Online Companion for

"Production and Inventory Models Using Net Present Value"

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Appendix

A Proofs of Theorems and Lemmas in section 1

Proof of Lemma 1.2 (Properties of $TCPV(\theta|k,v,w)$):

Part 1 can be proved by computing the second derivative and using the inequality $e^{\theta} > 1+\theta$ for all $\theta > 0$. Part 2 is an immediate result of Part 1 and the definition of Lambert's W Function:

$$\begin{aligned}
1+\theta^*+q &= e^{\theta^*}, \\
1 &= e^{-\theta^*}\left(\theta^*+1+q\right), \\
-e^{-1-q} &= e^{-\theta^*-1-q}\left(-\theta^*-1-q\right), \\
-\theta^*-1-q &= LambertW\left(-1,-e^{-1-q}\right), \\
\theta^* &= -1-q-LambertW\left(-1,-e^{-1-q}\right).
\end{aligned}$$

Part 3 is easy to verify. □

Proof of Lemma 1.3

We know that

$$\begin{aligned}
TCPV(\theta^*|k,v,w) &= \frac{2}{\alpha^2}\left(k(\alpha^2/2)f_1(\theta^*)+vf_2(\theta^*)-w\right) \ \geq\ 0, \\
TCPV(r\theta^*|k,v,w) &= \frac{2}{\alpha^2}\left(k(\alpha^2/2)f_1(r\theta^*)+vf_2(r\theta^*)-w\right) \ \geq\ 0,
\end{aligned}$$

and $TCPV(r\theta^*|k,v,w) \geq TCPV(\theta^*|k,v,w)$ means

$$k(\alpha^2/2)f_1(r\theta^*)+vf_2(r\theta^*) \ \geq\ k(\alpha^2/2)f_1(\theta^*)+vf_2(\theta^*). \tag{A.1}$$

It is easy to see that

$$\text{If } a \ \geq\ b \ \geq\ v,\ w\ \geq\ v, \qquad \text{then } \frac{a-w}{b-w} \ \leq\ \frac{a-v}{b-v}.$$

Based on inequalities (A.1) and (1.3b), we have the inequality in the following derivation.

$$\begin{aligned}
\frac{TCPV(r\theta^*|k,v,w)}{TCPV(\theta^*|k,v,w)} &= \frac{k(\alpha^2/2)f_1(r\theta^*)+vf_2(r\theta^*)-w}{k(\alpha^2/2)f_1(\theta^*)+vf_2(\theta^*)-w} \\
&\leq \frac{k(\alpha^2/2)f_1(r\theta^*)+vf_2(r\theta^*)-v}{k(\alpha^2/2)f_1(\theta^*)+vf_2(\theta^*)-v} = \frac{\dfrac{k(\alpha^2/2)+vr\theta^*}{1-e^{-r\theta^*}}-v}{\dfrac{k(\alpha^2/2)+v\theta^*}{1-e^{-\theta^*}}-v} \\
&= \frac{\dfrac{(k/v)(\alpha^2/2)+r\theta^*}{1-e^{-r\theta^*}}-1}{\dfrac{(k/v)(\alpha^2/2)+\theta^*}{1-e^{-\theta^*}}-1} = \frac{((k/v)(\alpha^2/2)+r\theta^*-1+e^{-r\theta^*})(1-e^{-\theta^*})}{((k/v)(\alpha^2/2)+\theta^*-1+e^{-\theta^*})(1-e^{-r\theta^*})}.
\end{aligned}$$

From equation (1.7), we may replace $(k/v)(\alpha^2/2)$ by $e^{\theta^*}-1-\theta^*$ and have

$$\begin{aligned}
\frac{TCPV(r\theta^*|k,v,w)}{TCPV(\theta^*|k,v,w)} &\leq \frac{(e^{\theta^*}-1-\theta^*+r\theta^*-1+e^{-r\theta^*})(1-e^{-\theta^*})}{(e^{\theta^*}-1-\theta^*+\theta^*-1+e^{-\theta^*})(1-e^{-r\theta^*})} \\
&= \frac{(e^{-\theta^*}-2-\theta^*+r\theta^*+e^{-r\theta^*})(1-e^{-\theta^*})}{(e^{\theta^*}-2+e^{-\theta^*})(1-e^{-r\theta^*})} \\
&= \frac{(e^{\theta^*}-2-\theta^*+r\theta^*+e^{-r\theta^*})(1-e^{-\theta^*})}{e^{\theta^*}(1-e^{-\theta^*})^2(1-e^{-r\theta^*})} \\
&= \frac{1+(r\theta^*-\theta^*-2+e^{-r\theta^*})e^{-\theta^*}}{(1-e^{-\theta^*})(1-e^{-r\theta^*})}. \qquad \text{(A.2)}
\end{aligned}$$

As

$$E(r,\theta^*) = \frac{1+(r\theta^*-\theta-2+e^{-r\theta^*})e^{-\theta^*}}{(1-e^{-\theta^*})(1-e^{-r\theta^*})}.$$

By using symbolic mathematical tools (such as Maple), we can easily get the following estimate.

$$\frac{E(r,\theta^*)}{\frac{1}{2}\left(r+\frac{1}{r}\right)} \leq (1+\epsilon), \quad \forall\theta^*>0 \text{ and } r\in[\sqrt{0.5},\sqrt{2}], \qquad \text{(A.3)}$$

where $\epsilon = 0.000570126$ and the left hand side of (A.3) reaches its maximum value 1.000570126 when $r=\sqrt{2}$ and $\theta^* = 0.28768268$. □

B Proofs of Theorems and Lemmas in section 3

Proof of Lemma 3.2 (Minimax Theorem for the continuous relaxation (RP) of (P)):

Let

$$\delta \triangleq \min\{K(N)-K(N\backslash\{a\}) : a\in N\}.$$

Figure 3: the graph of $E(r,\theta)/[(r+1/r)/2]$ for $r \in [\sqrt{0.5}, \sqrt{2}]$, $\theta > 0$

