Online Companion for

“Optimal Lot Sizes with Geometric Production Yield and Right Demand”

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APPENDIX A: PROOFS OF OBSERVATIONS FOR THE GENERAL CASE

Observation 1: For all D=1,2..., Max N(D)≤ D.

Furthermore, for all n≥D, U(n+1,D) - U(n,D) = C(n+1) - C(n) = β_{n+1}.

Proof: For all n≥D, V(D-n) = V(D-n-1) = 0, and both parts of the observation follow immediately from equation (2). •

Observation 2: For all D = 1,2..., V(D) > V(D-1).

Proof: If $1 \in N(D)$ then from equation (1), $V(D) - V(D-1) = \dfrac{\alpha + \beta_1}{q_1} > 0$.

If $n+1 \in N(D)$ for some n≥1 , use V(D) = W(n+1,D) and V(D-1)≤ W(n,D-1) to get

V(D) - V(D-1)≥ W(n+1,D) - W(n,D-1)

$$= \{C(n+1) + \sum_{k=1}^{n+1} P[G_{n+1}=k]\, V(D-k)\} /q_1$$

$$- \{C(n) + \sum_{k=1}^{n} P[G_n =k]\, V(D-1-k)\} /q_1$$

$$= \frac{\beta_{n+1}}{q_1} + \frac{1}{q_1} \{ \sum_{k=1}^{n-1} [Q_k-Q_{k+1}][V(D-k) - V(D-1-k)]$$

$$+ [Q_n-Q_{n+1}]V(D-n) + Q_{n+1}V(D-1-n) - Q_n V(D-1-n)\}$$

$$= \frac{\beta_{n+1}}{q_1} + \frac{1}{q_1} \sum_{k=1}^{n} [Q_k-Q_{k+1}][V(D-k) - V(D-1-k)] \geq \frac{\beta_{n+1}}{q_1} > 0$$ •

Observation 3:

(a) Suppose marginal costs are bounded away from zero, i.e., $\beta_i \geq \beta > 0$ for all i = 1,2,..., and suppose further that $Q_i \to 0$ as $i \to \infty$. Then

for all D = 1,2,... , we have that max N(D) ≤ M

where M = max{i = 1,2,... : $\beta_i \leq Q_i V(1)$}.

(b) If, in addition, the hazard rates are bounded away from zero, i.e., $q_i \leq q < 1$ for all i = 1,2,..., then

for all D = 1,2,... , we have that max N(D) ≤ θ

where $\theta = \dfrac{\log V_1 - \log \beta}{\log(1/q)}$, i.e., $q^{\theta} V(1) = \beta$.

Proof:

(a) First we note that for all D_1 and D_2, $D_1 \geq 1$, $D_2 \geq 1$, we have that $V(D_1+D_2) \leq V(D_1)+V(D_2)$, since it is always possible to satisfy a rigid demand for D_1+D_2 units by delivery of D_1 units and then D_2 units. In particular, we have, for all n and D,

$V(D-n) - V(D-n-1) \leq V(1)$, and hence from equation (2)

$U(n+1,D) - U(n,D) \geq \beta_{n+1} - Q_{n+1}V(1)$

For all $n \geq M$, $\beta_{n+1} - Q_{n+1}V(1) > 0$, hence $U(n+1,D) > U(n,D)$, and the result follows.

(b) For $n \geq \theta$, $Q_{n+1}V(1) \leq q^{n+1} V(1) < \beta \leq \beta_{n+1}$, hence $U(n+1,D) > U(n,D)$, as above. •

Observation 4: Suppose that marginal costs are non-decreasing, i.e., $\beta_i \leq \beta_{i+1}$ for all i. Then for D = 2,3,... , max N(D)≤ min N(D-1) + 1.

