the analysis to obtain an upper bound becomes problematic.

The proof of the upper bound relies on the Lyapunov drift argument in (11), whose first-order term explicitly depends on the current queue length $Q(t)$. Summing this drift over a window $\{t, t+1, \ldots, t+\omega(t)-1\}$ gives rise to terms of the form

$$\sum_{s=t}^{t+\omega(t)-1} \sum_{i=1}^{n} Q_i(s)\big(\lambda_i(s) - d_i(s)\big). \tag{EC.1}$$

Unlike several existing papers (Yang et al. 2023, Liang and Modiano 2018), we do not assume bounded arrivals or departures, and hence, we do not have a uniform bound on $Q(s)$. Moreover, even if such boundedness assumptions were imposed, the resulting bound would likely be too coarse to recover the sharp finite-time guarantees established in (14) and (16). As a result, because each summand depends on $Q(s)$, the term in (EC.1) cannot be directly controlled under the window-based condition. Hence, our current proof technique does not readily generalize to this weaker setting. Existing works under such window-based conditions typically establish stability results or derive comparatively coarse finite-time guarantees, rather than the sharper finite-time upper bounds obtained here.

D. Lower Bound Inequalities

This section presents several auxiliary results supporting the analysis in Section 3. Among them, Lemma EC.3 is used directly in the main proof, while the first two lemmas are utilized in its derivation.

We begin with a lower bound for the expected maximum of a drifted Brownian motion.

LEMMA EC.1 **(Lower Bound for the Maximum of a Drifted Brownian Motion)**. *Let* $Y_t = \sigma W_t - \theta t$ *with* $\theta > 0$, $\sigma > 0$, *and* W *a standard Brownian motion. For* $t > 0$, *define* $M_t := \sup_{0 \le s \le t} Y_s$. *Then*

$$\mathbb{E}[M_t] \ge \underline{c} \min\left\{\sigma\sqrt{t}, \sigma^2/\theta\right\}, \qquad \underline{c} = f_{\mathcal{N}}(1) + 2F_{\mathcal{N}}(1) - \tfrac{3}{2} \approx 0.425, \tag{EC.2}$$

where $f_{\mathcal{N}}$ *and* $F_{\mathcal{N}}$ *denote the standard normal density and cumulative distribution functions, respectively.*

Proof The distribution of the maximum of a Brownian motion with drift (see, e.g., Section 1.9 of Harrison (2013)) is given by

$$\mathbb{P}(M_t \le x) = F_{\mathcal{N}}\Big(\frac{x + \theta t}{\sigma\sqrt{t}}\Big) - e^{-\frac{2\theta x}{\sigma^2}} F_{\mathcal{N}}\Big(\frac{-x + \theta t}{\sigma\sqrt{t}}\Big), \quad x \ge 0.$$

Integrating the tail probability yields, for $r = \theta\sqrt{t}/\sigma$,

$$\mathbb{E}[M_t] = \sigma\sqrt{t}A(r) + \frac{\sigma^2}{\theta}B(r), \text{ where } A(r) := f_{\mathcal{N}}(r) - r(1 - F_{\mathcal{N}}(r)), \; B(r) := F_{\mathcal{N}}(r) - 1/2.$$

Define the functions

$$g(r) := \frac{\mathbb{E}[M_t]}{\sigma\sqrt{t}} = A(r) + \frac{B(r)}{r}, \qquad h(r) := \frac{\mathbb{E}[M_t]}{\sigma^2/\theta} = r\,A(r) + B(r), \qquad r > 0.$$

We will show that $g(r)$ is decreasing on the interval $(0,1]$ and $h(r)$ is increasing on the interval $[1,\infty)$.

Since $f_{\mathcal{N}}$ decreases on $[0,\infty)$,

$$B(r) = \int_0^r f_{\mathcal{N}}(u)\mathrm{d}u \geq r f_{\mathcal{N}}(r), \qquad r > 0. \tag{EC.3}$$

Differentiating $g(r)$ gives

$$g'(r) = A'(r) + \frac{B'(r)r - B(r)}{r^2} = -(1 - F_{\mathcal{N}}(r)) + \frac{f_{\mathcal{N}}(r)r - B(r)}{r^2},$$

where the last equality follows from the identity $f'_{\mathcal{N}}(r) = -r f_{\mathcal{N}}(r)$. By (EC.3), $f_{\mathcal{N}}(r)r - B(r) \leq 0$. Thus $g(r)$ is decreasing on $(0,1]$, and $\inf_{0<r\leq 1} g(r) = g(1)$.

Let the Mills ratio be $m(r) := \frac{1-F_{\mathcal{N}}(r)}{f_{\mathcal{N}}(r)}$ for $r > 0$. Then

$$h'(r) = A(r) + rA'(r) + B'(r) = 2f_{\mathcal{N}}(r)(1 - rm(r)).$$

By the Mills inequality $rm(r) \leq 1$, we have $h'(r) \geq 0$ for all $r > 0$. Hence $h(r)$ is increasing on $[1,\infty)$, and $\inf_{r\geq 1} h(r) = h(1)$. A direct computation gives

$$g(1) = h(1) = A(1) + B(1) = f_{\mathcal{N}}(1) + 2F_{\mathcal{N}}(1) - \tfrac{3}{2}.$$

Set

$$\underline{c} := g(1) = h(1) = f_{\mathcal{N}}(1) + 2F_{\mathcal{N}}(1) - \tfrac{3}{2} \approx 0.425.$$

For $r \leq 1$, i.e., $t \leq (\sigma/\theta)^2$,

$$\mathbb{E}[M_t] \geq \sigma\sqrt{t}g(r) \geq \underline{c}\sigma\sqrt{t}.$$

For $r \geq 1$, i.e., $t \geq (\sigma/\theta)^2$,

$$\mathbb{E}[M_t] \geq (\sigma^2/\theta)h(r) \geq \underline{c}\sigma^2/\theta.$$

Combining the two regimes yields (EC.2).

The following lemma follows from the Komlós–Major–Tusnády (KMT) coupling theorem, which couples partial sums of i.i.d. random variables with a Brownian motion. For a random variable Z, define

$$\bar{\delta} := \sup\left\{\delta \geq 0 : \delta\mathbb{E}\left[|Z|^3 e^{\delta|Z|}\right] \leq \mathrm{Var}(Z)\right\}. \tag{EC.4}$$

If $\bar{\delta} > 0$, we say that Z admits a Sakhanenko parameter $\bar{\delta}$.

LEMMA EC.2 (**KMT Coupling**). *Let $\{Z(k)\}_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with a non-normal distribution satisfying*

$$\mathbb{E}[Z(1)]=0, \quad \mathbb{E}[Z(1)^2]=\sigma^2>0,$$

and suppose that $Z(1)$ admits a Sakhanenko parameter $\bar{\delta}>0$. Then there exist i.i.d. random variables $\{\widetilde{Z}(k)\}_{k\in\mathbb{N}}$ each with the same distribution as $Z(1)$, and a standard Brownian motion $W=\{W_t\}_{t\geq 0}$ such that the corresponding partial sum $\widetilde{S}(k)=\sum_{i=1}^k \widetilde{Z}(i)$ satisfy,

$$\mathbb{E}\left[\max_{0\leq k\leq t}\left|\widetilde{S}(k)-\sigma W_k\right|\right]\leq \bar{c}\log(1+t), \tag{EC.5}$$

where $\bar{c}$ is a positive constant depending only on σ and $\bar{\delta}$.

Moreover, if for each $\alpha\in\mathcal{A}$, the sequence $Z^\alpha=\{Z^\alpha(k)\}_{k\in\mathbb{N}}$ satisfies the above conditions with uniformly bounded variance and a uniformly positive lower bound on the Sakhanenko parameter across α, then the same constant $\bar{c}$ can be chosen uniformly over α.

REMARK EC.1. Scaling of the KMT constant The constant $\bar{c}$ can be taken to scale on the order

$$\bar{c}=O\left(\frac{1}{\bar{\delta}}+\sigma\right).$$

Proof of Lemma EC.2 By Theorem 2.4 in Waudby-Smith et al. (2025), there exist random variables $\{\widetilde{Z}(k)\}_{k\in\mathbb{N}}$ and i.i.d. standard normal random variables $\{V(k)\}_{k\in\mathbb{N}}$ such that, for any $z>0$ and $0<\delta<\bar{\delta}$,

$$\mathbb{P}\left(\sup_{k\geq 4}\frac{|\widetilde{S}(k)-\sum_{i=1}^k\sigma V(k)|}{\log k}\geq 4z\right)\leq 2\sum_{n\geq 1}(1+\delta\sigma 2^{2^n-1})\exp\{-c\delta z 2^n\log 2\}, \tag{EC.6}$$

where $c>0$ is an absolute constant. Define

$$\Lambda:=\sup_{k\geq 4}\frac{|\widetilde{S}(k)-\sum_{i=1}^k\sigma V(k)|}{\log k}.$$

Let $z_0=\frac{2}{c\delta}$. Then

$$\mathbb{E}[\Lambda]=4\int_0^\infty \mathbb{P}(\Lambda\geq 4z)\,\mathrm{d}z\leq 4z_0+4\int_{z_0}^\infty \mathbb{P}(\Lambda\geq 4z)\,\mathrm{d}z.$$

By (EC.6) and an application of Tonelli's theorem,

$$\begin{aligned}4\int_{z_0}^\infty \mathbb{P}(\Lambda\geq 4z)\,\mathrm{d}z &\leq 8\sum_{n\geq 1}\int_{z_0}^\infty (1+\delta\sigma 2^{2^n-1})e^{-c\delta z 2^n\log 2}\,\mathrm{d}z\\ &=\frac{8}{c\delta\log 2}\sum_{n\geq 1}\frac{1}{2^n}(1+\delta\sigma 2^{2^n-1})\,e^{-c\delta z_0 2^n\log 2}.\end{aligned}$$

With $z_0 = \frac{2}{c\delta}$, we have $e^{-c\delta z_0 2^n \log 2} = e^{-2\cdot 2^n \log 2} = 2^{-2^{n+1}}$. Hence,

$$4\int_{z_0}^{\infty} \mathbb{P}(\Lambda \geq 4z)\mathrm{d}z \leq \frac{8}{c\delta \log 2}\sum_{n\geq 1}\left(\frac{2^{-2^{n+1}}}{2^n} + \frac{\delta\sigma}{2}\cdot\frac{2^{2^n}2^{-2^{n+1}}}{2^n}\right)$$
$$= \frac{8}{c\delta\log 2}\sum_{n\geq 1}\frac{2^{-2^{n+1}}}{2^n} + \frac{4\sigma}{c\log 2}\sum_{n\geq 1}\frac{2^{-2^n}}{2^n}. \tag{EC.7}$$

Both series converge due to the super-exponential decay of 2^{-2^n}. Therefore, there exists a constant $\widetilde{c} > 0$ (depending only on σ and δ) such that

$$\mathbb{E}[\Lambda] \leq \widetilde{c}.$$

Since $\log k \leq \log t$ for all $4 \leq k \leq t$,

$$\sup_{4\leq k\leq t}\left|\widetilde{S}(k) - \sum_{i=1}^{k}\sigma V(k)\right| \leq \Lambda \log t.$$

Taking expectations, we obtain

$$\mathbb{E}\left[\sup_{4\leq k\leq t}\left|\widetilde{S}(k) - \sum_{i=1}^{k}\sigma V(k)\right|\right] \leq \widetilde{c}\log t.$$