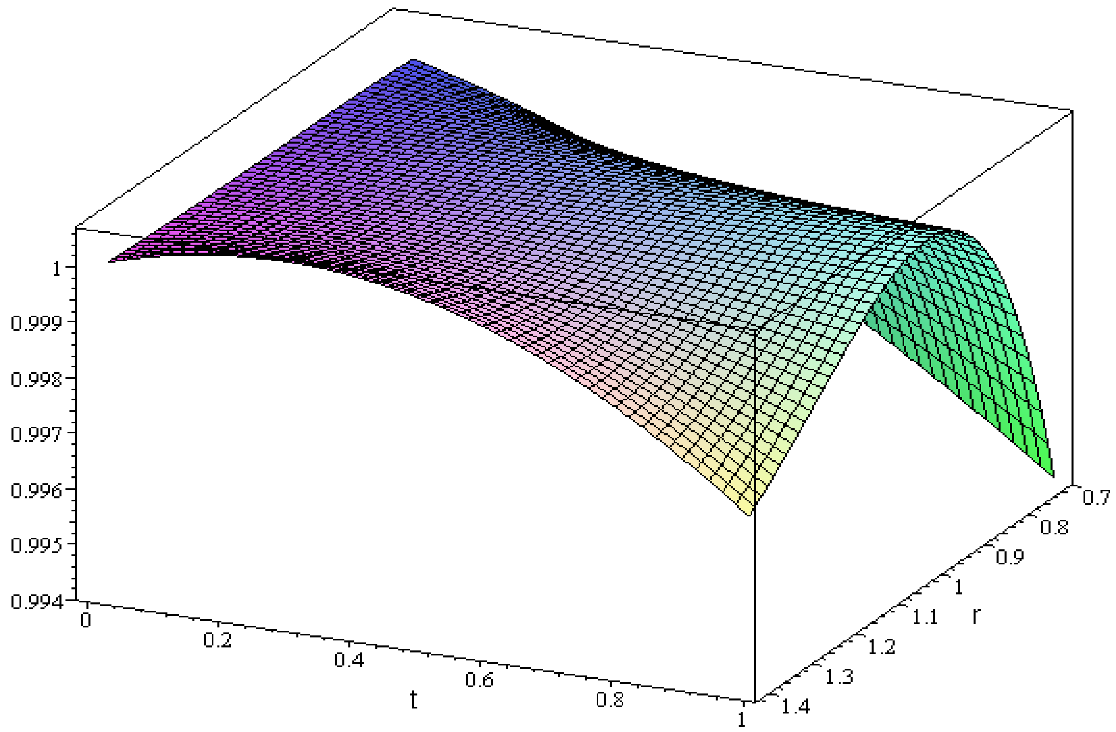
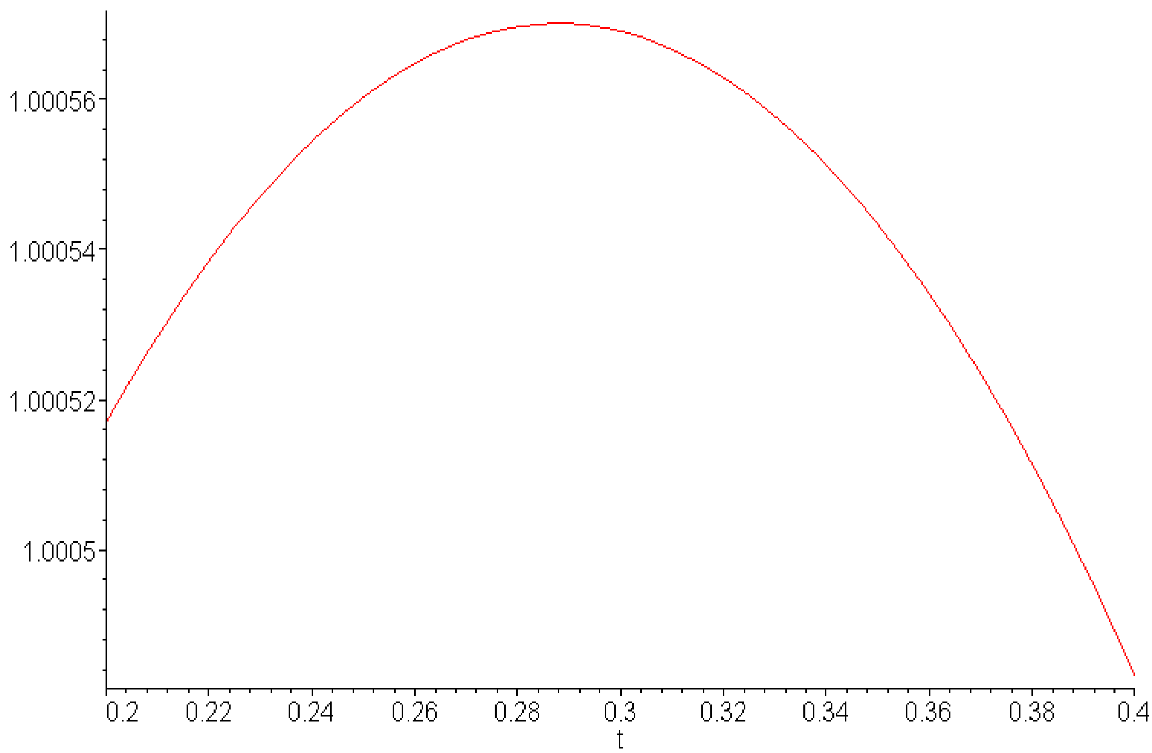


Figure 4: The graph of $E(r,\theta)/[(r+1/r)/2]$ for $r = \sqrt{2}$



Since $K(\cdot)$ is strictly mono-tone, we have $\delta > 0$. Note that for all $k \in K$ and $a \in N$, $k_a = K(N) - \sum_{i \neq a} k_i \geq K(N) - K(N \backslash \{a\}) \geq \delta$. Let $\overline{T}_L = T_L$ if T_L is fixed, and $\overline{T}_L$ be an arbitrary allowable value for T_L if T_L is variable. Defining $t^L \in T$ by $t_i^L = T_L$ for all $i \in N$. we have $\tau_r^L = \overline{T}_L$ for all $r \in R$. Thus for all $k \in K$ we have

$$f(t^L, k) \;=\; K(N) f_1(\alpha \overline{T}_L) + H(N) f_2(\overline{T}_L) = \frac{K(N) + H(N)\overline{T}_L}{1 - e^{-\alpha \overline{T}_L}} \triangleq c^0,$$

where $H(N) = \sum_{r \in R} \left(u_{i_r} + \frac{h_{i_r}}{\alpha} \right) \frac{d_r}{\alpha}$. As $t \searrow 0$, $f_1(t) \to \infty$, we may find t^- satisfy

$$f_1(\alpha t^-) \;\geq\; \frac{c^0}{\delta},$$

and t^+ satisfy

$$f_2(\alpha t^+) \;\triangleq\; \frac{c^0}{\min\limits_{r \in R} d_r \min\limits_{i \in N_0} \left(\frac{h_i}{\alpha} + u_i \right)}.$$

