Appendix

Proof of Proposition 3:

We first prove that U_{i0} *divides* d_i. By definition, $U_{i0}(t) = (jm_i \mod m_0)$ for some non-negative integer j. We may write $jm_i = km_0 + U_{i0}(t)$ for some non-negative integers j, k. Since $d_i = g.c.d.(m_i, m_0)$, $U_{i0}(t)$ *divides* d_i. Taking the limit $j \to \infty$ as $t \to \infty$, U_{i0} *divides* d_i.

Next we will prove that $U_{i0} \sim U\ [0, d_i, 2d_i, \ldots, m_0 - d_i]$. The arguments are based on the renewal approach and the i.i.d. interarrival times of demands. By definition $U_{i0}(0) = 0$. $U_{i0}(.)$ is renewed whenever $U_{i0}(t)$ reaches 0 for any $t \geq 0$. Equivalently, $U_{i0}(.)$ is renewed whenever the demand from retailer i (in terms of the number of batches of size Q) reaches $g = l.c.m.(m_i, m_0) = m_i m_0/d_i$. Since retailer i orders m_i of Q's per each replenishment order, this implies there are altogether $g/m_i = m_0/d_i$ states for all $U_{i0}(t)$. We claim that these m_0/d_i states have to be $[0, d_i, 2d_i, \ldots, m_0 - d_i]$. First, from above, U_{i0} *divides* d_i. Thus U_{i0} is always a multiple of d_i. Second, if $[0, d_i, 2d_i, \ldots, m_0 - d_i]$ are not all the states, then according to the total number of m_0/d_i states for $U_{i0}(t)$, there exists at least one state within $[0, d_i, 2d_i, \ldots, m_0 - d_i]$ that is reached at least twice before $U_{i0}(.)$ is renewed. However, this implies $U_{i0}(.)$ will be renewed before demand reaches gQ, and contradicts the definition of g as the *l.c.m*. Hence $U_{i0}(t)$ is distributed over $[0, d_i, 2d_i, \ldots, m_0 - d_i]$ for all $t \geq 0$. The proposition follows immediately by noting that interarrival times of demands from retailer i to the supplier are i.i.d.

QED

Proof of Proposition 5:

By Proposition 2 and Corollary 4.1, Proposition 5 holds if any of m_i and m_j is relatively prime with m_0. We shall prove that Proposition 5 is true even when this does not hold. Let $d_i = g.c.d.(m_i, m_0)$ and $d_j = g.c.d.(m_j, m_0)$. We have $d_i > 1$ and $d_j > 1$. Let $g = l.c.m.(d_i, d_j)$. Since m_i and m_j are relatively prime, $g = d_i d_j$ and $d_i \neq d_j$.

We shall prove that the proposition is true both when (i) $m_0 = g$ and (ii) $m_0 \neq g$.

(i) We first prove that $(D_{i0} + D_{j0})$ *mod* $g \sim U\,[0, 1, 2, \ldots, g - 1]$. The approach is similar to the proof of Proposition 3. For simplicity, rewrite (D_{i0} *mod* g) as X and (D_{j0} *mod* g) as Y. By Proposition 1, it is equivalent to prove that $(X + Y)$ *mod* $g \sim U\,[0, 1, 2, \ldots, g - 1]$. From Proposition 3, $X \sim U\,[0, d_i, 2d_i, \ldots, g - d_i]$. Similarly $Y \sim U\,[0, d_j, 2d_j, \ldots, g - d_j]$. Let $(X_1, Y_1) = (a_1 d_i, a_2 d_j)$ and $(X_2, Y_2) = (b_1 d_i, b_2 d_j)$ be any two different feasible combinations for (X, Y) where a_1, b_1 $[0, 1, 2, \ldots, (g - d_i)/d_i]$, and a_2, b_2 $[0, 1, 2, \ldots, (g - d_j)/d_j]$,. We claim that $(X_1 + Y_1)$ *mod* $g \neq (X_2 + Y_2)$ *mod* g. Suppose this is not true; then algebraically it can be shown that the fact that $g = l.c.m.(d_i, d_j)$ is contradicted (the proof is available from the authors). Now recall there are altogether $d_i d_j = g$ combinations of (X, Y) and each has a different value for $(X + Y)$ *mod* g. Also, $0 \leq (X + Y)$ *mod* $g <$ g. So the values should be $[0, 1, 2, \ldots, g - 1]$. Furthermore, $(X + Y)$ *mod* g should be uniformly distributed over this set since both of X and Y are uniformly distributed. We conclude $(D_{i0} + D_{j0})$ *mod* $g \sim U\,[0, 1, 2, \ldots, g - 1]$. The proof is complete if $m_0 = d_i d_j = g$.