We now control the first three terms. Note that

$$\mathbb{E}\left[\max_{0\leq k\leq 3}\left|\widetilde{S}(k) - \sum_{i=1}^{k}\sigma V(k)\right|\right] \leq \mathbb{E}\left[\max_{0\leq k\leq 3}|\widetilde{S}(k)|\right] + \mathbb{E}\left[\max_{0\leq k\leq 3}\left|\sum_{i=1}^{k}\sigma V(i)\right|\right].$$

The partial sums $\{\widetilde{S}_k\}_{k\in\mathbb{N}}$ and $\{\sum_{i=1}^{k} V(i)\}_{k\in\mathbb{N}}$ form square-integrable martingales. Applying Jensen's inequality and Doob's maximal inequality yields

$$\mathbb{E}\left[\max_{0\leq k\leq 3}|\widetilde{S}(k)|\right] \leq \sqrt{\mathbb{E}\left[\max_{0\leq k\leq 3}\widetilde{S}(k)^2\right]} \leq \sqrt{4\mathbb{E}[\widetilde{S}(3)^2]} = 2\sqrt{3}\sigma,$$
$$\mathbb{E}\left[\max_{0\leq k\leq 3}\left|\sum_{i=1}^{k}V(i)\right|\right] \leq \sqrt{\mathbb{E}\left[\max_{0\leq k\leq 3}\left(\sum_{i=1}^{k}V(i)\right)^2\right]} \leq \sqrt{4\mathbb{E}\left[\sum_{i=1}^{3}V(i)^2\right]} = 2\sqrt{3}.$$

Summing the two parts gives

$$\mathbb{E}\left[\max_{0\leq k\leq 3}\left|\widetilde{S}_k - \sum_{i=1}^{k}\sigma V(i)\right|\right] \leq 4\sqrt{3}\sigma.$$

Thus, there exists a constant $\bar{c}$ depending only on σ and $\bar{\delta}$ such that (EC.5) holds, using the fact that the sequence $\{V(k)\}_{k\in\mathbb{N}}$ can be embedded into a standard Brownian motion $W = \{W_t\}_{t\geq 0}$. The second part follows directly from the choice of $\bar{c}$.

Finally, combining Lemmas EC.1 and EC.2, we derive the following lower bound for the maximum of a drifted random walk. This is the key result used in the main proof of Section 3.

LEMMA EC.3 **(Lower Bound for the Maximum of a Drifted Random Walk).** *Let $\{Z(k)\}_{k\in\mathbb{N}}$ satisfy the same conditions as in Lemma EC.2. Define the partial sum $S(t)=\left(\sum_{k=1}^{t} Z(k)\right)-\theta t$ with $\theta \geq 0$. Then, for all $t \geq \psi^2$ and for all θ satisfying $\theta \leq \sigma/\psi$, the expected maximum satisfies*

$$\mathbb{E}\left[\max_{0\leq k\leq t} S(k)\right] \geq \frac{\underline{c}}{2}\min\left\{\sigma\sqrt{t}, \frac{\sigma^2}{\theta}\right\}, \tag{EC.8}$$

where ψ is the unique positive solution to $\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+x)+\frac{8}{\underline{c}}=x$; $\underline{c}$ and $\bar{c}$ are the constants defined in Lemmas EC.1 and EC.2 respectively.

REMARK EC.2. The equation $\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+x)+\frac{8}{\underline{c}}=x$ admits a unique positive root.

Proof of Lemma EC.3 Let $X_s := \sigma W_s - \theta s$ and $M_s := \sup_{0\leq u\leq s} X_u$ for $s>0$, where W denotes a standard Brownian motion. In the following, the index k ranges over integers, while s and u vary over real numbers. For an integer $t \geq 1$, note that

$$\max_{0\leq k\leq t} S(k) \geq \max_{0\leq k\leq t} X_k - \max_{0\leq k\leq t}\left|S(k)-X_k\right| = \max_{0\leq k\leq t} X_k - \max_{0\leq k\leq t}\left|\sum_{i=1}^{k} Z(i)-\sigma W_k\right|.$$

Taking expectations and using Lemma EC.2 yields

$$\mathbb{E}\left[\max_{0\leq k\leq t} S(k)\right] \geq \mathbb{E}\left[\max_{0\leq k\leq t} X_k\right] - \bar{c}\log(1+t).$$

Moreover, for all $t\geq 1$,

$$\mathbb{E}\left[\max_{0\leq k\leq t} X_k\right] \geq \mathbb{E}\left[\max_{0\leq s\leq t} X_s\right] - \sigma\left(\sqrt{2\log(4t)}+\frac{1}{\sqrt{2\log(4t)}}\right), \tag{EC.9}$$

whose proof will be completed later. Since $\sqrt{2y}\leq y+1$ for all $y\geq 0$, we obtain

$$\mathbb{E}\left[\max_{0\leq k\leq t} S(k)\right] \geq \mathbb{E}[M_t] - ((\bar{c}+\sigma)\log(1+t)+4\sigma).$$

We first consider $\theta>0$. Let $u := t\wedge(\sigma/\theta)^2$. Then

$$\mathbb{E}\left[\max_{0\leq k\leq t} S(k)\right] \geq \mathbb{E}\left[\max_{0\leq k\leq u} S(k)\right] \geq \mathbb{E}[M_u] - ((\bar{c}+\sigma)\log(1+u)+4\sigma), \quad u=t\wedge(\sigma/\theta)^2. \tag{EC.10}$$

By Lemma EC.1, for every $u>0$,

$$\mathbb{E}[M_u] \geq \underline{c}\,\min\left\{\sigma\sqrt{u}, \frac{\sigma^2}{\theta}\right\} = \underline{c}\min\left\{\sigma\sqrt{t}, \frac{\sigma^2}{\theta}\right\}. \tag{EC.11}$$

Combining (EC.10) and (EC.11) gives the additive-error inequality

$$\mathbb{E}\left[\max_{0\le k\le t} S(k)\right] \ge \underline{c}\min\left\{\sigma\sqrt{t}, \frac{\sigma^2}{\theta}\right\} - \left((\bar{c}+\sigma)\log\left(1+t\wedge\left(\frac{\sigma}{\theta}\right)^2\right)+4\sigma\right).$$

To obtain (EC.8), it suffices to ensure

$$(\bar{c}/\sigma+1)\log\left(1+t\wedge(\frac{\sigma}{\theta})^2\right)+4\le\frac{\underline{c}}{2}\min\left\{\sqrt{t}, \frac{\sigma}{\theta}\right\}.$$

By the definition of ψ, for $t\ge\psi^2$, we have

$$\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+\sqrt{t})+\frac{8}{\underline{c}}\le\sqrt{t}. \tag{EC.12}$$

Hence,

$$(\bar{c}/\sigma+1)\log\left(1+t\wedge(\tfrac{\sigma}{\theta})^2\right)+4\le(\bar{c}/\sigma+1)\log(1+t)+4\le2(\bar{c}/\sigma+1)\log(1+\sqrt{t})+4\le\frac{\underline{c}}{2}\sqrt{t}.$$

where the second inequality follows from $(1+\sqrt{t})^2\ge1+t$.

Let $r:=\sigma/\theta$. Under the assumption on θ, we have $r\ge\psi$, and thus

$$\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+r)+\frac{8}{\underline{c}}\le r.$$

Then

$$(\bar{c}/\sigma+1)\log\left(1+t\wedge(\tfrac{\sigma}{\theta})^2\right)+4\le(\bar{c}/\sigma+1)\log(1+r^2)+4\le2(\bar{c}/\sigma+1)\log(1+r)+4\le\frac{\underline{c}}{2}r.$$

Combining the two cases completes the proof for case $\theta>0$. When $\theta=0$,

$$\mathbb{E}[M_t]=\sigma\sqrt{2t/\pi}\ge\sigma\underline{c}\sqrt{t},$$

and, by applying (EC.12), the proof proceeds in the same manner.

Now we prove (EC.9). Let $\widetilde{X}_s:=X_k$ and $\widetilde{W}_s:=W_k$ for $k\le s<k+1$. Note that

$$\max_{0\le k\le t}\widetilde{X}_k=\max_{0\le s\le t}\widetilde{X}_s\ge\max_{0\le s\le t}X_s-\max_{0\le s\le t}\left|X_s-\widetilde{X}_s\right|.$$

It therefore suffices to show that

$$\max_{0\le s\le t}\left|W_s-\widetilde{W}_s\right|\le\sqrt{2\log(4t)}+\frac{1}{\sqrt{2\log(4t)}}.$$

On each interval $[k,k+1)$, write $s=k+u$ with $u\in[0,1)$. Let $t\geq 1$ be an integer. By the stationary and independent increments of Brownian motion,

$$\sup_{0\leq s\leq t}|\widetilde{W}_s-W_s|=\max_{0\leq k\leq t-1}\sup_{0\leq u\leq 1}|W_{k+u}-W_k|\stackrel{d}{=}\max_{1\leq j\leq t}Y_j,$$

where $Y_1,\ldots,Y_t$ are i.i.d. copies of $Y:=\sup_{0\leq u\leq 1}|W_u|$. By the reflection principle and the standard Gaussian tail bound $1-F_{\mathcal{N}}(x)\leq e^{-x^2/2}$ which holds for all $x\geq 0$, we have

$$\mathbb{P}(Y>x)=\mathbb{P}\left(\sup_{0\leq u\leq 1}|W_u|>x\right)\leq 2\mathbb{P}\left(\sup_{0\leq u\leq 1}W_u>x\right)=4(1-F_{\mathcal{N}}(x))\leq 4e^{-x^2/2},\quad x\geq 0.$$

Let $\widetilde{Y}_t:=\max_{1\leq j\leq t}Y_j$. For any $a>0$, we decompose

$$\mathbb{E}[\widetilde{Y}_t]=\int_0^\infty\mathbb{P}(\widetilde{Y}_t\geq x)\mathrm{d}x\leq a+\int_a^\infty\mathbb{P}(\widetilde{Y}_t\geq x)\mathrm{d}x.$$

Since

$$\mathbb{P}(\widetilde{Y}_t\geq x)\leq\sum_{k=1}^t\mathbb{P}(Y_k\geq x)=t\mathbb{P}(Y\geq x)\leq 4te^{-x^2/2},$$

we have

$$\mathbb{E}[\widetilde{Y}_t]\leq a+4t\int_a^\infty e^{-x^2/2}\mathrm{d}x\leq a+\frac{4t}{a}e^{-a^2/2},$$

where the last inequality follows from integration by parts. Choosing $a:=\sqrt{2\log(4t)}$ yields

$$\mathbb{E}[\widetilde{Y}_t]\leq\sqrt{2\log(4t)}+\frac{1}{\sqrt{2\log(4t)}}.$$

Therefore, we complete the proof.

