A Online Appendix - Proofs (Supporting Document)

A.1 Proof of Proposition 1

Let $\beta := \frac{\xi}{\alpha+\xi} - \left(\frac{\eta}{\eta+\xi} - \frac{\alpha}{\alpha+\xi}\right) p$. It is then easy to show that $0 < \beta < 1$. Also, by assumption, we have $\xi << \alpha < \eta$ so that β is linearly decreasing in p, i.e., $\frac{\eta}{\eta+\xi} - \frac{\alpha}{\alpha+\xi} \geq 0$, and, hence, $\frac{d\beta(p)}{dp} = -\left(\frac{\eta}{\eta+\xi} - \frac{\alpha}{\alpha+\xi}\right) < 0$.

Using (1), it is easy to show that

$$\Lambda_1 \quad = \quad \frac{\lambda}{\beta}, \qquad \frac{d\Lambda_1(p)}{dp} = -\frac{\lambda \frac{d\beta(p)}{dp}}{\beta^2} > 0, \qquad \frac{d^2\Lambda_1(p)}{dp} = \frac{2\lambda\left(\frac{d\beta(p)}{dp}\right)^2}{\beta^3} > 0. \tag{16}$$

Thus, Λ_1 is strictly increasing and convex in p. Also, using (1), we have

$$\frac{d\Lambda_3(p)}{dp} \quad = \quad \Lambda_1 + p\frac{d\Lambda_1(p)}{dp}, \qquad \frac{d^2\Lambda_3(p)}{dp} = 2\frac{d\Lambda_1(p)}{dp} + p\frac{d^2\Lambda_1(p)}{dp},$$

and combining the latter with (16), we conclude that Λ_3 is strictly increasing and convex in p. Further, using (1) again, we have

$$\frac{d\Lambda_2(p)}{dp} = -\Lambda_1 + (1-p)\frac{d\Lambda_1(p)}{dp} = -\frac{\lambda}{\beta} - (1-p)\frac{\lambda\frac{d\beta(p)}{dp}}{\beta^2} = -\frac{\lambda}{\beta^2}\left(\frac{d\beta(p)}{dp} + \beta - \frac{d\beta(p)}{dp}p\right),$$

where

$$\frac{d\beta(p)}{dp} + \beta - \frac{d\beta(p)}{dp}p = \frac{\xi}{\eta+\xi} > 0, \tag{17}$$

so that $\frac{d\Lambda_2(p)}{dp} < 0$, i.e., Λ_2 is strictly decreasing in p. Additionally,

$$\frac{d^2\Lambda_2(p)}{dp} = -\frac{d\Lambda_1(p)}{dp} - \frac{d\Lambda_1(p)}{dp} + (1-p)\frac{d^2\Lambda_1(p)}{dp} = \frac{2\lambda}{\beta^3}\frac{d\beta(p)}{dp}\left(\frac{d\beta(p)}{dp} + \beta - \frac{d\beta(p)}{dp}p\right) < 0,$$

so that Λ_2 is concave in p. Since $\Lambda_4 = \Lambda_2$ (see (1)), Λ_4 is also strictly decreasing and concave in p. Furthermore, utilizing (1), it is easy to verify that:

- $\Lambda_1 + \Lambda_2 = (2-p)\Lambda_1$ so that

$$\begin{aligned}
\frac{d(\Lambda_1+\Lambda_2)(p)}{dp} \quad &= \quad -\Lambda_1 + (2-p)\frac{d\Lambda_1(p)}{dp} \overset{(16)}{=} -\frac{\lambda}{\beta} - (2-p)\frac{\lambda\frac{d\beta(p)}{dp}}{\beta^2} = -\frac{\lambda}{\beta^2}\left(2\frac{d\beta(p)}{dp} + \beta - \frac{d\beta(p)}{dp}p\right) \\
&= \quad -\frac{\lambda}{\beta^2}\left[\frac{d\beta(p)}{dp} + \left(\frac{d\beta(p)}{dp} + \beta - \frac{d\beta(p)}{dp}p\right)\right] \\
&\overset{(17)}{=} \quad -\frac{\lambda}{\beta^2}\left(\frac{d\beta(p)}{dp} + \frac{\xi}{\eta+\xi}\right) = -\frac{\lambda}{\beta^2}\left(\frac{\xi}{\eta+\xi} - \frac{\eta}{\eta+\xi} + \frac{\alpha}{\alpha+\xi}\right).
\end{aligned}$$