Define $\theta = \{t : t^- 1 \leq t \leq t^+ 1, \text{ satisfy (2.4b)}\}$ with $1 \in R^{|N|}$ a vector of ones. Since θ is compact and convex and f has the properties required in Lemma 1 we have for some $(t^*, k^*) \in \theta \times K$:

$$\max_{k \in K} \min_{t \in \theta} TCPV(t, k) = \min_{t \in \theta} \max_{k \in K} TCPV(t, k) = TCPV(t^*, k^*),$$

since

$$\begin{aligned} \frac{c^0}{f_1(\alpha t^-)} \leq \delta \;&\leq\; K(N) \leq (K(N) + H(N)\overline{T}_L) = c^0(1 - e^{-\alpha \overline{T}_L}) = \frac{c^0}{f_1(\alpha \overline{T}_L)}, \\ f_1(\alpha t^-) \;&\geq\; f_1(\alpha \overline{T}_L). \end{aligned}$$

As $f_1(t)$ is a decreasing function of t, we have $t^- \leq \overline{T}_L$. Since

$$(\min_{r \in R} d_r) \min_{i \in N_0} \left(\frac{h_i}{\alpha} + u_i \right) \frac{\overline{T}_L}{1 - e^{-\alpha \overline{T}_L}} \;\leq\; \frac{H(N)\overline{T}_L}{1 - e^{-\alpha \overline{T}_L}} \leq c^0 = \min_{r \in R} d_r \min_{i \in N_0} \left(\frac{h_i}{\alpha} + u_i \right) \frac{t^+}{1 - e^{-\alpha t^+}}.$$

Therefore,

$$\frac{\overline{T}_L}{1 - e^{-\alpha \overline{T}_L}} \;\leq\; \frac{t^+}{1 - e^{-\alpha t^+}},$$

or $f_2(\overline{T}_L) \leq f_2(\alpha t^+)$. As $f_2(t)$ is an increasing function of t, we have $\overline{T}_L \leq t^+$.

Therefore, vector $\overline{T}_L 1 \in \theta$ and we have

$$TCPV(t^*, k^*) \;\; \leq \;\; c^0. \tag{B.1}$$

In view of (B.1), it suffices to show that

$$TCPV(t, k) \;\; \geq \;\; c^0 \text{ for all } t \in T \backslash \theta \text{ and } k \in K. \tag{B.2}$$

Thus, fix $t \in T \backslash \theta$ and $k \in K$. There are two cases to consider:

(1) $t_a = \min_{i \in N} t_i < t^-$ for some $a \in N$.

Clearly,

$$TCPV(t, k) \;\; \geq \;\; k_a f_1(\alpha t_a) \geq \delta f_1(\alpha t_a) \geq \delta f_1(\alpha t^-) = c^0. \tag{B.3}$$

(2) $\max_{r \in R} \tau_r > t^+$.

Note, in view of (2.7), that max τ_r is achieved for some route ℓ whose initial node $a \in N_0$. Hence,

$$\begin{aligned} TCPV(t, k) \;\; &\geq \;\; \sum_{r \in R} \left(u_{i_r} + \frac{h_{i_r}}{\alpha} \right) \frac{d_r}{\alpha} f_2(\alpha \tau_r) \;\; \geq \;\; H_\ell f_2(t^+) & \text{(B.4)} \\ &\geq \;\; (\min_{r \in R} d_r) \min_{i \in N_0} \left(\frac{h_i}{\alpha} + u_i \right) \alpha / f_2(t^+) = c^0. & \text{(B.5)} \end{aligned}$$

Thus (B.2) follows from (B.3) and (B.5). □

Proof of Lemma 3.3 (Another expression of the continuous relaxation (RP)):

For any $r \in R$ and $i \in R$. let x_{ri} be a Lagrange multiplier associated with constraint (2.8). Fix $k^0 \in K$, let (RP_{k^0}) denote the convex (inner) minimization problem to the right of (3.2) with $k_i = k_i^0 (i \in N)$ and let $TCPV^*(k^0)$ denote its optimal value. That is, the problem (RP_{k^0}) is

$$TCPV^*(k^0) \;\; = \;\; \min f(t, k^0),$$

subject to

$$f_2(\alpha \tau_r) \;\; \geq \;\; f_2(\alpha t_i), \quad \forall \, r \in R, i \in r,$$

$$\tau, t \;\; > \;\; 0.$$

Let $f(t,k^0) = \alpha^2/2 \times TCPV(t,k^0)$, where $TCPV(t,k^0)$ is defined in (2.6). The associated Lagrangian of (RP_{k^0}) is given by

$$L(x,t) = \sum_{i\in N} \frac{k_i^0\alpha^2}{2} f_1(\alpha t_i) + \sum_{r\in R} H_r f_2(\alpha\tau_r) - \frac{1}{2}\sum_{r\in R} h_{i_r} d_r + \sum_{r\in R}\sum_{i\in r} x_{ri}\left[f_2(\alpha t_i) - f_2(\alpha\tau_r)\right].$$

By definition, see Geoffrion [9] (1971), the Lagrangian dual of (RP_{k^0}) can be written as

$$D = \sup_{x\geq 0}\inf_{t>0} L(x,t). \tag{B.6}$$

Regrouping the terms of $L(x,t)$, we have

$$L(x,t) = \sum_{i\in N}\left[\frac{k_i^0\alpha^2}{2} f_1(\alpha t_i) + \left(\sum_{r:i\in r} x_{ri}\right) f_2(\alpha t_i)\right] + \sum_{r\in R}\left[H_r - \sum_{i\in r} x_{ri}\right] f_2(\alpha\tau_r) - \frac{1}{2}\sum_{r\in R} h_{i_r} d_r. \tag{B.7}$$

