Online Companion for

“The Censored Newsvendor and the Optimal Acquisition of Information”

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Appendix for "The Censored Newsvendor and the Optimal Acquisition of Information" by Xiaomei Ding, Martin L. Puterman and Arnab Bisi

Proof of Proposition 1. The proof consists of three steps. First, we show that the posterior $\pi_{n+1}(\theta|x_n)$ is stochastically increasing in x_n. Second, we show $\pi^c_{n+1}(\theta|x_n) \geq_{st} \pi_{n+1}(\theta|x_n)$, $n = 1, 2, \ldots, N$. Then we show that this implies the stochastic ordering of the marginals.

[*Step 1*] For any x_n and x'_n such that $x_n \leq x'_n$, we have

$$\pi_{n+1}(\theta|x_n) = \frac{f(x_n|\theta)\pi'_n(\theta)}{m'_n(x_n)}, \quad \pi_{n+1}(\theta|x'_n) = \frac{f(x'_n|\theta)\pi'_n(\theta)}{m'_n(x'_n)}.$$

Since $f(x|\theta)$ is likelihood ratio increasing in θ, for all $\theta \leq \theta'$,

$$\frac{f(x'_n|\theta)}{f(x_n|\theta)} \leq \frac{f(x'_n|\theta')}{f(x_n|\theta')}.$$

Therefore, for all $\theta \leq \theta'$,

$$\frac{\pi_{n+1}(\theta|x'_n)}{\pi_{n+1}(\theta|x_n)} = \frac{f(x'_n|\theta)m'_n(x_n)}{f(x_n|\theta)m'_n(x'_n)} \leq \frac{f(x'_n|\theta')m'_n(x_n)}{f(x_n|\theta')m'_n(x'_n)} = \frac{\pi_{n+1}(\theta'|x'_n)}{\pi_{n+1}(\theta'|x_n)}. \tag{1}$$

Thus $\boldsymbol{\theta}|[X_n = x'_n] \geq_{LR} \boldsymbol{\theta}|[X_n = x_n]$ for any $x'_n \geq x_n$. Since the likelihood ratio ordering is stronger than the stochastic ordering, $\pi_{n+1}(\theta|x_n)$ increases stochastically in x_n.

[*Step 2*]

$$\pi^c_{n+1}(\theta|x_n) = \frac{\int_{x_n}^{\infty} f(x|\theta)dx\pi'_n(\theta)}{\int_\Theta \int_{x_n}^{\infty} f(x|\theta)\pi'_n(\theta)dxd\theta} = \frac{\int_{x_n}^{\infty} m'_n(x)\frac{f(x|\theta)\pi'_n(\theta)}{m'_n(x)}dx}{\int_{x_n}^{\infty} m'_n(x)dx}.$$

That is,

$$\pi^c_{n+1}(\theta|x_n) = \frac{\int_{x_n}^{\infty} m'_n(x)\pi_{n+1}(\theta|x)dx}{\int_{x_n}^{\infty} m'_n(x)dx}. \tag{2}$$

By (1), for all $\theta \leq \theta'$,

$$\begin{aligned}\frac{\pi^c_{n+1}(\theta|x_n)}{\pi_{n+1}(\theta|x_n)} &= \frac{\int_{x_n}^{\infty} m'_n(x)\pi_{n+1}(\theta|x)dx}{\int_{x_n}^{\infty} m'_n(x)dx} \Big/ \pi_{n+1}(\theta|x_n) \\ &\leq \frac{\int_{x_n}^{\infty} m'_n(x)\pi_{n+1}(\theta'|x)dx}{\int_{x_n}^{\infty} m'_n(x)dx} \Big/ \pi_{n+1}(\theta'|x_n) \\ &= \frac{\pi^c_{n+1}(\theta'|x_n)}{\pi_{n+1}(\theta'|x_n)}\end{aligned}$$

Thus $\boldsymbol{\theta}|[X_n \geq x_n] \geq_{LR} \boldsymbol{\theta}|[X_n = x_n]$. Therefore, $\pi^c_{n+1}(\theta|x_n) \geq_{st} \pi_{n+1}(\theta|x_n)$ for arbitrary x_n.

[*Step 3*] We write the marginals as

$$M_n(x|x_{n-1}) = \int_0^x \int_\Theta f(t|\theta)\pi_n(\theta|x_{n-1})d\theta dt \ . \tag{3}$$

By changing the order of integral, (3) becomes,

$$M_n(x|x_{n-1}) = \int_\Theta F(x|\theta)\pi_n(\theta|x_{n-1})d\theta = E_{\boldsymbol{\theta}}[F(x|\boldsymbol{\theta})] \tag{4}$$

where $\boldsymbol{\theta}$ has density $\pi_n(\theta|x_{n-1})$.

Similarly,

$$M^c_n(x|x_{n-1}) = \int_\Theta F(x|\theta)\pi^c_n(\theta|x_{n-1})d\theta = E_{\boldsymbol{\theta'}}[F(x|\boldsymbol{\theta'})] \tag{5}$$

where $\boldsymbol{\theta'}$ is a random variable with density $\pi^c_n(\theta|x_{n-1})$.