E. Proofs of Theorems and Supporting Results for Section 3

E.1. Auxiliary Lemma

LEMMA EC.4 **(Lower Bound on Total Queue Length)**. *Given the arrival process $\{A(t)\}_{t\in\mathbb{N}}$ and the departure process $\{D(t)\}_{t\in\mathbb{N}_0}$, the total queue length at time $T\geq 1$ satisfies*

$$\sum_{i=1}^n Q_i(T)\geq\max\Bigg\{\sum_{t=1}^{T-1}\Big(\sum_{i=1}^n A_i(t)-\sum_{i=1}^n D_i(t)\Big),\sum_{t=2}^{T-1}\Big(\sum_{i=1}^n A_i(t)-\sum_{i=1}^n D_i(t)\Big),\ldots,$$
$$\sum_{i=1}^n A_i(T-1)-\sum_{i=1}^n D_i(T-1),0\Bigg\}+\sum_{i=1}^n A_i(T).\qquad\text{(EC.13)}$$

Proof Let $\Delta(t) = A(t) - D(t)$ for all $t \geq 1$. By (1), for $1 \leq i \leq n$ and $T \geq 1$, we have

$$
\begin{aligned}
& Q_i(T) - A_i(T) \\
& = \max\{Q_i(T-1) - D_i(T-1), 0\} \\
& = \max\{\max\{Q_i(T-2) - D_i(T-2), 0\} + A_i(T-1) - D_i(T-1), 0\} \\
& = \max\{Q_i(T-2) - D_i(T-2) + \Delta_i(T-1), \Delta_i(T-1), 0\} \\
& = \max\left\{Q_i(1) - D_i(1) + \sum_{t=2}^{T-1} \Delta_i(t), \sum_{t=2}^{T-1} \Delta_i(t), \ldots, \Delta_i(T-1), 0\right\} \\
& = \max\left\{\sum_{t=1}^{T-1} \Delta_i(t), \sum_{t=2}^{T-1} \Delta_i(t), \ldots, \Delta_i(T-1), 0\right\},
\end{aligned}
$$

where the last equation follows that $Q_i(0) = 0$ and then $Q_i(1) = A_i(1)$. Therefore, we have

$$
\begin{aligned}
\sum_{i=1}^{n} Q_i(T) &= \sum_{i=1}^{n} \max\left\{\sum_{t=1}^{T-1} \Delta_i(t), \sum_{t=2}^{T-1} \Delta_i(t), \ldots, \Delta_i(T-1), 0\right\} + \sum_{i=1}^{n} A_i(T) \\
&\geq \max\left\{\sum_{t=1}^{T-1} \sum_{i=1}^{n} \Delta_i(t), \sum_{t=2}^{T-1} \sum_{i=1}^{n} \Delta_i(t), \ldots, \sum_{i=1}^{n} \Delta_i(T-1), 0\right\} + \sum_{i=1}^{n} A_i(T),
\end{aligned}
$$

which completes the proof.

E.2. Proof of Theorem 1

Proof of Theorem 1. Fix a time-invariant scheduling sequence $\mathcal{D}$ with $\mathcal{D}_t = \mathcal{D}_\star$ for all $t \in \mathbb{N}_0$, and assume that $\mathcal{D}_\star$ satisfies the condition in the model class $\mathcal{M}^\rho(B, C_\mathrm{a}, C_\mathrm{d})$. Let

$$
M = \max\left\{\sum_{i=1}^{n} d_i : d \in \mathcal{D}_\star\right\}
$$

denote the maximum total service capacity in a single time slot. In the following, we derive a lower bound for the right-hand-side of inequality (EC.13) by constructing a family of hard instances (A, S). Here the arrival process A and the random departure field S are constructed independently. We focus on the case $C_\mathrm{a}^2 + C_\mathrm{d}^2 > 0$; as the case $C_\mathrm{a}^2 + C_\mathrm{d}^2 = 0$ is straightforward.

Arrival Process: We construct an i.i.d. sequence $\{A(t)\}_{t \in \mathbb{N}}$ over time. At time t,

$$
\begin{aligned}
& \mathbb{P}(A_1(t) = K_\mathrm{a}\lambda_1, A_2(t) = K_\mathrm{a}\lambda_2, \ldots, A_n(t) = K_\mathrm{a}\lambda_n) = 1/K_\mathrm{a}, \\
& \mathbb{P}(A_1(t) = 0, A_2(t) = 0, \ldots, A_n(t) = 0) = (K_\mathrm{a} - 1)/K_\mathrm{a},
\end{aligned}
$$

where

$$\frac{\lambda}{\rho} \in \underset{\substack{\gamma \in \Pi(\mathcal{D}_\star) \\ \sum_{i=1}^n \gamma_i = M}}{\arg\min} \left\{ \sum_{i=1}^n \gamma_i^2 \right\} \quad \text{and} \quad K_\mathrm{a} = \frac{nC_\mathrm{a}^2}{\sum_{i=1}^n \lambda_i^2} + 1. \tag{EC.14}$$

Note that

$$\mathbb{E}[A(t)] = \lambda, \quad \text{and} \quad \mathrm{Var}(A_i(t)) = \frac{(K_\mathrm{a}-1)^2\lambda_i^2 + \lambda_i^2(K_\mathrm{a}-1)}{K_\mathrm{a}} = (K_\mathrm{a}-1)\lambda_i^2. \tag{EC.15}$$

The choice of K_a ensures that $\frac{1}{n}\sum_{i=1}^n \mathrm{Var}(A_i(t)) = C_\mathrm{a}^2$.

Random Departure Field: We construct an independent sequence $\{S_t\}_{t\in\mathbb{N}_0}$ over time. At time t, for each $d \in \mathcal{D}_t$,

$$\mathbb{P}(S_d(t) = K_\mathrm{d} d) = 1/K_\mathrm{d}, \text{ and } \mathbb{P}(S_d(t) = 0) = (K_\mathrm{d}-1)/K_\mathrm{d},$$

where

$$K_\mathrm{d} = \frac{nC_\mathrm{d}^2}{\max_{d\in\mathcal{D}_\star} \sum_{i=1}^n d_i^2} + 1. \tag{EC.16}$$

If the scheduling policy selects $d(t) \in \mathcal{D}_t$, the actual departure $D(t)$ is distributed as

$$\mathbb{P}(D_1(t) = K_\mathrm{d} d_1(t), D_2(t) = K_\mathrm{d} d_2(t), \ldots, D_n(t) = K_\mathrm{d} d_n(t)) = 1/K_\mathrm{d},$$
$$\mathbb{P}(D_1(t) = 0, D_2(t) = 0, \ldots, D_n(t) = 0) = (K_\mathrm{d}-1)/K_\mathrm{d}.$$

Similarly, the choice of K_d ensures S satisfies the conditions of the model class $\mathcal{M}^\rho(B, C_\mathrm{a}, C_\mathrm{d})$.

The departures satisfy $\mathbb{E}[D(t)] = d(t)$, and $\{\sum_{i=1}^n D_i(t)\}_{t\in\mathbb{N}_0}$ forms a sequence of binary random variables:

$$\mathbb{P}\left(\sum_{i=1}^n D_i(t) = K_\mathrm{d}\sum_{i=1}^n d_i(t)\right) = 1/K_\mathrm{d}, \text{ and } \mathbb{P}\left(\sum_{i=1}^n D_i(t) = 0\right) = (K_\mathrm{d}-1)/K_\mathrm{d}.$$

Let $\{G(t)\}_{t\in\mathbb{N}}$ be an i.i.d. sequence of random variables defined by

$$G(t) = \frac{M}{\sum_{i=1}^n d_i(t)} \sum_{i=1}^n D_i(t).$$

Since $\sum_{i=1}^n d_i(t) \le M$ for any scheduling policy, then $G(t) \ge \sum_{i=1}^n D_i(t)$. Let $X(t) = \sum_{i=1}^n A_i(t)$. By (EC.13), the queue length satisfies

$$\sum_{i=1}^n Q_i(T) \ge \max\left\{ \sum_{t=1}^{T-1}\Big(X(t) - G(t)\Big), \sum_{t=2}^{T-1}\Big(X(t) - G(t)\Big), \ldots, X(T-1) - G(T-1), 0 \right\} + X(T).$$

Observe that

$$X(t) - G(t) = (X(t) - \rho M) + (M - G(t)) - (1-\rho)M.$$

Note that $\{(X(t) - \rho M) - (M - G(t))\}_{t\in\mathbb{N}}$ forms an i.i.d. sequence of random variables, satisfying

$$\begin{aligned} &\mathbb{E}[(X(t) - \rho M) - (M - G(t))] = 0, \\ &\mathrm{Var}((X(t) - \rho M) + (M - G(t))) = M^2(\rho^2(K_{\mathrm{a}} - 1) + (K_{\mathrm{d}} - 1)), \end{aligned}$$

where the variance follows from the independence between A and S. Thus, by Lemma EC.3, we obtain the following lower bound: for all $T \geq \psi_1^2 + 1$ and $(1-\rho)M \leq \sigma/\psi_1$,

$$\begin{aligned} &\mathbb{E}\left[\max\left\{\sum_{t=1}^{T-1}\Big(X(t) - G(t)\Big), \sum_{t=2}^{T-1}\Big(X(t) - G(t)\Big), \ldots, X(T-1) - G(T-1), 0\right\}\right] \\ &= \mathbb{E}\left[\max\left\{0, X(1) - G(1), \sum_{t=1}^{2}\Big(X(t) - G(t)\Big), \ldots, \sum_{t=1}^{T-1}\Big(X(t) - G(t)\Big)\right\}\right] \\ &\geq \frac{\underline{c}}{2}\min\left\{M\sqrt{(\rho^2(K_{\mathrm{a}} - 1) + (K_{\mathrm{d}} - 1))(T-1)}, \frac{M(\rho^2(K_{\mathrm{a}} - 1) + (K_{\mathrm{d}} - 1))}{(1-\rho)}\right\}, \end{aligned} \tag{EC.17}$$

where ψ_1 is the unique positive solution to $\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+x) + \frac{8}{\underline{c}} = x$; $\sigma = M\sqrt{\rho^2(K_{\mathrm{a}} - 1) + (K_{\mathrm{d}} - 1)}$, $\underline{c}$ and $\bar{c}$ are the constants defined in Lemmas EC.1 and EC.2 respectively.

We now construct a specific scheduling set:

$$\mathcal{D}_\star = \{(x_1, x_2, \ldots, x_n) : 0 \leq x_i \leq B \text{ for } 1 \leq i \leq n\}. \tag{EC.18}$$

Note that this construction of $\mathcal{D}_\star$ ensures that scheduling sequence $\mathcal{D}$ satisfies the condition in the model class $\mathcal{M}^\rho(B, C_{\mathrm{a}}, C_{\mathrm{d}})$. In this case, we choose $\lambda = (\rho B, \rho B, \ldots, \rho B)$ so that

$$M = nB, \ K_{\mathrm{a}} - 1 = \frac{C_{\mathrm{a}}^2}{\rho^2 B^2}, \text{ and } K_{\mathrm{d}} - 1 = \frac{C_{\mathrm{d}}^2}{B^2}. \tag{EC.19}$$

Substitute (EC.19) into (EC.17). Observe that for $\rho \in [1/2, 1]$, the random variables $(X(t) - \rho M) + (M - G(t))$ have uniformly bounded support across all ρ. Indeed,

$$|(X(t) - \rho M) + (M - G(t))| \leq \big(\rho((K_{\mathrm{a}} - 1)\vee 1) + (K_{\mathrm{d}} - 1)\vee 1\big)M \leq nB\left(\frac{2C_{\mathrm{a}}^2}{B^2}\vee 1 + \frac{C_{\mathrm{d}}^2}{B^2}\vee 1\right)$$

and the variance of $(X(t) - \rho M) + (M - G(t))$ is $\sigma^2 = n^2(C_{\mathrm{a}}^2 + C_{\mathrm{d}}^2)$. Consequently, their Sakhanenko parameters are uniformly bounded away from zero by a positive constant depending only on n, C_{a}, C_{d} and B. By Lemma EC.2, the constant $\bar{c}$ can therefor be chosen to depend only on n, C_{a}, C_{d} and B.

Let $c_1 = \underline{c}/2$ and define $\psi_1(n, B, C_{\mathrm{a}}, C_{\mathrm{d}})$ as the unique positive solution to $\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+x) + \frac{8}{\underline{c}} = x$, we recover the result in (4).

