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APPENDIX

A-1. Normal Approximation of $p_{mn}(r,t)$

Using the central limit theorem, the joint distribution of $X_m(r)$ and $X_n(t)$ can be approximated by a bivariate normal distribution. Thus,

$$p_{mn}(r,t) \quad \int_{a_m(r)}^{\infty}\int_{a_n(t)}^{\infty} \phi_2(z_1,z_2;\rho_{mn}(r,t))dz_1dz_2 \tag{A-1}$$

where

$$a_j(s) = -\mathrm{E}[X_j(s)]/\sqrt{\mathrm{V}[X_j(s)]} \tag{A-2}$$

$$\mathrm{E}[X_j(s)] = \theta_s - \sum_{i=1}^{j} c_i p_i \tag{A-3}$$

$$\mathrm{V}[X_j(s)] = \sigma_s^2 + \sum_{i=1}^{j} c_i^2 p_i q_i \tag{A-4}$$

$$\rho_{mn}(r,t) = \mathrm{Cov}[X_m(r),X_n(t)]/\sqrt{\mathrm{V}[X_m(r)]\times \mathrm{V}[X_n(t)]} \tag{A-5}$$

$$\mathrm{Cov}[X_m(r),X_n(t)] = \sigma_{r,t} + \sum_{i=1}^{\min(m,n)} c_i^2 p_i q_i \mathrm{e}^{-\delta_i|t-r|} \tag{A-6}$$

A-2. Edgeworth (Gram-Charlier) Approximation of $p_{mn}(r,t)$

The approximate expression for $p_{mn}(r,t)$ using the bivariate Edgeworth expansion for the joint distribution function of $X_m(r)$ and $X_n(t)$ in which only the terms involving cumulants up to the third order is retained is given by:

$$\begin{aligned} p_{mn}(r,t) \quad & \int_{a_m(r)}^{\infty}\int_{a_n(t)}^{\infty} \phi_2[z_1,z_2;\rho_{mn}(r,t)]\times\left\{1+\frac{1}{6}\frac{\mathrm{K}_{30}}{\mathrm{K}_{20}^{3/2}}\mathrm{H}_{30}(z_1,z_2;\rho_{mn}(r,t))\right. \\ & +\frac{1}{2}\frac{\mathrm{K}_{21}}{\mathrm{K}_{20}K_{02}^{1/2}}\mathrm{H}_{21}(z_1,z_2;\rho_{mn}(r,t))+\frac{1}{2}\frac{\mathrm{K}_{12}}{\mathrm{K}_{20}^{1/2}K_{02}}\mathrm{H}_{12}(z_1,z_2;\rho_{mn}(r,t)) \\ & \left.+\frac{1}{6}\frac{\mathrm{K}_{03}}{\mathrm{K}_{02}^{1/2}}\mathrm{H}_{30}(z_1,z_2;\rho_{mn}(r,t))\right\}\mathrm{d}z_1\mathrm{d}z_2 \end{aligned} \tag{A-7}$$

Here, K_{kl} is the cumulant of order (k,l) of $[X_m(r),\ X_n(t)]$, and H_{ij} the bivariate Hermite polynomial. When only cumulants up to the third order are considered, the Gram-Charlier and the Edgeworth series give identical expansions.

A-3. Large Deviation Approximation of $p_{mn}(r,t)$

When tail probabilities are involved, a better approximation is given by the large deviation approximation. Exponential tilting is used here to convert the tail region into a central region and then approximate the tilted distribution by a bivariate normal distribution. Let the exponentially tilted distribution of $\mathbf{X}=[X_m(r),\ X_n(t)]$ for a given vector $\mathbf{S}=(s_1,s_2)$ be:

$$\mathrm{dF}^{\mathbf{S}}(\mathbf{X};m,n,r,t)=e^{\mathbf{S}'\mathbf{X}-\mathrm{K}(\mathbf{S};m,n,r,t)}\,\mathrm{dF}(\mathbf{X};m,n,r,t) \qquad \text{(A-8)}$$

where $F(x;m,n,r,t)$ is the bivariate c.d.f. of the random vector $\mathbf{X}$, and

$$\mathrm{K}(\mathbf{S};m,n,r,t)=\ln \mathrm{E}[e^{\mathbf{S}'\mathbf{X}}] \qquad \text{(A-9)}$$