The condition that $f(x|\theta)$ is increasing in likelihood ratio as θ increases implies that $f(x|\theta)$ is also stochastically increasing in θ since the likelihood ratio ordering is stronger than the stochastic ordering. Therefore, $1 - F(x|\theta)$ is non-decreasing in θ. By Proposition 8.1.2 in Ross (1983, p. 252), we have

$$E_{\boldsymbol{\theta}}[1 - F(x|\boldsymbol{\theta})] \leq E_{\boldsymbol{\theta'}}[1 - F(x|\boldsymbol{\theta'})] \tag{6}$$

if $\boldsymbol{\theta}'$ is stochastically greater than $\boldsymbol{\theta}$. Hence by step 2 it follows from (6) that

$$M_n^c(x|x_{n-1}) \leq M_n(x|x_{n-1}).$$

■

Proof of Proposition 2. Since $u_{N+1}(\pi'_{N+1}) = 0$ by the assumption of the boundary condition,

$$u_N(\pi'_N) = \min_{y_N}\{R_B(\pi'_N, y_N) + 0\}. \tag{7}$$

As we have pointed out earlier, $R_B(\pi'_N, y_N)$ is convex in y_N. So $u_N(\pi'_N)$ is minimized at

$$y_N^{*e} = y_N^e = [M_N(k|x_{N-1})]^{-1} \qquad \text{if } \pi'_N(\cdot|x_{N-1}) = \pi_N(\cdot|x_{N-1}),$$

$$i.e., \text{ when } X_{N-1} = x_{N-1} < y_{N-1}, \tag{8}$$

$$y_N^{*c} = y_N^c = [M_N^c(k|y_{N-1})]^{-1} \qquad \text{if } \pi'_N(\cdot|x_{N-1}) = \pi_N^c(\cdot|y_{N-1}),$$

$$i.e., \text{ when } X_{N-1} \geq x_{N-1} = y_{N-1}. \tag{9}$$

That is,

$$y_N^* = y_N^{BN}.$$

Since both $M_N(x|x_{N-1})$ and $M_N^c(x|x_{N-1})$ are monotonically increasing in x, and $M_N^c(x|x_{N-1}) = M_N^c(x|y_{N-1})$, using Proposition 1 for $n = N$, it is obvious that

$$y_N^e = [M_N(k|x_{N-1})]^{-1} \leq y_N^c = [M_N^c(k|y_{N-1})]^{-1}. \tag{10}$$

■

Proof of Theorem A. We divide the proof into three steps. In step 1, we introduce some notation and develop a few relations for use in the subsequent steps. In steps 2 and 3, we use backward induction to complete the proof.

[*Step 1*] Recall that $\pi'_n(\cdot|x_{n-1}), n = 2, \ldots, N$, are calculated differently depending on whether X_{n-1} is fully observed or censored. Hence,

$$\begin{aligned} & u_n(\pi'_n(\cdot|x_{n-1})) \\ \equiv & \begin{cases} u_n(\pi_n(\cdot|x_{n-1})) & \text{if } \pi'_n(\cdot|x_{n-1}) = \pi_n(\cdot|x_{n-1}),\ i.e.,\ \text{when } X_{n-1} = x_{n-1} < y_{n-1} \\ u_n(\pi^c_n(\cdot|y_{n-1})) & \text{if } \pi'_n(\cdot|x_{n-1}) = \pi^c_n(\cdot|y_{n-1}),\ i.e.,\ \text{when } X_{n-1} \geq x_{n-1} = y_{n-1}. \end{cases} \end{aligned} \quad (11)$$

Since $\pi_n(\cdot|x_{n-1})$ can be considered as a function of π'_{n-1} and x_{n-1}, and $\pi^c_n(\cdot|y_{n-1})$ as a function of π'_{n-1} and y_{n-1}, we redefine $u_n(\pi'_n(\cdot|x_{n-1}))$ as

$$u_n(\pi'_n(\cdot|x_{n-1})) \equiv \begin{cases} u_n(\pi'_{n-1}, x_{n-1}) & \text{if } x_{n-1} < y_{n-1} \\ u^c_n(\pi'_{n-1}, y_{n-1}) & \text{if } x_{n-1} = y_{n-1}. \end{cases}$$

For notational convenience, we denote

$$u_n(\pi'_{n-1}, x_{n-1}) \equiv u_n(x_{n-1}), \qquad u^c_n(\pi'_{n-1}, y_{n-1}) \equiv u^c_n(y_{n-1}).$$

Then the optimality equations can be written as

$$u_n(\pi'_n) = \min_{y_n \in R_+} \left\{ R_B(\pi'_n, y_n) + \int_0^{y_n} u_{n+1}(x) m'_n(x) dx + u^c_{n+1}(y_n))[1 - M'_n(y_n)] \right\}, \quad (12)$$

for $n = 1, \ldots, N$, with the boundary condition $u_{N+1}(\pi'_{N+1}) = 0$ for all π'_{N+1}. The second term on the right hand side of (12) is the expected cost at $n+1$ if X_n is fully observed (i.e. $X_n < y_n$) and the third term is the expected cost if X_n is censored at y_n (i.e. $X_n \geq y_n$).