E.3. Proof of Remark 1

Proof of Remark 1. Recall that Remark 1 claims that the quantity $\psi_1(n,B,C_{\mathrm{a}},C_{\mathrm{d}})$ can be chosen independent of n. In fact, we show that it satisfies

$$\psi_1(n,B,C_{\mathrm{a}},C_{\mathrm{d}}) = O\left((1+R^3)\log(1+R^3)\right) \quad \text{where} \quad R = \frac{B}{\sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2}}\left(\frac{2C_{\mathrm{a}}^2}{B^2}\vee 1+\frac{C_{\mathrm{d}}^2}{B^2}\vee 1\right).$$

Moreover, in the regimes in which $\sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2} \ll B$ and $\sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2} \gg B$, the above scaling simplifies to

$$\psi_1(n,B,C_{\mathrm{a}},C_{\mathrm{d}}) = O\left(\tilde{R}^3\log\tilde{R}^3\right),$$

where

$$\tilde{R} = \begin{cases} \frac{B}{\sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2}}, & \sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2} \ll B, \\ \frac{\sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2}}{B} & \sqrt{C_{\mathrm{a}}^2+C_{\mathrm{d}}^2} \gg B. \end{cases}$$

Now we prove the above claims. Let

$$\chi = nB\left(\frac{2C_{\mathrm{a}}^2}{B^2}\vee 1+\frac{C_{\mathrm{d}}^2}{B^2}\vee 1\right).$$

In fact, the Sakhanenko parameters of random variables $(X(t)-\rho M)+(M-G(t))$ are lower bounded by δ_0, where δ_0, by definition of the Sakhanenko parameter given in (EC.4), is the unique solution to

$$\delta\chi^3 e^{\delta\chi} = \sigma^2.$$

Solving this equation yields $\delta_0 = \frac{1}{\chi}W\left(\frac{\sigma^2}{\chi^2}\right)$, where $W(\cdot)$ denotes the Lambert W function defined by $W(u)e^{W(u)} = u$. By Lemma EC.2, the constant $\bar{c}$ can therefor be chosen as

$$\bar{c} = \frac{\omega}{\delta_0}+\omega\sigma$$

for some absolute constant $\omega > 0$. Consequently, ψ_1 is the unique solution of

$$\psi = H\log(1+\psi)+\frac{8}{\underline{c}},$$

where

$$H = \frac{4}{\underline{c}}\left(1+\omega+\frac{\omega}{(\sigma/\chi)W(\sigma^2/\chi^2)}\right).$$

In fact, we can upper bound ψ_1 as

$$\psi_1 \leq 2H\log H + \frac{8}{\underline{c}}. \tag{EC.20}$$

Let

$$x_0 := 2H\log H + \frac{8}{\underline{c}}.$$

We claim that for all $H \geq e^{1/2}$,

$$x_0 \geq H\log(1+x_0) + \frac{8}{\underline{c}}. \tag{EC.21}$$

Since ψ_1 is the unique solution to $\psi = H\log(1+\psi) + \frac{8}{\underline{c}}$, it is the unique zero of

$$F(\psi) := \psi - H\log(1+\psi) - \frac{8}{\underline{c}}.$$

Moreover, F is increasing on $[\psi_1, \infty)$. Hence, (EC.21) implies $F(x_0) \geq 0$ and therefore $\psi_1 \leq x_0$, which establishes (EC.20). We now verify (EC.21). Using the inequality $1 + 2s\log s \leq s^2$ for $s \geq 1$, we obtain

$$1 + 2H\log H \leq H^2.$$

Therefore,

$$H\log(1 + 2H\log H) \leq 2H\log H.$$

Next, we use the bound $\log(a+t) = \log a + \log(1+t/a) \leq \log a + t/a$ for $a, t > 0$. Take $a = 1 + 2H\log H$ and $t = \frac{8}{\underline{c}}$, we have

$$\begin{aligned} H\log(1+x_0) &\leq H\log(1+2H\log H) + \frac{8H}{\underline{c}(1+2H\log H)} \\ &\leq 2H\log H + \frac{4}{\underline{c}\log H} \leq x_0. \end{aligned}$$

where the last inequality uses $\log H \geq 1/2$ (e.g., $H \geq e^{1/2}$). This proves (EC.21) and completes the proof of (EC.20).

It suffices to characterize the order of H. Let $u = \frac{\sigma^2}{\chi^2}$. We first consider the case $u \in (0, 1]$ (i.e., $\sigma \leq \chi$). Since $W(u) \leq 1$ for $u \leq 1$, we have $u = W(u)e^{W(u)} \leq eW(u)$, which implies $W(u) \geq \frac{u}{e}$. Therefore,

$$\frac{1}{(\sigma/\chi)W(u)} \leq e\left(\frac{\chi}{\sigma}\right)^3.$$

Next, consider $u \geq 1$ (i.e., $\sigma \geq \chi$). The Lambert W function is increasing and satisfies $W(1) \approx 0.567 > \frac{1}{2}$. Hence $W(u) \geq \frac{1}{2}$. Consequently,

$$\frac{1}{(\sigma/\chi)W(u)} \leq \frac{2\chi}{\sigma} \leq 2.$$

Combining the two cases yields

$$H = O(1 + (\chi/\sigma)^3).$$

Consequently,

$$\psi_1 = O\left(\left(1 + (\chi/\sigma)^3\right) \log\left(1 + (\chi/\sigma)^3\right)\right).$$

Substituting the definitions of χ and σ then yields the claim stated in Remark 1.

F. Proofs of Theorems and Supporting Results for Section 4

F.1. Formal Proof of the Upper Bound in (13)

For any scheduling sets, arrival processes and random departure fields in $\mathcal{M}^\rho(B, C_\mathrm{a}, C_\mathrm{d})$, and any scheduling policy Φ, consider the one-step Lyapunov drift. Here for notational simplicity and without risk of ambiguity, we write $\mathbb{E}^\Phi_{(A,S)}$ as $\mathbb{E}$. First, observe that

$$\mathbb{E}\left[\sum_{i=1}^n (\max\{Q_i(t) - D_i(t), 0\})^2 \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[\sum_{i=1}^n (\max\{Q_i(t) - d_i(t), 0\})^2 \middle| \mathcal{F}_t\right] + \sum_{i=1}^n \mathrm{Var}(D_i(t)). \tag{EC.22}$$

To see this, define $g(d_i) = (\max\{x_i - d_i, 0\})^2$. Since $|g'(d_i) - g'(s_i)| \leq 2|d_i - s_i|$, the descent lemma yields for any D_i,

$$g(D_i) \leq g(d_i) + g'(d_i)(D_i - d_i) + (D_i - d_i)^2.$$

Taking conditional expectations given $\mathcal{F}_t$ gives

$$\mathbb{E}[g(D_i)|\mathcal{F}_t] \leq \mathbb{E}[g(d_i)|\mathcal{F}_t] + \mathrm{Var}(D_i).$$

Taking the summation over i completes the proof of (EC.22).

The one-step Lyapunov drift satisfies

$$\mathbb{E}\left[V\big(Q(t+1) - A(t+1)\big) - V\big(Q(t) - A(t)\big) \mid \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{i=1}^{n}(\max\{Q_i(t)-D_i(t),0\})^2\middle|\mathcal{F}_t\right] - \sum_{i=1}^{n}(Q_i(t)-A_i(t))^2 \\
&\le \sum_{i=1}^{n}\mathbb{E}\left[(\max\{Q_i(t)-d_i(t),0\})^2\middle|\mathcal{F}_t\right] + \sum_{i=1}^{n}\text{Var}(D_i(t)) - \sum_{i=1}^{n}(Q_i(t)-A_i(t))^2 \\
&\le \sum_{i=1}^{n}\left(\mathbb{E}\left[(Q_i(t)-d_i(t))^2\middle|\mathcal{F}_t\right] - Q_i(t)^2 + 2Q_i(t)A_i(t) - A_i(t)^2\right) + \sum_{i=1}^{n}\text{Var}(D_i(t)) \\
&= \sum_{i=1}^{n}\mathbb{E}\left[d_i(t)^2 - A_i(t)^2 + 2Q_i(t)\big(A_i(t)-d_i(t)\big)\middle|\mathcal{F}_t\right] + \sum_{i=1}^{n}\text{Var}(D_i(t)) \\
&= f(Q(t),d(t)) + r(Q(t),A(t)) + \sum_{i=1}^{n}\text{Var}(D_i(t)),
\end{aligned}
$$

where

$$
f(Q(t),d) := \underbrace{\mathbb{E}\left[2\sum_{i=1}^{n}Q_i(t)\big(\lambda_i(t)-d_i\big)\middle|\mathcal{F}_t\right]}_{\text{first-order term}} + \underbrace{\sum_{i=1}^{n}\left(d_i^2-\lambda_i(t)^2\right)}_{\text{second-order term}},
$$

and

$$
r(Q(t),A(t)) = \sum_{i=1}^{n}\left[2Q_i(t)(A_i(t)-\lambda_i(t)) + \lambda_i(t)^2 - A_i(t)^2\right].
$$

Note that

$$
\begin{aligned}
&\mathbb{E}\big[Q_i(t)A_i(t)\big] \\
&= \mathbb{E}\left[\big(Q_i(t)-A_i(t)\big)A_i(t)\right] + \mathbb{E}\big[A_i(t)^2\big] \\
&\overset{(a)}{=} \mathbb{E}\big[Q_i(t)-A_i(t)\big]\mathbb{E}\big[A_i(t)\big] + \mathbb{E}\big[A_i(t)^2\big] \\
&\overset{(b)}{=} \mathbb{E}\big[Q_i(t)\big]\lambda_i(t) + \text{Var}(A_i(t)),
\end{aligned}
$$

where (a) follows from the independence between $Q_i(t)-A_i(t)$ and $A_i(t)$, and (b) follows from the relation $\mathbb{E}\big[A_i(t)^2\big] = \lambda_i(t)^2 + \text{Var}(A_i(t))$.

Then we have

$$
\mathbb{E}[r(Q(t),A(t))] = \sum_{i=1}^{n}\text{Var}(A_i(t)).
$$

By taking expectation and summing over for $1 \le t \le T-1$, we have

$$
\begin{aligned}
&\mathbb{E}[V(Q(T)-A(T)] \\
&\le \mathbb{E}[V(Q(1)-A(1))] + \sum_{t=1}^{T-1}\mathbb{E}[f(Q(t),d(t))] + \sum_{t=1}^{T-1}\sum_{i=1}^{n}(\text{Var}(A_i(t)) + \text{Var}(D_i(t)))
\end{aligned}
$$

$$\leq \sum_{t=1}^{T-1} \mathbb{E}[f(Q(t),d(t))] + (T-1)n(C_{\mathrm{a}}^2 + C_{\mathrm{d}}^2), \tag{EC.23}$$

where the last inequality follows from the relation $Q(1) = A(1)$. Note that

$$\begin{aligned}
\left(\mathbb{E}\Big[\sum_{i=1}^{n} \big(Q_i(T) - A_i(T)\big)\Big]\right)^2 &\overset{(c)}{\leq} \mathbb{E}\Big[\Big(\sum_{i=1}^{n} \big(Q_i(T) - A_i(T)\big)\Big)^2\Big] \\
&\overset{(d)}{\leq} n\mathbb{E}\left[\sum_{i=1}^{n} \big(Q_i(T) - A_i(T)\big)^2\right] \\
&= n\mathbb{E}\big[V\big(Q(T) - A(T)\big)\big].
\end{aligned}$$