Thus, $\frac{d(\Lambda_1+\Lambda_2)(p)}{dp} > 0$, if and only if

$$\frac{\xi}{\eta+\xi} < \frac{\eta}{\eta+\xi} - \frac{\alpha}{\alpha+\xi}. \tag{18}$$

Noting that (18) is equivalent to $\alpha+\xi<\eta-\alpha$, we conclude that $(\Lambda_1+\Lambda_2)$ is strictly increasing in p if and only if $\alpha+\xi<\eta-\alpha$; and it is strictly decreasing, otherwise. Additionally,

$$\begin{aligned}\frac{d^2(\Lambda_1+\Lambda_2)(p)}{dp} &= -2\frac{d\Lambda_1(p)}{dp}+(2-p)\frac{d^2\Lambda_1(p)}{dp} \overset{(16)}{=} \frac{\lambda\frac{d\beta(p)}{dp}}{\beta^2}+(2-p)\frac{2\lambda(\frac{d\beta(p)}{dp})^2}{\beta^3}\\ &= \frac{2\lambda\frac{d\beta(p)}{dp}}{\beta^3}\left(2\frac{d\beta(p)}{dp}+\beta-\frac{d\beta(p)}{dp}p\right).\end{aligned}$$

Combining this result with (17), we conclude that $(\Lambda_1+\Lambda_2)$ is convex in p if and only if $\alpha+\xi<\eta-\alpha$; and it is concave, otherwise.

textcolorblueWe next use (2) and conclude that the offered loads (ρ_i) are strictly increasing and convex functions of the corresponding aggregate arrival rates, Λ_i, $i=1,2,3,4$. Combining this result with the above analysis, we conclude that $\rho_1(p)$ and $\rho_3(p)$ are strictly increasing and convex in p, while $\rho_2(p)$ and $\rho_4(p)$ are strictly decreasing and concave in p. Likewise, the result for the revisit ratios follows from (3) and the above. ■

A.2 Proof of Corollary 1

Let $\%\Delta M$ denote the percent difference encountered on quantity M. Suppose that an increase in p by c percent leads to an increase in Λ_1 by y percent. That is:

$$\text{If } \%\Delta p=\frac{p'-p}{p}=c, \text{ then } \%\Delta\Lambda_1=\frac{\Lambda_1^{'}-\Lambda_1}{\Lambda_1}=y, \text{ so that } p'=p+cp \text{ and } \frac{\Lambda_1^{'}}{\Lambda_1}=y+1, \tag{19}$$

where p', $\Lambda_1^{'}$ denote the new values of p and Λ_1, respectively. Then, the percent decrease in Λ_2 is

$$\%\Delta\Lambda_2 = \frac{\Lambda_2^{'}-\Lambda_2}{\Lambda_2} \overset{(1)}{=} \frac{(1-p')\Lambda_1^{'}-(1-p)\Lambda_1}{(1-p)\Lambda_1} \overset{(19)}{=} \frac{1-p-cp}{1-p}(y+1)-1=y-\frac{cp}{1-p}(y+1)<y.$$

Hence, the decrease in Λ_2 is $\frac{cp}{1-p}(y+1)$ times lower relative to the corresponding increase in Λ_1. ■

A.3 Proof of Theorem 2

A.3.1 Part (a) of Theorem 2

Since a patient is forwarded for dialysis with probability $(1-p)$, the probability that she completes one p-loop without exiting the system equals $\frac{\eta}{\eta+\xi}p$. Consequently, the probability that she completes n such p-loops before transitioning to DR without exiting the system is given by (11).

A.3.2 Part (b) of Theorem 2

Combining (11) and the definition of $E[R]$, i.e., $E[R]=\sum_{n=0}^{\infty}n\Pr[R=n]$, the computation of the expected number of p-loops of a patient is straightforward.

