Technical Appendix to: Asymmetric Bertrand-Edgeworth-Chamberlin Competition with Linear Demand:

A Pediatric Vaccine Pricing Model

Banafsheh Behzad
Department of Information Systems
California State University, Long Beach

Sheldon H. Jacobson
Department of Computer Science
University of Illinois at Urbana-Champaign

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Proof of Theorem 1. If (a) holds, then no capacity constraint exists and the equilibrium price is the Bertrand-Chamberlin equilibrium $p^B = \dfrac{a}{2b - c(n-1)}$. If (b) holds, then $P_i(k_i, \sum_{j=1, j\neq i}^{n} k_j)$ is an equilibrium for manufacturer i; since if i charges less than that price it still sells k_i but at a lower price, and hence, earns less. If manufacturer i charges a higher price, it has some residual demand but the optimal response is to charge $r_i(\sum_{j=1, j\neq i}^{n} k_j)$ since k_i is less than or equal to optimal response function $r_i(\sum_{j=1, j\neq i}^{n} k_j)$ in Cournot competition and $r_i(\sum_{j=1, j\neq i}^{n} k_j)$ is decreasing, and hence, the firm is constrained by k_i. $\square$

Proof of Theorem 2. **(a) and (b)** Define F as the set of manufacturers $i \leq n$ such that $\overline{p}_i = \overline{p}$. Arguing along the lines of [2], note that when any manufacturer $j \in F$ charges $\overline{p}$, its sales fall to less than its capacity k_j. If the latter does not hold, then $\alpha - \overline{p} \geq \sum_{i=1}^{n} k_i > \gamma \sum_{i=1}^{n} k_i$, which denotes that for any manufacturer i, such that $Pr(p_i < \overline{p}) > 0$, manufacturer i has not made the optimal response as it sold less than its capacity when charging $\overline{p}$. Furthermore, at most one manufacturer charges $\overline{p}$ with positive probability. In fact, if $Pr(p_j = \overline{p}) > 0$, because of the jump in manufacturer i's residual demand, when manufacturer j charges $\overline{p}$, charging slightly less than $\overline{p}$ is better for any manufacturer $i \neq j$. Therefore, at least one manufacturer $j \in F$ exists in the region of the mixed strategy equilibrium, where no manufacturer $i \neq j$ charges $\overline{p}$ with positive probability. Hence, there exists at least one manufacturer that charges a lower price than manufacturer j when it charges $\overline{p}$. In addition, $\overline{p}$ must be an optimal response to manufacturer $i \neq j$ equilibrium strategies. Therefore, it must be $\overline{p} = \arg\max_p p \left(\alpha - p - \gamma \sum_{i=1, i \neq j}^{n} k_i \right) = P_j \left(r_j \left(\sum_{i=1, i \neq j}^{n} k_i \right), \sum_{i=1, i \neq j}^{n} k_i \right)$ and $\pi_j^* = \overline{p} \left(\alpha - \overline{p} - \gamma \sum_{i=1, i \neq j}^{n} k_i \right) = R_j(\sum_{i=1, i \neq j}^{n} k_i)$, for any $j \in \{1, 2, \cdots, n\}$ with the largest capacity.

Now it is required to show that when manufacturer $j \in F$ and no manufacturer $i \neq j$ charges $\overline{p}$ with positive probability, $k_j < k_1$ cannot hold. Note that at the equilibrium, manufacturer 1 will not charge a price of $P_1 \left(r_1 \left(\sum_{i=2}^{n} k_i \right), \sum_{i=2}^{n} k_i \right)$, and hence, $R_1 \left(\sum_{i=2}^{n} k_i \right) = r_1 \left(\sum_{i=2}^{n} k_i \right) P_1 \left(r_1 \left(\sum_{i=2}^{n} k_i \right), \sum_{i=2}^{n} k_i \right) \leq \pi_1^*$. Moreover, the assumption $k_j < k_1$ results in $k_1 R_j(\sum_{i=1, i \neq j}^{n} k_i) < k_j R_1(\sum_{i=1, i \neq j}^{n} k_i)$ (the proof is given for homogeneous products in [2] and is similar to the case of the differentiated products). The following equation is the immediate result of the two previous equations:

$$\pi_j^* = R_j\left(\sum_{i=1,i\neq j}^{n} k_i\right) < \frac{k_j}{k_1} R_1\left(\sum_{i=2}^{n} k_i\right) \leq \frac{k_j}{k_1}\pi_1^* < \pi_1^*, \quad j = 1, 2, \cdots, n. \tag{1}$$