Proof: Suppose m = min N(D-1) and m+1 < n+1 ≤ D. Then from equation (2)

$$U(n,D-1) - U(m,D-1) = {}_{k=m}^{n-1} [U(k+1,D-1) - U(k,D-1)]$$

$$= {}_{k=m}^{n-1} \{\beta_{k+1} - Q_{k+1} [V(D-k-1)-V(D-k-2)]\} \geq 0$$

where the non-negativity derives from m   N(D-1). Similarly,

$$U(n+1,D) - U(m+1,D) = {}_{k=m+1}^{n} \{\beta_{k+1} - Q_{k+1} [V(D-k)-V(D-k-1]\}$$

$$= {}_{k=m}^{n-1} \{\beta_{k+2} - Q_{k+2} [V(D-k-1)-V(D-k-2)]\}.$$

Now $\beta_{k+2} \geq \beta_{k+1}$, $Q_{k+2} < Q_{k+1}$, and $V(D-k-1)-V(D-k-2) > 0$ for all $k \leq n-1 \leq D-2$, together with n > m, imply U(n+1,D) - U(m+1,D) > 0, hence n+1   N(D), and the observation follows. •

Observation 5: Suppose that marginal costs are non-decreasing, i.e., $\beta_i \leq \beta_{i+1}$ for all i.

Then (i) D   N(D) implies N(d) = {d} for d = 1,...,D-1; and

(ii) D   N(D) implies d   N(d) for d = D+1, D+2,....

Proof In view of Observation 1 and Observation 4, we have

(i) D   N(D)   max N(D)=D   min N(D-1)≥ max N(D)-1=D-1   N(D-1)={D-1} etc.

(ii) D $N(D)$ $\min N(D) \le D - 1$ $\max N(D+1) \le D$ $D+1$ $N(D+1)$ etc. •

APPENDIX B: PROOF OF LEMMA 3.2

Lemma 3.2: Suppose that hazard rates are constant, i.e., $q_i = q$ for all i, and that costs are such that ϕ exists. Then,

(a) for D = 1,2,..., and n = 1,...,D,

$|S(n,D)| \leq [V(1) - \phi]\, Q^D$

(b) $V(D+1) - V(D) \rightarrow \phi$ as $D \rightarrow \infty$.

Proof of Lemma 3.2

Part (a):

First recall equation (5):

$$U(n,D) = V(D) + [\, f(n) - \varphi]\ \sum_{k=1}^{n} q^k \ - S(n,D) \qquad (5)$$

This equation implies the following relationships:

(i) For all n, $U(n,D) \geq V(D)$ implying $S(n,D) \leq [\, f(n) - \phi]\ \sum_{k=1}^{n} q^k$.

For r N(D), $U(r,D) = V(D)$ implying $S(r,D) = [\, f(r) - \phi]\ \sum_{k=1}^{r} q^k$.

(ii) For all n, $[\, f(n) - \phi] \geq 0$; (by definition of ϕ)

For m N_o, $[\, f(m) - \phi] = 0$; (by definition of N_o)

Hence:

For r N(D), $S(r,D) \geq 0$

For m N_o, $S(m,D) \leq 0$.

The statement in part (a) of the lemma holds for n = D = 1. We assume inductively that it holds for d = 1,...,D-1, and n = 1,...,d, and prove that it holds for D and n = 1,...,D.

For convenience, define also S(n,D) = 0 if n = 0 or D = 0. From the definition of S, we have for i = 1,…,j:

$$S(j,D) = S(i,D) + q^i\, S(j-i,D-i)$$

In what follows, let r   N(D), m N_o and n   {1,...,D}. To avoid messier notation, we consider separately each of the following four cases.