If, for a given $x \geq 0$, $\left(H_r - \sum_{i\in r} x_{ri}\right) < 0$, then $\inf_{t>0} L(x,t) = \lim_{t\to\infty} L(x,t) = -\infty$; likewise, if $\left(H_r - \sum_{i\in r} x_{ri}\right) \geq 0$, $\inf_{\tau_r>0}\left[H_r - \sum_{i\in r} x_{ri}\right] f_2(\alpha\tau_r) = 0$. To achieve the supremum in (B.6), we may thus restrict ourselves to vectors x for which

$$H_r \geq \sum_{i\in r} x_{ri}, \quad r\in R. \tag{B.8}$$

Note that the last summation $\frac{1}{2}\sum_{r\in R} h_{i_r} d_r$ in (B.7)is a constant. We will allocate it to individual item i by w_i as defined in (3.3f). Note also that in an optimal solution (x^*, v^*), (B.8) must be satisfied as *equalities*, we have

$$\sum_{r\in R}\frac{h_{i_r} d_r}{2} = \sum_{r\in R}(H_r y_r) = \sum_{r\in R}\left(\sum_{i\in r} x_{ri} y_r\right) = \sum_{i\in N}\left(\sum_{r:i\in r} x_{ri} y_r\right) = \sum_{i\in N} w_i.$$

Also we note that this allocation will not affect the optimal solution. However, w_i has to satisfy the inequality (3.3f) in order to use the estimation of the inventory cost for power-of-two policies. Let

$$y_i \triangleq \max_{r:i\in r} y_r, \quad \forall\, i\in N.$$

Obviously, $y_r < 1$, $y_i < 1$, and we can see that w_i satisfying the inequality (3.3f) due to the following inequality:

$$w_i \leq \sum_{r:i\in r} x_{ri} y_i = v_i y_i < v_i.$$

Imposing these additional constraints and substituting (3.3c), we may rewrite (B.6) as:

$$\begin{aligned} D &= \sup_{\{x\geq 0, v\geq 0: (3.3c),(B.8)\}} \inf_{t>0} \sum_{i\in N} \left[k_i^0(\alpha^2/2)f_1(\alpha t_i) + v_i f_2(\alpha t_i) - w_i\right] \\ &= \sup_{\{x\geq 0, v\geq 0: (3.3c),(B.8)\}} \sum_{i\in N} \min_{t>0} \left[\frac{k_i^0\alpha^2/2 + v_i\alpha t_i}{1-e^{-\alpha t_i}} - w_i\right] \\ &= \sup_{\{x\geq 0, v\geq 0: (3.3c),(B.8\})} \frac{\alpha^2}{2} \sum_{i\in N} TCPV(\alpha t_i^* | k_i^0, v_i, w_i), \end{aligned} \tag{B.9}$$

where $TCPV(\alpha t_i^*|k_i^0, v_i, w_i)$ is defined in (1.6). *Strong duality*, i.e., $TCPV^*(k^0) = D$, and the existence of a pair (x^*, v^*) achieving the supremum in (B.9) all follow from (RP_k) being a *stable convex* program, see Theorem 3 in Geoffrion [9] (1971). Stability may be verified by Slater's condition, i.e., there exists a vector $t > 0$, with $\tau_r > T_i$ for all $r \in R$ and $i \in r$. (Let $\tau_r = 2$, all $r \in R$ and $T_i = 1$, all $i \in N$.) Finally note that in an optimal solution (x^*, v^*), (B.8) must be satisfied as *equalities* and may thus be replaced by (3.3b). The lemma thus follows from (B.9) and the subsequent observations. □

C Proofs of Theorems and Lemmas in section 4

Proof of Theorem 4.1 (The Lower Bound Theorem):

Assume first that $K(\cdot)$ is strictly monotone so that Lemmas 3.2 and 3.3 apply. Let t^* achieve the minimum in (RP), see Lemma 3.2. It follows from Lemma 3.3 that a vector $k^* \in K$, and vectors x^*, v^* exist which satisfy (3.3b), (3.3c), (3.3d) and with $TCPV^* = \sum_{j\in N} TCPV(\alpha t_j^*|k^*, v^*, w_i)$.

For a feasible policy π, let c be the NPV of the total cost incurred by the policy. We show that $c \geq TCPV^*$. We evaluate the total setup costs, production cost and holding costs separately.

Let η^m $(m = 0, 1, 2, \ldots)$ be all the instants at which items are replenished, and S_m be the set of items replenished at instant η^m. Let η_j^ℓ (or $0 = \eta_j^0, \eta_j^1, \eta_j^2, \cdots$) be the instant at which item j is replenished at its ℓth time and the order quantity for item j is Q_j^ℓ. Let τ_r^ℓ (or $0 = \tau_r^0, \tau_r^1, \tau_r^2, \cdots$) be the instant at which echelon inventory on route r is replenished at its ℓth time.

Note that η_j^ℓ is simply the renumbering of the replenish instant of η^m in terms of item j in order to make the summation over product j easier to express.

The NPV of the total setup cost is

$$\begin{aligned} K^{tot} &= \sum_{m=0}^{\infty} K(S_m) e^{-\alpha\eta^m} \geq \sum_{m=0}^{\infty}\left(\sum_{j\in S_m} k_j\right) e^{-\alpha\eta^m} = \sum_{j\in N}\left(\sum_{\ell=0}^{\infty} k_j e^{-\alpha\eta_j^\ell}\right) \\ &= \sum_{j\in N} k_j \left(\sum_{\ell=0}^{\infty} e^{-\alpha\eta_j^\ell}\right). \end{aligned}$$

Note that the first node of route r is i_r. The holding cost on route r is:

$$\begin{aligned} &\sum_{\ell=0}^{\infty}\left(h_{i_r} d_r \int_0^{\eta_r^{\ell+1}-\eta_r^\ell}\left(\eta_r^{\ell+1}-\eta_r^\ell - t\right) e^{-\alpha t} dt\right) e^{-\alpha\eta_r^\ell} \\ &= \frac{h_{i_r} d_r}{\alpha}\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell - \frac{1-e^{-\alpha(\eta_r^{\ell+1}-\eta_r^\ell)}}{\alpha}\right) e^{-\alpha\eta_r^\ell} \\ &= \frac{h_{i_r} d_r}{\alpha}\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell\right) e^{-\alpha\eta_r^\ell} - \frac{h_{i_r} d_r}{\alpha^2}\sum_{\ell=0}^{\infty}\left(e^{-\alpha\eta_r^\ell} - e^{-\alpha\eta_r^{\ell+1}}\right) \\ &= \frac{h_{i_r} d_r}{\alpha}\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell\right) e^{-\alpha\eta_r^\ell} - \frac{h_{i_r} d_r}{\alpha^2}. \end{aligned}$$

