Online Companion for

"Analysis of the Effects of Uncertainty, Risk-pooling and Subcontracting Mechanisms on Project Performance"

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1 Appendix

1.1 Proof of Theorem 2

Let $\{x_1, x_2, x_3\}$ be the support of each of the random variables and assume that $x_1 < x_2 < x_3$ without any loss in generality.

Theorem 2 *(i) Suppose Supp(.)=3 and* $E(X) = E(\hat{X})$*. Then* $Var(\hat{X}) \geq Var(X)$ *implies* $E(\hat{Z}) \geq E(Z)$*.*

(ii) Given any integer N *greater than 3, there exist random variables* $X_N, \hat{X}_N$ *and* Y_N *defined on a support of* N *points such that* $E(\hat{X_N}) = E(X_N), Var(\hat{X_N}) > Var(X_N)$ *and* $E(\hat{Z_N}) < E(Z_N)$*.*

Proof:(i) Since the mean is just a weighted sum of x_1, x_2, and x_3, it follows that for $\hat{X}$ there must be less weight (probability) attached to x_2 and more weight attached to x_1 and x_3, compared with X. Now it follows from Fact 2 in Section 3 (by considering the distribution functions of X and $\hat{X}$) that $\hat{X} \geq_{cx} X$. Hence, by Theorem 1, it follows that $E(\hat{Z}) \geq E(Z)$.

(ii) We know the claim is true for $N = 4$ by the Counterexample to Sconberger's claim in Section 3. Suppose the claim is true for $N = k$. Then, by adding one point carrying a sufficiently small probability to the support, it follows that the claim is true for $N = k + 1$. Hence, the claim is true for all integral N greater than or equal to 4. □

1.2 Proof of Proposition 4

We treat the case when there are two contractors and two serial subprojects and show that the results obtained in Section 5 under the assumption of constant probabilities carry over to the case when the win probabilities are realizations of a random vector (P_1, P_2) with some distribution H. In the case of Splitting, let the realizations be p_1,p_2 for subproject 1 and q_1, q_2 for subproject 2. Let the realizations be r_1, r_2 in the case of Pooling. Forming the MGFs and differentiating in a parallel fashion to that in Section 5, we obtain the *conditional* expectation of project completion time in the case of Splitting to be $q_1\mu_1^2 + q_2\mu_2^2 + p_1\mu_1^1 + p_2\mu_2^1$. Unconditioning and letting E_P denote the mean of P_1 (and therefore $1 - E_P$ the mean of P_2), we obtain $\dot{F}(0)$ $= E_P\{(\mu_1^2 + \mu_1^1) - (\mu_2^2 + \mu_2^1)\}$. A little algebra shows that the expected *conditional* project completion time in the case of Pooling works out to $r_1(\mu_1^2+\mu_1^1)+r_2(\mu_2^2+\mu_2^1)$. Unconditioning, we obtain $\dot{G}(0)=\dot{F}(0)$.

Differentiating once more and unconditioning, we obtain:
$\ddot{F}(0)=2(E_P\mu_1^1+(1-E_P)\mu_2^1)(E_P\mu_1^2+(1-E_P)\mu_2^2)+E_P(\phi_1^1+\phi_1^2)+(1-E_P)(\phi_2^2+\phi_2^1)$ and
$\ddot{G}(0)=E_P(2\mu_1^1\mu_1^2+\phi_1^1+\phi_1^2)+(1-E_P)(2\mu_2^1\mu_2^2+\phi_2^2+\phi_2^1)$.
Now a little algebra shows that $\ddot{F}(0)\leq\ddot{G}(0)$ if the mean completion times of the contractors are ordered.

Recall the conditional variance formula: $Var(X)=E[Var(X/Y)]+Var(E[X/Y])$. Now $\ddot{F}(0)-[\dot{F}(0)]^2$ represents the first term of the conditional variance formula; likewise for G. We need to compute the second term to compare the variance of completion time under Splitting and Pooling.
For Splitting, the second term is $\{(\mu_1^2)^2+(\mu_2^2)^2+(\mu_1^1)^2+(\mu_2^1)^2)\}\{Var(P)\}$. For Pooling, the second term is $\{(\mu_1^2+\mu_1^1)^2+(\mu_2^2+\mu_2^1)^2\}\{Var(P)\}$, which is identically larger than the second term under Splitting.
Thus, if the mean completion times of the contractors are ordered (or equivalently, if the contractors are consistent), Splitting and Pooling result in the same mean completion time but the variance of completion time under Splitting is smaller. $\Box$

1.3 Proof of Proposition 5

The following simple counter-example with 2 contractors and a 2-activity parallel project substantiates the claim.
We have: $w \;=\; p\,\max(X_1,Y_1)+(1-p)\,\max(X_2,Y_2)$ and
$v \;=\; p^2\,\max(X_1,Y_1)+(1-p)^2\,\max(X_2,Y_2)+p(1-p)[\max(X_1,Y_2)+\max(X_2,Y_1)]$
We define our random variables as follows:
$X_1=10\ w.p\ \frac{1}{2}$ and $20\ w.p\ \frac{1}{2}$
$X_2=10\ w.p\ \frac{1}{2}$ and $30\ w.p\ \frac{1}{2}$
$Y_1=10\ w.p\ 1$
$Y_2=25\ w.p\ 1$
Clearly, X_2 is stochastically greater than X_1 and Y_2 is stochastically greater than Y_1 and hence

the contractors are pointwise consistent.