(i) $n \leq \min\{m,r\}$:

$S(r,D) = S(n,D) + q^n S(r-n,D-n) \geq 0$

$S(n,D) \geq - q^n S(r-n,D-n) \geq - q^n [V(1) - \phi] q^{D-n} = - [V(1) - \phi] q^D$

$S(m,D) = S(n,D) + q^n S(m-n,D-n) \leq 0$

$S(n,D) \leq - q^n S(m-n,D-n) \leq q^n [V(1) - \phi] q^{D-n} = [V(1) - \phi] q^D$

Thus, $|S(n,D)| \leq [V(1) - \phi] q^D$

(ii) $n \geq \max\{m,r\}$:

$S(r,D) = S(n,D) - q^r S(n-r,D-r) \geq 0$

$S(n,D) \geq q^r S(n-r,D-r) \geq - q^r [V(1) - \phi] q^{D-r} = - [V(1) - \phi] q^D$

$S(m,D) = S(n,D) - q^m S(n-m,D-m) \leq 0$

$S(n,D) \leq q^m S(n-m,D-m) \leq q^m [V(1) - \phi] q^{D-m} = [V(1) - \phi] q^D$

Similarly for,

(iii) $m \leq n \leq r$:

$S(r,D) = S(n,D) + q^n S(r-n,D-n) \geq 0$ exactly as in (i)

$S(m,D) = S(n,D) - q^m S(n-m,D-m) \leq 0$ as in (ii)

(iv) $r \leq n \leq m$:

$S(r,D) = S(n,D) - q^r S(n-r,D-r) \geq 0$ as in (ii)

$S(m,D) = S(n,D) + q^n S(m-n,D-n) \leq 0$ as in (i)

In all four cases, $|S(n,D)| \leq [V(1) - \phi] q^D$, proving the lemma by induction.

Part (b):

Follows immediately from part (a), as $V(D+1) - V(D) - \phi = S(1,D+1)/q \to 0$. ♦

APPENDIX C: PROOFS OF SOME RESULTS FOR THE STANDARD CASE

We start with a technical observation and a lemma.

Observation C-1 $$\sum_{j=i}^{k} j\, q^j = \frac{iq^i}{1-q} + q^{i+1}\,\frac{1-q^{k-i}}{(1-q)^2} - \frac{kq^{k+1}}{1-q}$$

To verify this, define $\sigma = \sum_{j=i}^{k} j\, q^j = i\, q^i + (i+1)\, q^{i+1} + \ldots + k\, q^k$

and thus, $q\sigma = \sum_{j=i}^{k} j\, q^{j+1} = i\, q^{i+1} + \ldots + (k-1)\, q^k + k\, q^{k+1}$

$$(1-q)\sigma = i\, q^i + [\, q^{i+1} + q^{i+2} + \ldots + q^k\,] - k\, q^{k+1}$$

Dividing by (1-q), and summing the geometric-series, gives the stated observation.

The following lemma is used in the proof of Theorem 2.

DefineA = $\frac{1-q}{q}\alpha + 1$, and B = $\frac{1-q}{q}$.

Lemma C.1

If n-1 N(d-1) and n N(d),

then V(d) - V(d-1) = A + Bn.

Proof

According to equation (1), under the postulated conditions

$$V(d-1) = U(n-1,d-1) = V(d-1) + \alpha + n - 1 - \sum_{k=1}^{n-1} q^k\, [V(d-k) - V(d-1-k)]$$

hence $\alpha + n - 1 = \sum_{k=1}^{n-1} q^k\, [V(d-k) - V(d-1-k)]$ (C.1)

Similarly,

$$V(d) = U(n,d) = V(d) + \alpha + n - \sum_{k=1}^{n} q^k\, [V(d+1-k) - V(d-k)]$$

$$\alpha + n = \sum_{k=1}^{n} q^k\, [V(d+1-k) - V(d-k)] \qquad \text{(C.2)}$$

Subtracting (C.1) from (C.2), we get

$$1 = q[V(d) - V(d-1)] + \sum_{k=2}^{n} q^k\, [V(d+1-k) - V(d-k)] - \sum_{k=1}^{n-1} q^k\, [V(d-k) - V(d-1-k)]$$

$$= q[V(d) - V(d-1)] + \sum_{k=1}^{n-1} q^{k+1}\, [V(d-k) - V(d-1-k)] - \sum_{k=1}^{n-1} q^k\, [V(d-k) - V(d-1-k)]$$

implying

$$V(d) - V(d\text{-}1) = \frac{1}{q} + \frac{1-q}{q} \sum_{k=1}^{n-1} q^k \, [V(d\text{-}k) - V(d\text{-}1\text{-}k)]$$

and using (C.1) once more, we finally get

$$V(d) - V(d\text{-}1) = \frac{1}{q} + \frac{1-q}{q}(\alpha + n - 1) \quad = A + Bn. \qquad \blacklozenge$$