Let $q_{1_{(p_1=\underline{p}_1)}}$ denote manufacturer 1's expected sale when it charges $\underline{p}_1$. Manufacturer 1's expected profit at equilibrium can then be written as $\pi_1^* = \underline{p}_1 q_{1_{(p_1=\underline{p}_1)}}$. Define $x \equiv \frac{q_{1_{(p_1=\underline{p}_1)}}}{k_1} \leq 1$. Then $\frac{k_j}{k_1}\pi_1^* = \underline{p}_1 k_j x$. Equation (1) then leads to $\pi_j^* < k_j \underline{p}_1 x$. Note that if manufacturer j charges a price slightly less than $\underline{p}_1$, say $\underline{p}_1^-$, it sells its whole capacity k_j. Therefore, $\pi_j(p_j = \underline{p}_1^-) = \underline{p}_1 k_j \geq k_j \underline{p}_1 x > \pi_j^*$, which is a contradiction. Therefore, $\overline{p} = \arg\max_p \, p\left(\alpha - p - \gamma\sum_{i=2}^{n} k_i\right) = P_1\left(r_1\left(\sum_{i=2}^{n} k_i\right), \sum_{i=2}^{n} k_i\right)$ and $\pi_j^* = R_j(\sum_{i=1,i\neq j}^{n} k_i) = \overline{p}\left(\alpha - \overline{p} - \gamma\sum_{i=2}^{n} k_i\right)$ for any j such that $k_j = k_1$. These hold true when $k_1 > k_j$ for every $j \neq 1$ as then it must be that $1 \in F$ and no manufacturer $j \neq 1$ charges $\overline{p}$ with positive probability. The case remaining is when $k_j = k_1$ for some $j \neq 1$. By setting $k_j = k_1$ for any $j \in F$ for which no $i \neq j$ charges $\overline{p}$ with positive probability, the aforementioned contradiction does not arise. Moreover, $\pi_j^* = R_j(\sum_{i=1,i\neq j}^{n} k_i)$ for some j: $k_j = k_1$. In fact, if this does not hold, and hence, $1 \in F$, but $\pi_j^* > \pi_1^* = R_1(\sum_{i=2}^{n} k_i)$ for some j: $k_j = k_1$. Likewise, by charging $\underline{p}_j^-$, manufacturer 1 would sell k_1, and hence, $\pi_1(p_1 = \underline{p}_j) = \underline{p}_j k_1 \geq \pi_j^* > \pi_1^*$.

(c) If $\underline{p} < p < \max\left\{\hat{p}, \hat{\hat{p}}\right\}$, then $\pi_1(p) \leq p\min\{(\alpha - p), k_1\} < \pi_1^* = \hat{p}k_1 = \hat{\hat{p}}(\alpha - \hat{\hat{p}})$. Therefore, $\underline{p} \geq \max\left\{\hat{p}, \hat{\hat{p}}\right\}$. Furthermore, if $\underline{p} > \max\left\{\hat{p}, \hat{\hat{p}}\right\}$, then $\pi_1(\underline{p}^-) > \pi_1^*$. The reason is as follows: if $\hat{p} > \hat{\hat{p}}$, then $\alpha - \hat{\hat{p}} > k_1$, and since $\underline{p} > \hat{p} > \hat{\hat{p}}$, then $\alpha - \underline{p} < \alpha - \hat{\hat{p}}$. Then it is either $\alpha - \underline{p} \geq k_1$, which results in $\pi_1(\underline{p}^-) = \underline{p}k_1 > \hat{p}k_1 = \pi_1^*$, or $\alpha - \underline{p} < k_1$, which results in $\pi_1(\underline{p}^-) = \underline{p}(\alpha - \underline{p}) > \hat{\hat{p}}(\alpha - \hat{\hat{p}}) = \pi_1^*$. The last inequality is correct as $p(\alpha - p)$ is increasing for $p \in [0, \overline{p}]$. If $\hat{\hat{p}} > \hat{p}$, then $k_1 > \alpha - \hat{\hat{p}}$, which results in $\pi_1(\underline{p}^-) = \underline{p}(\alpha - \underline{p}) > \hat{\hat{p}}(\alpha - \hat{\hat{p}}) = \pi_1^*$.

