Appendix A: Algorithms

Algorithm 1: PSA for queue length

Input: time-varying arrival rates of customers and bicycles $\lambda(t), \gamma(t)$
Output: time-varying queue lengths of customers and bicycles $X_1(t), X_2(t)$

for $t = 0$ *to* T **do**
  $\lambda \leftarrow \lambda(t)$; $\gamma \leftarrow \gamma(t)$;
  if $0 < q < 1$ **then**
    derive $\bar{X}_1 = E(X_1(t))$ and $\bar{X}_2 = E(X_2(t))$ with (19) ;
  else
    derive $\bar{X}_1 = E(X_1(t))$ and $\bar{X}_2 = E(X_2(t))$ with (20) ;
  $X_1(t) \leftarrow \bar{X}_1$; $X_2(t) \leftarrow \bar{X}_2$;

Algorithm 2: Waiting time and sojourn time

Input: $t_0, N_0, \lambda(t), \gamma(t)$
Output: total waiting time and sojourn time

if $\lambda(t) > \gamma(t)$ **then**
  find t_T with (27);
  derive the number of bicycle arrivals from other stations during time $[t_0, t_T]$ with (C-9);
  derive the time of the ith customer arrival t_i with (C-8);
  derive the time of the jth bicycle arrival from other stations t'_j with (C-12);
  derive the time of the jth bicycle hired by a customer y_j with (C-13);
  obtain the total bicycle sojourn time S during time interval $[t_0, t_T]$ with (29);
  for $t = t_T$ *to* T **do**
    $\lambda \leftarrow \lambda(t)$; $\gamma \leftarrow \gamma(t)$; ;
    derive the mean waiting time $E[W]$ with (24);
    derive the mean sojourn time $E[S]$ with (25);
    $E_t[W] \leftarrow E[W]$; $E_t[S] \leftarrow E[S]$;
  $W \leftarrow$ total waiting time during time interval $[t_T, T]$;
  $S \leftarrow$ total sojourn time during time interval $[t_T, T]$ plus S;
else
  for $x = t_0$ *to* T **do**
    $X(x) \leftarrow$ queue length of bicycles at time x with (31);
    $S \leftarrow$ total sojourn time during time interval $[t_0, t_T]$ with (30);
    $W \leftarrow 0$;

Algorithm 3: Nonhomogeneous Time Interval Dispatch plan

Input: opportunity cost and dispatch cost
Output: the exact dispatch plan and its cost

while $i \neq \infty$ **do**
  for t=$DT(i)$ *to* T **do**
    obtain cumulative Opportunity cost and Dispatch cost at time t;
    if *cumulative opportunity cost* $\geq$ *Dispatch cost* **then**
      Let DT(i) $\leftarrow t$; i $\leftarrow i + 1$;
      return to step 2;
    else
      Let i $\leftarrow \infty$;

Appendix B: Figures

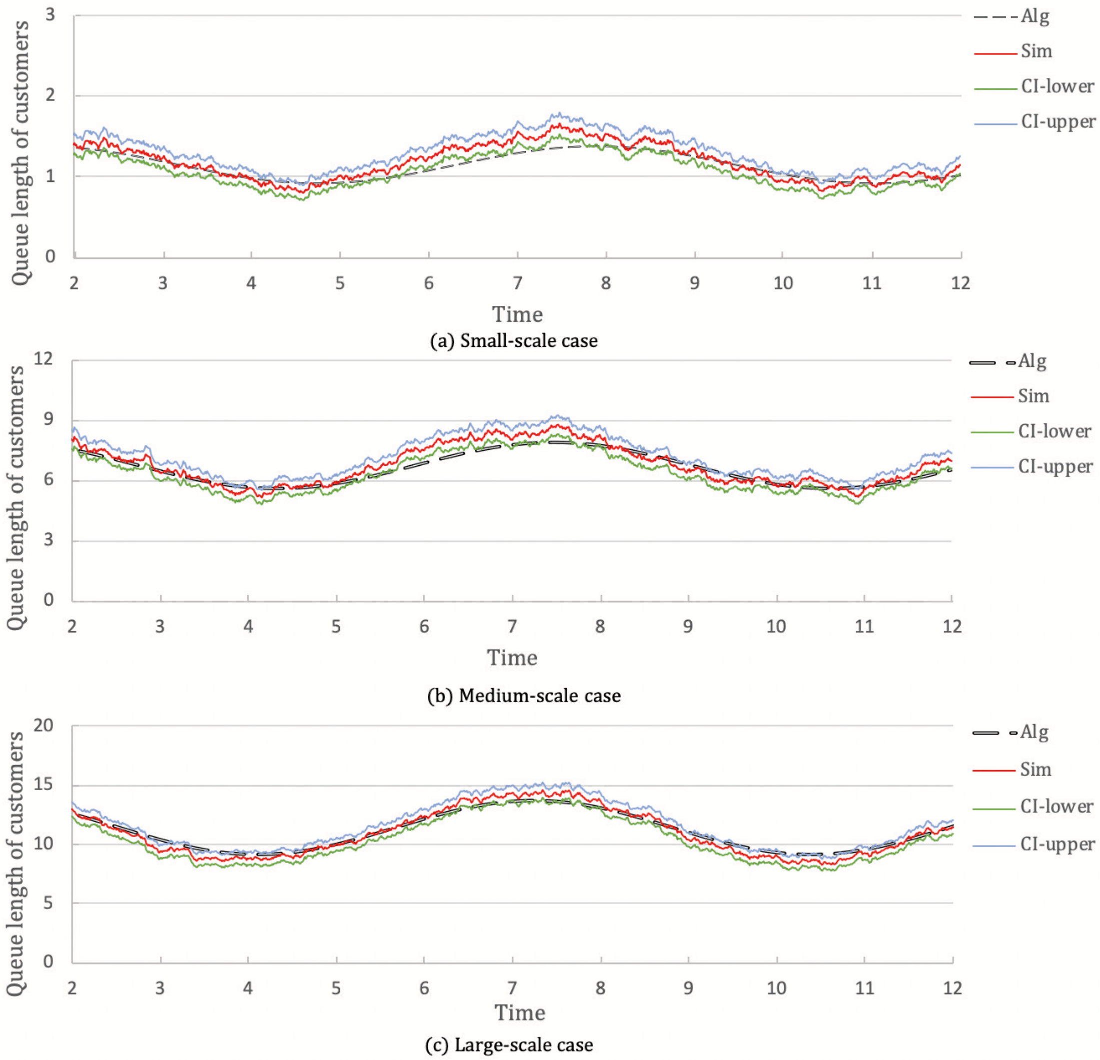

Figure B-1: Comparisons for $K = 1$, $\theta = 5$, $\varrho = 5$ and $p = 0.2$ with a 95% confidence interval.