Let

$$\begin{aligned} J(\pi'_n, y_n) &= \left\{ R_B(\pi'_n, y_n) + \int_0^{y_n} u_{n+1}(x) m'_n(x) dx + u^c_{n+1}(y_n))[1 - M'_n(y_n)] \right\} \\ &\equiv R_B(\pi'_n, y_n) + I_n(y_n). \end{aligned} \quad (13)$$

Then the optimal policy is obtained from

$$y^*_n \in arg \min_{y_n} J(\pi'_n, y_n), \text{for } n = 1, 2, \ldots, N.$$

Now, for continuous demand, the differentiability of $u_{n+1}^c(y_n)$ in y_n can be verified by using backward induction on equations (12). This will become clear later from (22)-(23) in step 2 and (39) in step 3 where we explicitly calculate the derivatives. Therefore, $J(\pi_n', y_n)$ is also differentiable in y_n (refer to (13)). Thus, to show $y_n^* \geq y_n^{BN}$ for any $n = 1, 2, \ldots, N-1$, it suffices to show that $\frac{dJ(\pi_n', y_n)}{dy_n}|_{y_n = y_n^{BN}} \leq 0$ where

$$\begin{aligned} & \frac{dJ(\pi_n', y_n)}{dy_n} \\ = \quad & \frac{dR_B(\pi_n', y_n)}{dy_n} + \frac{dI_n(y_n)}{dy_n} \qquad (14) \\ = \quad & (c-p) + (p-h)M_n'(y_n) + \frac{du_{n+1}^c(y_n)}{dy_n}[1 - M_n'(y_n)] - (u_{n+1}^c(y_n) - u_{n+1}(y_n))m_n'(y_n). \end{aligned}$$

Since $\frac{dR_B(\pi_n', y_n)}{dy_n}|_{y_n = y_n^{BN}} = 0$, to show $y_n^* \geq y_n^{BN}$, we only need to show that at $y_n = y_n^{BN}$,

$$\frac{dI_n(y_n)}{dy_n} = \frac{du_{n+1}^c(y_n)}{dy_n}[1 - M_n'(y_n)] - (u_{n+1}^c(y_n) - u_{n+1}(y_n))m_n'(y_n) \leq 0. \qquad (15)$$

[*Step 2*] From Proposition 2 we know $y_N^* = y_N^{BN}$. We will now prove the theorem for $n = N-1$. That is, by (15), we will show that at $y_{N-1} = y_{N-1}^{BN}$,

$$\begin{aligned} & \frac{dI_{N-1}(y_{N-1})}{dy_{N-1}} \\ = \quad & \frac{du_N^c(y_{N-1})}{dy_{N-1}}[1 - M_{N-1}'(y_{N-1})] - (u_N^c(y_{N-1}) - u_N(y_{N-1}))m_{N-1}'(y_{N-1}) \leq 0. \qquad (16) \end{aligned}$$

As a matter of fact, we will show that (16) is true at any value of y_{N-1}.

We first write out $u_N(y_{N-1})$ and $u_N^c(y_{N-1})$ below.

$$\begin{aligned} u_N(y_{N-1}) &= R_B(\pi_N, y_N^e) = E_{\pi_N}[r(X, y_N^e)] \\ &= p\int_{y_N^e}^{\infty} x m_N(x|y_{N-1})dx + h\int_0^{y_N^e} x m_N(x|y_{N-1})dx \\ &= p\left[\int_{y_N^c}^{\infty} x m_N(x|y_{N-1})dx + \int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx\right] \\ &\quad + h\left[\int_0^{y_N^c} x m_N(x|y_{N-1})dx - \int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx\right]. \qquad (17) \end{aligned}$$

Similarly,

$$\begin{aligned} u_N^c(y_{N-1}) &= R_B(\pi_N^c, y_N^c) = E_{\pi_N^c}[r(X, y_N^c)] \\ &= p\int_{y_N^c}^{\infty} x m_N^c(x|y_{N-1})dx + h\int_0^{y_N^c} x m_N^c(x|y_{N-1})dx. \end{aligned} \tag{18}$$