(ii) If $m_0 \neq g$, then there exists an integer $\alpha > 1$ such that $m_0 = \alpha d_i d_j$, since m_0 *divides* both d_i and d_j, and $d_i \neq d_j$. From Proposition 3, D_{i0} *mod* $m_0 \sim U\,[0, d_i, 2d_i, \ldots, m_0 - d_i]$ and D_{j0} *mod* $m_0 \sim U\,[0, d_j, 2d_j, \ldots, m_0 - d_j]$, which can be rewritten as D_{i0} *mod* $m_0 \sim \{U\,[0, g, 2g, \ldots, (\alpha - 1)g] + U\,[0, d_i, 2d_i, \ldots, g - d_i]\}$ and D_{j0} *mod* $m_0 \sim \{U\,[0, g, 2g, \ldots, (\alpha - 1)g] + U\,[0, d_j, 2d_j, \ldots, g - d_j]\}$. In other words, D_{i0} *mod* $m_0 = ((D_{i0}$ *mod* $m_0)$ *div* $g)g + (D_{i0}$ *mod* $g)$.

Similarly, $D_{j0}\ mod\ m_0 = ((D_{j0}\ mod\ m_0)\ div\ g)g + (D_{j0}\ mod\ g)$. We write $(D_{i0} + D_{j0})\ mod\ m_0 = (((D_{i0}\ mod\ m_0)\ div\ g)g + ((D_{j0}\ mod\ m_0)\ div\ g)g + (((D_{i0}\ mod\ g) + (D_{j0}\ mod\ g))\ div\ g)g + ((D_{i0}\ mod\ g) + (D_{j0}\ mod\ g))\ mod\ g)\ mod\ m_0$.

From above, $(((D_{i0}\ mod\ m_0)\ div\ g)g)\ mod\ m_0 \sim U\ [0, g, 2g, . . . , (\alpha - 1)g]$. Hence by Proposition 2, $(((D_{i0}\ mod\ m_0)\ div\ g)g + ((D_{j0}\ mod\ m_0)\ div\ g)g + (((D_{i0}\ mod\ g) + (D_{j0}\ mod\ g))\ div\ g)g)\ mod\ m_0 \sim U\ [0, g, 2g, . . . , (\alpha - 1)g]$. We also proved $((D_{i0}\ mod\ g) + (D_{j0}\ mod\ g))\ mod\ g \sim U\ [0, 1, 2, . . . , g - 1]$, which implies $(((D_{i0}\ mod\ g) + (D_{j0}\ mod\ g))\ mod\ g)\ mod\ m_0 \sim U\ [0, 1, 2, . . . , g - 1]$. Hence, from Proposition 1, $(D_{i0} + D_{j0})\ mod\ m_0 \sim \{U\ [0, g, 2g, . . ., (\alpha - 1)g] + U\ [0, 1, 2, . . . , g - 1]\}\ mod\ m_0$. We conclude that $(D_{i0} + D_{j0})\ mod\ m_0 \sim U\ [0, 1, 2, . . . , \alpha g - 1]\}$. QED

Proof of Proposition 6:

Recall $U_i(t) = D_i(0, t)\ mod\ Q_i$. It is well known that $U_i \sim U\ [0, 1, . . . , Q_i - 1]$. From Proposition 3, $U_{i0} \sim U\ [0, d_i, . . . , m_0 - d_i]$ where $d_i = g.c.d.(m_i, m_0)$.