Here (c) follows from Jensen's inequality and (d) follows from the Cauchy–Schwartz inequality. Substituting this expression into (EC.23), we have

$$\mathbb{E}\left[\sum_{i=1}^{n} Q_i(T)\right] \leq n\sqrt{\sum_{t=1}^{T-1} \frac{1}{n}\,\mathbb{E}[f(Q(t),d(t))] + (T-1)(C_{\mathrm{a}}^2 + C_{\mathrm{d}}^2)} + \sum_{i=1}^{n} \mathbb{E}[A_i(T)].$$

F.2. Formal Proof of Theorem 4

Proof of Theorem 4 **Step 1.** Fix an instance $(\mathcal{D}, A, S)$ in the model class $\mathcal{M}^1(B, C_{\mathrm{a}}, C_{\mathrm{d}})$. For $\rho \in (0,1]$, let $\{Q^{(\rho)}(t)\}_{t\in\mathbb{N}_0}$ and $\{d^{(\rho)}(t)\}_{t\in\mathbb{N}_0}$ denote the queue length process and the corresponding MaxWeight decisions when the arrival process $\{A(t)\}_{t\in\mathbb{N}}$ is replaced by the scaled process $\{\rho A(t)\}_{t\in\mathbb{N}}$. For brevity, when $\rho = 1$, we write $\{Q(t)\}_{t\in\mathbb{N}_0}$ and $\{d(t)\}_{t\in\mathbb{N}_0}$ in place of $\{Q^{(1)}(t)\}_{t\in\mathbb{N}_0}$ and $\{d^{(1)}(t)\}_{t\in\mathbb{N}_0}$, respectively. Suppose that each scheduling set $\mathcal{D}_t$ is a polytope and that the MaxWeight policy yields a unique maximizer of $\langle Q(t), d\rangle$ at every time (with the exception of $t = 0$, where we fix an arbitrary schedule). Under this assumption, we establish that

$$\mathbb{E}_{(A,S)}\left[\sum_{i=1}^{n} Q_i^{(\rho)}(T)\right]$$

is continuous in ρ on an interval $(1-\delta, 1]$ for some $\delta > 0$.

By definition, the MaxWeight policy always selects an extreme point of the scheduling set. Since the extreme points of the polytope is finite, there exists $\delta > 0$ such that for $\rho \in (1-\delta, 1]$, the maximizer of $\langle Q^{(\rho)}(t), d\rangle$ remains unchanged, that is, $d^{(\rho)}(t) = d(t)$ for all $t \leq T$ when $\rho \in (1-\delta, 1]$. Consequently, for every sample path, $\sum_{i=1}^{n} Q_i^{(\rho)}(t)$ is continuous in ρ on the interval $(1-\delta, 1]$. Moreover, because

$$\sum_{i=1}^{n} Q_i^{(\rho)}(T) \leq \sum_{t=1}^{T} \sum_{i=1}^{n} A_i(T),$$

the family $\{\sum_{i=1}^n Q_i^{(\rho)}(T)\}_{\rho\in(1-\delta,1]}$ is uniformly integrable. By the dominated convergence theorem, the expectation $\mathbb{E}_{(A,S)}[\sum_{i=1}^n Q_i^{(\rho)}(T)]$ is continuous in ρ on the interval $(1-\delta,1]$.

Step 2. We begin by considering a family of instances $\mathcal{I}_{b,1}$, which is given in (21) and illustrated in Figure 3 for the case $q=1$. For the instance $\mathcal{I}_{b,1}$ with $b\geq 6$, we can show (as done in the proof following Step 5) that for all time horizons T satisfying $\left\lceil \frac{b^2}{b-1}\right\rceil \leq T \leq \lceil (\frac{b}{2}-1)^3\rceil + 1$,

$$\mathbb{E}_{(A,S)}\left[\sum_{i=1}^2 Q_i(T)\right] \geq \frac{\sqrt{bT}}{2\sqrt{2}}. \tag{EC.24}$$

Step 3. We now consider the family of instances $\mathcal{I}_{b,q}$, which is given in (21) and illustrated in Figure 3. To ensure that the instance lies within the model class $\mathcal{M}^1(B,C_{\mathrm{a}},C_{\mathrm{d}})$, we require

$$\frac{q^2}{2}\leq B^2, \text{ and } \frac{(qb)^2}{2}\leq B^2.$$

Recalling that $b>1$, both conditions are satisfied by setting $q=\frac{\sqrt{2}B}{b}$. It then follows that $\mathcal{I}_{b,\frac{\sqrt{2}B}{b}} \in \mathcal{M}^1(B,C_{\mathrm{a}},C_{\mathrm{d}})$ for all $b>1$. Consequently, the queue lengths under the MaxWeight policy for the instance $\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}$ are exactly the queue lengths for the instance $\mathcal{I}_{b,1}$ multiplied by the scaling factor $q=\frac{\sqrt{2}B}{b}$, which can be shown by induction. Suppose $\widetilde{Q}(t)=qQ(t)$. Since MaxWeight selects $d(t)\in\arg\max_{d\in\mathcal{D}_t}\langle Q(t),d\rangle$, it follows that

$$qd(t)\in\arg\max_{d\in q\mathcal{D}_t}\langle Q(t),d\rangle = \arg\max_{d\in q\mathcal{D}_t}\langle \widetilde{Q}(t),d\rangle.$$

Therefore, recalling $D(t)=d(t)$, the queue dynamics satisfy

$$\widetilde{Q}(t+1)=\max\{\widetilde{Q}(t)-qD(t),\mathbf{0}\}+qA(t)=q(\max\{Q(t)-D(t),\mathbf{0}\}+A(t))=qQ(t+1).$$

By induction, this holds for all $t\in\mathbb{N}$. As a result, applying inequality (EC.24), for any instance $\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}$ with $b\geq 6$, and for all time horizons satisfying $\left\lceil \frac{b^2}{b-1}\right\rceil \leq T \leq \lceil (\frac{b}{2}-1)^3\rceil + 1$, we have

$$\mathbb{E}_{(A,S)}\left[\sum_{i=1}^2 Q_i(T)\right] \geq \frac{B\sqrt{T}}{2\sqrt{b}}. \tag{EC.25}$$

Step 4. For any $T\geq 9$, we can choose the instance $\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}$ with $b\geq 6$ such that $T=\lceil (\frac{b}{2}-1)^3\rceil+1$. For this instance, the total queue length at time T satisfies inequality (EC.25). Moreover, since $T=\lceil (\frac{b}{2}-1)^3\rceil+1$, it follows that $b\leq 3T^{\frac{1}{3}}$, and hence

$$\mathbb{E}_{(A,S)}\left[\sum_{i=1}^2 Q_i(T)\right] \geq \frac{BT^{\frac{1}{3}}}{2\sqrt{3}}.$$

Therefore, for all $T \geq 9$,

$$\sup_{\{\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}:b\geq 6\}} \mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i(T)\right] \geq \frac{BT^{\frac{1}{3}}}{2\sqrt{3}}.$$

By the continuity of $\mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i^{(\rho)}(T)\right]$ in ρ on the interval $(1-\delta,1]$ for each instance $\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}$, we further obtain

$$\liminf_{\rho\uparrow 1} \sup_{\{\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}:b\geq 6\}} \mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i^{(\rho)}(T)\right] \geq \sup_{\{\mathcal{I}_{b,\frac{\sqrt{2}B}{b}}:b\geq 6\}} \lim_{\rho\uparrow 1} \mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i^{(\rho)}(T)\right] \geq \frac{BT^{\frac{1}{3}}}{2\sqrt{3}}.$$

Since $\mathcal{I}_{b,\frac{\sqrt{2}B}{b}} \in \mathcal{M}^1(B,C_{\mathrm{a}},C_{\mathrm{d}})$ for all $b>1$, we conclude that for all $T\geq 9$,

$$\liminf_{\rho\uparrow 1} \sup_{\mathcal{M}^{\rho}(B,C_{\mathrm{a}},C_{\mathrm{d}})} \mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i(T)\right] \geq \liminf_{\rho\uparrow 1} \sup_{\mathcal{M}^{1}(B,C_{\mathrm{a}},C_{\mathrm{d}})} \mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i^{(\rho)}(T)\right] \geq \frac{BT^{\frac{1}{3}}}{2\sqrt{3}}, \tag{EC.26}$$

where the first inequality uses that scaling the arrival process of any instance $(\mathcal{D},A,S)\in\mathcal{M}^1(B,C_{\mathrm{a}},C_{\mathrm{d}})$ by ρ yields an instance in $\mathcal{M}^{\rho}(B,C_{\mathrm{a}},C_{\mathrm{d}})$.

Step 5. Combining the bound in (EC.26) with the lower bound in (4) from Theorem 1, we obtain the following: for all $T\geq \psi_1(n,B,C_{\mathrm{a}},C_{\mathrm{d}})^2+9$, we have

$$\begin{aligned}\liminf_{\rho\uparrow 1} \sup_{\mathcal{M}^{\rho}(B,C_{\mathrm{a}},C_{\mathrm{d}})} \mathbb{E}_{(A,S)}\left[\sum_{i=1}^{2} Q_i(T)\right] &\geq \max\left\{2c_1\sqrt{(C_{\mathrm{a}}^2+C_{\mathrm{d}}^2)(T-1)}, \frac{BT^{\frac{1}{3}}}{2\sqrt{3}}\right\}\\ &\geq c_1\sqrt{(C_{\mathrm{a}}^2+C_{\mathrm{d}}^2)(T-1)}+\frac{BT^{\frac{1}{3}}}{4\sqrt{3}},\end{aligned}$$

which completes the proof of the theorem.

We now prove inequality (EC.24). Before presenting the formal argument, we first outline the main idea.

Proof Idea for Inequality (EC.24). The choices of b and ε ensure that no two schedules tie under MaxWeight at finite time, so the policy always selects either $(0,1)$ or $(b,0)$. Under MaxWeight, one can show: At each time t, MaxWeight selects $(b,0)$ unless $\frac{Q_2(t)}{Q_1(t)}\geq b$. As a result, $Q_2(t)$ must accumulate to b before it forces the policy to use $(0,1)$; however, doing so causes $Q_1(t)$ to increase to 2, requiring $Q_2(t)$ to build up to $2b$ before $(0,1)$ can be used again more frequently. More precisely, let $t_0=0$ and

$$t_k=\min\{t:Q_2(t)\geq kb\} \text{ for } 1\leq k\leq \left\lfloor\frac{b}{2}\right\rfloor.$$

Under the MaxWeight policy, we observe the following behavior: when $t_0 \le t < t_1$, $d(t) = (b,0)$, so the average growth rate of Q_2 is $\frac{b-1}{b}$; when $t_1 \le t < t_2$, $Q_2(t) \ge b-1$, and $d(t)$ alternates between $(b,0)$ and $(0,1)$ until $Q_2(t) = 2b$, the average growth rate of Q_2 is $\frac{b-1}{b} - \frac{1}{2}$; when $t_2 \le t < t_3$, $Q_2(t) \ge 2b-1$, and each use of $(b,0)$ is followed by two uses of $(0,1)$ until $Q_2(t) = 3b$, the average growth rate of Q_2 is $\frac{b-1}{b} - \frac{2}{3}$.

More generally, for all $k \le \lfloor b/2 \rfloor$, during the interval $t_{k-1} \le t < t_k$, $Q_2(t) \ge (k-1)b - 1$, and each use of $(b,0)$ is followed by k uses of $(0,1)$ until $Q_2(t) = kb$, the average growth rate of Q_2 is $\frac{b-1}{b} - \frac{k-1}{k}$. Moreover, since the interval length satisfies $t_k - t_{k-1} = \Theta(kb)$, it follows that $t_k = \Theta(k^2 b)$. Note that for $t_k \le t < t_{k+1}$, $Q_2(t) \ge kb - 1$. Combining these observations yields $Q_2(t) = \Theta(\sqrt{bt})$.