Subtracting (17) from (18) gives

$$\begin{aligned} & u_N^c(y_{N-1}) - u_N(y_{N-1}) \\ = \; & p\int_{y_N^c}^{\infty} x[m_N^c(x|y_{N-1}) - m_N(x|y_{N-1})]dx + h\int_0^{y_N^c} x[m_N^c(x|y_{N-1}) - m_N(x|y_{N-1})]dx \\ & - (p-h)\int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx \\ = \; & D_N^1 - (p-h)\int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx, \end{aligned} \tag{19}$$

where

$$D_N^1 = p\int_{y_N^c}^{\infty} x[m_N^c(x|y_{N-1}) - m_N(x|y_{N-1})]dx + h\int_0^{y_N^c} x[m_N^c(x|y_{N-1}) - m_N(x|y_{N-1})]dx,$$

with $m_N(x|y_{N-1})$ and $m_N^c(x|y_{N-1})$ given by

$$m_N(x|y_{N-1}) = \int_\Theta f(x|\theta)\pi_N(\theta|y_{N-1})d\theta = \frac{\int_\Theta f(x|\theta)f(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta}{m'_{N-1}(y_{N-1})}, \tag{20}$$

$$m_N^c(x|y_{N-1}) = \frac{m'_{N-1}(x) - \int_\Theta f(x|\theta)F(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta}{1 - M'_{N-1}(y_{N-1})}. \tag{21}$$

We now calculate $\frac{du_N^c(y_{N-1})}{dy_{N-1}}$. From (18), we have the following by Leibniz's rule.

$$\begin{aligned} \frac{du_N^c(y_{N-1})}{dy_{N-1}} &= p\int_{y_N^c}^{\infty} x\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}}dx - py_N^c m_N^c(y_N^c|y_{N-1})\frac{dy_N^c}{dy_{N-1}} \\ &\quad + h\int_0^{y_N^c} x\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}}dx + hy_N^c m_N^c(y_N^c|y_{N-1})\frac{dy_N^c}{dy_{N-1}} \\ &= D_N^2 - (p-h)y_N^c m_N^c(y_N^c|y_{N-1})\frac{dy_N^c}{dy_{N-1}}, \end{aligned} \tag{22}$$

where

$$D_N^2 = p\int_{y_N^c}^{\infty} x\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}}dx + h\int_0^{y_N^c} x\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}}dx,$$

with

$$\begin{aligned}\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}} &= \frac{m'_{N-1}(y_{N-1})[m'_{N-1}(x) - \int_\Theta f(x|\theta)F(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta]}{[1-M'_{N-1}(y_{N-1})]^2} \\ &\quad -\frac{\int_\Theta f(x|\theta)f(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta}{1-M'_{N-1}(y_{N-1})}. \qquad (23)\end{aligned}$$

Now, using (19) and (22), from (16) we write $\frac{dI_{N-1}(y_{N-1})}{dy_{N-1}}$ as

$$\begin{aligned}&\frac{dI_{N-1}(y_{N-1})}{dy_{N-1}} \\ &= \Big[[1-M'_{N-1}(y_{N-1})]D_N^2 - m'_{N-1}(y_{N-1})D_N^1\Big] + \Big\{m'_{N-1}(y_{N-1})(p-h)\int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx \\ &\quad - [1-M'_{N-1}(y_{N-1})](p-h)y_N^c m_N^c(y_N^c|y_{N-1})\frac{dy_N^c}{dy_{N-1}}\Big\}. \qquad (24)\end{aligned}$$

From (23),

$$\begin{aligned}&[1-M'_{N-1}(y_{N-1})]\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}} \\ &= \frac{m'_{N-1}(y_{N-1})m'_{N-1}(x)}{1-M'_{N-1}(y_{N-1})} - \frac{\int_\Theta f(x|\theta)F(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta}{1-M'_{N-1}(y_{N-1})}m'_{N-1}(y_{N-1}) \\ &\quad - \int_\Theta f(x|\theta)f(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta. \qquad (25)\end{aligned}$$

And from (21),

$$\begin{aligned}&m'_{N-1}(y_{N-1})m_N^c(x|y_{N-1}) \\ &= \frac{m'_{N-1}(y_{N-1})m'_{N-1}(x)}{1-M'_{N-1}(y_{N-1})} - \frac{\int_\Theta f(x|\theta)F(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta}{1-M'_{N-1}(y_{N-1})}m'_{N-1}(y_{N-1}). \qquad (26)\end{aligned}$$

Using (25) and (26), we have

$$\begin{aligned}&[1-M'_{N-1}(y_{N-1})]\frac{dm_N^c(x|y_{N-1})}{dy_{N-1}} - m'_{N-1}(y_{N-1})[m_N^c(x|y_{N-1}) - m_N(x|y_{N-1})] \\ &= -\int_\Theta f(x|\theta)f(y_{N-1}|\theta)\pi'_{N-1}(\theta)d\theta + m'_{N-1}(y_{N-1})m_N(x|y_{N-1}) = 0. \qquad (27)\end{aligned}$$

Since the integrand is zero,

$$[1 - M'_{N-1}(y_{N-1})]D_N^2 - m'_{N-1}(y_{N-1})D_N^1 = 0. \tag{28}$$

Therefore (24) reduces to

$$\begin{aligned}\frac{dI_{N-1}(y_{N-1})}{dy_{N-1}} &= m'_{N-1}(y_{N-1})(p-h)\int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx \\ &\quad - [1 - M'_{N-1}(y_{N-1})](p-h)y_N^c m_N^c(y_N^c|y_{N-1})\frac{dy_N^c}{dy_{N-1}}. \end{aligned}\tag{29}$$