To state and prove Theorem 2, recall the following definitions from Sections 2 and 4.

$$\theta \quad = \frac{\ln \frac{1+\alpha}{q}}{\ln \frac{1}{q}}, \qquad \text{i.e.,} \quad q^{\theta} \, \frac{1+\alpha}{q} = 1$$

$$\xi \;=\; \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha / \ln(1/q)}}{2}, \qquad \text{i.e.,} \quad \xi(\alpha+\xi)\ \ln(1/q) = \alpha$$

$$\lfloor \xi \rfloor \;=\; \text{Max}\ \{n = 0,1,2,... \ : n \le \xi\ \}$$

$$\kappa \;= \lfloor \xi \rfloor \quad \text{if} \quad q \le \frac{\lfloor \xi \rfloor}{(\lfloor \xi \rfloor + 1)} \, \frac{(\alpha + \lfloor \xi \rfloor + 1)}{(\alpha + \lfloor \xi \rfloor)}$$

$$= \lfloor \xi \rfloor + 1 \quad \text{otherwise} \qquad (\kappa \text{ is a strictly positive integer})$$

$$L_1 = \theta \qquad \text{if } \kappa = 1$$

$$= \kappa + \frac{\log \frac{\alpha + \kappa}{\kappa}}{\log \frac{1}{q}} \qquad \text{if } 1 < \kappa \le \frac{q}{1-q}\alpha$$

$$= \frac{q}{1-q}\alpha \; + 1 \qquad \text{if } \kappa > \frac{q}{1-q}\alpha \text{ and } \kappa > 1$$

$$L \;= \lfloor L_1 \rfloor$$

Theorem 2

For D = 1,2,..., D N(D) if and only if D ≤ L.

Proof of Theorem 2

Note that θ > 1. We first analyze the case θ < 2: By Observation 3, max N(D) ≤ θ for all D, implying N(D) = {1} for all D, and thus D N(D) if and only if D = 1. Therefore, for this case it remains to show that L=1 or equivalently, that $1 \le L_1 < 2$. Rewriting the expression for θ, we get that $q^{\theta-1}(1+\alpha)=1$. Combining this equality with $1 < \theta < 2$ we obtain that $q(\alpha+1)<1$, i.e., $\alpha < \frac{1}{q} - 1$, i.e., $\frac{q}{1-q}\alpha < 1$. Hence, by definition of L_1, L_1 can either be equal to θ or $L_1 = \frac{q}{1-q}\alpha + 1$, and in both cases we get $1 < L_1 < 2$ which implies L = 1.

For θ ≥ 2, the theorem is proved by induction. 1 D (1) and $1 < L_1$ hold trivially. Assuming inductively that d N(d) for d = 1, ..., D, we now show that D+1 N(D+1) if and only if D+1≤L. (To complete the proof, we will show later that for all D > L, D N(D).)

Clearly,

D+1 N(D+1) ⇔ U (D+1, D+1) ≤ U (m,D+1) for m = 1, ..., D.