Then, the value of $p_{mn}(r,t)=\int_0^\infty\int_0^\infty \mathrm{dF}(\mathbf{X};m,n,r,t)$ can be expressed as a function of the distribution function $F^{\mathbf{S}}(\mathbf{X};m,n,r,t)$ as

$$p_{mn}(r,t)=e^{\mathrm{K}(\mathbf{S};m,n,r,t)}\int_0^\infty\int_0^\infty e^{-\mathbf{S}'\mathbf{X}}\,\mathrm{dF}^{\mathbf{S}}(\mathbf{X};m,n,r,t) \qquad \text{(A-10)}$$

Next, the central limit theorem is used to approximate $F^{\mathbf{S}}(\mathbf{X};m,n,r,t)$ by a normal distribution $\Phi_2[\mathbf{X};\mathbf{B},\Sigma]$ with appropriately determined mean vector $\mathbf{B}$ and covariance matrix Σ. Thus, we obtain

$$p_{mn}(r,t)\quad e^{\mathrm{K}(\mathbf{S};m,n,r,t)}\int_0^\infty\int_0^\infty e^{-\mathbf{S}'\mathbf{X}}\,\mathrm{d}\Phi(\mathbf{X};\mathbf{B},\mathbf{Ó}) \qquad \text{(A-11)}$$

The constant vector **S** is now chosen so that the lower limits of the integrals is the expected value of the random variable $\mathbf{X}^{\mathbf{S}}$ whose distribution function is $F^{S}(\mathbf{X};m,n,r,t)$. This reduces to the following system of equations (assuming $m>n$):

$$\theta_r+\sigma_r^2 s_1+\sigma_{r,t}s_2-\sum_{i=1}^{n}\frac{c_i p_i \mathrm{g}_{r,t}(q_i,q_i,\delta_i)e^{-s_1c_i}-c_i p_i \mathrm{f}_{r,t}(p_i,q_i,\delta_i)e^{-(s_1+s_2)c_i}}{q_i\mathrm{f}_{r,t}(q_i,p_i,\delta_i)+p_i\mathrm{g}_{r,t}(q_i,q_i,\delta_i)e^{-s_1c_i}+q_i\mathrm{g}_{r,t}(p_i,p_i,\delta_i)e^{-s_2c_i}+p_i\mathrm{f}_{r,t}(p_i,q_i,\delta_i)e^{-(s_1+s_2)c_i}} - \sum_{i=n+1}^{m}\frac{c_i p_i e^{s_1c_i}}{q_i+p_i e^{-s_1c_i}} = 0 \qquad \text{(A-12)}$$

$$\theta_t+\sigma_t^2 s_2+\sigma_{r,t}s_1-\sum_{i=1}^{n}\frac{c_i p_i \mathrm{g}_{r,t}(p_i,p_i,\delta_i)e^{-s_2c_i}-c_i p_i \mathrm{f}_{r,t}(p_i,q_i,\delta_i)e^{-(s_1+s_2)c_i}}{q_i\mathrm{f}_{r,t}(q_i,p_i,\delta_i)+p_i\mathrm{g}_{r,t}(q_i,q_i,\delta_i)e^{-s_1c_i}+q_i\mathrm{g}_{r,t}(p_i,p_i,\delta_i)e^{-s_2c_i}+p_i\mathrm{f}_{r,t}(p_i,q_i,\delta_i)e^{-(s_1+s_2)c_i}} = 0 \qquad \text{(A-13)}$$

where $\mathrm{f}_{r,t}(a_i,b_i,\delta_i)=a_i+b_i\mathrm{e}^{-\delta_i|r-t|}$ and $\mathrm{g}_{r,t}(a_i,b_i,\delta_i)=a_i-b_i\mathrm{e}^{-\delta_i|r-t|}$

Let $\mathbf{S}_0 = [s_1, s_2]$ denote the roots of equations A-12 and A-13. Then, Equation A-11 can be rewritten as

$$p_{mn}(r,t) \quad \mathrm{e}^{\mathrm{K}(\mathbf{S}_0;m,n,r,t)}\int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-\mathbf{S}_0'\mathbf{X}}\,\mathrm{d}\Phi(\mathbf{X};\mathbf{0},\mathbf{\acute{O}}) \qquad \text{(A-14)}$$