The NPV of the total holding cost is

$$H^{tot} = \sum_{r\in R}\left[\frac{h_{i_r} d_r}{\alpha}\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell\right) e^{-\alpha\eta_r^\ell}\right] - \sum_{r\in R}\frac{h_{i_r} d_r}{\alpha^2}.$$

The NPV of the total production cost is

$$U^{tot} = \sum_{r\in R}\left[u_{i_r} d_r \sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell\right) e^{-\alpha\eta_r^\ell}\right].$$

And the NPV of the total holding cost and production cost is

$$H^{tot} + U^{tot} = \sum_{r\in R}\left[\left(\frac{h_{i_r}}{\alpha} + u_{i_r}\right)\frac{d_r}{\alpha}\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell\right)\alpha e^{-\alpha\eta_r^\ell}\right] - \sum_{r\in R}\frac{h_{i_r} d_r}{\alpha^2}.$$

If $t_3 > t_2 > t_1 \geq 0$, then it is obvious that

$$(t_3 - t_1) e^{-\alpha t1} \geq (t_2 - t_1) e^{-\alpha t_1} + (t_3 - t_2) e^{-\alpha t_2}.$$

This means whenever we add extra replenishment instants, the summation $\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^\ell\right) e^{-\alpha\eta_r^\ell}$ is getting smaller. Therefore, for any node $i \in r$ the replenishment set $(\eta_r^0, \eta_r^1, \eta_r^2, \cdots)$ of route r is the subset of the replenishment set $(\eta_i^0, \eta_i^1, \eta_i^2, \cdots)$ of node i. And we have

$$\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^{\ell}\right)e^{-\alpha\eta_r^{\ell}} \quad \geq \quad \sum_{\ell=0}^{\infty}\left(\eta_i^{\ell+1}-\eta_i^{\ell}\right)e^{-\alpha\eta_i^{\ell}}, \quad \forall i \in r.$$

Therefore, we can get the lower bound of the NPV of the total holding cost and production cost

$$\begin{aligned}
H^{tot}+U^{tot} &= \sum_{r\in R}\left[\left(\frac{h_{i_r}}{\alpha}+u_{i_r}\right)\frac{d_r}{\alpha}\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^{\ell}\right)\alpha e^{-\alpha\eta_r^{\ell}}\right]-\sum_{r\in R}\frac{h_{i_r}d_r}{\alpha^2}\\
&= \sum_{r\in R}\left[\left(\sum_{i\in r}x_{ri}^*\right)\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^{\ell}\right)\alpha e^{-\alpha\eta_r^{\ell}}\right]-\sum_{r\in R}\frac{h_{i_r}d_r}{\alpha^2}\\
&= \sum_{i\in N}\sum_{r:i\in r}\left\{x_{ri}^*\left[\sum_{\ell=0}^{\infty}\left(\eta_r^{\ell+1}-\eta_r^{\ell}\right)\alpha e^{-\alpha\eta_r^{\ell}}\right]\right\}-\sum_{r\in R}\frac{h_{i_r}d_r}{\alpha^2}\\
&\geq \sum_{i\in N}\sum_{r:i\in r}\left\{x_{ri}^*\left[\sum_{\ell=0}^{\infty}\left(\eta_i^{\ell+1}-\eta_i^{\ell}\right)\alpha e^{-\alpha\eta_i^{\ell}}\right]\right\}-\sum_{r\in R}\frac{h_{i_r}d_r}{\alpha^2}\\
&= \sum_{i\in N}\left\{\left(\sum_{r:i\in r}x_{ri}^*\right)\left[\sum_{\ell=0}^{\infty}\left(\eta_i^{\ell+1}-\eta_i^{\ell}\right)\alpha e^{-\alpha\eta_i^{\ell}}\right]\right\}-\sum_{r\in R}\frac{h_{i_r}d_r}{\alpha^2}\\
&= \sum_{i\in N}\left\{v_i^*\sum_{\ell=0}^{\infty}\left(\eta_i^{\ell+1}-\eta_i^{\ell}\right)\alpha e^{-\alpha\eta_i^{\ell}}-w_i\right\}.
\end{aligned}$$

The NPV c of the total cost is therefore,

$$\begin{aligned}
c &= K^{tot}+H^{tot}+U^{tot} \geq \sum_{i\in N}\left\{k_i\left(\sum_{\ell=0}^{\infty}e^{-\alpha\eta_i^{\ell}}\right)+v_i^*\sum_{\ell=0}^{\infty}\left(\eta_i^{\ell+1}-\eta_i^{\ell}\right)\alpha e^{-\alpha\eta_i^{\ell}}-w_i\right\} \qquad \text{(C.1)}\\
&\geq \sum_{i\in N}TCPV(\alpha t_i^*|k_i, v_i^*, w_i) = TCPV^*.
\end{aligned}$$

The last inequality is based on that $TCPV(\alpha t_i^*|k_i, v_i^*, w_i)$ is the optimal solution for each term in (C.1) satisfying (3.3f), which is concluded from the result on the single product expressed in (1.2) and satisfies (1.3b). Therefore, the optimal value $TCPV^*$ of (RP) is a lower bound on the total cost (NPV) for any feasible policy. □

Proof of Lemma 4.2:

Let

$$f(\beta) = \frac{1}{\beta\ln 2},$$

then we have

$$\int_{\sqrt{0.5}}^{\sqrt{2}} f(\beta)d\beta = 1.$$

Based on equation (4.1), we have

$$\frac{\overline{t}_i}{t_i} = \begin{cases} \dfrac{2^{p_i}\beta T_0}{b_i T_0 2^{p_i}\sqrt{0.5}} = \dfrac{\beta\sqrt{2}}{b_i}, & \text{if } \sqrt{0.5} \le \beta < b_i, \\ \dfrac{2^{p_i-1}\beta T_0}{b_i T_0 2^{p_i}\sqrt{0.5}} = \dfrac{\beta\sqrt{0.5}}{b_i}, & \text{if } b_i \le \beta < \sqrt{2}. \end{cases}$$