We shall prove that the stationary distribution $P\{U_{i0} = a_1, U_i = a_2\} = d_i/m_0Q_i$ holds for all a_1 $[0, d_i, . . . , m_0 - d_i]$ and a_2 $[0, 1, . . . , Q_i - 1]$. Independence between U_{i0} and U_i will then be implied. Let $P\{U_{i0}(t) = a_1, U_i(t) = a_2\} = d_i/m_0Q_i$. Consider a time $t_1 > t$; we want to determine $P\{U_{i0}(t_1) = b_1, U_i(t_1) = b_2\}$ for any b_1 $[0, d_i, . . . , m_0 - d_i]$ and b_2 $[0, 1, . . . , Q_i - 1]$. Let $D_i(t, t_1) = \varepsilon$; $(U_{i0}(t_1) = b_1, U_i(t_1) = b_2 \mid D_i(t, t_1) = \varepsilon)$ holds if and only if $U_{i0}(t) = a_1$ and $U_i(t) = a_2$, where the values of a_1 and a_2 will be specified below.

Because $U_i(t_1) = (U_i(t) + D_i(t, t_1))\ mod\ Q_i$, $b_2 = (a_2 + \varepsilon)\ mod\ Q_i$. Since $0 \le a_2 < Q_i$, $a_2 = c_2Q_i + b_2 - \varepsilon$, where c_2 is the smallest (and the only) non-negative integer such that $0 \le a_2 < Q_i$. Also, $U_{i0}(t_1) = (U_{i0}(t) + ((U_i(t) + D_i(t, t_1))\ div\ Q_i)m_i)\ mod\ m_0$. So $b_1 = (a_1 + ((c_2Q_i + b_2 - \varepsilon + \varepsilon)$

$div\ Q_i)m_i)\ mod\ m_0 = (a_1 + c_2m_i)\ mod\ m_0$. Since $0 \leq a_1 < m_0$, $a_1 = c_1m_0 + b_1 - c_2m_i$ where c_1 is the smallest (and the only) non-negative integer such that $0 \leq a_1 < m_0$. In addition, a_1 $[0, d_i, . . . , m_0 - d_i]$ since $a_1 = c_1m_0 + b_1 - c_2m_i$ in which c_1m_0, b_1 and c_2m_i all *divides* d_i.

The values of a_1 and a_2 are thus available. Hence , $P\{U_{i0}(t_1) = b_1, U_i(t_1) = b_2 \mid D_i(t, t_1) = \varepsilon\} = P\{U_{i0}(t) = a_1, U_i(t) = a_2\} = d_i/m_0Q_i$. By summing up all the probabilities of $P\{D_i(t, t_1) = \varepsilon\}$, the proof is complete.

QED

Table 1: Parameter values for 46 test problems (m_0 = 4 for cases 1-23, m_0 = 8 for parallel cases 24-46)

Case	N	λ_1	λ_2	λ_3	λ_4	Q_1	Q_2	Q_3	Q_4	$\sum_{i=1}^{N} \lambda_i / Q_i$
1	2	2	4	-	-	4	8	-	-	1
2	2	2	6	-	-	4	12	-	-	1
3	2	4	6	-	-	8	12	-	-	1
4	4	1	1	2	2	4	4	8	8	1
5	4	1	1	3	3	4	4	12	12	1
6	4	2	2	3	3	8	8	12	12	1
7	2	4	8	-	-	4	8	-	-	1
8	4	2	2	4	4	4	4	8	8	2
9	4	2	2	6	6	4	4	12	12	2
10	4	4	4	6	6	8	8	12	12	2
11	8	1	1	2	2	4	4	8	8	2
12	8	1	1	3	3	4	4	12	12	2
13	8	2	2	3	3	8	8	12	12	2
14	2	8	8	-	-	4	4	-	-	4
15	4	4	4	8	8	4	4	8	8	4
16	4	8	4	4	4	4	4	8	8	4
17	4	8	4	6	6	4	4	12	12	4
18	8	2	2	4	4	4	4	8	8	4
19	8	2	2	6	6	4	4	12	12	4
20	8	4	4	6	6	8	8	12	12	4
21	8	4	4	8	8	4	4	8	8	8
22	8	8	4	4	4	4	4	8	8	8
23	8	8	4	6	6	4	4	12	12	8

Note: When N = 8, retailers 5-8 are identical to retailers 1-4 respectively.