Formal Proof for Inequality (EC.24). Now we present the formal proof. Consider the MaxWeight policy under the instance (21) with $q = 1$, where $b \ge 6$ is rational and we choose $0 < \varepsilon < \frac{b-1}{b}$ to be a small irrational constant.

By the queue dynamics, $Q(1) = (1, \frac{b-1}{b} - \varepsilon)$. MaxWeight repeatedly selects $(b,0)$ until $Q_2(t) \ge b$. Thus $Q(t) = (1, \frac{b-1}{b}t - \varepsilon)$ for $1 \le t < \lceil \frac{b(b+\varepsilon)}{b-1} \rceil$. When $t = \lceil \frac{b(b+\varepsilon)}{b-1} \rceil$, we have $Q_2(t) > b = bQ_1(t)$ so MaxWeight choose $(0,1)$. There is no tie in this decision, since $Q_2(t)$ is irrational while $Q_1(t)$ is rational. Note that once $Q_2(t)$ falls below b, MaxWeight will always choose $(b,0)$. Consequently, for all $t \ge \lceil \frac{b(b+\varepsilon)}{b-1} \rceil$, $Q_2(t)$ can never drop below $b-1$. This ensures that the $-\varepsilon$ term is always preserved in $Q_2(t)$, making it irrational for all t. In the following, we consider only time horizons where $Q_1(t) < b$ holds throughout, ensuring that $Q_1(t)$ always remains an integer. Consequently, there is never a tie in MaxWeight decisions, and it either selects $(b,0)$ or $(0,1)$. Therefore, we henceforth restrict our analysis to the MaxWeight policy under

$$\begin{aligned}
&\mathcal{D}_\star = \{(b,0), (0,1)\};\\
&A(t) = \left(1, \frac{b-1}{b}\right) \text{ for all } t \ge 2, \text{ and } A(1) = \left(1, \frac{b-1}{b} - \varepsilon\right);\\
&S_d(t) = d, \qquad \forall d \in \mathcal{D}_\star,
\end{aligned} \tag{EC.27}$$

where $b \ge 6$ is rational and $0 < \varepsilon < \frac{b-1}{b}$ is an irrational constant. In addition, we choose ε small enough so that $\left\lceil \frac{b(b+\varepsilon)}{b-1} \right\rceil = \left\lceil \frac{b^2}{b-1} \right\rceil$. This is possible because $\frac{b^2}{b-1}$ is not an integer for any $b \ge 6$.

At each time t, the MaxWeight policy will choose $(0,1)$ only when $Q_2(t) > bQ_1(t)$ and $(b,0)$ when $bQ_1(t) > Q_2(t)$. Let $t_0 = 0$ and

$$t_k = \min\{t : Q_2(t) \ge kb\} \text{ for } 1 \le k \le \left\lfloor \frac{b}{2} \right\rfloor.$$

We will show, by induction, that

(i) t_k is well defined;

(ii) $Q_1(t_k) = 1$;

(iii) For $t_{k-1} \leq t < t_k$, $1 \leq Q_1(t) \leq k$; $(k-1)b - (k-1)/b \leq Q_2(t) < kb$.

For $k = 1$, by the above discussion, $t_1 = \lceil \frac{b(b+\varepsilon)}{b-1} \rceil$ is well defined, $Q_1(t_1) = 1$, and (iii) naturally holds.

Now suppose that (i)-(iii) hold for some $k-1$ with $k \geq 2$. We show (i)-(iii) also hold for k. Note the fact that at any time t, choosing $(0,1)$ gives

$$Q_1(t+1) = Q_1(t) + 1, Q_2(t+1) = Q_2(t) - \frac{1}{b},$$

whereas choosing $(b,0)$ yields

$$Q_1(t+1) = \max\{Q_1(t) - b, 0\} + 1 = 1, Q_2(t+1) = Q_2(t) + 1 - \frac{1}{b}.$$

Thus Q_1 increases strictly only when $(0,1)$ is used, and Q_2 increases strictly only when $(b,0)$ is used. Starting from $t \geq t_{k-1}$, and noting $Q_1(t_{k-1}) = 1$, consider the next k time slots. There must be at least one use of $(b,0)$, because if the first $k-1$ departures were all $(0,1)$, then $Q_1(t_{k-1}+k-1) = k$, while $Q_2(t_{k-1}+k-1) < (k-1)b + 1 - (k-1)/b < kb$, forcing the kth departure to be $(b,0)$. Moreover, if only consider $k-1$ time slots after t_{k-1}, at most one $(b,0)$ can be used, because for $s \leq k$, $Q_2(t_{k-1}+s) \geq (k-1)b - s/b > (k-2)b$. Note that $Q_2(t_{k-1}+k) \geq (k-1)b + 1 - k/b > (k-1)b$. By the same reasoning and the fact that $Q_2(t_{k-1}+k+s) \geq (k-1)b + 1 - (k+s)/b > (k-1)b$ for $s \leq k$, in each subsequent block of k slots, Q_2 increases by exactly $1 - k/b > 0$. Hence t_k is well defined, and since Q_2 only increases when $(b,0)$ is applied, we conclude $Q_1(t_k) = 1$. Finally, (iii) follows directly from the analysis above. And we complete the induction. In addition, we also have the following properties:

1. For all $t_{k-1} \leq t \leq t_k$,

$$Q_1(t) + Q_2(t) \geq (k-1)b. \tag{EC.28}$$

2. Each interval length satisfies

$$t_1 - t_0 = \left\lceil \frac{b(b+\varepsilon)}{b-1} \right\rceil,$$
$$\left\lceil \frac{k(b-1)}{1-k/b} \right\rceil - 1 \leq t_k - t_{k-1} \leq \left\lceil \frac{kb}{1-k/b} \right\rceil, \; 2 \leq k \leq \left\lfloor \frac{b}{2} \right\rfloor.$$

By inequality (EC.28), for all $t_{k-1} \le t \le t_k$,

$$Q_1(t) + Q_2(t) \ge (k-1)b = \sqrt{t}b\frac{k-1}{\sqrt{t}} \ge \sqrt{t}b\frac{k-1}{\sqrt{t_k}}.$$

Note that

$$t_k = \sum_{i=1}^{k}(t_i - t_{i-1}) \le \left\lceil \frac{b(b+\varepsilon)}{b-1} \right\rceil + \sum_{i=2}^{k} \left\lceil \frac{ib}{1-i/b} \right\rceil \le \sum_{i=1}^{k} \left\lceil \frac{ib}{1-i/b} \right\rceil + 1.$$

The remainder of the proof relies on two technical claims.

Claim 1. For $b \ge 4$ and $k \ge 2$,

$$\frac{k-1}{\sqrt{t_k}} \ge \frac{1}{2\sqrt{2b}}.$$

Claim 2. For $b \ge 4$,

$$t_{\lfloor \frac{b}{2} \rfloor} \ge (\frac{b}{2} - 1)^3 + 1.$$

With the help of Claim 1, for $t_1 \le t \le t_{\lfloor \frac{b}{2} \rfloor}$,

$$Q_1(t) + Q_2(t) \ge \frac{\sqrt{bt}}{2\sqrt{2}}.$$

Since $t_1 = \left\lceil \frac{b(b+\varepsilon)}{b-1} \right\rceil \le \frac{b^2}{b-1} + 1$, and when $b \ge 6$, we have $\frac{b^2}{b-1} + 1 \le \lceil (\frac{b}{2} - 1)^3 \rceil + 1$. Thus, for $b \ge 6$, for all $\left\lceil \frac{b^2}{b-1} \right\rceil \le t \le \lceil (\frac{b}{2} - 1)^3 \rceil + 1$,

$$Q_1(t) + Q_2(t) \ge \frac{\sqrt{bt}}{2\sqrt{2}}.$$

Substituting $b = \sqrt{2}B$ completes the proof.

We now proceed to prove these two claims.

Proof of Claim 1. For $2 \le k \le \lfloor b/2 \rfloor$ and each $1 \le i \le k$, we have $1 - i/b \ge 1/2$. Hence

$$\left\lceil \frac{ib}{1-i/b} \right\rceil \le \frac{ib}{1-i/b} + 1 \le 2ib + 1.$$

Summing over $i = 1, 2, \ldots, k$ yields

$$t_k \le \sum_{i=1}^{k}(2ib + 1) + 1 = bk(k+1) + k + 1 \le 2bk^2.$$

The last inequality holds under the assumptions $b \ge 4$ and $k \ge 2$. Hence

$$\frac{k-1}{\sqrt{t_k}} \ge \frac{k-1}{k\sqrt{2b}} = \frac{1-\frac{1}{k}}{\sqrt{2b}} \ge \frac{1}{2\sqrt{2b}},$$

where the final inequality uses $k \ge 2$, which implies $1 - \frac{1}{k} \ge \frac{1}{2}$.

Proof of Claim 2. Note that

$$t_{\lfloor \frac{b}{2} \rfloor} = \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor} (t_i - t_{i-1}) \geq \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor} \left(\left\lceil \frac{i(b-1)}{1-i/b} \right\rceil - 1 \right) + 1.$$

And

$$\sum_{i=1}^{k} \left(\left\lceil \tfrac{i(b-1)}{1-i/b} \right\rceil - 1 \right) + 1 \geq \sum_{i=1}^{k} \left(\frac{ib(b-1)}{b-i} - 1 \right) + 1 \geq \frac{bk(k+1)}{2} - k + 1.$$

Then

$$\sum_{i=1}^{\lfloor b/2 \rfloor} \left(\left\lceil \tfrac{i(b-1)}{1-i/b} \right\rceil - 1 \right) + 1 \geq \frac{b\lfloor b/2 \rfloor (\lfloor b/2 \rfloor + 1)}{2} - \lfloor b/2 \rfloor + 1 \geq (\frac{b}{2} - 1)^3 + 1.$$

F.3. Formal Proof of Corollary 1

Proof of Corollary 1 We continue to use the same construction as in (21) with $q = 1$ and $B = b/\sqrt{2}$, under which the lower bound for MaxWeight is given by (EC.24). For $\mathrm{LyapOpt}$, since the arrival rate lies in the scheduling set $\mathcal{D}_\star$ and $C_{\mathrm{a}} = C_{\mathrm{d}} = 0$, Theorem 2 and (16) yield

$$\sum_{i=1}^{2} Q_i^{\mathrm{LyapOpt}}(T) = \sum_{i=1}^{2} A_i(T) = 2 - \frac{1}{b} - \varepsilon \leq 2 - \frac{1}{\sqrt{2}B},$$

which completes the proof.

G. Proofs of Theorems and Supporting Results for Section 5

G.1. Proof of Theorem 5

We invoke the following Foster–Lyapunov lemma (see, e.g., Meyn and Tweedie (2012)).

LEMMA EC.5 **(Foster–Lyapunov Criterion)**. *Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a Markov chain on a countable state space $\mathcal{X}$. Suppose there exists a function $V : \mathcal{X} \to [0, \infty)$, a finite subset $W \subset \mathcal{X}$, and constants $\epsilon > 0$ and $b < \infty$ such that*

$$\mathbb{E}[V(X_{t+1}) - V(X_t) \mid X_t = x] \leq -\epsilon \mathbf{1}_{\{x \notin W\}} + b\mathbf{1}_{\{x\in W\}}.$$

Then the Markov chain $\{X_t\}_{t\in\mathbb{N}_0}$ is stochastically stable in the sense that it reaches the recurrent states with probability 1, and every recurrent state is positive recurrent.