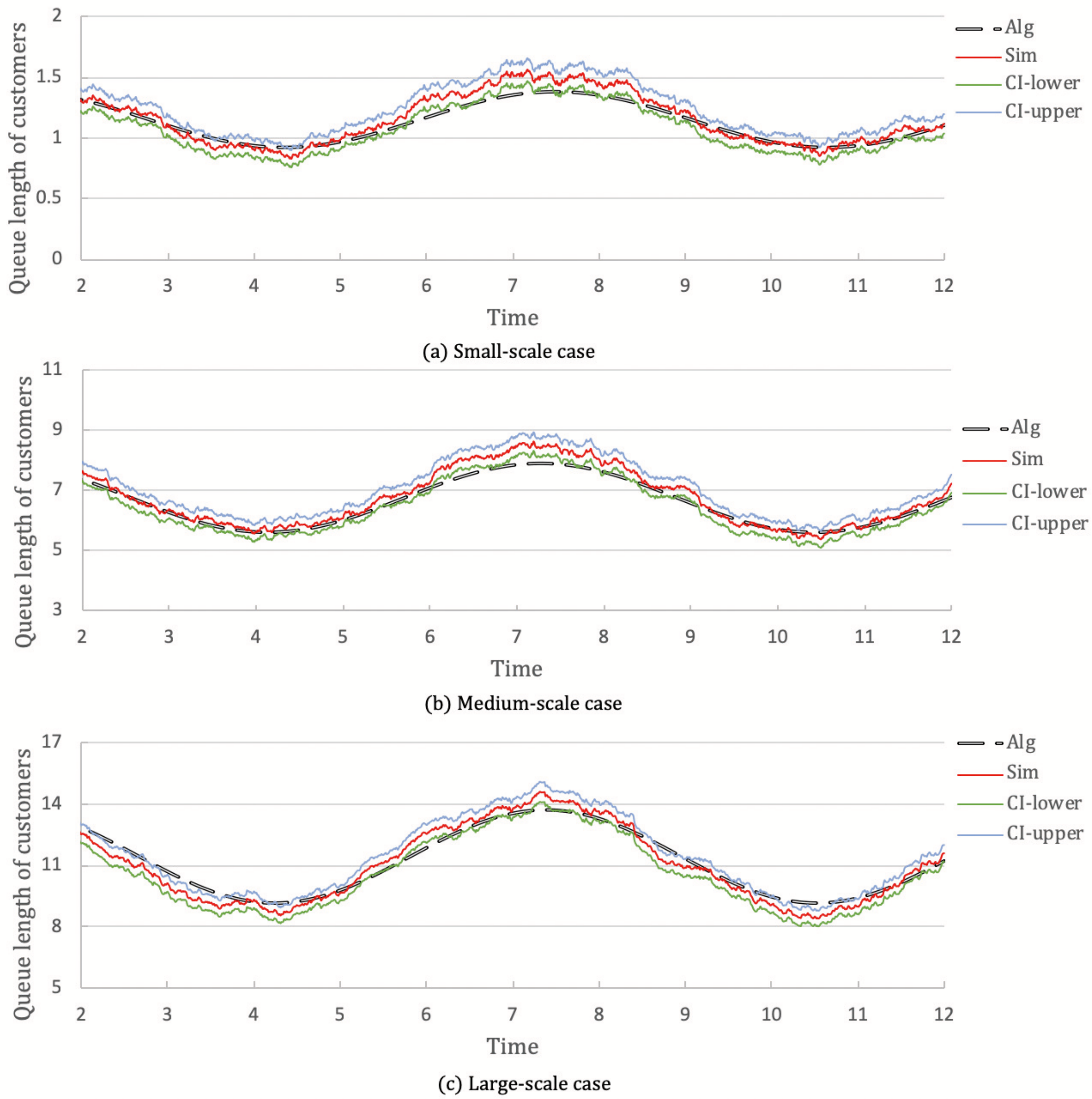

Figure B-2: Comparisons for $K = 2$, $\theta = 5$, $\varrho = 5$ and $p = 0.2$ with a 95% confidence interval.

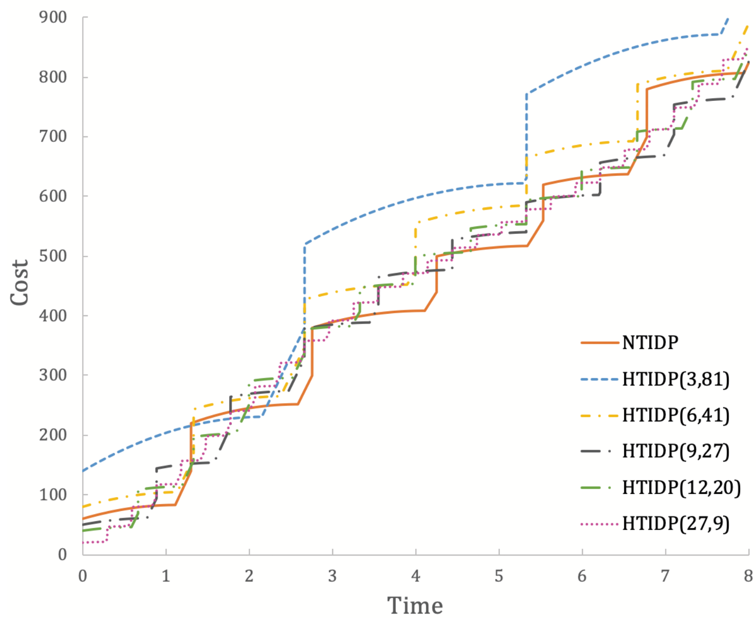

Figure B-3: Cost comparison between NTIDP and HTIDPs with similar total dispatch quantum