A.3.3 Part (c) of Theorem 2

The duration of a single p-loop corresponds to the time between two successive visits in the diamond-shaped decision point seen in Figure 1. Since the completion of a p-loop signals that the patient's revisit time expires sooner that her exit time, Y consists of three components: (1) The conditional time spent in PTO given that a revisit occurs, denoted by the random variable $X_3|(R_3 < E_3)$, (2) the waiting time in ER, W_1, and (3) the service time in ER, X_1. Consequently, Y can be written as follows

$$Y = X_3|(R_3 < E_3) + Q_1 + X_1. \tag{20}$$

We next prove that $E[X_3|(R_3 < E_3)] = \frac{1}{\eta+\xi}$.

$$\begin{aligned} E[X_3|(R_3 < E_3)] &= E[\min(R_3, E_3)|R_3 < E_3] = \int_x x \Pr[\min(R_3, E_3) = x|R_3 < E_3]dx \\ &= \int_0^\infty \frac{x \Pr[\min(R_3, E_3) = x, R_3 < E_3]}{\Pr[R_3 < E_3]}dx = \int_0^\infty \frac{x \Pr[(R_3 = x, E_3 > x]}{\frac{\eta}{\eta+\xi}}dx \\ &= \int_0^\infty \frac{x\eta e^{-\eta x} e^{-\xi x}}{\frac{\eta}{\eta+\xi}}dx = \int_0^\infty x(\eta+\xi)e^{-(\eta+\xi)x}dx = \frac{1}{\eta+\xi}, \end{aligned}$$

Next, we remind that $E[X_1] = \frac{1}{\mu_1}$. Combining this with the latter, we conclude with (13).

A.3.4 Part (d) of Theorem 2

The total delay consists of (1) an initial waiting time in ER (Q_1), (2) an initial service time in ER (X_1), (3) the cumulative delay caused by the random number of p-loops (denoted by $\sum_{i=0}^{N} Y_i$), and (4) a waiting time in DR (Q_2). Hence,

$$D = Q_1 + X_1 + \sum_{i=0}^{N} Y_i + Q_2. \tag{21}$$

We remind that the random variable R is independent of the random variables Y_i, and, hence, taking expectations yields

$$E[D] \quad = \quad W_1 + E[X_1] + E[R]E[Y_i] + W_2.$$

Plugging $E[X_1] = \frac{1}{\mu_1}$ and (13) into the latter yields (14).

A.3.5 Part (e) of Theorem 2

A patient leaving the hospital without receiving treatment implies that a patients is rejected at each once. Assuming the patient gets rejected n consequent times, the probability that he/she leaves the system without treatment implies that exit from Node 3 occurs faster than the corresponding revisit on the n^{th} rejection, whereas the reverse is true for all the former rejections. Therefore,

$$q = \sum_{n=1}^{\infty} \frac{\xi}{\eta+\xi} \left(\frac{\eta}{\eta+\xi}\right)^{n-1} p^n = \frac{\xi p}{\eta+\xi} \sum_{n=1}^{\infty} \left(\frac{\eta p}{\eta+\xi}\right)^{n-1} = \frac{\xi p}{\eta+\xi} \cdot \frac{1}{1-\frac{\eta p}{\eta+\xi}} = \frac{\xi p}{\eta+\xi-\eta p}.$$

This completes the proof. ■

A.4 Proof of Theorem 3

A.4.1 Part (a) of Theorem 3

We first note that in a certain queueing system the expected waiting time, the expected queue length, and the probability of an empty system are performance measures that are strictly increasing functions of the arrival rate of the system. This is a direct conclusion of using a coupling argument. As a result, in our system, W_i, L_{qi} and γ_i are strictly increasing functions of the aggregate arrival rate, i.e., Λ_i. Moreover, according to [50], the same measures are also convex functions of the aggregate arrival rate, Λ_i. The above results will be used for the proof of parts (b)-(c) as well.

A careful look at (7) reveals that the $Pr(Q_i > t)$ consists of two terms, γ_i and an exponential term, $e^{-\mu_i(s_i-\rho_i)t}$. Using (2), we see that ρ_i is an strictly increasing and convex function of Λ_i. Consequently, the exponential term is also a strictly increasing and convex function of Λ_i. Combining this with the fact that γ_i is also a strictly increasing and convex function of Λ_i, and taking into account that both terms are positive, we conclude that the $\Pr(Q_i > t)$ is a strictly increasing and convex function of Λ_i given by

$$\Pr(Q_i > t) = f(\Lambda_i(p)), \tag{22}$$

where the function $f(x)$ is a strictly increasing and convex function of x. Taking the first derivative in p yields

$$\frac{d\Pr(Q_i > t)}{dp} = \frac{df(\Lambda_i(p))}{dp} \frac{d\Lambda_i(p)}{dp}.$$