□

Lemma 1. *In a duopoly setting with linear demand, when no pure strategy equilibrium exists for manufacturer i,* $\underline{p}_i \geq P_i(k_i, k_j)$ *for* $i, j = 1, 2$, $i \neq j$.

Proof. If a price p less than $P_i(k_i, k_j)$ is charged, then the maximum profit for manufacturer i is pk_i. If price $P_i(k_i, k_j)$ is charged, then the minimum profit for manufacturer i is $P_i(k_i, k_j)(k_i + k_j - k_j) = P_i(k_i, k_j)k_i$, which is higher than the maximum profit gained if a price p (less than $P_i(k_i, k_j)$) is charged. Therefore the minimum price charged by manufacturer i should be greater than or equal to $P_i(k_i, k_j)$. □

Lemma 2. *In a duopoly setting with linear demand, when no pure strategy equilibrium exists for manufacturers 1 and 2, assume that either* $\overline{p}_1 > \overline{p}_2$, *or that* $\overline{p}_1 = \overline{p}_2$ *and* $\overline{p}_2$ *is not charged by manufacturer 2. Then:*

- *(a)* $\overline{p}_1 = P_1(r_1(k_2), k_2) = \overline{p}$;
- *(b)* $k_1 > r_1(k_2)$
- *(c)* $\underline{p}_1 = \underline{p}_2 = \underline{p}$;
- *(d) Neither* $\underline{p}_1$ *and* $\underline{p}_2$ *is charged by manufacturers 1 and 2, respectively.*

Proof. **(a) and (b)** Assume that manufacturer 1 charges price p and manufacturer 2 charges a price less than p. As in [3], define the revenue of manufacturer 1 as

$$\Xi(p) = p\left[min(k_1, max(0, \alpha - \gamma k_2 - p))\right]. \tag{2}$$

If manufacturer 1 charges $\overline{p}_1$, its profit is $\Xi(\overline{p}_1)$ (given the assumption $\overline{p}_1 > \overline{p}_2 > p_2$). If manufacturer 1 charges any price $p > \overline{p}_1$, its profit is $\Xi(p)$. If manufacturer 1 charges any price $p < \overline{p}_1$, its profit is at least $\Xi(p)$. Therefore, if an equilibrium exists, $\overline{p}_1$ must be the maximization point of $\Xi(p)$.

By assumption, $\gamma k_2 < a$, and since $a < \alpha = a(1+\gamma)$, then $\gamma k_2 < \alpha$. To maximize $\Xi(p)$, $\alpha - p < \gamma k_2$ cannot hold. To see this, assume that $\alpha - p < \gamma k_2$, then $\Xi(p) = 0$ holds. Therefore, $p \in [P_1(k_1, k_2), P_1(0, k_2)]$. There exists a level of k for each p in this interval, indicated by $k(p) = \alpha - p - \gamma k_2$, such that $\Xi(p) = k(p)P_1(k(p), k_2)$. Clearly, $k(p) \in [0, k_1]$. By the strict concavity of $k(p)P_1(k(p), k_2)$ and given that $\underset{k(p)}{\arg\max}\ \{k(p)P_1(k(p), k_2)\}$ is equal to the Cournot optimal response function for manufacturer 2 ($r_1(k_2)$),

$$\underset{k(p)\in[0,k_1]}{\arg\max}\ \{k(p)P_1(k(p), k_2)\} = min(r_1(k_2), k_1). \tag{3}$$

If $k_1 < r_1(k_2)$, then $\overline{p}_1 = P_1(k_1, k_2)$, which is a contradiction (by Lemma 1). Therefore, $k_1 > r_1(k_2)$, and hence, $\overline{p}_1 = P_1(r_1(k_2), k_2)$ and manufacturer 1 earns profit $R_1(k_2) = r_1(k_2)P_1(r_1(k_2), k_2)$. Furthermore, it is shown in Theorem 2 that $\overline{p} = \underset{p}{\arg\max}\, p\,(\alpha - p - \gamma k_2) = \dfrac{\alpha - \gamma k_2}{2} = P_1(r_1(k_2), k_2) = \overline{p}_1$, and (a) is proven.