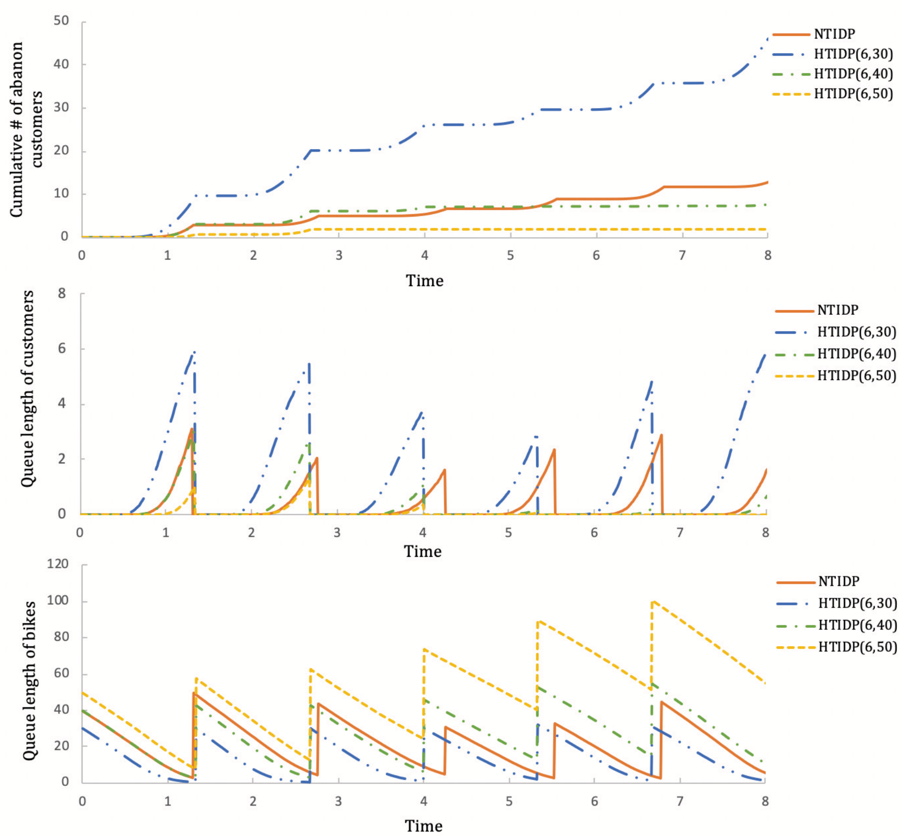

Figure B-4: Performance comparison between NTIDP and HTIDPs in Case 1

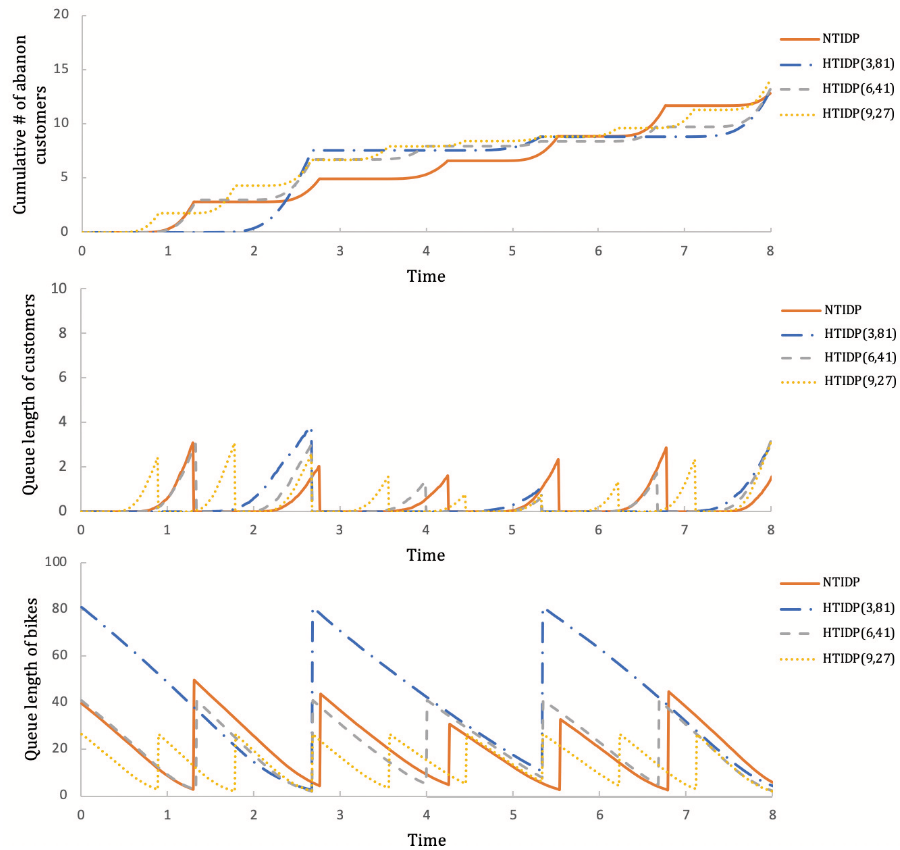

Figure B-5: Performance comparison between NTIDP and HTIDPs with similar total dispatch quantum

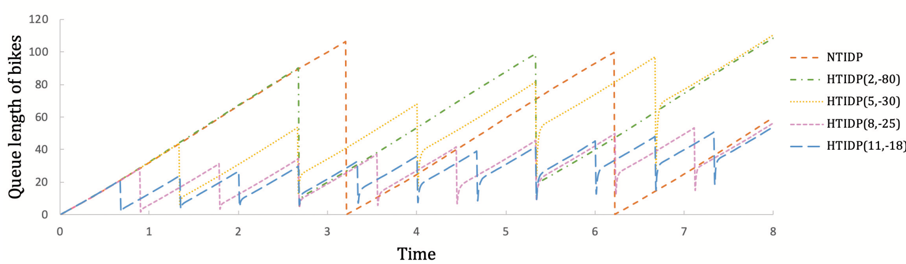

Figure B-6: Performance comparison between NTIDP and HTIDPs in Case 2

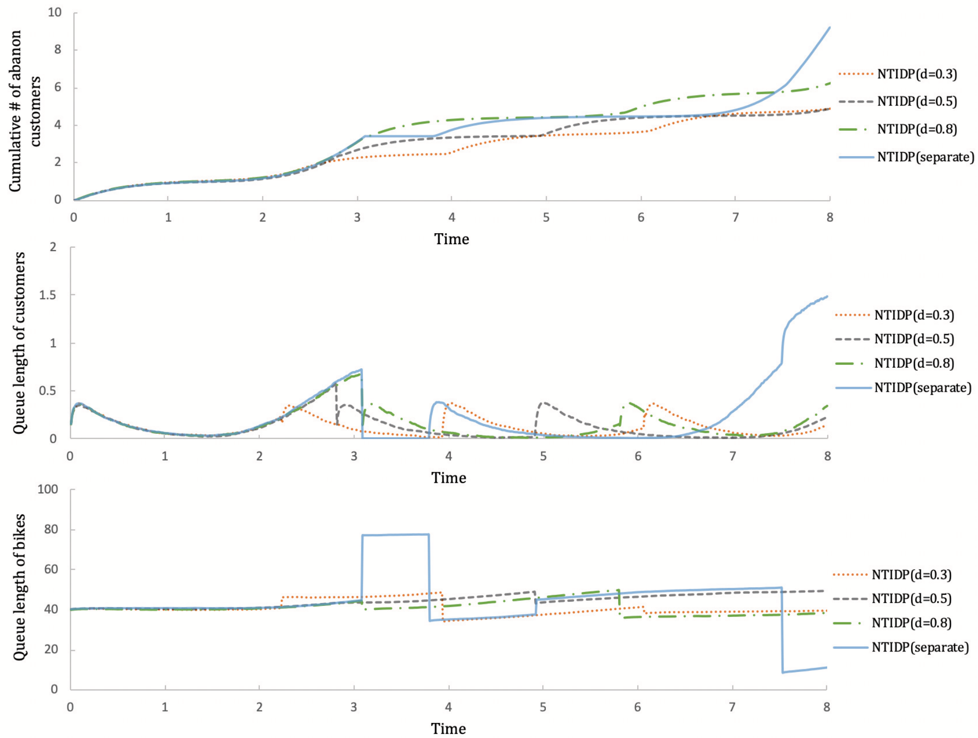

Figure B-7: Performance comparison between NTIDPs(d) and NTIDP(separate) in Case 3