Thus, we will show that U (D+1, D+1) ≤ U (m,D+1) for m = 1, ..., D if and only if D+1≤L.We partition the proof of this statement into three steps: In the first step of the proof we will show that the condition U(D+1, D+1)≤U(m,D+1) for m = 1, ..., D is equivalent to the following condition: $q^{D+1} \ge \max_{x=1,\ldots,D} y(x)$ where $y(x)=\frac{xq^x}{\alpha + x}$. As a result we prove in the next two steps that $q^{D+1} \ge \max_{x=1,\ldots,D} y(x)$ if and only if D+1≤L. In the second step we will show that the function y(x) is unimodal on x≥0 and it is maximized at x=ξ. In addition, we will show that κ, which is equal to either ë ξûor to ë ξû+1, maximizes y(x) on x=1,2,...Therefore,

$$\max_{x=1,\ldots,D} y(x) = y(D) \quad \text{if } D \le \kappa$$

$$= y(\kappa) \quad \text{if } D > \kappa$$

In the third step we will prove the desired result, i.e., $q^{D+1} \geq \max_{x=1,\ldots,D} y(x)$ if and only if $D+1 \leq L$.

We start with the first step of the proof and note that

$$U(D+1, D+1) - U(m, D+1) = \sum_{j=m}^{D} [U(j+1, D+1) - U(j, D+1)]$$
$$= \sum_{j=m}^{D} \{1 - q^{j+1}[V(D+1-j) - V(D-j)]\}.$$

For j = 1,..., D-1, $2 \leq D+1-j \leq D$, hence by the induction assumption both D-j   N(D-j) and D+1-j   N(D+1-j) hold, and we use Lemma C.1 to get

$$U(D+1, D+1) - U(m,D+1) = D-m+1 - \sum_{j=m}^{D-1} q^{j+1}[A+B(D+1-j)] - q^{D+1}\frac{\alpha+1}{q}$$

where $A = \frac{1-q}{q}\alpha + 1$ and $B = \frac{1-q}{q}$. Thus,

$$U(D+1, D+1) - U(m, D+1) = D-m+1 - [A+B(D+1)] \sum_{j=m}^{D-1} q^{j+1} + B\sum_{j=m}^{D-1} j\, q^{j+1} - q^{D}(\alpha+1)$$

Observe that, $\sum_{j=m}^{D-1} q^{j+1} = q^{m+1}\frac{1-q^{D-m}}{1-q}$

and from Observation C-1

$$\sum_{j=m}^{D-1} j\, q^{j+1} = \frac{mq^{m+1}}{1-q} + q^{m+2}\frac{1-q^{D-m-1}}{(1-q)^2} - \frac{(D-1)q^{D+1}}{1-q}.$$

Thus,

$$U(D+1, D+1) - U(m, D+1) = D-m+1 - \{\frac{1-q}{q}\alpha + 1 + \frac{1-q}{q}(D+1)\}\, q^{m+1}\frac{1-q^{D-m}}{1-q}$$
$$+ \frac{1-q}{q}\left\{\frac{mq^{m+1}}{1-q} + q^{m+2}\frac{1-q^{D-m-1}}{(1-q)^2} - \frac{(D-1)q^{D+1}}{1-q}\right\} - q^{D}(\alpha+1)$$

$$U(D+1, D+1) - U(m, D+1) = D - m + 1 - \alpha\, q^{m}[1-q^{D-m}] - q^{m+1}\frac{1-q^{D-m}}{1-q}$$

$$- Dq^m \, [1-q^{D-m}] - q^m [1- q^{D-m}] + mq^m + q^{m+1} \frac{1-q^{D-m-1}}{1-q} - (D-1)\, q^D - q^D (\alpha+1)$$

$$= D-m+1 - \alpha q^m + \alpha q^D - \frac{q^{m+1}}{1-q} + \frac{q^{D+1}}{1-q} - Dq^m + Dq^D - q^m + q^D + mq^m + \frac{q^{m+1}}{1-q}$$

$$- \frac{q^D}{1-q} - Dq^D + q^D - \alpha\, q^D - q^D$$

$$= D - m + 1 - \alpha\, q^m - q^D - Dq^m - q^m + q^D + mq^m$$

This reduces to:

$$U(D+1, D+1) - U(m,D+1) = D-m+1 - \alpha\, q^m - (D-m+1)\, q^m$$

Therefore, $U(D+1, D+1) \leq U(m, D+1)$ for $m = 1, ..., D$

is equivalent to

$D - m + 1 \leq \alpha\, q^m + (D-m+1)\, q^m$ for $m = 1, ..., D$.