Proof of Theorem 5 Under the $\mathrm{LyapOpt}$ policy, the one-step Lyapunov drift

$$
\begin{aligned}
&\mathbb{E}^{\mathrm{LyapOpt}}_{(A,S)}\left[V(Q(t+1))-V(Q(t))\,|\,Q(t)=x\right]\\
&=\mathbb{E}_{(A,S)}\left[\sum_{i=1}^n(\max\{x_i(t)-D_i^{\mathrm{LyapOpt}}(t),0\}+A_i(t+1))^2\,\middle|\,Q(t)=x\right]-\sum_{i=1}^n x_i(t)^2\\
&\le\sum_{i=1}^n\left(\mathbb{E}\left[(\max\{x_i(t)-D_i^{\mathrm{LyapOpt}}(t),0\})^2\middle|Q(t)=x\right]+2x_i(t)\mathbb{E}[A_i(t+1)]+\mathbb{E}[A_i(t+1)^2]-x_i(t)^2\right),
\end{aligned}
$$

where the inequality uses the independence between $A(t+1)$ and $Q(t)$. Use the same argument as in (EC.22) of Appendix F.1, we have

$$
\begin{aligned}
&\sum_{i=1}^n\mathbb{E}\left[(\max\{x_i(t)-D_i^{\mathrm{LyapOpt}}(t),0\})^2\middle|Q(t)=x\right]\\
&\le\sum_{i=1}^n\mathbb{E}\left[(\max\{x_i(t)-d_i^{\mathrm{LyapOpt}}(t),0\})^2\middle|Q(t)=x\right]+\sum_{i=1}^n\mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t))\\
&=\min_{d\in\mathcal{D}_\star}\sum_{i=1}^n(\max\{x_i(t)-d_i,0\})^2+\sum_{i=1}^n\mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t))\\
&\le\min_{d\in\mathcal{D}_\star}\sum_{i=1}^n(x_i(t)-d_i)^2+\sum_{i=1}^n\mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t))\\
&\le\min_{d\in\mathcal{D}_\star}\sum_{i=1}^n-2x_i(t)d_i+\max_{d\in\mathcal{D}_\star}\sum_{i=1}^n d_i^2+\sum_{i=1}^n x_i^2+\sum_{i=1}^n\mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t)),
\end{aligned}
$$

where the equality uses the definition of $\mathrm{LyapOpt}$, and the last inequality is by the fact that $\min(f+g)\le\min f+\max g$. Therefore,

$$
\begin{aligned}
&\mathbb{E}^{\mathrm{LyapOpt}}_{(A,S)}\left[V(Q(t+1))-V(Q(t))\,|\,Q(t)=x\right]\\
&\le\min_{d\in\mathcal{D}_\star}\sum_{i=1}^n 2x_i(t)(\lambda_i(t+1)-d_i)+2nB^2+nC_{\mathrm{a}}^2+nC_{\mathrm{d}}^2\\
&\le-2(1-\rho)\max_{d\in\mathcal{D}_\star}\sum_{i=1}^n x_i(t)d_i+2nB^2+nC_{\mathrm{a}}^2+nC_{\mathrm{d}}^2,
\end{aligned}
$$

where the last inequality follows from $\lambda(t+1)\in\rho\Pi(\mathcal{D}_\star)$, which implies

$$\sum_{i=1}^n x_i(t)\lambda_i(t+1)\le\max_{d\in\mathcal{D}_\star}\rho\sum_{i=1}^n x_i(t)d_i.$$

Then we choose

$$W=\left\{x\in\mathbb{Z}_+^n:\max_{d\in\mathcal{D}_t}\sum_{i=1}^n x_id_i\le\frac{\epsilon+n(2B^2+C_{\mathrm{a}}^2+C_{\mathrm{d}}^2)}{2(1-\rho)}\right\}.$$

Since W is finite, it satisfies the condition in Lemma EC.5, which completes the proof.

G.2. Proof of Proposition 1 and Related Lemma

The next lemma follows directly from Assumption 1.

LEMMA EC.6. *Under Assumption 1, for all $t \in \mathbb{N}_0$ and all $x \in \mathbb{R}_+^n$,*

$$\max_{d \in \mathcal{D}_t} \langle x, d \rangle \geq \frac{\alpha}{n} \sum_{i=1}^{n} x_i. \tag{EC.29}$$

Proof of Lemma EC.6 Fix $t \in \mathbb{N}_0$ and $x \in \mathbb{R}_+^n$. By Assumption 1, for each j there exists a schedule vector $d^{(j)}(t) \in D_t$ such that its j-th component $d_j^{(j)}(t) \geq \alpha$. Hence

$$\max_{d \in \mathcal{D}_t} \langle x, d \rangle \geq \max_{1 \leq j \leq n} \langle x, d^{(j)}(t) \rangle \geq \max_{1 \leq j \leq n} x_j d_j^{(j)}(t) \geq \alpha \max_{1 \leq j \leq n} x_j.$$

Since $\max_i x_i \geq \frac{1}{n} \sum_{i=1}^{n} x_i$, inequality (EC.29) follows.

Proof of Proposition 1 By inequality (10) and the definition of $\mathrm{LyapOpt}$, the one-step Lyapunov drift under the $\mathrm{LyapOpt}$ policy satisfies

$$\begin{aligned}
&\mathbb{E}_{(A,S)}^{\mathrm{LyapOpt}} \left[V(Q(t+1) - A(t+1)) - V(Q(t) - A(t)) \mid \mathcal{F}_t \right] \\
&\leq \min_{d \in \mathcal{D}_t} f(Q(t), d) + r(Q(t), A(t)) + \sum_{i=1}^{n} \mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t)) \\
&= \min_{d \in \mathcal{D}_t} \left(2 \sum_{i=1}^{n} Q_i(t) \big(\lambda_i(t) - d_i\big) + \sum_{i=1}^{n} \big(d_i^2 - \lambda_i(t)^2\big) \right) + r(Q(t), A(t)) + \sum_{i=1}^{n} \mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t)) \\
&\leq \min_{d \in \mathcal{D}_t} 2 \sum_{i=1}^{n} Q_i(t) \big(\lambda_i(t) - d_i\big) + nB^2 + r(Q(t), A(t)) + \sum_{i=1}^{n} \mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t)) \\
&\leq -2(1-\rho) \max_{d \in \mathcal{D}_t} \sum_{i=1}^{n} Q_i(t) d_i + nB^2 + r(Q(t), A(t)) + \sum_{i=1}^{n} \mathrm{Var}(D_i^{\mathrm{LyapOpt}}(t)),
\end{aligned}$$

where the last inequality follows from $\lambda(t) \in \rho \Pi(\mathcal{D}_t)$, which implies

$$\sum_{i=1}^{n} Q_i(t) \lambda_i(t) \leq \max_{d \in \mathcal{D}_t} \rho \sum_{i=1}^{n} Q_i(t) d_i.$$

Taking expectations and summing over t yields

$$\mathbb{E}_{(A,S)}^{\mathrm{LyapOpt}} [V(Q(T) - A(T))] \leq -2(1-\rho) \sum_{t=1}^{T-1} \max_{d \in \mathcal{D}_t} \sum_{i=1}^{n} \mathbb{E}_{(A,S)}^{\mathrm{LyapOpt}} [Q_i(t)] d_i + n(T-1)(B^2 + C_{\mathrm{a}}^2 + C_{\mathrm{d}}^2).$$

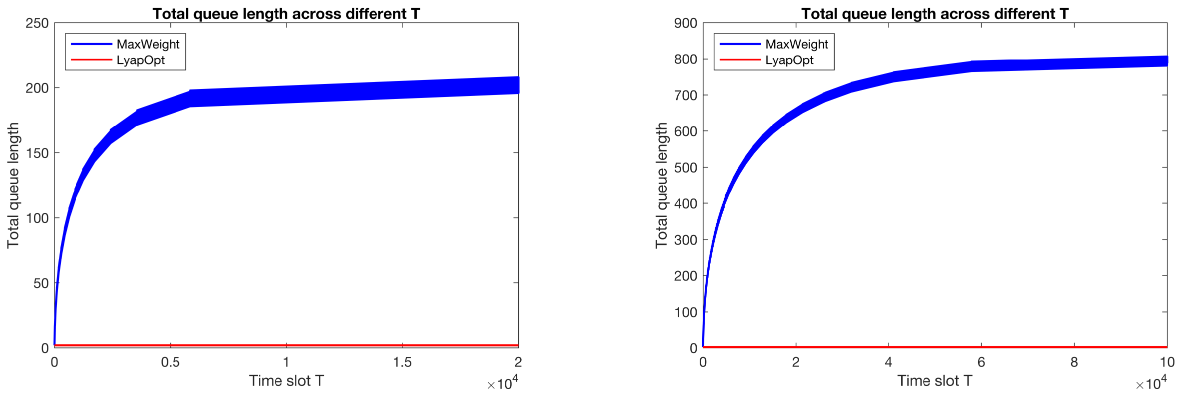
By Lemma EC.6,

$$\max_{d \in \mathcal{D}_t} \sum_{i=1}^{n} \mathbb{E}_{(A,S)}^{\mathrm{LyapOpt}} [Q_i(t)] d_i \geq \frac{\alpha}{n} \sum_{i=1}^{n} \mathbb{E}_{(A,S)}^{\mathrm{LyapOpt}} [Q_i(t)].$$

Then we have the uniform stability bound (22).

H. Supplementary Experiments in Section 6.1

The following two figures confirm that, in the setting of the Corollary 1, the total queue length under MaxWeight eventually becomes bounded (i.e., $O(1)$ in T) as T increases.



(a) Total queue length when $B = 10$, $C_{\mathrm{a}} = C_{\mathrm{d}} = 0$ **(b)** Total queue length when $B = 20$, $C_{\mathrm{a}} = C_{\mathrm{d}} = 0$

Figure EC.1 Performance comparison of MaxWeight and LyapOpt policies versus B

I. Formal Proofs of Results in Section 7

I.1. Proof of Proposition 2

Proof of Proposition 2 We focus on the case $C_{\mathrm{a}}^2 + C_{\mathrm{d}}^2 > 0$; as the case $C_{\mathrm{a}}^2 + C_{\mathrm{d}}^2 = 0$ is straightforward. Fix a time-invariant scheduling sequence $\mathcal{D}$ with $\mathcal{D}_t = \mathcal{D}_\star$ for all $t \in \mathbb{N}$, where $\mathcal{D}_\star$ is defined in (EC.18). Let

$$M = \max\left\{\sum_{i=1}^{n} d_i : d \in \mathcal{D}_\star\right\} = nB$$

denote the maximum total service capacity in a single time slot. We construct arrival process A and random departure field S are mutually independent. For each t, the components $A_i(t)$ are independent across i and distributed as

$$\mathbb{P}\Big(A_i(t) = K_{\mathrm{a}}\lambda_i\Big) = \frac{1}{K_{\mathrm{a}}}, \quad \mathbb{P}\Big(A_i(t) = 0\Big) = \frac{K_{\mathrm{a}} - 1}{K_{\mathrm{a}}},$$

where the choices of λ and K_{a} follow those specified in (EC.14). Thus, the mean and variance conditions for A in (EC.15) still holds. In particular, the total variance across all queues is given by

$$\sum_{i=1}^{n} \mathrm{Var}(A_i(t)) = \sum_{i=1}^{n} (K_{\mathrm{a}} - 1)\lambda_i^2 = nC_{\mathrm{a}}^2.$$