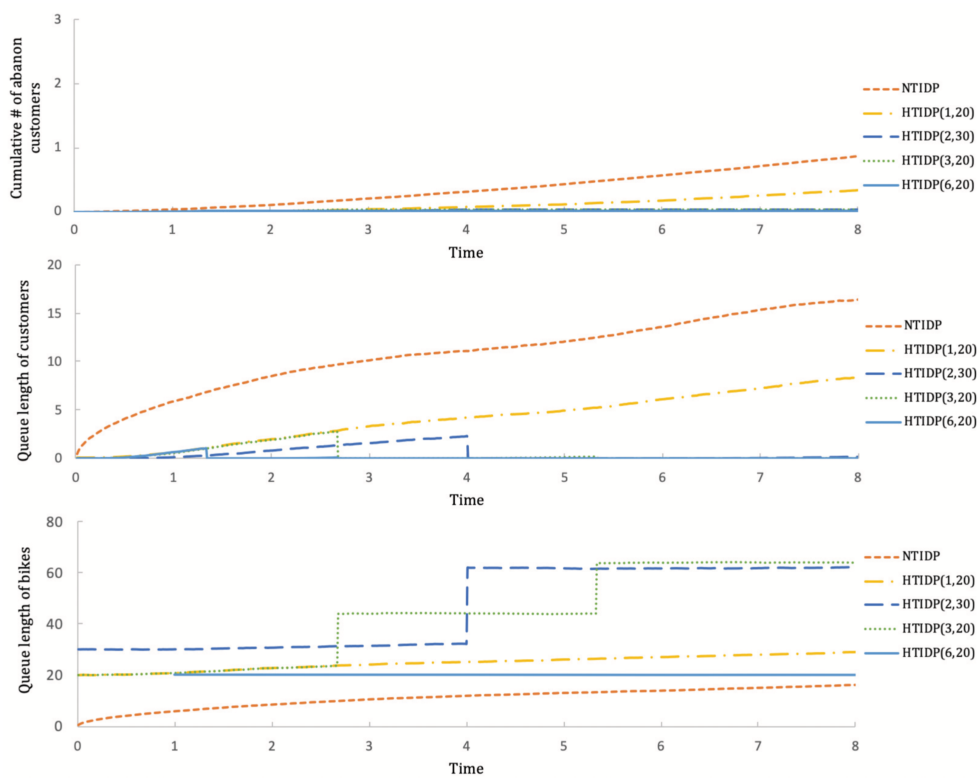

Figure B-8: Performance comparison between NTIDP and HTIDPs in Case 4

Appendix C: Supplement

Proof of Theorem 2.1

In Eick et al. (1993), according to $M_t/G/\infty$ queueing model with NHPP arrivals, $Q(t)$ is a random variable subject to Poisson distribution with mean

$$E[Q(t)] = \int_0^w \lambda(x)\bar{G}(t-x)dx, \tag{C-1}$$

and the departure process $\{D(t), t \geq 0\}$ is an NHPP with rate

$$\delta(t) = \int_0^t \lambda(x)g(t-x)dx. \tag{C-2}$$

However, there are finite bicycles in FFBSS, so arriving customers may find no bicycles and have to wait in the queue. If the waiting time exceeds the patience time, the abandonment occurs. Liu and Whitt (2012) considered a sequence of $M_t/GI/s_t + GI$ model with the many-server heavy-traffic scaling, the expected number of busy servers is

$$E[BS(t)] = \bar{F}(w)\int_0^{t-w} \bar{G}(x)\lambda(t-w-x)dx \cdot 1_{\{t>w\}}.$$

We consider the behavior of returning by customers initially abandoning and an empty system denoted by the $(M_t/(GI,GI)/s_t + (GI,GI)) + (GI/\infty)_2$ queueing model of region A with no initial bicycles or customers which could be approximated by five IS queues. The performance of each queue should be obtained, and it is a direct application of equations above. Hence, the time-dependent queue lengths in waiting room with mean $E[Q_{\mathrm{A}i}(t)]$ and the time-dependent number of hired bicycles with mean $E[Bs_{\mathrm{A}i}(t)]$ are as follows.

$$E[Q_{\mathrm{A}i}(t)] = \int_0^{w_{\mathrm{A}}} \lambda_{\mathrm{A}i}(t-x)\bar{F}_{\mathrm{A}i}(x)dx,$$

$$E[BS_{Ai}(t)] = \bar{F}_{Ai}(w_A)\int_0^{(t-w_A)^+} \lambda_{Ai}(t-w_A-x)\bar{G}_{Ai}(x)dx, \qquad i = 1, 2, 3.$$

The process counting the number of customer retrial after abandonments from the waiting room is an independent Poisson process with rate function $\lambda_{\mathrm{A}2}(t)$, where

$$\lambda_{\mathrm{A}2}(t) = p_{\mathrm{A}1}\int_0^t \xi_{\mathrm{A}1}(t-x)dH_{\mathrm{A}1}(x). \tag{C-3}$$

The process counting the number of customer abandonments from the waiting room is independent Poisson process with rate function $\xi_{\mathrm{A}i}(t), i = 1, 2, 3$, where

$$\xi_{\mathrm{A}i}(t) = \int_0^{t\wedge w_{\mathrm{A}}} \lambda_{\mathrm{A}i}(t-x)dF_{\mathrm{A}}(x). \tag{C-4}$$

The process counting the number of customer arriving after abandonments from region B is an independent Poisson process with rate function $\lambda_{\mathrm{A}3}(t)$, which equals to the rate of customers abandoning from another region, derived by

$$\lambda_{\mathrm{A}3}(t) = \eta_{\mathrm{A}}(t) = p_{\mathrm{B}2} \int_0^t \xi_{\mathrm{B}1}(t-x) dH_{\mathrm{B}2}(x). \tag{C-5}$$

Notations above with superscript in region B have similar derivation with those in the selected region.□

Proof of Proposition 2.1

As mentioned before, customers arrive with the rate of $\lambda_{\mathrm{A}1}$ ($\lambda_{\mathrm{B}1}$) to hire bicycles in region A (B) for the first time according to an external NHPP. Based on (C-4), customers abandon in the first situation according to a NHPP. What's more, (C-3) and (C-5) give that customers arrive in other two situations both according to NHPPs. So the integrated arrival rate function is the sum of arrival rate functions in all situations and (3) is obtained.