(c) Assume that $\underline{p}_i < \underline{p}_j$, for $i, j = 1, 2$. If manufacturer i charges $\underline{p}_i$, its profit is $\underline{p}_i(min(\alpha - \underline{p}_i), k_i)$. If manufacturer i increases the price to any $p \in (\underline{p}_i, \underline{p}_j)$, its profit is $p(min(\alpha - p), k_i)$. Since both $k_1P_1(k_1, 0)$ and $k_2P_2(0, k_2)$ are strictly concave, moving from $\underline{p}_i$ towards the monopoly price will increase the profit, and hence, an equilibrium exists only if $\alpha - \underline{p}_i < k_i$ and $\underline{p}_i$ is the monopoly price. This means that $\underline{p}_i = P_i(r_i(0), 0) = \dfrac{\alpha}{2}$. However, it is known that, $\underline{p}_i < \overline{p}_1 = P_1(r_1(k_2), k_2) = \dfrac{\alpha - \gamma k_2}{2} < P_1(r_1(0), 0)$, which is a contradiction. Therefore $\underline{p}_1 = \underline{p}_2 = \underline{p}$ and its value is given by Theorem 2.

(d) As shown in Lemma 1, $\underline{p} \geq P_1(k_1, k_2)$. However, $\underline{p} = P_1(k_1, k_2)$ does not hold since if manufacturer 1 selects a price sufficiently close to $\underline{p}$, its profit is at most $k_1 P_1(k_1, k_2)$. However, since $k_1 > r_1(k_2)$ and the profit gained by manufacturer 1 at equilibrium is $R_1(k_2)$, this is not possible. Therefore, $\underline{p} > P_1(k_1, k_2)$. Now assume that, manufacturer 2 with lower capacity charges $\underline{p}$. Then manufacturer 1 could increase its profit by charging a price marginally less than $\underline{p}$. Therefore, it is irrational for manufacturer 2 to charge $\underline{p}$. Note that $\underline{p}$ is the infimum of the support of the interval of prices charged by manufacturer 2, thus this manufacturer must charge prices greater than $\underline{p}$. However, if manufacturer 1 charges $\underline{p}$, manufacturer 2 profits more if it charges a price lower than $\underline{p}$ than to charge a price higher than $\underline{p}$. Therefore, neither manufacturer 1, nor manufacturer 2, will charge $\underline{p}$.

□

Lemma 3. *In a duopoly setting with linear demand, when no pure strategy equilibrium exists for manufacturers 1 and 2,* $\overline{p}_1 = \overline{p}_2 = \overline{p} = P_1(r_1(k_2), k_2)$.

Proof. It is shown in Lemma 2 that $\overline{p}_1 = P_1(r_1(k_2), k_2) = \overline{p}$. Manufacturer 2 cannot earn a profit more than $R_2(k_1)$, and hence, it has no incentive to charge a price greater than $\overline{p}_1$. So $\overline{p}_1 = \overline{p}_2 = \overline{p} = P_1(r_1(k_2), k_2)$. □

Proof of Theorem 3. Assume that manufacturer 1 charges a price $p \in [\underline{p}, \overline{p}]$ and manufacturer 2 either charges a price less than p or greater than p. The expected profit of manufacturer 1 is given by

$$E_1(p) = \phi_2(p)p\left[max(\alpha - p - \gamma k_2, 0)\right] + (1 - \phi_2(p))p\left[min((\alpha - p), k_1)\right],$$

where $\phi_2(p)$ is the probability that manufacturer 2 charges a price less than p. As shown in the proof of Lemma 2, $\alpha - p > \gamma k_2$ and hence

$$E_1(p) = \phi_2(p)p\left[\alpha - p - \gamma k_2\right] + (1 - \phi_2(p))p\left[min((\alpha - p), k_1)\right]. \tag{4}$$

Note that $E_1(p) = R_1(k_2) = \underline{p}\left[min((\alpha - \underline{p}), k_1)\right]$.

