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Appendix

Proof of Lemma 3.1. From (3.2) it follows that

$$\sigma_{Ci}^2 = (1\text{-}i/n)^2\Sigma_{j\ \ [n]}(1+\alpha(m+n\text{-}j))^2\sigma_d^2 + \Sigma_{j\ \ [m+i]}(1+\alpha(m+i\text{-}j))^2\sigma_d^2. \qquad \text{(A.1)}$$

Substituting the following identify for a quadratic series

$$\Sigma_{j\ \ [i]}[1+\alpha(i+k\text{-}j)]^2\sigma_d^2 = i\sigma_d^2 + [i^3\alpha^2/3 + i^2(\alpha\text{-}\alpha^2/2) - i(\alpha\text{-}\alpha^2/6) + i^2k\alpha^2 + i(\alpha^2k^2+2\alpha k\text{-}\alpha^2k)]\sigma_d^2 \qquad \text{(A.2)}$$

into (A.1) and letting x = i/n, and y = m/n we get

$$\sigma_{Ci}^2 = [(n^3\alpha^2/3)H_3(x,y) + n^2(\alpha\text{-}\alpha^2/2)H_2(x,y) + n(1\text{-}\alpha+\alpha^2/6)H_1(x,y)]\sigma_d^2 \qquad \text{(A.3)}$$

where

$$H_k(x,y) = (1+y)^k – [(1+y)^k – y^k](x\text{-}x^2) – (x\text{-}x^k) \text{ for } k \quad \{1,2,3\}. \qquad \text{(A.4)}$$

If $\alpha = 0$, then

$1 \geq \theta_i = [1 – (x\text{-}x^2)/(1+y)]^{1/2} \geq [\min_{x\ \ [0,1]}\{1 – (x - x^2)/(1+y)\}]^{1/2} = \{1 – 1/[4(1+y)]\}^{1/2} \geq (3/4)^{1/2} \approx .866.$

If $\alpha > 0$, it follows from $1 \geq 1 – (x\text{-}x^2)/(1+y) \geq H_k(x,y)/H_k(1,y) \geq 1 – (x\text{-}x^2) – (x\text{-}x^3)$ for k {1,2,3} that

$1 \geq [1 – (x\text{-}x^2)/(1+y)]^{1/2} \geq \theta_i \geq [1 – (x\text{-}x^2) – (x\text{-}x^3)]^{1/2} \geq [\min_{x\ \ [0,1]}\{1 –(x\text{-}x^2) – (x\text{-}x^3)\}]^{1/2} \approx .607.$ ❑

Proof of Theorem 3.1. If we let $z = I_T/\sigma_{Cn}$ and $\theta_i = \sigma_{Ci}/\sigma_{Cn}$, then $EBH(n,I_T)$ can be written as

$$EBH(n,I_T) = I_T C_h + \sigma_{Cn}(C_h+C_b)(1/n)\Sigma_{i\ \ [n]}G(z/\theta_i)\theta_i. \qquad (A.5)$$

From Lemma 3.1, it follows that for any given n and I_T, there exists θ [.607, 1] (or θ [.866, 1] if $\alpha = 0$) such that

$$EBH(n,I_T) = I_T C_h + \sigma_{Cn}(C_h+C_b)G(z/\theta)\theta. \qquad (A.6)$$

Thus, if I_T^* is the optimal target buffer stock for given n, then expected backorder and holding cost can be expressed as

$$EBH(n,I_T^*) = [(z/\theta)C_h + (C_h+C_b)G(z/\theta)]\theta\sigma_{Cn}, \qquad (A.7)$$

which can be interpreted as the expected backorder and holding cost in a single period with the standard deviation in period-ending inventory equal to $\theta\sigma_{Cn}$. Let F(x) denoted the cumulative standard normal probability distribution function. From standard newsvendor analysis, it follows that

$$\theta F^{-1}(C_b/(C_h+C_b)) = \text{argmin}_z\{[(z/\theta)C_h + (C_h+C_b)G(z/\theta)]\theta\sigma_{Cn}\}, \qquad (A.8)$$

which in turn implies I_T^* [$.607z_{SL}\sigma_{Cn}$, $z_{SL}\sigma_{Cn}$] (or I_T^* [$.866z_{SL}\sigma_{Cn}$, $z_{SL}\sigma_{Cn}$] if $\alpha = 0$).

Let n^* denoted the optimal cycle length for a given target buffer stock I_T. From (A.6), the minimum expected cost per period can be expressed as

$$EC_1(n^*,I_T) = S_o G(0)\sigma_\Delta/n^* + [zC_h + (C_h+C_b)G(z/\theta)\theta]\sigma_{Cn} \qquad (A.9)$$

for some θ [.607, 1] (or θ [.866, 1] if $\alpha = 0$). Letting $r = m/n^*$, σ_{Cn} and σ_Δ can be written as

$$\sigma_{Cn} = \{(\alpha^2/3)[n^*(1+r)]^3 + (\alpha-\alpha^2/2)[n^*(1+r)]^2+(1-\alpha+\alpha^2/6)[n^*(1+r)]\}^{1/2}\sigma_d \qquad (A.10)$$

$$\sigma_\Delta = \{(8\alpha^2/3 + 4\alpha^2 r + 2\alpha^2 r^2)n^* + (4\alpha+2\alpha r-2\alpha^2) + (2-2\alpha+\alpha^2/3)/n^*\}^{1/2}\sigma_d. \qquad (A.11)$$

The coefficients on $(n^*)^k$ for k {1,2,3} in (A.10) are all nonnegative and sum to $U^2 = (\alpha^2/3)(1+r)^3 + (\alpha-\alpha^2/2)(1+r)^2+(1-\alpha+\alpha^2/6)(1+r)$; the coefficients on $(n^*)^k$ for k {-1,0,1} are all nonnegative and sum to $V^2 = \alpha^2(1+4r+2r^2) + \alpha(2+2r) + 2$. Therefore, there exists β [1/2,3/2) and γ (1/2,3/2] such that

$$EC_1(n^*,I_T) = S_oG(0)V\sigma_d(n^*)^{-\gamma} + [zC_h + (C_h+C_b)G(z/\theta)\theta]U\sigma_d(n^*)^{\beta}. \quad (A.12)$$

A function of the form $f(n) = An^a + Bn^b$ with $A > 0$, $B > 0$, $a < 0$, and $b > 0$, is minimized at

$n^* = [(-aA)/(bB)]^{1/(b-a)}$ and the ratio of $A(n^*)^a$ to $B(n^*)^b$ is $-b/a$ (Duffin, Peterson, and Zener 1967). Therefore, n^* is the optimal value of n and $EC_1(n^*,I_T)$ is the optimum for the following alternative problem

$$\mathrm{Min}_n\{[S_oG(0)V\sigma_d]n^{-\gamma} + [zC_h + (C_h+C_b)G(z/\theta)\theta]U\sigma_d n^{\beta}\} \quad (A.13)$$

for which

$$\{S_oG(0)V\sigma_d(n^*)^{-\gamma}\}/\{[zC_h + (C_h+C_b)G(z/\theta)\theta]U\sigma_d(n^*)^{\beta}\} = EPC_1(n^*)/EBH(n^*,I_T) = \beta/\gamma \quad [1/3,3). \quad (A.14)$$

When $\alpha = 0$, it follows from (A.10) and (A.11) that $U = (1+r)^{1/2}$, $V = 2^{1/2}$, $\beta = 1/2$, and $\gamma = 3/2$. ❑