Then we could integrate other property functions. The integrated OL function is obtained by

$$\begin{aligned} m_{\mathrm{A}}(t) &= \int_0^{(t-w_{\mathrm{A}})^+} \bar{F}_{\mathrm{A}}(w_{\mathrm{A}}) \cdot \lambda_{\mathrm{A}}(t-w_{\mathrm{A}}-x)\bar{G}_{\mathrm{A}}(x)dx \\ &= \sum_{i=1}^{3} \int_0^{(t-w_{\mathrm{A}})^+} \bar{F}_{\mathrm{A}i}(w_{\mathrm{A}}) \cdot \lambda_{\mathrm{A}i}(t-w_{\mathrm{A}}-x)\bar{G}_{\mathrm{A}i}(x)dx \\ &= \sum_{i=1}^{3} m_{\mathrm{A}i}(t). \end{aligned}$$

Also, equation (5) could be expanded by

$$\begin{aligned} \xi_{\mathrm{A}}(t) &= \int_0^{t\wedge w_{\mathrm{A}}} \lambda_{\mathrm{A}}(t-x)dF_{\mathrm{A}}(x) \\ &= \sum_{i=1}^{3} \int_0^{t\wedge w_{\mathrm{A}}} \lambda_{\mathrm{A}i}(t-x)dF_{\mathrm{A}i}(x) \\ &= \sum_{i=1}^{3} \xi_{\mathrm{A}i}(t) \end{aligned}$$

All customers getting in service depart by bike at the rate of

$$\begin{aligned} \delta_{\mathrm{A}}(t) &= \int_0^{(t-w_{\mathrm{A}})^+} \bar{F}_{\mathrm{A}}(w_{\mathrm{A}}) \cdot \lambda_{\mathrm{A}}(t-w_{\mathrm{A}}-x)dG_{\mathrm{A}}(x) \\ &= \sum_{i=1}^{3} \int_0^{(t-w_{\mathrm{A}})^+} \bar{F}_{\mathrm{A}i}(w_{\mathrm{A}}) \cdot \lambda_{\mathrm{A}i}(t-w_{\mathrm{A}}-x)dG_{\mathrm{A}i}(x) \\ &= \sum_{i=1}^{3} \delta_{\mathrm{A}i}(t). \end{aligned}$$

The proposition of additivity is proved. □

Additional analysis of cross-region riding in Section 2.3

The service zone is then divided into 50 regions with K-means cluster, and we present the top 10 Max-CRR regions in Table C-1 for demonstration. We can find the not big Max-CRR and the rare Min-CRR of ten cluster regions. Besides, we show that the Avg-CRR and the degree for each region. With 7 origins of 50 separated regions in Figure C-1, the dark blue dots mean the origins in selected region, while the orange dots are in the same region with the selected origin and the red dots represent cross-region riding records. Other colors are the connected regions related with selected origin. Obviously, the number of orange dots is more than red dots' and the orange dots nearly cover most of the dark blue dots, which shows the small amount of cross-region riding trajectory.

Table C-1: The cross region bicycle hiring among the top 10 of the 50 virtual regions

Region	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10
Max-CRR	0.115	0.108	0.091	0.082	0.077	0.074	0.071	0.070	0.068	0.067
Min-CRR	0.005	0.001	0.002	0.014	0	0	0	0	0	0
Avg-CRR	0.045	0.022	0.028	0.048	0.030	0.020	0.026	0.018	0.019	0.023
Degree	3	7	5	2	8	9	7	8	7	9

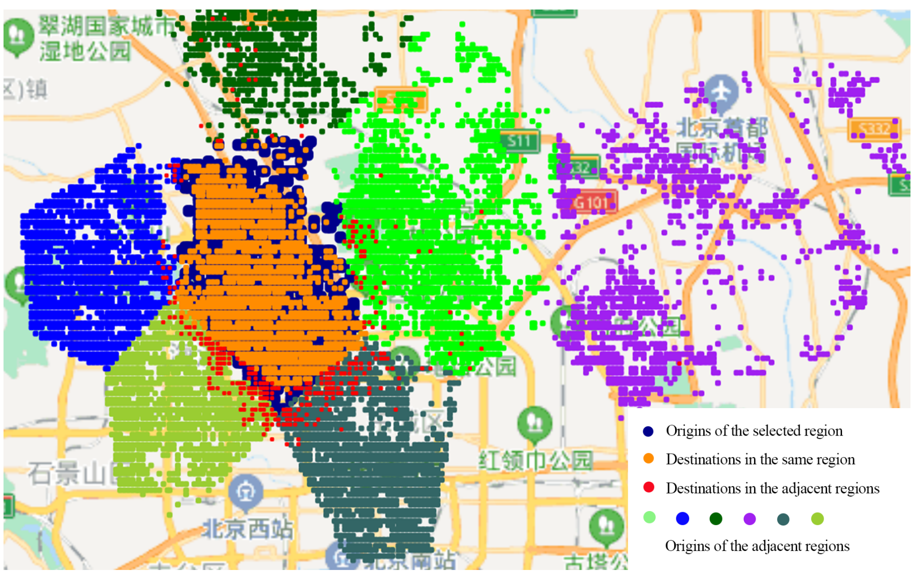

Figure C-1: Cross-region bicycle hiring with 50 clusters

Similarly, the service zone is divided into 100 regions with K-means cluster, and we present the top 10 Max-CRR regions in Table C-2. We can find that the Max-CRR no more than one in five and that the Min-CRR is negligible of top ten cluster regions. Besides, the values of Max-CRR in 100 regions are much big than 10 region Max-CRR, which is intuitive for the reason that the smaller division the lager CRR. Instead, we still show that the Avg-CRR and the degree are small for each region. With 7 origins of 100 separated regions in Figure C-2, the dark blue dots mean the origins in selected region, while the orange dots are in the same region with the selected origin and the red dots represent cross-region riding records. Other colors are the connected regions related with selected origin. Obviously, the number of orange dots is overwhelmingly more than the red dots. And it is observed that only a small amount of red cross-region dots are around the selected region.

Table C-2: The cross region bicycle hiring among the top 10 of the 100 virtual regions

Region	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10
Max-CRR	0.242	0.230	0.191	0.153	0.152	0.142	0.134	0.130	0.127	0.125
Min-CRR	0.001	0	0.001	0	0	0	0.002	0	0	0.001
Avg-CRR	0.082	0.040	0.044	0.030	0.023	0.028	0.042	0.021	0.031	0.041
Degree	3	10	5	10	14	12	7	12	11	5