By setting $k = D-m+1$ (so that $m = D-k+1$) we can rewrite the above condition as

$k \leq (\alpha + k)\, q^{D+1-k}$ for $k = 1, ..., D$, or equivalently, that :

$q^{D+1} \geq \frac{kq^k}{\alpha + k}$ for $k = 1, ..., D$. By defining $y(x)$ as $\frac{xq^x}{\alpha + x}$, we get that the condition $U(D+1,D+1) \leq U(m,+1)$ for $m = 1, ..., D$ is equivalent to $q^{D+1} \geq \max_{x=1,...,D}$ $y(x)$, which completes the first step of the proof. As a result we conclude that it is sufficient to show that $q^{D+1} \geq \max_{x=1,...,D}\, y(x)$ if and only if $D+1 \leq L$.

For the second step of the proof we consider the derivative of y(x):

$$y'(x) = \frac{1}{(\alpha + x)^2} \{[q^x + x \ln(q)\, q^x]\, (\alpha + x) - xq^x\} = \frac{q^x}{(\alpha + x)^2} \{[1 + x \ln q]\, (\alpha + x) - x\}$$

$$= \frac{q^x}{(\alpha + x)^2} \{\alpha - x\, (\alpha + x)\, \ln (\frac{1}{q})\}$$

y is unimodal on $x \geq 0$ since:

$$y'(x) \quad > 0 \qquad \text{if} \quad x(\alpha + x)\ \ln\left(\frac{1}{q}\right) < \alpha$$

$$= 0 \qquad \text{if} \quad x(\alpha + x)\ \ln\left(\frac{1}{q}\right) = 0$$

$$< 0 \qquad \text{if} \quad x(\alpha + x)\ \ln\left(\frac{1}{q}\right) > \alpha.$$

Hence ξ , as defined above, maximizes y(x) on $x \geq 0$, and κ , as defined above, maximizes y(x) on x = 1,2, ... This completes the second step in the proof.

In the third step of the proof we show that $q^{D+1} \geq \max_{x=1,\ldots,D} y(x)$ if and only if $D+1 \leq L$.

We note that since D is integer, $D \leq L$ if and only if $D \leq L_1$. Consider the three cases in the definition of L_1:

(i) Suppose $\kappa = 1$.

Then x = 1 maximizes y(x) on x = 1, ..., D and the condition $q^{D+1} \geq \max_{x=1,\ldots,D} y(x)$ becomes $q^{D+1} \geq \frac{q}{\alpha + 1}$, or equivalently $q^D(\alpha + 1) \geq 1$. In view of the fact that $q^{\theta-1}(\alpha+1)=1$ (see definition of θ) we get that $q^D(\alpha + 1) \geq 1$ if and only if $D \leq \theta - 1$ and thus $q^{D+1} \geq \max_{x=1,\ldots,D} y(x)$ is satisfied if and only if $D+1 \leq \theta = L_1$, i.e., if and only if $D+1 \leq L$.

(ii) Suppose $1 < \kappa \leq \frac{q}{1-q}\ \alpha$.

If $D+1 \leq L_1 = \kappa + \dfrac{\log \frac{\alpha + \kappa}{\kappa}}{\log \frac{1}{q}}$, then $\left(\frac{1}{q}\right)^{D+1} \leq \left(\frac{1}{q}\right)^{\kappa} \frac{\alpha + \kappa}{\kappa}$. Hence,

$q^{D+1} \geq \frac{\kappa q^{\kappa}}{\alpha + \kappa} = y(\kappa) = \max_{x=1,2,\ldots} y(x) \geq \max_{x=1,\ldots,D} y(x)$, and the condition $q^{D+1} \geq \max_{x=1,\ldots,D} y(x)$ is satisfied, so that D+1   N(D+1).