Note that $M_N^c(x_N|x_{N-1})$ is a function of both x_N and x_{N-1}. It is a function of x_{N-1} through π_N^c. Since $M_N^c(y_N^c|y_{N-1}) = k$ implicitly defines the function $y_N^c(y_{N-1})$, it follows from the properties of the partial derivatives that

$$\frac{dy_N^c}{dy_{N-1}} = -\frac{\frac{\partial M_N^c(y_N^c|y_{N-1})}{\partial y_{N-1}}}{m_N^c(y_N^c|y_{N-1})}.$$

This gives

$$[1 - M'_{N-1}(y_{N-1})]y_N^c m_N^c(y_N^c|y_{N-1})\frac{dy_N^c}{dy_{N-1}} = -[1 - M'_{N-1}(y_{N-1})]y_N^c\frac{\partial M_N^c(y_N^c|y_{N-1})}{\partial y_{N-1}}. \tag{30}$$

Since from (21),

$$M_N^c(y_N^c|y_{N-1}) = \int_0^{y_N^c} m_N^c(x|y_{N-1})dx = \frac{M'_{N-1}(y_N^c) - \int_\Theta F(y_{N-1}|\theta)F(y_N^c|\theta)\pi'_{N-1}(\theta)d\theta}{1 - M'_{N-1}(y_{N-1})},$$

(30) becomes

$$\begin{aligned}&-[1 - M'_{N-1}(y_{N-1})]y_N^c\frac{\partial M_N^c(y_N^c|y_{N-1})}{\partial y_{N-1}} \\ =\ & y_N^c m'_{N-1}(y_{N-1})[M_N(y_N^c|y_{N-1}) - M_N^c(y_N^c|y_{N-1})].\end{aligned}\tag{31}$$

By Proposition 2, we have $y_N^e \leq y_N^c$ so that

$$m'_{N-1}(y_{N-1})\int_{y_N^e}^{y_N^c} x m_N(x|y_{N-1})dx$$

$$
\begin{aligned}
&\leq\ m'_{N-1}(y_{N-1})y_N^c \int_{y_N^e}^{y_N^c} m_N(x|y_{N-1})dx \\
&=\ m'_{N-1}(y_{N-1})y_N^c[M_N(y_N^c|y_{N-1}) - M_N(y_N^e|y_{N-1})]. \qquad (32)
\end{aligned}
$$

From (8)-(9), we know $M_N^c(y_N^c|y_{N-1}) = M_N(y_N^e|y_{N-1}) = k$. Replacing (31) and (32) in (29), we get

$$
\begin{aligned}
&\frac{dI_{N-1}(y_{N-1})}{dy_{N-1}} \\
&\leq\ (p-h)y_N^c\left[m'_{N-1}(y_{N-1})(M_N(y_N^c|y_{N-1}) - k) - m'_{N-1}(y_{N-1})(M_N(y_N^c|y_{N-1}) - k)\right] = 0
\end{aligned}
$$

which gives $\frac{dJ(\pi'_{N-1}, y_{N-1})}{dy_{N-1}}|_{y_{N-1}=y_{N-1}^{BN}} \leq 0$. Hence the theorem is true for $n = N-1$.

[*Step 3*] We assume that the theorem holds for $n = k$, i.e., $y_k^* \geq y_k^{BN}$, for $k < N-1$. We will prove that it holds for $n = k-1$. Thus, in view of (15) for $n = k$, we are given that at $y_k = y_k^{BN}$,

$$
\frac{dI_k(y_k)}{dy_k} = \frac{du_{k+1}^c(y_k)}{dy_k}[1 - M'_k(y_k)] - (u_{k+1}^c(y_k) - u_{k+1}(y_k))m'_k(y_k) \leq 0. \qquad (33)
$$

Moreover, following backward induction it can be checked that $J(\pi'_k, y_k)$ is convex in y_k (*e.g.*, modify the proof of Theorem 1 in Treharne and Sox (1999) for 0-convexity). Therefore, at $y_k = y_k^*$, we have

$$
\begin{aligned}
\frac{dJ(\pi'_k, y_k)}{dy_k} &= (c-p) + (p-h)M'_k(y_k) + \frac{du_{k+1}^c(y_k)}{dy_k}[1 - M'_k(y_k)] \\
&\quad -(u_{k+1}^c(y_k) - u_{k+1}(y_k))m'_k(y_k) = 0. \qquad (34)
\end{aligned}
$$

We want to show $y_{k-1}^* \geq y_{k-1}^{BN}$, that is, by (15), at $y_{k-1} = y_{k-1}^{BN}$,

$$
\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}} = \frac{du_k^c(y_{k-1})}{dy_{k-1}}[1 - M'_{k-1}(y_{k-1})] - (u_k^c(y_{k-1}) - u_k(y_{k-1}))m'_{k-1}(y_{k-1}) \leq 0. \qquad (35)
$$

We will actually show that (35) holds at any y_{k-1}.