Ordering terms, completing squares, and evaluating the integrals gives:

$$p_{mn}(r,t) \quad \mathrm{e}^{\mathrm{K}(s_1,s_2;m,n,r,t)+\mathrm{G}(s_1,s_2;m,n,r,t,\rho)}\Phi_2(\alpha_1,\alpha_2,\rho(s_1,s_2;m,n,r,t)) \qquad \text{(A-15)}$$

Expressions for $\rho(\,)$, $\mathrm{G}(\,)$, α_1, and α_2 have been derived to be

$$\rho(s_1,s_2;m,n,r,t)=\frac{\mathrm{K}_{11}(s_1,s_2;m,n,r,t)}{\sqrt{\mathrm{K}_{02}(s_1,s_2;m,n,r,t)\,\mathrm{K}_{20}(s_1,s_2;m,n,r,t)}} \qquad \text{(A-16)}$$

$$\mathrm{G}(\,)=\frac{1}{2[1-\rho^2]}\Big\{[\sigma_m(r)s_1+\rho\sigma_n(t)s_2]^2+[\rho\sigma_m(r)s_1+\sigma_n(t)s_2]^2+ 2\rho[\rho\sigma_m(r)s_1+\sigma_n(t)s_2][\sigma_m(r)s_1+\rho\sigma_n(t)s_2]\Big\} \qquad \text{(A-17)}$$

$$\alpha_1=\sigma_m(r)s_1+\rho\sigma_n(t)s_2 \qquad \text{(A-18)}$$

$$\alpha_2 = \rho\sigma_m(r)s_1 + \sigma_n(t)s_2 \qquad \text{(A-19)}$$

where

$$\sigma_m(r) = \sqrt{K_{20}(s_1,s_2;m,n,r,t)}\ ,\sigma_n(t) = \sqrt{K_{02}(s_1,s_2;m,n,r,t)}\ \text{, and }\ \rho = \rho(s_1,s_2;m,n,r,t)$$

The bivariate normal approximation to $F^{\mathbf{S}}(\mathbf{X};m,n,r,t)$ is not likely to be very accurate in the tails. However, if s_1 and s_2 are positive the multiplier, $\exp(-s_1x_1-s_2x_2)$, reduces the error of the bivariate normal approximation in the tails in Equation A-14. If either s_1 or s_2 turns out to be negative, the probability of the complement of the region is approximated instead. Three cases are identified: s_1 is positive and s_2 is negative; s_1 is negative and s_2 is positive; s_1 and s_2 are negative. The same change of variable previously used in the one-dimensional case is also used here, which gives the following equations:

<u>Case 1</u>: if s_1 is positive and s_2 is negative redefine s_2 as - s_2, and use Equation A-20:

$$p_{mn}(r,t) \quad \Pr[X_m(\mathrm{r}) \geq 0] - e^{K(\mathbf{S}_0;m,n,r,t)+G(\mathbf{S}_0;m,n,r,t,\rho)}\Phi_2(\alpha_1,-\alpha_2,-\rho) \qquad \text{(A-20)}$$

<u>Case 2</u>: if s_1 is negative and s_2 is positive redefine s_1 as – s_1, and use Equation A-21

$$p_{mn}(r,t) \quad \Pr[X_m(\mathrm{r}) \geq 0] + \Pr[X_n(\mathrm{t}) \geq 0] + e^{K(\mathbf{S}_0;m,n,r,t)+G(\mathbf{S}_0;m,n,r,t,\rho)}\Phi_2(-\alpha_1,\alpha_2,-\rho) \qquad \text{(A-21)}$$

<u>Case 3</u>: if both s_1 and s_2 are negative redefine s_1 as $-s_1$ and s_2 as $-s_2$, and use Equation A-22

$$p_{mn}(r,t) \quad \Pr[X_n(\mathrm{t}) \geq 0] - e^{K(\mathbf{S}_0;m,n,r,t)+G(\mathbf{S}_0;m,n,r,t,\rho)}\Phi_2(-\alpha_1,-\alpha_2,\rho) \qquad \text{(A-22)}$$