Combining part (a) of Proposition 1 with the structural properties of the function $f(x)$, we conclude that $\Pr(Q_1 > t)$ is strictly increasing in p, while $\Pr(Q_2 > t)$ is strictly decreasing in p, respectively. Taking, now, the second derivative yields

$$\frac{d^2\Pr(Q_i > t)}{dp^2} = \frac{d^2 f(\Lambda_i(p))}{dp^2} \left(\frac{d\Lambda_i(p)}{dp}\right)^2 + \frac{df(\Lambda_i(p))}{dp} \frac{d^2\Lambda_i(p)}{dp^2}. \tag{23}$$

Combining again part (a) of Proposition 1 with the structural properties of the function $f(x)$, we conclude that $\Pr(Q_1 > t)$ is convex in p, simply because all the four terms in (23) are positive. However, we also note that the corresponding behavior of the behavior of the second derivative of $\Pr(Q_2 > t)$ seems to be more complex. Specifically, the first three terms in (23) are positive, but the last one is negative (see part (a) of Proposition 1). As a result, the sign of the second derivative of $\Pr(Q_2 > t)$ is impacted by the system parameters and the function $f(x)$ in a non-trivial way. Given the complexity of the formula (7), we can reach no certain conclusion with regards to the behavior of the second derivative of $\Pr(Q_2 > t)$.

Following a similar reasoning, we conclude that W_1 is strictly increasing and convex in p, while W_2 is strictly decreasing in p.

Following a similar reasoning, we conclude that L_{q1} is strictly increasing and convex in p, while L_{q2} is strictly decreasing in p.

A.4.2 Part (b) of Theorem 3

Taking the first and second derivatives of 11 yields after some algebra

$$\frac{d\Pr(R=n)}{dp} = \left(\frac{\eta}{\eta+\xi}\right)^n p^{n-1}\left(-p+n-np\right), \quad \frac{d^2\Pr(R=n)}{dp^2} = n\left(\frac{\eta}{\eta+\xi}\right)^n p^{n-2}\left[-2p+(n-1)(1-p)\right].$$

Using the latter, we can immediately conclude that $\Pr(R=n)$ is strictly increasing in p if $p < \frac{n}{n+1}$ and convex in p if $p < \frac{n-1}{n+1}$. Noting that $\frac{n-1}{n+1} < \frac{n}{n+1}$ yields the desired result.

A.4.3 Part (c) of Theorem 3

First, note that (12) can be written as follows:

$$E[R] = f(p)\cdot g(p), \tag{24}$$

where

$$f(p) = \theta p(1-p), \quad g(p) = \frac{1}{(1-\theta p)^2}, \quad \text{and} \quad \theta = \frac{\eta}{\eta+\xi}. \tag{25}$$

A closer look at $f(p)$ reveals that $f(p) > 0$, given that $\theta \in [0,1]$, and $p \in [0,1)$. Moreover, it is increasing in p if and only if $p \in \left[0, \frac{1}{2}\right]$, since $\frac{df(p)}{dp} = \theta(1-2p)$. Additionally, $f(p)$ is concave in p, since $\frac{d^2 f(p)}{dp} = -2\theta < 0$. Similarly, $g(p) > 0$, $g(p)$ is strictly increasing in p, since $\frac{dg(p)}{dp} = \frac{2\theta}{(1-\theta p)^3}$, and convex in p, since $\frac{d^2 g(p)}{dp} = \frac{6\theta^2}{(1-\theta p)^4} > 0$. Taking the first derivative of (24) in p yields

$$\frac{dE[R](p)}{dp} = \frac{\theta}{(1-\theta p)^3}\cdot\left(1-(2-\theta)p\right).$$

Thus, $E[R]$ is strictly increasing in p if $1-(2-\theta)p \geq 0$. Plugging θ from (25) yields that $E[R]$ is strictly increasing in p if $p < \frac{\eta+\xi}{\eta+2\xi}$. Taking the second derivative of (24) with respect to p yields

$$\frac{d^2 E[R](p)}{dp} = \frac{2\theta}{(1-\theta p)^4}\cdot\left((2\theta-1)-\theta(2-\theta)p\right).$$

Thus, $E[R]$ is convex in p if and only if $(2\theta-1)-\theta(2-\theta)p \geq 0$. Again, plugging θ from (25) yields that $E[R]$ is convex in p if $p < \frac{\eta^2-\xi^2}{\eta(\eta+2\xi)}$, equivalently if $p < (1-\frac{\xi}{\eta})\frac{\eta+\xi}{\eta+2\xi}$. Noting that $(1-\frac{\xi}{\eta})\frac{\eta+\xi}{\eta+2\xi} < \frac{\eta+\xi}{\eta+2\xi}$ and combining the monotonicity and convexity behavior of $E[R]$ proved above yields the desired result.