Table 2: Exact and approximate solutions for expected backorders $E(B_0)$ when $m_0 = 4$ and $Q = 4$

$E(B_0)$ Values

S_0: Case:	0	$2Q$	$4Q$	$6Q$	$8Q$	$10Q$
1	2.5000	0.8182	0.0683	0.0000	0.0000	0.0000
2	2.5000	0.8175	0.0675	0.0000	0.0000	0.0000
3	2.5000	0.8139	0.0639	0.0000	0.0000	0.0000
4	2.5000	0.8432	0.0943	0.0011	0.0000	0.0000
5	2.5000	0.8432	0.0943	0.0011	0.0000	0.0000
6	2.5000	0.8428	0.0938	0.0010	0.0000	0.0000
Poisson Approximation	2.5000	0.8679	0.1248	0.0071	0.0002	0.0000
7	3.5000	1.5773	0.3361	0.0088	0.0000	0.0000
8	3.5000	1.6152	0.3865	0.0214	0.0000	0.0000
9	3.5000	1.6144	0.3850	0.0206	0.0000	0.0000
10	3.5000	1.6107	0.3777	0.0170	0.0000	0.0000
11	3.5000	1.6423	0.4373	0.0463	0.0013	0.0000
12	3.5000	1.6423	0.4373	0.0463	0.0013	0.0000
13	3.5000	1.6418	0.4363	0.0457	0.0013	0.0000
Poisson Approximation	3.5000	1.6692	0.4925	0.0804	0.0075	0.0004
14	5.5000	3.5020	1.6318	0.4116	0.0322	0.0005
15	5.5000	3.5037	1.6390	0.4180	0.0331	0.0005
16	5.5000	3.5046	1.6558	0.4422	0.0418	0.0008
17	5.5000	3.5045	1.6550	0.4408	0.0411	0.0007
18	5.5000	3.5122	1.7049	0.5083	0.0681	0.0025
19	5.5000	3.5120	1.7034	0.5057	0.0666	0.0023
20	5.5000	3.5111	1.6961	0.4930	0.0593	0.0013
Poisson Approximation	5.5000	3.5321	1.8144	0.6838	0.1810	0.0337
21	9.5000	7.5000	5.5008	3.5253	1.7491	0.5622
22	9.5000	7.5000	5.5012	3.5326	1.7792	0.6024
23	9.5000	7.5000	5.5012	3.5322	1.7777	0.6002
Poisson Approximation	9.5000	7.5009	5.5201	3.6465	2.0725	0.9789

Table 3: Exact and approximate solutions for expected backorders $E(B_0)$ when $m_0 = 8$ and $Q = 4$

$E(B_0)$ Values

S_0: Case:	0	$2Q$	$4Q$	$6Q$	$8Q$	$10Q$
1	4.5000	2.6591	1.2841	0.4091	0.0341	0.0000
2	4.5000	2.6588	1.2838	0.4088	0.0338	0.0000
3	4.5000	2.6569	1.2819	0.4069	0.0319	0.0000
4	4.5000	2.6716	1.2972	0.4222	0.0472	0.0005
5	4.5000	2.6716	1.2971	0.4221	0.0471	0.0005
6	4.5000	2.6714	1.2969	0.4219	0.0469	0.0005
Poisson Approximation	4.5000	2.6839	1.3124	0.4375	0.0625	0.0036
7	5.5000	3.5386	1.9180	0.7930	0.1680	0.0044
8	5.5000	3.5576	1.9432	0.8183	0.1933	0.0107
9	5.5000	3.5572	1.9425	0.8175	0.1925	0.0103
10	5.5000	3.5554	1.9389	0.8139	0.1886	0.0085
11	5.5000	3.5712	1.9686	0.8443	0.2193	0.0231
12	5.5000	3.5712	1.9686	0.8443	0.2193	0.0231
13	5.5000	3.5709	1.9681	0.8438	0.2188	0.0228
Poisson Approximation	5.5000	3.5846	1.9962	0.8748	0.2500	0.0404
14	7.5000	5.5010	3.5659	1.9568	0.8320	0.2060
15	7.5000	5.5018	3.5695	1.9608	0.8361	0.2092
16	7.5000	5.5023	3.5779	1.9734	0.8488	0.2215
17	7.5000	5.5023	3.5775	1.9727	0.8481	0.2208
18	7.5000	5.5061	3.6025	2.0103	0.8865	0.2554
19	7.5000	5.5060	3.6017	2.0089	0.8850	0.2540
20	7.5000	5.5055	3.5980	2.0020	0.8777	0.2472
Poisson Approximation	7.5000	5.5160	3.6572	2.1079	0.9977	0.3587
21	11.500	9.5000	7.5004	5.5126	3.6250	2.0438
22	11.500	9.5000	7.5006	5.5163	3.6402	2.0675
23	11.500	9.5000	7.5006	5.5161	3.6394	2.0662
Poisson Approximation	11.500	9.5005	7.5100	5.5737	3.7963	2.3127