Recall the notation

$$y_k^* = \begin{cases} y_k^{*e} & \text{if } X_{k-1} = x_{k-1} < y_{k-1} \\ y_k^{*c} & \text{if } X_{k-1} \geq x_{k-1} = y_{k-1}. \end{cases}$$

We divide the rest of the proof into two cases: case $y_k^{*e} \leq y_k^{*c}$, and case $y_k^{*e} \geq y_k^{*c}$.

First we consider the case $y_k^{*e} \leq y_k^{*c}$. From (12) we can write $u_k(y_{k-1})$ and $u_k^c(y_{k-1})$ as below.

$$\begin{aligned} u_k(y_{k-1}) &= [(c-p) + (p-h)M_k(y_k^{*e}|y_{k-1})]\, y_k^{*e} \\ &\quad + p\int_{y_k^{*e}}^{\infty} x m_k(x|y_{k-1})dx + h\int_0^{y_k^{*e}} x m_k(x|y_{k-1})dx \\ &\quad + \int_0^{y_k^{*e}} u_{k+1}(x) m_k(x|y_{k-1})dx + u_{k+1}^c(y_k^{*e})[1 - M_k(y_k^{*e}|y_{k-1})], \end{aligned} \tag{36}$$

and

$$\begin{aligned} u_k^c(y_{k-1}) &= [(c-p) + (p-h)M_k^c(y_k^{*c}|y_{k-1})]\, y_k^{*c} \\ &\quad + p\int_{y_k^{*c}}^{\infty} x m_k^c(x|y_{k-1})dx + h\int_0^{y_k^{*c}} x m_k^c(x|y_{k-1})dx \\ &\quad + \int_0^{y_k^{*c}} u_{k+1}(x) m_k^c(x|y_{k-1})dx + u_{k+1}^c(y_k^{*c})[1 - M_k^c(y_k^{*c}|y_{k-1})]. \end{aligned} \tag{37}$$

Subtracting (36) from (37) gives

$$\begin{aligned} & u_k^c(y_{k-1}) - u_k(y_{k-1}) \\ = \; & D_k^1 + [(c-p) + (p-h)M_k^c(y_k^{*c}|y_{k-1})]\, y_k^{*c} - [(c-p) + (p-h)M_k(y_k^{*e}|y_{k-1})]\, y_k^{*e} \\ & -(p-h)\int_{y_k^{*e}}^{y_k^{*c}} x m_k(x|y_{k-1})dx + \int_0^{y_k^{*c}} u_{k+1}(x) m_k^c(x|y_{k-1})dx - \int_0^{y_k^{*e}} u_{k+1}(x) m_k(x|y_{k-1})dx \\ & + u_{k+1}^c(y_k^{*c})[1 - M_k^c(y_k^{*c}|y_{k-1})] - u_{k+1}^c(y_k^{*e})[1 - M_k(y_k^{*e}|y_{k-1})], \end{aligned} \tag{38}$$

where

$$D_k^1 = p\int_{y_k^{*c}}^{\infty} x[m_k^c(x|y_{k-1}) - m_k(x|y_{k-1})]dx + h\int_0^{y_k^{*c}} x[m_k^c(x|y_{k-1}) - m_k(x|y_{k-1})]dx.$$

We now calculate $\frac{du_k^c(y_{k-1})}{dy_{k-1}}$. Using Leibniz's rule we get from (37),

$$\begin{aligned}\frac{du_k^c(y_{k-1})}{dy_{k-1}} &= D_k^2 + [(c-p) + (p-h)M_k^c(y_k^{*c}|y_{k-1})]\,\frac{dy_k^{*c}}{dy_{k-1}} \\ &+(p-h)y_k^{*c}\frac{dM_k^c(y_k^{*c}|y_{k-1})}{dy_{k-1}} - (p-h)y_k^{*c}m_k^c(y_k^{*c}|y_{k-1})\frac{dy_k^{*c}}{dy_{k-1}} \\ &+\int_0^{y_k^{*c}} u_{k+1}(x)\frac{dm_k^c(x|y_{k-1})}{dy_{k-1}}dx + u_{k+1}(y_k^{*c})m_k^c(y_k^{*c}|y_{k-1})\frac{dy_k^{*c}}{dy_{k-1}} \\ &+\frac{du_{k+1}^c(y_k^{*c})}{dy_{k-1}}[1 - M_k^c(y_k^{*c}|y_{k-1})] - u_{k+1}^c(y_k^{*c})\frac{dM_k^c(y_k^{*c}|y_{k-1})}{dy_{k-1}}, \end{aligned} \tag{39}$$

where

$$D_k^2 = p\int_{y_k^{*c}}^{\infty} x\frac{dm_k^c(x|y_{k-1})}{dy_{k-1}}dx + h\int_0^{y_k^{*c}} x\frac{dm_k^c(x|y_{k-1})}{dy_{k-1}}dx.$$