For each t and $d \in \mathcal{D}_t$, the components of $(S_d(t))_i$ are independent across i and distributed as

$$\mathbb{P}\Big((S_d(t))_i = K_\mathrm{d} d_i\Big) = \frac{1}{K_\mathrm{d}}, \quad \mathbb{P}\Big((S_d(t))_i = 0\Big) = \frac{K_\mathrm{d}-1}{K_\mathrm{d}},$$

where the choice of K_d follows those specified in (EC.16). This confirms that the constructed pair (A,S) belongs to the model class $\mathcal{M}^\rho_\mathcal{I}(B, C_\mathrm{a}, C_\mathrm{d})$. Recall that $D(t) = S_{d(t)}(t)$. So the components $D_i(t)$ are independent across i and distributed as

$$\mathbb{P}\Big(D_i(t) = K_\mathrm{d} d_i(t)\Big) = \frac{1}{K_\mathrm{d}}, \quad \mathbb{P}\Big(D_i(t) = 0\Big) = \frac{K_\mathrm{d}-1}{K_\mathrm{d}}.$$

Let $\{G(t)\}_{t\in\mathbb{N}}$ be an i.i.d. sequence of random variables defined by $G(t) = \sum_{i=1}^n Y_i(t)$, where $Y_i(t)$ are independent across i and distributed as

$$\mathbb{P}\Big(Y_i(t) = K_\mathrm{d} B\Big) = \frac{1}{K_\mathrm{d}}, \quad \mathbb{P}\Big(Y_i(t) = 0\Big) = \frac{K_\mathrm{d}-1}{K_\mathrm{d}},$$

so that

$$\mathbb{P}(G(t) \geq y) \geq \mathbb{P}(\sum_{i=1}^n D_i(t) \geq y) \ \text{ for each } y, \ \mathbb{E}[G(t)] = M, \ \text{ and } \mathrm{Var}(G(t)) = nC_\mathrm{d}^2.$$

The following argument follows the same idea as in the proof of Theorem 1. Let $X(t) = \sum_{i=1}^n A_i(t)$. By (EC.13), the queue length satisfies

$$\sum_{i=1}^n Q_i(T) \geq \max\left\{\sum_{t=1}^{T-1}\Big(X(t) - G(t)\Big), \sum_{t=2}^{T-1}\Big(X(t)-G(t)\Big), \ldots, X(T-1) - G(T-1), 0\right\} + X(T).$$

Observe that

$$X(t) - G(t) = (X(t) - \rho M) + (M - G(t)) - (1-\rho)M.$$

Note that $\{(X(t) - \rho M) - (M - G(t))\}_{t\in\mathbb{N}}$ forms an i.i.d. sequence of random variables, satisfying

$$\mathbb{E}[(X(t) - \rho M) - (M - G(t))] = 0, \mathrm{Var}((X(t) - \rho M) + (M - G(t))) = nC_\mathrm{a}^2 + nC_\mathrm{d}^2,$$

where the variance follows from the independence of all components. Thus, by Lemma EC.3, we obtain the following lower bound: for all $T \geq \psi_2^2 + 1$ and $(1-\rho)M \leq \sigma/\psi_2$,

$$\begin{aligned}
&\mathbb{E}\left[\max\left\{\sum_{t=1}^{T-1}\Big(X(t)-G(t)\Big), \sum_{t=2}^{T-1}\Big(X(t)-G(t)\Big), \ldots, X(T-1)-G(T-1), 0\right\}\right] \\
&= \mathbb{E}\left[\max\left\{0, X(1)-G(1), \sum_{t=1}^{2}\Big(X(t)-G(t)\Big), \ldots, \sum_{t=1}^{T-1}\Big(X(t)-G(t)\Big)\right\}\right] \\
&\geq \frac{c}{2}\min\left\{\sqrt{n(C_\mathrm{a}^2 + C_\mathrm{d}^2)(T-1)}, \frac{C_\mathrm{a}^2 + C_\mathrm{d}^2}{B(1-\rho)}\right\},
\end{aligned}$$

where ψ_2 is the unique positive solution to $\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+x)+\frac{8}{\underline{c}}=x$; $\sigma=\sqrt{nC_{\mathrm{a}}^2+nC_{\mathrm{d}}^2}$; $\underline{c}$ and $\bar{c}$ are the constants defined in Lemmas EC.1 and EC.2 respectively.

Observe that for $\rho\in[1/2,1]$, the random variables $(X(t)-\rho M)+(M-G(t))$ have uniformly bounded support across ρ,

$$|(X(t)-\rho M)+(M-G(t))|\le nB\left(\frac{2C_{\mathrm{a}}^2}{B^2}\vee 1+\frac{C_{\mathrm{d}}^2}{B^2}\vee 1\right)$$

and variance $\sigma^2=n(C_{\mathrm{a}}^2+C_{\mathrm{d}}^2)$. Consequently, their Sakhanenko parameters are uniformly bounded away from zero by a positive constant depending only on n, C_{a}, C_{d} and B. By Lemma EC.2, the constant $\bar{c}$ can therefore be chosen to depend only on n, C_{a}, C_{d} and B.

Let $c_1=\underline{c}/2$ and define $\psi_2(n,B,C_{\mathrm{a}},C_{\mathrm{d}})$ as the unique positive solution to $\frac{4(\bar{c}/\sigma+1)}{\underline{c}}\log(1+x)+\frac{8}{\underline{c}}=x$, we recover the result in (23).

I.2. Proof of Proposition 5

Proof of Proposition 5 Under the assumption that $\mathbb{E}[A(t)]=\lambda\in\mathcal{D}_\star$, we have

$$\sum_{i=1}^{n}\max\{\widehat{A}_i(t)-d_i^{\mathrm{EP}}(t),0\}\le\sum_{i=1}^{n}\max\{\widehat{A}_i(t)-\lambda_i,0\}. \tag{EC.30}$$

Let $\Delta(t)=A(t)-D^{\mathrm{EP}}(t)$. Then

$$\begin{aligned}\Delta(t)&=(A(t)-\lambda)+(\lambda-\widehat{A}(t))+(\widehat{A}(t)-d^{\mathrm{EP}}(t))+(d^{\mathrm{EP}}(t)-D^{\mathrm{EP}}(t))\\&\triangleq\Delta^{(1)}(t)+\Delta^{(2)}(t)+\Delta^{(3)}(t)+\Delta^{(4)}(t).\end{aligned}$$

Under this policy, for each i,

$$\begin{aligned}Q_i(T)-A_i(T)&=\max\left\{\sum_{t=1}^{T-1}\Delta_i(t),\sum_{t=2}^{T-1}\Delta_i(t),\ldots,\Delta_i(T-1),0\right\}\\&\le\sum_{j=1}^{4}\max\left\{\sum_{t=1}^{T-1}\Delta_i^{(j)}(t),\sum_{t=2}^{T-1}\Delta_i^{(j)}(t),\ldots,\Delta_i^{(j)}(T-1),0\right\}.\end{aligned}$$

For the first term, let $S_i^{(1)}(t)=\sum_{s=T-t}^{T-1}\Delta_i^{(1)}(s)$ and $M_i^{(1)}(t)=\max_{1\le s\le t}S_i^{(1)}(s)$ for $t\in\mathbb{N}$. Specifically, let $S_i^{(1)}(0)=M_i^{(1)}(0)=0$. since the variables $\{\Delta_i^{(1)}(s)\}_{s\ge1}$ are independent across s and satisfy $\mathbb{E}[\Delta_i^{(1)}(s)]=0$, $\{S_i^{(1)}(t)\}_{t=0}^{T-1}$ is a martingale. Thus, we have

$$\mathbb{E}\left[\max\left\{\sum_{t=1}^{T-1}\Delta_i^{(1)}(t),\sum_{t=2}^{T-1}\Delta_i^{(1)}(t),\ldots,\Delta_i^{(1)}(T-1),0\right\}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[M_i^{(1)}(T-1)\Big] \le \sqrt{\mathbb{E}\Big[\big(M_i^{(1)}(T-1)\big)^2\Big]} \\
&\le \sqrt{4\mathbb{E}\Big[\big(S_i^{(1)}(T-1)\big)^2\Big]} = 2\sqrt{\sum_{t=1}^{T-1} \mathrm{Var}(A_i(1))},
\end{aligned}
$$

where the first inequality follows from the Jensen's inequality and the second inequality follows from Doob's maximal inequality. Using the same argument, for the forth term, we also have

$$
\mathbb{E}\left[\max\left\{\sum_{t=1}^{T-1}\Delta_i^{(4)}(t), \sum_{t=2}^{T-1}\Delta_i^{(4)}(t), \ldots, \Delta_i^{(4)}(T-1), 0\right\}\right] \le 2\sqrt{\sum_{t=1}^{T-1}\mathrm{Var}(D_i(t))}.
$$

For the second term,

$$
\max\left\{\sum_{t=1}^{T-1}\Delta_i^{(2)}(t), \sum_{t=2}^{T-1}\Delta_i^{(2)}(t), \ldots, \Delta_i^{(2)}(T-1), 0\right\} \le \sum_{t=1}^{T-1}\max\{\Delta_i^{(2)}(t), 0\} \le \sum_{t=1}^{T-1}\big|\Delta_i^{(2)}(t)\big|.
$$

Note that

$$
\mathbb{E}\Big[\big|\Delta_i^{(2)}(t)\big|\Big] \le \sqrt{\mathbb{E}\Big[\big(\Delta_i^{(2)}(t)\big)^2\Big]} = \sqrt{\frac{\mathrm{Var}(A_i(1))}{t}}.
$$

Then

$$
\sum_{t=1}^{T-1}\big|\Delta_i^{(2)}(t)\big| \le \sqrt{\mathrm{Var}(A_i(1))}\sum_{s=1}^{T-1}\frac{1}{\sqrt{t}} \le \sqrt{\mathrm{Var}(A_i(1))}\int_{s=1}^{T-1}\frac{1}{\sqrt{t}}\mathrm{d}t \le 2\sqrt{(T-1)\,\mathrm{Var}(A_i(1))}.
$$

For the third term,

$$
\begin{aligned}
&\sum_{i=1}^{n}\max\left\{\sum_{t=1}^{T-1}\Delta_i^{(3)}(t), \sum_{t=2}^{T-1}\Delta_i^{(3)}(t), \ldots, \Delta_i^{(3)}(T-1), 0\right\} \\
&\le \sum_{i=1}^{n}\sum_{t=1}^{T-1}\max\{\Delta_i^{(3)}(t), 0\} = \sum_{t=1}^{T-1}\sum_{i=1}^{n}\max\{\widehat{A}_i(s) - d_i^{\mathrm{EP}}(t), 0\} \\
&\le \sum_{t=1}^{T-1}\sum_{i=1}^{n}\max\{\widehat{A}_i(s) - \lambda_i, 0\} \le \sum_{t=1}^{T-1}\sum_{i=1}^{n}|\widehat{A}_i(s) - \lambda_i| \\
&= \sum_{t=1}^{T-1}\sum_{i=1}^{n}\big|\Delta_i^{(2)}(t)\big| \le \sum_{i=1}^{n}2\sqrt{(T-1)\,\mathrm{Var}(A_i(t))}.
\end{aligned}
$$

In sum, we have

$$
\begin{aligned}
\mathbb{E}\Big[\sum_{i=1}^{n}Q_i(T)\Big] - \mathbb{E}\Big[\sum_{i=1}^{n}A_i(T)\Big] &\le \sum_{i=1}^{n}6\sqrt{(T-1)\,\mathrm{Var}(A_i(1))} + 2\sum_{i=1}^{n}\sqrt{\sum_{t=1}^{T-1}\mathrm{Var}(D_i(t))} \\
&\le n\sqrt{T-1}(6C_{\mathrm{a}} + 2C_{\mathrm{d}}).
\end{aligned}
$$

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