$Z_1 = \max(X_1, Y_1) = 10 \ w.p \ \frac{1}{2} \ and \ 20 \ w.p \ \frac{1}{2}$

$Z_2 = \max(X_2, Y_2) = 25 \ w.p \ \frac{1}{2} \ and \ 30 \ w.p \ \frac{1}{2}$

$\xi = \max(X_1, Y_2) = 25 \ w.p \ 1$

$\phi = \max(X_2, Y_1) = 10 \ w.p \ \frac{1}{2} \ and \ 30 \ w.p \ \frac{1}{2}$

$\bar{Z_1} = 15; \bar{Z_2} = 55/2; \bar{\xi} = 25; \bar{\phi} = 20.$

$Var(Z_1) = 25; Var(Z_2) = 6.25; Var(\xi) = 0; Var(\phi) = 100.$

$Cov(Z_1, \xi) = Cov(Z_2, \xi) = 0$ since $Var(\xi) = 0$.

It can be verified that $Pr(Z_1 = a, \phi = b) = 1/4$ for all 4 possible combinations of values of a and b and consequently we have $Cov(Z_1, \phi) = 0$.

To compute $Cov(Z_2, \phi)$ we need the joint density function of Z_2 and ϕ.

$$\begin{aligned}
P(Z_2 = 25; \phi = 10) &= P(\max(X_2, Y_2) = 25; \max(X_2, Y_1) = 10) \\
&= P(\max(X_2, Y_2) = 25; X_2 = 10) \\
&= P(\max(10, Y_2) = 25; X_2 = 10) \\
&= P(X_2 = 10) \\
&= \frac{1}{2}.
\end{aligned}$$

$P(Z_2 = 25; \phi = 30) = P(X_2 = 10; \max(X_2, Y_1) = 30) = 0$

$P(Z_2 = 30; \phi = 10) = P(X_2 = 30; \max(X_2, Y_1) = 10) = 0$

$P(Z_2 = 30; \phi = 30) = P(X_2 = 30; \max(X_2, Y_1) = 30) = 1/2$

$Cov(Z_2, \phi) = E(Z_2\phi) - E(Z_2)E(\phi) = 25$

$Var(w) = p^2 Var(Z_1) + (1-p)^2 Var(Z_2) = 25p^2 + \frac{25}{4}(1-p)^2$

$Var(v) = (25)p^4 + (1-p)^4 \frac{25}{4} + 100p^2(1-p)^2 + 2p(1-p)^3.25$

$Var(w) - Var(v) = 25[4p^2 + (1-p)^2 - 4p^4 - (1-p)^4 - 16p^2(1-p)^2 - 8p(1-p)^3] = 25f(p)$

Simplifying, $f(p) = -6p + 7p^2 + 12p^3 - 13p^4$

$f(1/2) = -9/16$ while $f(0.8) = 0.4992$

This proves the claim. □

1.4 Counterexample for Property 3 of associated random variables (page 24)

Let W, X, Y and Z be random variables with the following marginal and joint distributions (t and s are as yet unspecified real variables) :

x	$Pr(X=x)$	y	$Pr(Y=y)$	z	$Pr(Z=z)$
1	1/3	2	1/3	3	1/3
2	1/3	3	1/3	s	1/3
t	1/3	4	1/3	4	1/3

$Pr(X=1, Y=2, Z=3)=1/3$

$Pr(X=2, Y=3, Z=s)=1/3$

$Pr(X=t, Y=4, Z=4)=1/3$

We have, by direct computation,:

$Cov(X,Y)=(t-1)/3$; $Cov(Y,Z)=1/3$; $Cov(X,Z)=(3s+19t-12-st)/9$ while

$E(XYZ)-E(X)E(Y)E(Z)=(3s+9t-15-st)/3$

Now setting s to 0 and t to any value between 1 and 15/9 forces $E(XYZ)-E(X)E(Y)E(Z)$ to a negative value and this implies, by taking the contrapositive of the statement of the theorem above, that X, Y and Z cannot be associated random variables. On the other hand, each of the three pairwise covariances remains positive. This justifies statement (3).