Now assume that manufacturer 2 charges a price $p \in \left[\underline{p}, \overline{p}\right]$ and manufacturer 1 either charges a price less than p or charges greater than p. The expected revenue of manufacturer 2 is given by

$$E_2(p) = \phi_1(p)p\left[max(\alpha - p - \gamma k_1, 0)\right] + (1 - \phi_1(p))p\left[min((\alpha - p), k_2)\right], \tag{5}$$

where $\phi_1(p)$ is the probability that manufacturer 1 charges a price less than p. Note that $E_2(p) = R_2(k_1) = \underline{p}\left[min((\alpha - \underline{p}), k_2)\right]$. Equations (4) and (5) need to be solved for $\phi_2(p)$ and $\phi_1(p)$.

In solving for $\phi_2(p)$, three possible cases exist:

(1) If $k_1 > \alpha - \underline{p}$, then $k_1 > \alpha - p$. Therefore $\phi_2(p) = \dfrac{p(\alpha - p) - \underline{p}(\alpha - \underline{p})}{p\gamma k_2}$,

(2) If $k_1 < \alpha - \underline{p}$, and $k_1 < \alpha - p$, then $\phi_2(p) = \dfrac{(\underline{p} - p)k_1}{p\left[\alpha - p - \gamma k_2 - k_1\right]}$,

(3) If $k_1 < \alpha - \underline{p}$, and $k_1 > \alpha - p$, then $\phi_2(p) = \dfrac{p(\alpha - p) - \underline{p}k_1}{p\gamma k_2}$.

In solving for $\phi_1(p)$, six possible cases exist:

(1) If $k_2 > \alpha - \underline{p}$, and $\alpha - p > \gamma k_1$, then $k_2 > \alpha - p$. Therefore, $\phi_1(p) = \dfrac{p(\alpha - p) - \underline{p}(\alpha - \underline{p})}{p\gamma k_1}$.

(2) If $k_2 > \alpha - \underline{p}$, and $\alpha - p < \gamma k_1$, then $k_2 > \alpha - p$. Therefore, $\phi_1(p) = \dfrac{p(\alpha - p) - \underline{p}(\alpha - \underline{p})}{p(\alpha - p)}$.

(3) If $k_2 < \alpha - \underline{p}$, and $\alpha - p > \gamma k_1$, and $k_2 > \alpha - p$. Therefore, $\phi_1(p) = \dfrac{p(\alpha - p) - \underline{p}k_2}{p\gamma k_1}$.

(4) If $k_2 < \alpha - \underline{p}$, and $\alpha - p < \gamma k_1$, and $k_2 > \alpha - p$. Therefore, $\phi_1(p) = \dfrac{p(\alpha - p) - \underline{p}k_2}{p(\alpha - p)}$.

(5) If $k_2 < \alpha - \underline{p}$, and $\alpha - p > \gamma k_1$, and $k_2 < \alpha - p$. Therefore, $\phi_1(p) = \dfrac{(\underline{p} - p)k_2}{p\left[\alpha - p - \gamma k_1 - k_2\right]}$.

(6) If $k_2 < \alpha - \underline{p}$, and $\alpha - p < \gamma k_1$, and $k_2 < \alpha - p$. Therefore, $\phi_1(p) = \dfrac{(p - \underline{p})k_2}{pk_2}$.

The only forms of $\phi_1(p)$ and $\phi_2(p)$, which are consistent with [1] are case 2 for $\phi_2(p)$ and case 5 for $\phi_1(p)$, respectively. Therefore,

$$\phi_1(p) = \frac{(\underline{p} - p)k_2}{p\left[\alpha - p - \gamma k_1 - k_2\right]} \tag{6}$$

$$\phi_2(p) = \frac{(\underline{p} - p)k_1}{p\left[\alpha - p - \gamma k_2 - k_1\right]} \tag{7}$$

Note that if $k_1 = k_2$, then $\phi_1(p)$ and $\phi_2(p)$ are equal. $\square$

References

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