Table 4: Cost performance of the Poisson approximation when $K = 1$

Case				Approximate Solution			Exact Solution			Comparisons	
N	λ_i	Q	q	S_0/Q	Q_0/Q	TC_1	S_0/Q	Q_0/Q	TC_2	TC_3	Δ (%)
2	4	4	2	4	4	2.4774	4	3	2.0361	2.0557	0.96
			5	5	4	3.2855	4	3	2.4056	2.5543	6.18
			10	5	3	3.8240	4	2	2.7848	2.8569	2.59
2	4	8	2	2	2	1.7073	2	2	1.3876	1.3876	0.00
			5	3	3	2.3230	2	2	1.7752	1.8501	4.22
			10	3	2	2.6984	3	2	2.0230	2.0230	0.00
2	8	4	2	7	5	3.4920	6	4	2.7347	2.7888	1.98
			5	8	4	4.5862	7	4	3.3337	3.6933	10.79
			10	9	4	5.3967	7	3	3.8055	4.5536	19.66
2	8	8	2	4	4	2.4774	3	2	1.8916	1.9606	3.65
			5	5	4	3.2855	4	3	2.1892	2.3919	9.26
			10	5	3	3.8240	4	2	2.6147	2.7431	4.91
4	4	4	2	7	5	3.4920	6	4	2.8413	2.8754	1.20
			5	8	4	4.5862	7	4	3.4876	3.7659	7.98
			10	9	4	5.3967	7	3	3.9821	4.5918	15.31
4	4	8	2	4	4	2.4774	4	3	2.1188	2.1376	0.89
			5	5	4	3.2855	4	3	2.5710	2.6783	4.17
			10	5	3	3.8240	4	2	2.8897	2.9266	1.28
4	8	4	2	12	6	4.9769	11	5	3.9358	3.9485	0.32
			5	14	6	6.4379	12	5	4.7866	5.1972	8.58
			10	15	6	7.5014	13	5	5.4003	6.0320	11.70
4	8	8	2	7	5	3.4920	6	4	2.6640	2.7315	2.53
			5	8	4	4.5862	7	4	3.2100	3.6356	13.26
			10	9	4	5.3967	7	3	3.6647	4.5271	23.53
8	4	4	2	12	6	4.9769	12	6	4.0769	4.9769	0.00
			5	14	6	6.4379	13	6	5.0141	5.3129	5.96
			10	15	6	7.5014	13	5	5.6546	6.1331	8.46
8	4	8	2	7	5	3.4920	6	4	2.9868	2.9929	0.20
			5	8	4	4.5862	7	4	3.6986	3.8650	4.50
			10	9	4	5.3967	8	4	4.1692	4.6392	11.27
8	8	4	2	22	9	7.0218	21	8	5.5593	5.6095	0.90
			5	24	8	9.0576	22	7	6.8200	7.1681	5.13
			10	26	8	10.547	23	7	7.6865	8.7205	13.45
8	8	8	2	12	6	4.9769	11	5	3.8325	3.8612	0.75
			5	14	6	6.4379	12	5	4.6188	5.1105	10.65
			10	15	6	7.5014	13	5	5.2097	5.9601	14.40
										mean:	6.4%

TC_1: Approximate total cost using Poisson approximation for decision values and cost evaluation

TC_2: True optimal total cost

TC_3: True total cost from Poisson approximation decision values

Δ (%): $[(TC_3 - TC_2) / TC_2]100\%$