By the properties of the partial derivatives we can write

$$\frac{dM_k^c(y_k^{*c}|y_{k-1})}{dy_{k-1}} = \frac{\partial M_k^c(y_k^{*c}|y_{k-1})}{\partial y_{k-1}} + m_k^c(y_k^{*c}|y_{k-1})\frac{dy_k^{*c}}{dy_{k-1}}\ . \tag{40}$$

Now, using (38)-(40), from (35) we write $\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}}$ as

$$\begin{aligned}&\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}} \\ =\ &\Big[[1 - M'_{k-1}(y_{k-1})]D_k^2 - m'_{k-1}(y_{k-1})D_k^1\Big] \\ &+[1 - M'_{k-1}(y_{k-1})]\Big[[(c-p) + (p-h)M_k^c(y_k^{*c}|y_{k-1})]\,\frac{dy_k^{*c}}{dy_{k-1}} \\ &+(p-h)y_k^{*c}\frac{\partial M_k^c(y_k^{*c}|y_{k-1})}{\partial y_{k-1}} + \int_0^{y_k^{*c}} u_{k+1}(x)\frac{dm_k^c(x|y_{k-1})}{dy_{k-1}}dx \\ &+u_{k+1}(y_k^{*c})m_k^c(y_k^{*c}|y_{k-1})\frac{dy_k^{*c}}{dy_{k-1}} + \frac{du_{k+1}^c(y_k^{*c})}{dy_k^{*c}}[1 - M_k^c(y_k^{*c}|y_{k-1})]\frac{dy_k^{*c}}{dy_{k-1}} \\ &-u_{k+1}^c(y_k^{*c})\frac{\partial M_k^c(y_k^{*c}|y_{k-1})}{\partial y_{k-1}} - u_{k+1}^c(y_k^{*c})m_k^c(y_k^{*c}|y_{k-1})\frac{dy_k^{*c}}{dy_{k-1}}\Big] \\ &-m'_{k-1}(y_{k-1})\Big[[(c-p) + (p-h)M_k^c(y_k^{*c}|y_{k-1})]\,y_k^{*c} - [(c-p) + (p-h)M_k(y_k^{*e}|y_{k-1})]\,y_k^{*e} \\ &-(p-h)\int_{y_k^{*e}}^{y_k^{*c}} xm_k(x|y_{k-1})dx + \int_0^{y_k^{*c}} u_{k+1}(x)m_k^c(x|y_{k-1})dx - \int_0^{y_k^{*e}} u_{k+1}(x)m_k(x|y_{k-1})dx \\ &+u_{k+1}^c(y_k^{*c})[1 - M_k^c(y_k^{*c}|y_{k-1})] - u_{k+1}^c(y_k^{*e})[1 - M_k(y_k^{*e}|y_{k-1})]\Big]. \end{aligned} \tag{41}$$

To simplify the above expression of $\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}}$, we write several equations below.

Similar to (27) we can show

$$[1 - M'_{k-1}(y_{k-1})]\frac{dm_k^c(x|y_{k-1})}{dy_{k-1}} = m'_{k-1}(y_{k-1})[m_k^c(x|y_{k-1}) - m_k(x|y_{k-1})]. \tag{42}$$

Also, similar to (31) we have

$$[1 - M'_{k-1}(y_{k-1})]\frac{\partial M_k^c(y_k^{*c}|y_{k-1})}{\partial y_{k-1}} = -m'_{k-1}(y_{k-1})[M_k(y_k^{*c}|y_{k-1}) - M_k^c(y_k^{*c}|y_{k-1})]. \tag{43}$$

Using (42) we get

$$\begin{aligned}
& [1 - M'_{k-1}(y_{k-1})]\int_0^{y_k^{*c}} u_{k+1}(x)\frac{dm_k^c(x|y_{k-1})}{dy_{k-1}}dx \\
& -m'_{k-1}(y_{k-1})\left[\int_0^{y_k^{*c}} u_{k+1}(x)m_k^c(x|y_{k-1})dx - \int_0^{y_k^{*e}} u_{k+1}(x)m_k(x|y_{k-1})dx\right] \\
= & -m'_{k-1}(y_{k-1})\int_{y_k^{*e}}^{y_k^{*c}} u_{k+1}(x)m_k(x|y_{k-1})dx. \qquad (44)
\end{aligned}$$

Using (43) we get

$$\begin{aligned}
& -[1 - M'_{k-1}(y_{k-1})]u_{k+1}^c(y_k^{*c})\frac{\partial M_k^c(y_k^{*c}|y_{k-1})}{\partial y_{k-1}} - m'_{k-1}(y_{k-1})u_{k+1}^c(y_k^{*c})[1 - M_k^c(y_k^{*c}|y_{k-1})] \\
= & -m'_{k-1}(y_{k-1})u_{k+1}^c(y_k^{*c})[1 - M_k(y_k^{*c}|y_{k-1})]. \qquad (45)
\end{aligned}$$