And therefore,

$$\begin{aligned} \int_{\sqrt{0.5}}^{\sqrt{2}} TC(\overline{t}_i)\frac{d\beta}{\beta \ln 2} &= \int_{\sqrt{0.5}}^{\sqrt{2}} TC(\overline{t}_i) f(\beta) d\beta \\ &= \int_{\sqrt{0.5}}^{b_i} TC\left(\frac{\beta\sqrt{2}}{b_i}t_i\right) f(\beta)d\beta + \int_{b_i}^{\sqrt{2}} TC\left(\frac{\beta}{b_i\sqrt{2}}t_i\right) f(\beta)d\beta \qquad \text{(C.2)} \\ &= \int_{\frac{1}{b_i}}^{\sqrt{2}} TC(rt_i) f\left(\frac{rb_i}{\sqrt{2}}\right)\frac{b_i}{\sqrt{2}}dr + \int_{\sqrt{0.5}}^{\frac{1}{b_i}} TC(rt_i) f\left(rb_i\sqrt{2}\right) b_i\sqrt{2}dr \qquad \text{(C.3)} \\ &= \int_{\frac{1}{b_i}}^{\sqrt{2}} TC(rt_i)\frac{\sqrt{2}}{rb_i \ln 2}\frac{b_i}{\sqrt{2}}dr + \int_{\sqrt{0.5}}^{\frac{1}{b_i}} TC(rt_i)\frac{1}{rb_i\sqrt{2}\ln 2} b_i\sqrt{2}dr \\ &= \int_{\sqrt{0.5}}^{\sqrt{2}} TC(rt_i)\frac{dr}{r\ln 2}. \end{aligned}$$

Note that from (C.2) to (C.3) we make a replacement $r = \frac{\beta\sqrt{2}}{b_i}$ for the first integration while we make a replacement $r = \frac{\beta}{b_i\sqrt{2}}$ for the second integration. As we have

$$TC(rt_i) \le \frac{1+\epsilon}{2}\left(r + \frac{1}{r}\right) TC(t_i),$$

therefore,

$$\begin{aligned} \int_{\sqrt{0.5}}^{\sqrt{2}} TC(rt_i)\frac{dr}{r\ln 2} &\le \int_{\sqrt{0.5}}^{\sqrt{2}} \frac{1+\epsilon}{2}\left(r+\frac{1}{r}\right) TC(t_i)\frac{dr}{r\ln 2} \\ &= TC(t_i)\frac{1+\epsilon}{2}\int_{\sqrt{0.5}}^{\sqrt{2}}\left(r+\frac{1}{r}\right)\frac{dr}{r\ln 2} \\ &= \frac{1+\epsilon}{\sqrt{2}\ln 2} TC(t_i). \end{aligned}$$

This completes the proof. □

Proof of Theorem 4.3 (Worst case analysis of power-of-two policies):

We assume that $K(\cdot)$ is strictly monotone so that Lemma 3.3 applies. If it is not, the proof can be amended by considering a sequence of perturbed setup cost functions. In view of Lemma 3.3, let $t^* = \{t_1^*, t_2^*, \ldots, t_n^*\}$ denote an optimal solution to (RP) and $\overline{t}^* = (\overline{t}_1^*, \overline{t}_2^*, \ldots, \overline{t}_n^*)$ the power-of-two vector obtained by the rounding procedure. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a permutation of the node indices such that $t^{*-1}_{\gamma_1} \geq t^{*-1}_{\gamma_2} \geq \ldots \geq t^{*-1}_{\gamma_n}$. Note that the components of $\overline{t}^*$ may be ranked in the same way.

Based on Lemma 1.3, αt^*_i is the optimal solution to $TCPV(\alpha t_i|k_i, v_i, w_i)$. And $\overline{t}^*_i = r_i t^*_i$ the power-of-two rounding of t^*_i with $r_i \in (\sqrt{0.5}, \sqrt{2}]$ for all $i \in N$. Based on Lemma 1.3, we have

$$\frac{TCPV(r_i\alpha t^*_i|k_i, v_i, w_i)}{TCPV(\alpha t^*_i|k_i, v_i, w_i)} \leq \frac{1+\epsilon}{2}\left(r_i + \frac{1}{r_i}\right), \quad \forall r_i \in [\sqrt{0.5}, \sqrt{2}), \quad i \in N,$$

where $\epsilon = 0.000570126 > 0$.

It is easy to verify that

$$r + \frac{1}{r} \leq \frac{3}{\sqrt{2}} = 2.121320344, \quad \forall r \in [\sqrt{0.5}, \sqrt{2}).$$

And we have the following estimate for fixed base period T_L:

$$\begin{aligned}
\sum_{i\in N} TCPV(\alpha\overline{t}^*_i|k_i, v_i, w_i) &= \sum_{i\in N} TCPV(r_i\alpha t^*_i|k_i, v_i, w_i) \\
&\leq \sum_{i\in N} \frac{1+\epsilon}{2}\left(r_i + \frac{1}{r_i}\right) TCPV(\alpha t^*_i|k_i, v_i, w_i) \\
&\leq (1+\epsilon)\frac{2.121320344}{2} \sum_{i\in N} TCPV(\alpha t^*_i|k_i, v_i, w_i) \\
&\leq 1.061264882 \times TCPV^*.
\end{aligned}$$

Based on Lemma 4.2, we have the following estimate for variable base period T_L:

$$\begin{aligned}
\min_{T_L>0} \sum_{i\in N} TCPV(\alpha\overline{t}^*_i|k_i, v_i, w_i) &= \min_{T_L>0} \sum_{i\in N} TCPV(r_i\alpha t^*_i|k_i, v_i, w_i) \\
&\leq \sum_{i\in N} \int_{\sqrt{0.5}}^{\sqrt{2}} TCPV(\alpha t^*_i|k_i, v_i, w_i)\frac{dr_i}{r \ln 2} \\
&\leq \sum_{i\in N} \frac{1+\epsilon}{\sqrt{2}\ln 2} TCPV(\alpha t^*_i|k_i, v_i, w_i) \\
&\leq \frac{1+\epsilon}{\sqrt{2}\ln 2} \sum_{i\in N} TCPV(\alpha t^*_i|k_i, v_i, w_i) \\
&\leq 1.020721055 \times TCPV^*.
\end{aligned}$$

This completes our proof. □

D Proofs of Theorems and Lemmas in section 5

Proof of Lemma 5.1

Let T be the reorder interval. The total average cost TC for EOQ model is:

$$TC = \frac{k}{T} + \frac{1}{2}(h + \alpha u)dT = \frac{k}{T} + vdT,$$

where v is defined in (1.3a). We can easily find the optimal reorder interval T^{EOQ} below.

$$T^{EOQ} = \sqrt{k/v}. \tag{D.1}$$

If the reorder interval is T, the NPV of the total cost $TCPV(\alpha T|k, v, w)$ is defined in (1.5).