On the other hand, if $D+1 > L_1$, then by an identical argument $q^{D+1} < y(\kappa)$. Note that,

$$\kappa \le \frac{q}{1-q}\alpha \Leftrightarrow \kappa - q\kappa \le q\alpha \Leftrightarrow \kappa \le q(\alpha+\kappa) \Leftrightarrow \frac{1}{q} \le \frac{\alpha+\kappa}{\kappa} \Leftrightarrow \frac{\log\frac{\alpha+\kappa}{\kappa}}{\log\frac{1}{q}} \ge 1 \ .$$

Hence, if $D+1 > L_1$, then $D > \kappa$ (see definition of L_1), so that $y(\kappa) = \max_{x=1,\ldots,D} y(x)$. Thus, $q^{D+1} < \max_{x=1,\ldots,D} y(x)$, and therefore D+1   N(D+1).

(iii) Suppose $\kappa > \frac{q}{1-q}\alpha$, and $\kappa > 1$.

If $D+1 \le L_1 = \frac{q}{1-q}\alpha + 1$ then $D < \kappa$, and by the unimodality of y(x), the value x = D maximizes y(x) on x = 1,...,D , so that the condition $q^{D+1} \ge \max_{x=1,\ldots,D} y(x)$ becomes $q^{D+1} \ge \frac{Dq^D}{\alpha+D}$, which is equivalently to $q(\alpha+D) \ge D$, or alternatively, $D \le \frac{q}{1-q}\alpha$. Thus, $D+1 \le L_1$ implies that $q^{D+1} \ge \max_{x=1,\ldots,D} y(x)$ and therefore D+1   N(D+1). On the other hand, if $D+1 > \frac{q}{1-q}\alpha + 1$ then $D - qD > q\alpha$, i.e., $q < \frac{D}{\alpha+D}$, hence $q^{D+1} < \frac{Dq^D}{\alpha+D} = y(D)$, and clearly $y(D) \le \max_{x=1,\ldots,D} y(x)$, so that the condition $q^{D+1} \ge \max_{x=1,\ldots,D} y(x)$ is not satisfied and D+1   N(D+1).

The above proves by induction that D   N(D) whenever $D \le L$, and that L+1   N(L+1), i.e., max N(L+1) < L+1. It remains to show that max N(D) < D also for all D > L. This follows from Observation 4, which implies that max N(D) ≤max N(L+1) + D - (L+1) < D. ♦♦

Next we present Lemma C.2, which is used in the proof of Theorem 3 below.

Lemma C.2

For all D = 1,2,...., $\qquad V(D+1) - V(D) < A + B(L+1)$

Proof

We first show that $V(L+1) - V(L) < A + B(L+1)$.

By Theorem 2, L+1   N(L+1), hence from equation (1)

$$V(L+1) < U(L+1,L+1) = V(L+1) + \alpha + L + 1 - \sum_{k=1}^{L-1} q^k\,[V(L+2-k) - V(L-1-k)]$$

implying

$V(L+1) - V(L) < \frac{\alpha + L + 1}{q} - \sum_{k=1}^{L-1} q^k\,[V(L+1\text{-}k) - V(L\text{-}k)] - q^L \frac{\alpha + 1}{q}$. Note that when

k = 1,...,L-1, then $2 \le L+1\text{-}k \le L$, hence by Theorem 2, L+1-k   N(L+1-k), thus by Lemma C.1,

$V(L+1\text{-}k) - V(L\text{-}k) = A + B(L+1\text{-}k)$. Therefore,

$$V(L+1) - V(L) < \frac{\alpha + L + 1}{q} - \sum_{k=1}^{L-1} q^k\,[A + B(L+1\text{-}k)] - q^L \frac{\alpha + 1}{q}$$

$$V(L+1) - V(L) < \frac{\alpha + L + 1}{q} - [A + B(L+1)] \sum_{k=1}^{L-1} q^k + B \sum_{k=1}^{L-1} k\, q^k - q^L \frac{\alpha + 1}{q}.$$

In order to evaluate the right hand side of the above equation, we value the geometric-series summation in the second term, use Observation (C-1) to value the third term, and use the definitions of A and B, to get (details omitted)

$$\frac{\alpha + L + 1}{q} - [A + B(L+1)] \sum_{k=1}^{L-1} q^k + B \sum_{k=1}^{L-1} k\, q^k - q^L \frac{\alpha + 1}{q} = A + B(L+1)$$

and thus,

$V(L+1) - V(L) < A + B(L+1)$.