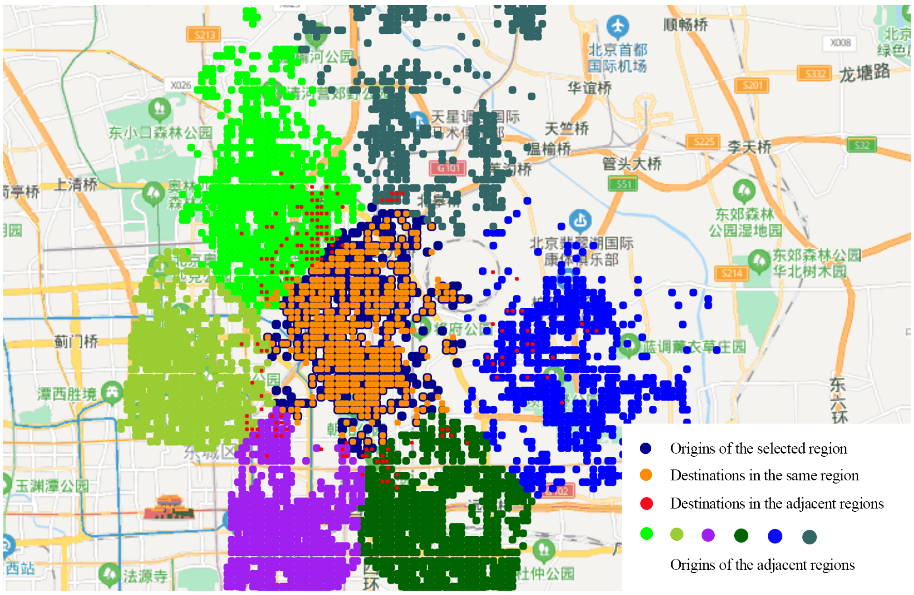

Figure C-2: Cross-region bicycle hiring with 100 clusters

Lemma C-1

Let $\bar{X}_i (i = 1, 2)$ be the unique solution to Equations (10)-(12), where $\xi > 0$ and $0 < q < 1$, then

$$\sup_{0 \le t < \infty} \bar{X}_1(t) < \frac{\lambda}{\xi}$$

$$\sup_{0 \le t < \infty} \bar{X}_2(t) < -\frac{1}{q} \log(1 - \frac{\gamma}{\lambda} e^{-\frac{\lambda q}{\gamma}}).$$

Proof. For any (x_1, x_2) such that $x_1 \ge \lambda/\xi, x_2 \ge -1/q\log(1 - \gamma/\lambda e^{-\lambda q/\gamma})$, we have

$$\begin{aligned} F_1(x_1, x_2) &= \lambda e^{-qx_2} - \gamma(1 - e^{-qx_1}) - \xi x_1 \\ &< \lambda - \xi x_1 \\ &\le 0 \\ F_2(x_1, x_2) &= \gamma e^{-qx_1} - \lambda(1 - e^{-qx_2}) \\ &\le \gamma e^{-qx_1} - \lambda(1 - e^{-\frac{\lambda q}{\xi}}) \\ &\le 0 \end{aligned}$$

With Equation (14), it is implied that $\bar{X}_1(t) < \lambda/\xi, \bar{X}_2(t) < -\log(1 - e^{-\lambda q/\gamma}\gamma/\lambda)/q$ for all t. Hence $\bar{X}_1(t)$ and $\bar{X}_2(t)$ are both bounded. □

Lemma C-2

For any $\delta > 0$ and $T > 0$, $\bar{X}_i$, $i = 1, 2$ is the unique solution of Equations (10)-(12),

$$\limsup_{n\to\infty} n^{-1} \log \mathbb{P}(\sup_{0\le t\le T} |\bar{X}_i^n(t) - \bar{X}_i(t)| > \delta) < 0$$

and as $n \to \infty$,

$$\bar{X}_i^n \xrightarrow{\text{a.s.}} \bar{X}_i \qquad \text{u.o.c..}$$

Proof. Similar proof could be found in Büke and Chen (2017) by showing three conditions in Proposition 5.1 in Darling and Norris (2005). □

Proof of Proposition 4.1

It is obvious that system of equations (18) has a unique solution. And we have

$$\begin{cases} x_1^* = \frac{\lambda-\gamma}{\xi}, \\ x_2^* = -\frac{1}{q}\log(1 - \frac{\gamma}{\lambda} e^{-\frac{q(\lambda-\gamma)}{\xi}}) \end{cases} \tag{C-6}$$

Then, we prove that $\bar{X}_i(t)$ converges to a fixed point x_i^* as $t \to \infty$. In the theory of ODEs, Lyapunov functions are scalar functions that are used to prove the stability of an equilibrium of an ODE. Strogatz (1994) proved that the stability holds if a Lyapunov function $V(x)$ can be found with the following properties:

(1)$V(x) > 0$ for all $x \neq x^*$ and $V(x^*) = 0$;

(2)$\frac{dV(\bar{X}(t))}{dt} < 0$ for all $x \neq x^*$.

Without loss of generality, we assume $\lambda \geq \gamma$, and let $V(x) = \lambda e^{-qx_2} - \gamma(1 - e^{-qx_1}) - \xi x_1 - (\gamma e^{-qx_1} - \lambda(1 - e^{-qx_2})) = \lambda - \gamma - \xi x_1$. Then, we have $V(x^*) = 0$ and $V(x) \neq 0$ for all $x \neq 0$. Applying first equation in Lemma C-1, we have $x_1 < \gamma/\xi$. Therefore, $V(x) > 0$ for all $x \neq x^*$. And

$$\frac{dV(\bar{X}(t))}{dt} = \frac{d(\bar{X}_1(t))}{dt} - \frac{d(\bar{X}_2(t))}{dt} = \gamma - \lambda + \xi\bar{X}_1(t) = -V(\bar{X}_1(t)) < 0.$$

Hence, the fluid limit $\bar{X}_i \to x_i^*$, $i = 1, 2$, as $t \to \infty$ holds for all initial system states. □

Additional analysis of stationary probabilities in Section 4.2.2

Figure C-3 depicts the birth-death process for the bicycle sharing model which is equivalent to a two-sided random walk. Then based on the empty initial FFBSS state, we have $c(0) = 1, a_m(0) = b_n(0) = 0$, for $m, n \geq 1$.

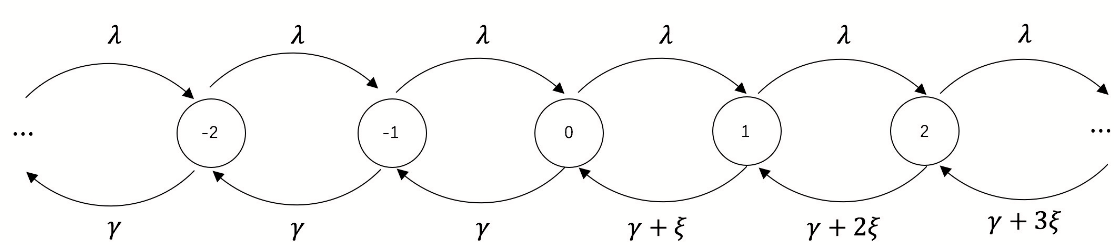

Figure C-3: Birth and death model for the FFBSS model

For each time t, arrival rates of customers and bicycles are of fixed values, and Kolmogorov Equations for the system states are formulated in the original form, and the derived probabilities a, b and c are all time-dependent. Thus, for $\gamma < \lambda$,

$$\frac{dc}{dt} + (\lambda + \gamma)c = (\gamma + \xi)a_1 + \lambda b_1,$$
$$\frac{da_1}{dt}(\lambda + \gamma + \xi)a_1 = (\gamma + 2\xi)a_2 + \lambda c,$$
$$\cdots$$
$$\frac{da_n}{dt} + (\lambda + \gamma + n\xi)a_n = (\gamma + (n+1)\xi)a_{n+1} + \lambda a_{n-1}, \quad n \geq 2$$
$$\frac{db_1}{dt} + (\lambda + \gamma)b_1 = \gamma c + \lambda b_2,$$
$$\cdots$$
$$\frac{db_n}{dt} + (\lambda + \gamma)b_n = \gamma b_{n-1} + \lambda b_{n+1}, \quad n \geq 2$$