$$TCPV(\alpha T|k, v, w) = \frac{2}{\alpha^2}\left(\frac{k(\alpha^2/2) + v\alpha T}{1 - e^{-\alpha T}} - w\right),$$

where w is defined in (1.3b). Then

$$\alpha T^{EOQ} = \sqrt{2q},$$

where q is defined in (1.4d). We have

$$\begin{aligned} TCPV(\alpha T^{EOQ}|k, v, w) &= TCPV(\sqrt{2q}|k, v, w) = \frac{2v}{\alpha^2}\left(\frac{(k/v)(\alpha^2/2) + \sqrt{2q}}{1 - e^{-\sqrt{2q}}} - \frac{w}{v}\right) \\ &= \frac{2v}{\alpha^2}\left(\frac{q + \sqrt{2q}}{1 - e^{-\sqrt{2q}}} - \frac{w}{v}\right). \end{aligned}$$

The optimal solution θ^* to $TCPV(\theta|k, v, w)$ and the optimal value of $TCPV(\theta|k, v, w)$ can be found in Lemma 1.2. And the optimal reorder interval is $T^{NPV} = \theta^*/\alpha$.

$$TCPV(\theta^*|k, v, w) = TCPV(\alpha T^{NPV}|k, v, w) = \frac{2v}{\alpha^2}\left(-LambertW\left(-1, -e^{-1-q}\right) - \frac{w}{v}\right).$$

Therefore, the ratio of $TCPV$s based on EOQ model and NPV model:

$$\begin{aligned} \frac{TCPV(\alpha T^{EOQ}|k, v, w)}{TCPV(\alpha T^{NPV}|k, v, w)} &= \frac{\frac{2v}{\alpha^2}\left(\frac{q + \sqrt{2q}}{1 - e^{-\sqrt{2q}}} - \frac{w}{v}\right)}{\frac{2v}{\alpha^2}\left(-LambertW\left(-1, -e^{-1-q}\right) - \frac{w}{v}\right)} \\ &\leq \frac{\frac{q + \sqrt{2q}}{1 - e^{-\sqrt{2q}}} - 1}{-LambertW\left(-1, -e^{-1-q}\right) - 1} < 1.09603. \end{aligned}$$

where $q^* = 28.75$ gives us the largest value of the ratio above and is the base for the last inequality of $q > 0$. □

Proof of Lemma 5.4

1. We have

$$\begin{aligned} c^{EOQ} &= \max \sum_{i \in N} 2(k_i v_i^*)^{1/2}, \qquad \text{(D.2)} \\ H_r &= \sum_{i \in r} x_{ri}^*, \; r \in R, \\ \sum_{r: i \in r} x_{ri}^* &= v_i^*, i \in N, \\ & k \geq 0, x \geq 0, v \geq 0. \end{aligned}$$

Note that $\tau_r \geq t_i$ for all $i \in r$. We have

$$\sum_{r \in R} H_r \tau_r = \sum_{r \in R} \sum_{i \in r} x_{ri}^* \tau_r = \sum_{i \in N} \sum_{r: i \in r} (x_{ri}^* \tau_r) \geq \sum_{i \in N} \sum_{r: i \in r} x_{ri}^* t_i = \sum_{i \in N} v_i^* t_i.$$

Therefore,

$$c^{EOQ} = \min_{t \in T} \max_{k \in K} \left\{ \sum_{i \in N} k_i / t_i + \sum_{r \in R} H_r \tau_r \right\} \geq \min_{t \in T} \sum_{i \in N} \left\{ k_i^* / t_i + v_i^* t_i \right\}.$$

Let

$$\begin{aligned} t_i^{EOQ} &\triangleq \sqrt{k_i^* / v_i^*} = \frac{1}{\alpha} \sqrt{2 q_i^*}, \\ \tau_r^{EOQ} &\triangleq \max_{i \in r} t_i^{EOQ}, \quad \forall\, r \in R, \end{aligned}$$

which minimize the equation above and q_i^* is defined in (5.1). Combining the equation (D.2), we have

$$c^{EOQ} \geq \min_{t \in T} \sum_{i \in N} \left\{ k_i^* / t_i + v_i^* t_i \right\} = \sum_{i \in N} 2(k_i^* v_i^*)^{1/2} = c^{EOQ}.$$

This means we can replace the equality for all the inequalities above. Then we have

$$\sum_{r:i\in r}\left(x_{ri}^*\tau_r^{EOQ}\right) = \sum_{r:i\in r} x_{ri}^* t_i^{EOQ}, \quad \forall\, i \in N.$$

As $\tau_r^{EOQ} \geq t_i^{EOQ}$ for all $i \in r$ and $x_{ri}^* \geq 0$, we conclude that

$$\text{If } \tau_r^{EOQ} > t_i^{EOQ}, \quad \text{then} \quad x_{ri}^* = 0, \tag{D.3a}$$

$$\text{If } x_{ri}^* > 0, \quad \text{then} \quad \tau_r^{EOQ} = t_i^{EOQ}. \tag{D.3b}$$

As

$$\begin{aligned}\sum_{r\in R} H_r\tau_r^{EOQ} &= \sum_{r\in R}\sum_{i\in r} x_{ri}^*\tau_r^{EOQ} = \sum_{i\in N}\sum_{r:i\in r} x_{ri}^*\tau_r^{EOQ} = \sum_{i\in N}\sum_{\substack{r:i\in r\\ x_{ri}^*>0}} x_{ri}^*\tau_r^{EOQ}\\ &= \sum_{i\in N}\sum_{\substack{r:i\in r\\ x_{ri}^*>0}} x_{ri}^* t_i^{EOQ} = \sum_{i\in N}\sum_{r:i\in r} x_{ri}^* t_i^{EOQ} = \sum_{i\in N} v_i^* t_i^{EOQ}.\end{aligned}$$

Therefore,

$$c^{EOQ} = \sum_{i\in N} k_i^*/t_i^{EOQ} + \sum_{r\in R} H_r\tau_r^{EOQ} = \sum_{i\in N}\left(k_i^*/t_i^{EOQ} + v_i^* t_i^{EOQ}\right) = \sum_{i\in N} 2(k_i^* v_i^*)^{1/2}.$$

That is, t_i^{EOQ} and τ_r^{EOQ} are the optimal solution to the average cost model. Now we would like to calculate the NPV of total cost in the NPV model using t_i^{EOQ} and τ_r^{EOQ}. Based on equation (2.6), the lower bound on the NPV of the total cost is as follows.