A.4.4 Part (d) of Theorem 3

Taking (13) into account yields that $E[Y]$ is strictly increasing and convex in W_1. Taking, next, part (a) of Theorem 3 into account, we conclude that $E[Y]$ is strictly increasing and convex in p.

A.4.5 Part (e) of Theorem 3

The proof is immediate and hence is omitted. ■

Appendix B - Alternative Proof of Theorem 1 & Proposition 2

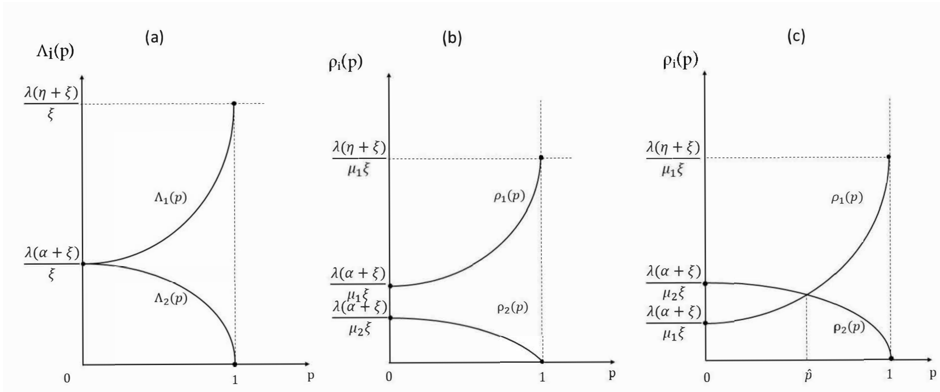

Figure 8: Stability conditions via an illustration of the aggregate arrival rates in ER (left) and DR (right).

ER Stability: For any given set of input parameters, ER stability is characterized analytically via the following mutually exclusive three cases, i.e., only one of the following three cases must be true:
Case I: $\mu_1 s_1 < \Lambda_1(0)$ iff $\frac{\xi}{\alpha+\xi} < \frac{\lambda}{\mu_1 s_1}$ iff $p_1 < 0$. **Case II:** $\Lambda_1(0) \leq \mu_1 s_1 \leq \Lambda_1(1)$ iff $\frac{\xi}{\eta+\xi} \leq \frac{\lambda}{\mu_1 s_1} \leq \frac{\xi}{\alpha+\xi}$ iff $0 \leq p_1 \leq 1$, and **Case III:** $\Lambda_1(1) < \mu_1 s_1$ iff $\frac{\lambda}{\mu_1 s_1} < \frac{\xi}{\eta+\xi}$ iff $p_1 > 1$.
In Case I, ER stability condition is never satisfied regardless of the choice of $p \in [0,1)$, i.e. ER does not have sufficient capacity at all regardless of the p-policy in place. In Case III, ER stability condition is never violated, i.e., ER has ample capacity. In Case II, however, ER stability condition is satisfied for $p \in [0, p_1]$ and violated for $p \in [p_1, 1)$. Hence, implementation of p-policy–when the value of p is not selected carefully, may threaten stability and lead to instability, i.e., very excessive crowding in ER.

DR Stability: Likewise, for any given set of input parameters, DR stability in terms of p is characterized analytically via the following mutually exclusive two cases:
Case a: $\Lambda_1(0) < \mu_2 s_2$ iff $\frac{\lambda}{\mu_2 s_2} < \frac{\xi}{\alpha+\xi}$ iff $p_2 < 0$ and **Case b:** $\mu_2 s_2 \leq \Lambda_1(0)$ iff $\frac{\xi}{\alpha+\xi} \leq \frac{\lambda}{\mu_2 s_2}$ iff $0 \leq p_2 \leq 1$.
In Case a, DR stability condition is always satisfied regardless of the choice of $p \in [0,1)$, i.e., DR has ample capacity. In Case b, DR stability condition is satisfied for $p \in [p_2, 1)$ and violated for $p \in [0, p_2]$. Hence, p-policy may alleviate excessive crowding in DR, when the value of p is selected with care. An interesting observation is that, while the ER stability threshold p_1 can take any value in the set of real numbers, the DR stability threshold $p_2 \leq 1$. This, in turn, implies that regardless of the values of input parameters (e.g., even when DR capacity is extremely limited), there exists a value of p, above which DR stability condition is satisfied.