By solving the equations above, the stationary probabilities in each system state are given by

$$\tilde{a}_n = \frac{r^n}{(v+1)(v+2)\cdots(v+n)}\tilde{c},$$
$$\tilde{b}_n = \tilde{c}(\frac{\gamma}{\lambda})^n,$$
$$\tilde{c} = \frac{1}{1 + \sum_{n\geq 1} \frac{r^n}{(v+1)(v+2)\cdots(v+n)} + \frac{v}{r-v}}, \quad n \geq 1.$$

where $v = \gamma/\xi, r = \lambda/\xi$.

Proof of Theorem 4.1

$$E[W](t) = \sum_{n\geq 0} E[W|X_1(t) = n, X_2(t) = 0](t)P[X_1(t) = n, X_2 = 0],$$
$$E[S](t) = \sum_{k\geq 0} E[S|X_1(t) = 0, X_2(t) = k](t)P[X_1(t) = 0, X_2 = k].$$

where

$$E[W|X_1(t) = 0, X_2(t) = 0](t) = \frac{1}{\gamma(t) + \xi(t)},$$
$$E[W|X_1(t) = n, X_2(t) = 0](t) = \frac{1}{\gamma(t) + (n+1)\xi(t)} + \frac{\gamma(t)}{\gamma(t) + (n+1)\xi(t)} \cdot E[W|X_1(t) = n-1, X_2(t) = 0](t) + \frac{(n+1)\xi(t)}{\gamma(t) + (n+1)\xi(t)} \cdot \left(\frac{1}{n+1} \cdot 0 + \frac{n}{n+1} \cdot (n+1)E[W|X_1(t) = n-1, X_2(t) = 0](t)\right),$$
$$n = 1, 2 \cdots.$$

In the same way, we have

$$E[S|X_1(t)=0, X_2(t)=0](t) = \frac{1}{\lambda(t)},$$
$$E[S|X_1(t)=0, X_2(t)=k](t) = \frac{1}{\lambda(t)} + E[S|X_1(t)=0, X_2(t)=k-1](t) = \frac{k+1}{\lambda(t)}.$$
$$k = 1, 2 \cdots.$$

Thus, by using Equations (21) - (23), the theorem is proved. □

Proof of Proposition 4.2

Since there are N_0 bicycles initially, and after time $(t_T - t_0)$, the system turns into the empty state for the first time. Namely, the number of customers arrive during time interval $[t_0, t_T]$ equal to the number of bicycles arrive during the same time interval plus the initial bicycle inventory at time t_0. Applying Equation (26), the equations are obtained as

$$\int_{t_0}^{t_T} \lambda(t)dt = N_0 + \int_{t_0}^{t_T} \gamma(t)dt,$$

$$\Lambda(t_T) - \Lambda(t_0) = N_0 + \Gamma(t_T) - \Gamma(t_0),$$

$$D(t_T) = N_0 + D(t_0).$$

□

Proof of Theorem 4.2

As for the first part, for $\{i \in N_0 | 0 \le i \le N_0 - 1\}$, it is assumed that there are $N_0 - i$ bicycles at time t_i. The initial bicycle inventory will wait for $t_{i+1} - t_i$, and move one forward in queue. Then $N_0 - i$ bicycles remain at the station. Based on Equation (26), a bicycle returns to the station at the time t_{i+1}. Therefore, until the initial bicycle inventory is depleted, the total sojourn time S_1 is obtained via

$$S_1 = \sum_{i=0}^{N_0-1} (N_0 - i)(t_{i+1} - t_i). \tag{C-7}$$

where

$$t_i = \Lambda^{-1}(1 + \Lambda(t_{i-1})), \quad i = 0, 1, \cdots N_0. \tag{C-8}$$

As for the other subproblem, N_b denotes the total number of bicycles from other stations during $[t_0, t_T]$, which is a direct deduction of Proposition 4.2 given t_T. By applying Equation (26), we have

$$N_b = \int_{t_0}^{t_T} \gamma(t)dt. \tag{C-9}$$

Meanwhile, Lemma C-1 is used to derive the time t'_i when the ith bicycle return occurs, and the bicycle queue length at time t'_i. Then the time gap $y_i - t'_i$ could be obtained when the bicycle is hired, which is equivalent to the sojourn time of the ith bicycle. Similarly, the total sojourn time S_2 of N_b bicycles can be derived as

$$S_2 = \sum_{j=1}^{N_b} (y_j - t'_j) \tag{C-10}$$

where t'_i, y_i are obtained via Equations (C-12) and (C-13), respectively, which are solutions of the following system of equations.

For $1 \leq j \leq N_b$, $j \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \int_{t'_{i-1}}^{t'_i} \gamma(t)dt = 1, \\ N_0 + D(t'_{i-1}) - D(t'_i) = \int_{t'_i}^{y_i} \lambda(t)dt. \end{array} \right\}. \tag{C-11}$$

Therefore, the solution to the system of equations (12) is reached as

$$t'_i = \Gamma^{-1}(1 + \Gamma(t'_{i-1})), \tag{C-12}$$

$$y_i = \Lambda^{-1}(\Lambda(t'_i) + X(t'_i)). \tag{C-13}$$

Eventually, by adding up S_1 and S_2, Theorem 4.2 is proved. □

Additional analysis and results of section 7

In the modified five node model, Ω is denoted as the state space of the closed-loop queueing network as shown in Figure 12. $\mathcal{N}$ is a stochastic variable indicating the number of customers in each node. In the initial state of the system, customers are all originated from the dummy node with a given total number of customers N in the endogenous system, which is preset according to the system features before running the queueing network. The variable X_i denotes the number of customers in the i_{th} ($i \in \{1, 2, 3, 4, 5\}$) node. Then the closed-loop network state space is expressed as,

$$\Omega = \{(X_1, X_2, X_3, X_4, X_5) : X_i \in \mathcal{N}, \sum_{i=1}^{5} X_i = N\}$$