$$TCPV(t,k) = \frac{2}{\alpha^2}\left\{\sum_{i\in N} k_i^*(\alpha^2/2) f_1(\alpha t_i^{EOQ}) + \sum_{r\in R} H_r f_2(\alpha\tau_r^{EOQ}) - \sum_{r\in R}\frac{h_{i_r} d_r}{2}\right\}.$$

Because

$$\begin{aligned}\sum_{r\in R} H_r f_2(\alpha\tau_r^{EOQ}) &= \sum_{r\in R}\sum_{i\in r} x_{ri}^* f_2(\alpha\tau_r^{EOQ}) = \sum_{i\in N}\sum_{r:i\in r} x_{ri}^* f_2(\alpha\tau_r^{EOQ})\\ &= \sum_{i\in N}\sum_{\substack{r:i\in r\\ x_{ri}^*>0}} x_{ri}^* f_2(\alpha\tau_r^{EOQ}) = \sum_{i\in N}\sum_{\substack{r:i\in r\\ x_{ri}^*>0}} x_{ri}^* f_2(\alpha t_i^{EOQ})\\ &= \sum_{i\in N}\left[\sum_{r:i\in r} x_{ri}^*\right] f_2(\alpha t_i^{EOQ}) = \sum_{i\in N} v_i^* f_2(\alpha t_i^{EOQ}),\end{aligned}$$

and let It is easy to verify that

$$\sum_{r\in R}\frac{h_{i_r}d_r}{2} \;=\; \sum_{i\in N} w_i^*.$$

Therefore, we have

$$\begin{aligned}
TCPV(t^{EOQ},k) &= \frac{2}{\alpha^2}\sum_{i\in N}\left\{k_i^*(\alpha^2/2)f_1(\alpha t_i^{EOQ})+v_i^*f_2(t_i^{EOQ})-w_i^*\right\}\\
&= \sum_{i\in N}\frac{2v_i^*}{\alpha^2}\left[\frac{(k_i^*/v_i^*)(\alpha^2/2)+\alpha t_i^{EOQ}}{1-e^{-\alpha t_i^{EOQ}}}-\frac{w_i^*}{v_i^*}\right]\\
&= \sum_{i\in N}\frac{2v_i^*}{\alpha^2}\left[\frac{q_i^*+\sqrt{2q_i^*}}{1-e^{-\sqrt{2q_i^*}}}-\frac{w_i^*}{v_i^*}\right].
\end{aligned}$$

2. Now we will find a feasible solution to NPV model based on x_{ri}^* and v_i^*.

Based on Lemma 1.2, we have

$$\begin{aligned}
t_i^{NPV} &= \frac{1}{\alpha}[-1-q_i^*-LambertW(-1,-e^{-1-q_i^*})],\\
TCPV(\alpha t_i^{NPV}|k,v_i^*,w_i^*) &= \frac{2v_i^*}{\alpha^2}\left[-LambertW\left(-1,-e^{-1-q_i^*}\right)-\frac{w_i^*}{v_i^*}\right],\\
\tau_r^{NPV} &\triangleq \max_{i\in r} t_i^{NPV}.
\end{aligned}$$

Because both t_i^{EOQ} and t_i^{NPV} are strictly increasing functions of q_i^*, the relationships in (D.3) will hold for t_i^{NPV} also. That is,

$$\text{If } \tau_r^{NPV} > t_i^{NPV}, \quad \text{then} \quad x_{ri}^* = 0, \tag{D.4a}$$

$$\text{If } x_{ri}^* > 0, \quad \text{then} \quad \tau_r^{NPV} = t_i^{NPV}. \tag{D.4b}$$

We know that k_i^*, t_i^{NPV}, τ_r^{NPV} are a feasible solution to continuous relaxation (RP). Therefore, based on the first equality in Lemma 3.2, we have

$$TCPV(t^{NPV},k) \;\geq\; TCPV^*. \tag{D.5}$$

However, we also have

$$\begin{aligned}\sum_{r\in R} H_r f_2(\alpha\tau_r^{NPV}) &= \sum_{r\in R}\sum_{i\in r} x_{ri}^* f_2(\alpha\tau_r^{NPV}) = \sum_{i\in N}\sum_{r:i\in r} x_{ri}^* f_2(\alpha\tau_r^{NPV}) \\ &= \sum_{i\in N}\sum_{\substack{r:i\in r\\ x_{ri}^*>0}} x_{ri}^* f_2(\alpha\tau_r^{NPV}) = \sum_{i\in N}\sum_{\substack{r:i\in r\\ x_{ri}^*>0}} x_{ri}^* f_2(\alpha t_i^{NPV}) \\ &= \sum_{i\in N}\left[\sum_{r:i\in r} x_{ri}^*\right] f_2(\alpha t_i^{NPV}) = \sum_{i\in N} v_i^* f_2(\alpha t_i^{NPV}).\end{aligned}$$

Therefore,

$$\begin{aligned}TCPV(t^{NPV},k) &= \frac{2}{\alpha^2}\left\{\sum_{i\in N} k_i^*(\alpha^2/2) f_1(\alpha t_i^{NPV}) + \sum_{r\in R} H_r f_2(\alpha\tau_r^{NPV}) - \sum_{r\in R}\frac{h_{i_r} d_r}{2}\right\} \\ &= \sum_{i\in N}\frac{2}{\alpha^2}\left\{k_i^*(\alpha^2/2) f_1(\alpha t_i^{NPV}) + v_i^* f_2(\alpha t_i^{NPV}) - w_i^*\right\} \\ &= \sum_{i\in N}\frac{2v_i^*}{\alpha^2}\left[\frac{q_i^* + \alpha t_i^{NPV}}{1-e^{\alpha t_i^{NPV}}} - \frac{w_i^*}{v_i^*}\right] \\ &= \sum_{i\in N}\frac{2v_i^*}{\alpha^2}\left[-LambertW\left(-1, -e^{-1-q_i^*}\right) - \frac{w_i^*}{v_i^*}\right].\end{aligned}$$

That is, k_i^*, x_i^*, v_i^* are also a feasible solution to the continuous relaxation (RP) in Lemma 3.3. That is,

$$TCPV(t^{NPV},k) \leq TCPV^*. \tag{D.6}$$

From (D.5) and (D.6), we conclude that

$$TCPV(t^{NPV},k) = TCPV^*.$$

This completes our proof. □