System Stability: For any given set of input parameters, we can then asses system stability in terms of p by examining the $3 \times 2 = 6$ possibilities based on the cases introduced above. These possibilities are referred as Cases I-a, I-b, II-a, II-b, III-a, and III-b. For example, in Case II-a, ER stability condition is satisfied for $p \in [0, p_1]$ whereas DR stability condition is satisfied for $p \in [0, 1)$. As a result, system stability conditions are satisfied for $p \in [0, p_1]$. We note that Cases I-a and I-b reduce to a single case. We, then, have an equivalent version of Theorem 1, seen in Theorem 4 below, which is stated without a proof as it directly follows from the formal results presented above.

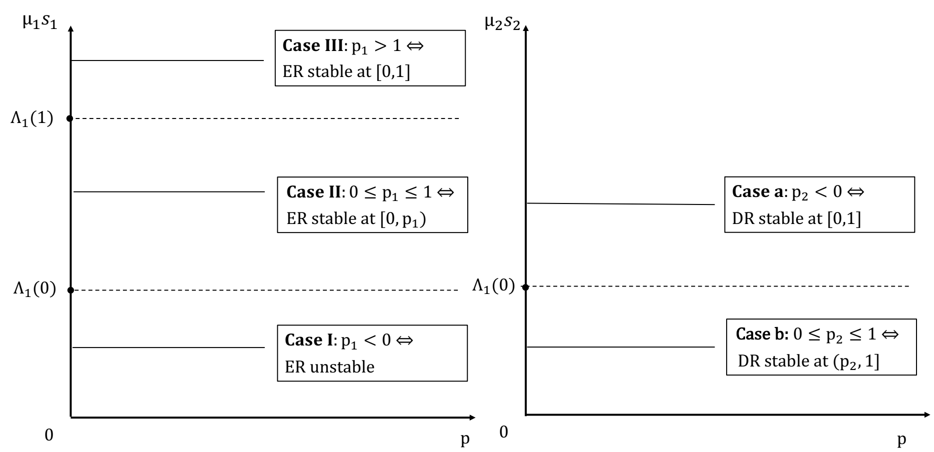

Figure 9: Stability conditions via an illustration of the aggregate arrival rates in ER (left) and DR (right).

Theorem 4 *For any given set of input parameters, system stability in terms of p is characterized analytically via the following mutually exclusive cases where p_1 and p_2 are given by (4).*

Case I: *If $p_1 < 0$, then ER stability, and hence system stability is violated regardless of the value of p.*

Case II: *If $0 \leq p_1 \leq 1$, then the following subcases arise:*

- **a:** *If $p_2 < 0$, then system stability is satisfied for $p \in [0, p_1)$.*
- **b:** *If $0 \leq p_2 \leq 1$, then the following subcases arise:*
  - **i:** *If $p_1 \leq p_2$ then system stability is violated regardless of the value of p.*
  - **ii:** *If $p_2 < p_1$, then system stability is satisfied for $p \in (p_2, p_1)$.*

Case III: *If $p_1 > 1$, then the following subcases arise:*

- **a:** *If $p_2 < 0$, then system stability is always satisfied for $p \in [0, 1)$.*
- **b:** *If $0 \leq p_2 \leq 1$, then system stability is satisfied for $p \in (p_2, 1)$.*

From now on, we refer to a given set of input parameters $\{\lambda, \eta, \alpha, \xi, \mu_1, \mu_2, s_1, s_2\}$ as a problem instance *and consider a set of problem instances denoted by $I = \{i : \{\lambda, \eta, \alpha, \xi, \mu_1, \mu_2, s_1, s_2\}_i\}$. Using Theorem*

1, we then have the framework of an algorithmic approach for classifying any set I into six mutually exclusive subsets $I_j, j = 1, \ldots, 6$ such that $I \equiv I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_6$ where each one of the six subsets represent the cases of Theorem 1, as illustrated in Figure **??**.