Rewriting (34) at $y_k = y_k^{*c}$ we get

$$\begin{aligned}
& (c-p) + (p-h)M_k^c(y_k^{*c}|y_{k-1}) + \frac{du_{k+1}^c(y_k^{*c})}{dy_k^{*c}}[1 - M_k^c(y_k^{*c}|y_{k-1})] \\
& -(u_{k+1}^c(y_k^{*c}) - u_{k+1}(y_k^{*c}))m_k^c(y_k^{*c}|y_{k-1}) = 0. \qquad (46)
\end{aligned}$$

Also, similar to (28) we can show

$$[1 - M'_{k-1}(y_{k-1})]D_k^2 - m'_{k-1}(y_{k-1})D_k^1 = 0. \tag{47}$$

Using (43)-(47) in (41) we get

$$
\begin{aligned}
&\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}} \\
= \;& -m'_{k-1}(y_{k-1})\Big[[(c-p)+(p-h)M_k(y_k^{*c}|y_{k-1})]\,y_k^{*c} - [(c-p)+(p-h)M_k(y_k^{*e}|y_{k-1})]\,y_k^{*e} \\
& -(p-h)\int_{y_k^{*e}}^{y_k^{*c}} x m_k(x|y_{k-1})dx + \int_{y_k^{*e}}^{y_k^{*c}} u_{k+1}(x) m_k(x|y_{k-1})dx \\
& +u_{k+1}^c(y_k^{*c})[1-M_k(y_k^{*c}|y_{k-1})] - u_{k+1}^c(y_k^{*e})[1-M_k(y_k^{*e}|y_{k-1})]\Big]. \qquad (48)
\end{aligned}
$$

Since $u_k(y_{k-1}) = J(\pi_k, y_k^{*e}) \le J(\pi_k, y_k^{*c})$, we have

$$
\begin{aligned}
& [(c-p)+(p-h)M_k(y_k^{*e}|y_{k-1})]\,y_k^{*e} + p\int_{y_k^{*e}}^{\infty} x m_k(x|y_{k-1})dx + h\int_0^{y_k^{*e}} x m_k(x|y_{k-1})dx \\
& + \int_0^{y_k^{*e}} u_{k+1}(x) m_k(x|y_{k-1})dx + u_{k+1}^c(y_k^{*e})[1-M_k(y_k^{*e}|y_{k-1})] \\
\le \;& [(c-p)+(p-h)M_k(y_k^{*c}|y_{k-1})]\,y_k^{*c} + p\int_{y_k^{*c}}^{\infty} x m_k(x|y_{k-1})dx + h\int_0^{y_k^{*c}} x m_k(x|y_{k-1})dx \\
& + \int_0^{y_k^{*c}} u_{k+1}(x) m_k(x|y_{k-1})dx + u_{k+1}^c(y_k^{*c})[1-M_k(y_k^{*c}|y_{k-1})], \qquad (49)
\end{aligned}
$$

which implies that the bracketed part on the right hand side of (48) is non-negative. Therefore, from (48) we get

$$
\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}} \le 0, \text{ when } y_k^{*e} \le y_k^{*c}. \qquad (50)
$$

For the case $y_k^{*e} \ge y_k^{*c}$, we can mimic the above arguments to get

$$
\begin{aligned}
&\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}} \\
= \;& -m'_{k-1}(y_{k-1})\Big[[(c-p)+(p-h)M_k(y_k^{*c}|y_{k-1})]\,y_k^{*c} - [(c-p)+(p-h)M_k(y_k^{*e}|y_{k-1})]\,y_k^{*e} \\
& +(p-h)\int_{y_k^{*c}}^{y_k^{*e}} x m_k(x|y_{k-1})dx - \int_{y_k^{*c}}^{y_k^{*e}} u_{k+1}(x) m_k(x|y_{k-1})dx \\
& +u_{k+1}^c(y_k^{*c})[1-M_k(y_k^{*c}|y_{k-1})] - u_{k+1}^c(y_k^{*e})[1-M_k(y_k^{*e}|y_{k-1})]\Big]. \qquad (51)
\end{aligned}
$$

By (49) we again see that the bracketed term on the right hand side of (51) is non-negative. Thus by (51) we have

$$\frac{dI_{k-1}(y_{k-1})}{dy_{k-1}} \leq 0, \text{ when } y_k^{*e} \geq y_k^{*c}. \tag{52}$$

Finally, (50) and (52) together implies the theorem for $n = k - 1$. ∎

Proof of Theorem B. Because of the convexity of the cost function $J(\pi_n', y_n)$ (defined in (13)), we can obtain the optimal order quantity by solving for y_n from $\frac{dJ(\pi_n', y_n)}{dy_n} = 0$ (refer to (14)). Setting $\frac{dJ(\pi_n', y_n)}{dy_n} = 0$ and rearranging terms we get the result. ∎

References

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