For D = 1,...,L-1, we have from Lemma C.1

$V(D+1) - V(D) = A + B(D+1) \le A + BL < A + B(L+1)$.

It remains to show that $V(D+1) - V(D) < A + B(L+1)$ for D = L+1, L+2,... We prove this by induction: assuming that $[V(d+1) - V(d) < A + B(L+1)$ for $d = 1,...,D - 1]$ holds for some D where $D - 1 \ge L$ (which we know holds for D = L+1), we show that

$V(D+1) - V(D) < A + B(L+1)$.

Let n   N(D). Since $D \ge L+1$, it follows from Theorem 2 that $n < D$. Using equation (1),

$$V(D+1) \le U(n,D+1) = V(D+1) + \alpha + n - \sum_{k=1}^{n} q^k\,[V(D+2\text{-}k) - V(D+1\text{-}k)]$$

$$0 \le \alpha + n - q[V(D+1) - V(D)] - \sum_{k=2}^{n} q^k\,[V(D+2\text{-}k) - V(D+1\text{-}k)]$$

$$V(D+1) - V(D) \le \frac{\alpha + n}{q} - \frac{1}{q} \sum_{j=1}^{n-1} q^{j+1}\,[V(D+1\text{-}j) - V(D\text{-}j)]$$

In order to evaluate α+n in the above expression, we use equation (1) again for U(n,D), recalling

n   N(D)

$$V(D) = U(n,D) = \alpha + n - \sum_{k=1}^{n} q^k [V(D+1-k) - V(D-k)]$$

$$\alpha + n = \sum_{k=1}^{n} q^k [V(D+1-k) - V(D-k)]$$

hence,

$$V(D+1) - V(D) \le \frac{1}{q}\sum_{k=1}^{n} q^k [V(D+1-k) - V(D-k)] - \frac{1}{q}\sum_{j=1}^{n-1} q^{j+1} [V(D+1-j) - V(D-j)]$$

$$V(D+1) - V(D) \le \frac{1-q}{q}\sum_{j=1}^{n-1} q^j [V(D+1-j) - V(D-j)] + q^n [V(D+1-n) - V(D-n)].$$

Since $n < D$ it follows from the induction assumption that $[V(D+1-j) - V(D-j)] < A + B(L+1)$ for $j = 1,...,n$. Thus,

$$V(D+1)-V(D)<\{ \frac{1-q}{q}\sum_{j=1}^{n-1} q^{j+1} + q^n\}\{A+B(L+1)\}=\{ \frac{1-q}{q}\ \frac{q-q^n}{1-q} + q^n \}<A +B(L+1)$$

which completes the proof. ♦

We conclude this appendix with the proof of Theorem 3.

<u>Theorem 3</u> max {k   N(D): D = 1,2,... } = L

<u>Proof</u>

From Theorem 2 we have L   N(L), hence max {k   N(D) , D = 1,2,... } ≥ L. Assume by contradiction that max{k   N(D): D = 1,2,... } > L. In view of Observation 4 (which states that a unit increase in the outstanding demand can induce no more than a unit increase in the optimal lot size) there exists d such that L   N(d) and L+1   N(d+1). Then from Lemma C.1 V(d+1) - V(d) = A + B(L+1), while from Lemma C.2, V(d+1) - V(d) < A + B(L+1)
which is a contradiction, forcing the conclusion that max {k   N(D): D = 1,2,...} = L, as stated.
♦♦