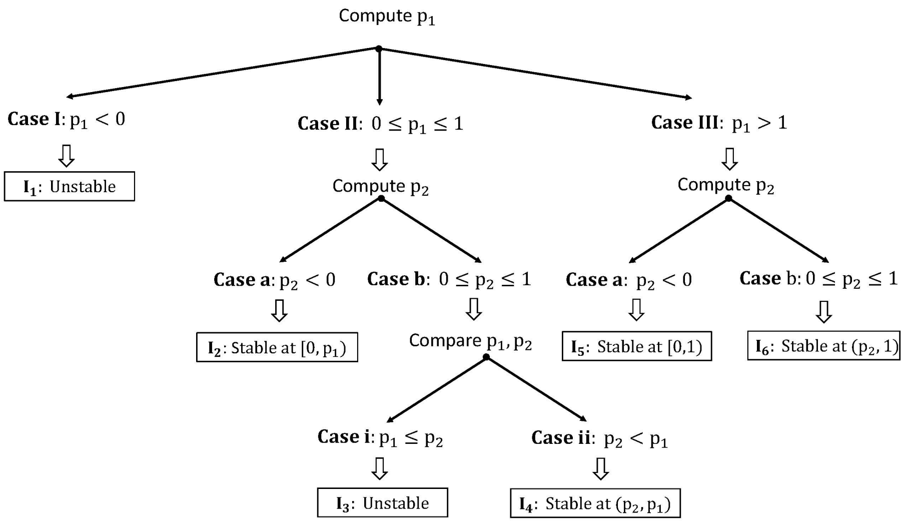

Figure 10: Stability conditions via an illustration of the aggregate arrival rates in ER (left) and DR (right).

Lemma 1 *The behavior of p_1 and p_2 in $1/\eta$ and $1/\xi$ is summarized as follows:*

(i) *p_i is strictly decreasing in $1/\eta$ iff $\frac{\xi}{\alpha+\xi} < \frac{\lambda}{\mu_i s_i}$ and strictly increasing otherwise.*

(ii) *p_1 is strictly decreasing in $1/\xi$ while p_2 is strictly decreasing in $1/\xi$ iff $1/\xi \in (0, 1/\xi_1)$ and increasing otherwise, where $1/\xi_1 = \left(1 - \sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}\right)/\alpha\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}$.*

A.5 Proof of Lemma 1

A.5.1 Part (i) of Lemma 1

Note that p_1 can be easily written as $p_1 = \left(\frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_1 s_1}\right)\frac{(\alpha+\xi)(\eta+\xi)}{(\eta-\alpha)\xi}$. Hence, its derivative with respect to η equals $\frac{dp_1}{d\eta} = -\left(\frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_1 s_1}\right)\frac{(\alpha+\xi)^2}{\xi(\eta-\alpha)^2}$. Hence, $\frac{dp_1}{d\eta} > 0$ (i.e., p_1 is strictly increasing in η, therefore strictly decreasing in $1/\eta$) iff $\frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_1 s_1} < 0$.

Similarly, the derivative of p_2 in η can be written as $\frac{dp_2}{d\eta} = \left(\frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_2 s_2}\right)\frac{\frac{-\xi}{(\eta+\xi)^2}}{\left(\frac{\eta}{\eta+\xi} - \frac{\alpha}{\alpha+\xi} - \frac{\lambda}{\mu_2 s_2}\right)^2}$. Therefore, $\frac{dp_2}{d\eta} > 0$ (i.e., p_2 is strictly increasing in η, therefore strictly decreasing in $1/\eta$) iff $\frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_2 s_2} < 0$.

A.5.2 Part (ii) of Lemma 1

Note that p_1 can be easily re-written as $p_1 = c_2\xi + c_3 + c_4/\xi$, where $c_2 = \left(1 - \frac{\lambda}{\mu_1 s_1}\right)\frac{1}{\eta-\alpha} > 0$ and $c_4 = -\frac{\eta\alpha\lambda}{\mu_1 s_1(\eta-\alpha)} < 0$ since $\eta - \alpha > 0$. Thus, $\frac{dp_1}{d\xi} = c_2 - c_4/\xi^2 > 0$, hence p_1 is strictly increasing in ξ, therefore strictly decreasing in $1/\xi$.