In reality, customers could be waiting or looking for bikes in each area, and the waiting space does not put any limited on the system capacity, due to the unique feature of the FFBSS. After waiting or searching for bikes in the nearby area for some time, customers are subject to abandonment with a probability $p_1(p_3)$. In the meantime, customers could also walk to another area to search for available bikes with a probability $p_{13}(p_{31})$, or directly leave for the dummy node with probability $p_{15}(p_{35})$. It is assumed that the customer would eventually find an available bicycle inside the same region with a probability of $p_{12}(p_{34})$. Then all the customers in the dummy node 5 will go to the node 1 with probability p_{51}, and others will ahead for node 3 with probability p_{52}. Within the FFBSS system, customers could travel among different nodes in the network with various routing probabilities, which is represented by the matrix $\mathcal{P}$.

$$\mathcal{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & p_{12} & p_1 \cdot p_{13} & 0 & p_1 \cdot p_{15} \\ 0 & 0 & 0 & 0 & 1 \\ p_3 \cdot p_{31} & 0 & 0 & p_{34} & p_3 \cdot p_{35} \\ 0 & 0 & 0 & 0 & 1 \\ p_{51} & 0 & p_{53} & 0 & 0 \end{pmatrix} \end{array}$$

where the constraint equations are $p_1 + p_{12} = 1$, $p_3 + p_{34} = 1$, $p_{13} + p_{15} = 1$, $p_{31} + p_{35} = 1$, and $p_{51} + p_{53} = 1$.

In the queueing network, the average number of customers arriving at a node i is named the visit ratio v_i. Since the input flow is equal to the output flow in the equilibrium queueing network, the system of equations for the traffic flow is obtained as below,

$$v_1 = v_3 p_3 p_{31} + v_5 p_{51}$$

$$v_2 = v_1 p_{12}$$

$$v_3 = v_1 p_1 p_{13} + v_5 p_{53}$$

$$v_4 = v_3 p_{34}$$

$$v_5 = v_1 p_1 p_{15} + v_2 + v_3 p_3 p_{35} + v_4$$

The above equations represent the relative visit ratios among each pair of nodes, and we can obtain the absolute value of the visit ratio v_i by assuming $v_1 = 1$.

The service rate of each node is related to the number of customers in the node, which could be expressed via the following formula,

$$\mu_i(X_i) = min\{X_i, s_i\} \cdot \mu_i, \ i \in \{1, 2, 3, 4, 5\}$$

where s_i is the number of servers, i.e. the shared bicycles, in each node. For node 1 and node 3, the number of servers is equal to the waiting space capacity. In particular, $\mu_1 = \theta_1$ and $\mu_3 = \theta_3$. Since the service processes of node 1 and node 3 are similar to the customer waiting/searching process, their service rates are related to the customer abandonment rate and the rate of cross-region searching for the bicycles.

Based on the analysis of Jackson closed queueing network, the steady-state distribution of the free-float bike sharing system could be obtained as follows.

$$\pi(X_1, X_2, X_3, X_4, X_5) = \frac{1}{G(M, N)} \prod_{i \in (\{1,2,3,4,5\})} f_i(X_i)$$

where $f_i(X_i)$ represents the marginal steady-state probability distribution of the i_{th} node, with its detailed form shown as follows,

$$f_i(X_i) = \begin{cases} \dfrac{v_i^{X_i}}{X_i!\mu_i^{X_i}}, & \text{if} \quad X_i \leq s_i \\ \dfrac{v_i^{X_i}}{N_i!s_i^{X_i-s_i}\mu_i^{X_i}}, & \text{if} \quad X_i > s_i \end{cases}$$

where $G(M, N)$ (M nodes, N customers) is the normalized constant that makes the sum of the probabilities of all the states equal to 1,

$$G(M, N) = \sum_{\sum X_i = N} \prod_{i=1}^{5} f_i(X_i)$$

Since the waiting space capacity is assumed to be infinite, the dummy node maintains an infinite number of servers. For the customers who succeed in locating bicycles in their respective region, the number of bikes could be considered infinite, thus each node is regarded as an infinite server queue. Therefore, the marginal steady-state probability distribution is shown as below,

$$f_i(X_i) = \frac{v_i^{X_i}}{X_i!\mu_i^{X_i}}$$

We let $G_i(M, N)$ denote the normalized constant matrix when node i is marked as the last node. Based on the above analysis, the relevant system performance indexes are obtained as below.

- Throughput:

$$T_i = v_i \frac{G_i(M, N-1)}{G_i(M, N)}$$

- Queue length:

  Firstly, the edge steady-state probability $p_i(X_i)$ is defined to represent the probability that the total number of customers of the i_{th} node is X_i,

$$p_i(X_i, N) = f_i(X_i) \frac{G_i(M, N - X_i)}{G_i(M, N)}$$

  Then the average queue length of the $i_t h$ node L_i could be obtained based on the edge steady-state probability,

$$L_i = \sum_{X_i=1}^{N} X_i p_i(X_i, N)$$

- Delay time:

  According to Little's Law, the average delay time for the customers at the i_{th} node is derived as follows,

$$D_i = \sum_{X_i=1}^{N} \frac{X_i}{\mu_i(X_i)} p_i(X_i - 1, N - 1)$$

Furthermore, in order to obtain $G(M,N)$, the convolution algorithm is applied. Define an auxiliary function

$$g(m,n) = \sum_{s_1+\ldots+s_m} \prod_{i\in(\{1,2,3,4,5\})} f_i(n_i)$$

Then $G(M,N) = g(M,N)$, and the recursive formula for solving $g(m,n)$ is presented as follows,

$$\begin{aligned} g(m,n) &= \sum_{i=0}^{n} \sum_{n_1+\ldots+n_{m-1}+i=n} \prod_{j=1}^{m} f_j(n_j) \\ &= \sum_{i=0}^{n} f_m(i) \sum_{n_1+\ldots+n_{m-1}=n-i} \prod_{j=1}^{m-1} f_j(n_j) \\ &= \sum_{i=0}^{n} g(m-1,n-i), \quad \text{for} \quad n = 0,1,\ldots,N. \end{aligned}$$

The boundary conditions are

$$g(1,n) = f_1(n),\ n = 0,1,\ldots,N$$

$$g(m,0) = 1, \qquad m = 1,\ldots,M$$

Here we consider two special examples to validate Remark 7.1.

- Case 1: Given $p_{51} = p_{53} = 0.5$, $p_{12} = p_{34} = 0.5$, $p_{13} = p_{31} = 0.5$, then $p_1 = p_3 = 0.5$, $p_{15} = p_{35} = 0.5$. And we have $v_1 = v_3 = 1$, $v_2 = v_4 = 0.5$, $v_5 = 1.5$, which conforms with Remark 7.1.
- Case 2: Given $p_{51} = p_{53} = 0.5$, $p_{12} = p_{34} = 0.2$, $p_{13} = p_{31} = 0.1$, then $p_1 = p_3 = 0.8$, $p_{15} = p_{35} = 0.9$. And we have $v_1 = v_3 = 1$, $v_2 = v_4 = 0.8$, $v_5 = 1.96$, which shows that the network symmetry of Remark 7.1 still holds.

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