With regards to p_2, note that it can be written as $p_2 = \frac{f(\xi)}{f(\xi)+g(\xi)}$, where $f(\xi) = \frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_2 s_2}$ and is strictly increasing in ξ, and $g(\xi) = -\frac{\xi}{\eta+\xi} < 0$ and is strictly decreasing in ξ. Then, the sign of $\frac{dp_2}{d\xi}$ is the same with the sign of $\frac{df}{d\xi}g(\xi) + \frac{dg}{d\xi}f(\xi)$, which equals (after some straightforward algebra) $\frac{(\eta-\alpha)\xi^2}{(\alpha+\xi)^2(\eta+\xi)^2} - \frac{\eta\lambda}{\mu_2 s_2(\eta+\xi)^2}$. Since $\eta - \alpha > 0$, we have that $\frac{dp_2}{d\xi} > 0$ iff $\frac{\xi}{\alpha+\xi} > \sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}$. Equivalently, $\frac{dp_2}{d\xi} > 0$ iff $\xi > \frac{\alpha\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}}{1-\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}}$ or $\frac{dp_2}{d\xi} > 0$ iff $1/\xi < \frac{1-\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}}{\alpha\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}}$. Therefore, $\frac{dp_2}{d1/\xi} > 0$ iff $1/\xi > \frac{1-\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}}{\alpha\sqrt{\frac{\eta}{\eta-\alpha}\frac{\lambda}{\mu_2 s_2}}}$.

We now note that in our setting $\mu_1 s_1 > \mu_2 s_2$ holds, hence $\frac{\lambda}{\mu_1 s_1} < \frac{\lambda}{\mu_2 s_2}$. We also note that the condition for p_1 seen in Lemma 1-(i) implies that $p_1 < 0$, according to (4). Likewise, the corresponding condition for p_2 implies $p_2 > 0$. Based on the above, as $1/\eta$ increases, the following cases arise:

(a) $\frac{\xi}{\alpha+\xi} < \frac{\lambda}{\mu_1 s_1}$: In this case, we have that $p_1 < 0$, $p_2 > 0$ and both are decreasing in $1/\eta$. Taking into account the conditions seen in Theorem 1, we conclude that the only I set that meets the above is I_1. Hence, the only option for an instance in this case is to keep the I_1 classification as $1/\eta$ increases.

(b) $\frac{\lambda}{\mu_1 s_1} \leq \frac{\xi}{\alpha+\xi} \leq \frac{\lambda}{\mu_2 s_2}$: In this case, we have that $p_1 \geq 0$ and $p_2 \geq 0$, hence the I_1, I_4, and I_5 sets are excluded (the instance cannot belong to these sets in this case). Moreover, p_1 is increasing and p_2 is decreasing in $1/\eta$ in this case. Hence, the only feasible transition for an instance is the $I_2 \to I_3$.

(c) $\frac{\xi}{\alpha+\xi} > \frac{\lambda}{\mu_2 s_2}$: In this case, we have that $p_1 > 0$, hence the I_1 set is excluded. Moreover, we have that $p_2 < 0$. This is because, if we re-write p_2 as $p_2 = \frac{\frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_2 s_2}}{-\frac{\xi}{\eta+\xi} + \frac{\xi}{\alpha+\xi} - \frac{\lambda}{\mu_2 s_2}}$, condition $p_2 < 0$ is equivalent to $\frac{\xi}{\alpha+\xi} < \frac{\lambda}{\mu_2 s_2} + \frac{\xi}{\eta+\xi}$. After some algebra, the latter can be written as $a\xi^2 + b\xi + c > 0$, where $a = \lambda$, $b = (\eta+\alpha)\lambda - (\eta-\alpha)\mu_2 s_2$, and $c = \eta\lambda\alpha$. Since $\mu_2 s_2 < \lambda$, we have that $b < 2\alpha\lambda$, hence, the determinant of the polynomial equals $\Delta = b^2 - 4ac < 4\alpha^2\lambda^2 - 4\lambda^2\eta\alpha = 4\alpha\lambda^2(\alpha-\eta) < 0$. Therefore, the polynomial always keeps a positive sign, hence indeed $p_2 < 0$. Consequently, the I_2, and I_3 sets are also excluded. Since, now, both p_1 and p_2 are increasing in $1/\eta$ in this case, we conclude that the only feasible transition for an instance is the $I_4 \to I_5$.

One can use a similar approach to prove the corresponding transitions as $1/\xi$ increases.