Electronic Companion: Approximations and Optimal Control for State-dependent Limited Processor Sharing Queues

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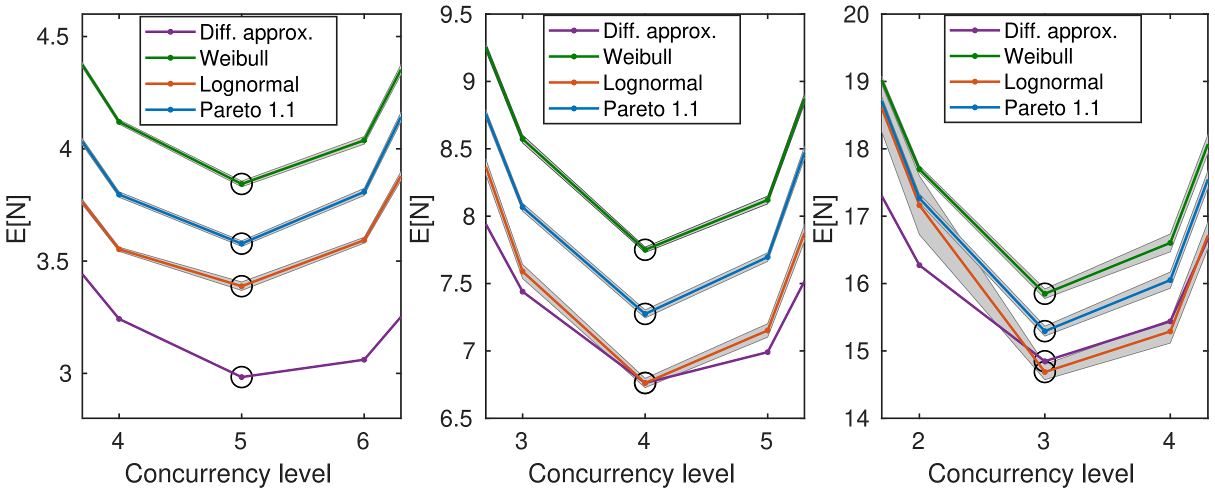
Appendix A: Addenda to Experiments



(a) $\lambda = 0.7$ (b) $\lambda = 0.8$ (c) $\lambda = 0.9$

Figure 4 Simulation results with steady-state mean number of jobs in the system and 95% confidence intervals for Figure 2. The y-axis shows mean number of jobs in the system and the x-axis shows the concurrency level for the service rate function shown in Figure 1 for various job size distributions. Also shown is the diffusion approximation from equation (13). The confidence intervals highlight that the optimal concurrency levels (shown with circles) are indeed the unique optimal with at least 90% confidence.

Appendix B: Diffusion and Steady State Analysis for the Workload Processes

Following from the dynamic equation (7), the diffusion-scaled workload is

$$\hat{W}^{(r)}(t) = \hat{W}^{(r)}(0) + \frac{1}{r} \sum_{i=1}^{\Lambda^{(r)}(r^2 t)} v_i^{(r)} - \frac{1}{r} \int_0^{r^2 t} \mu^{(r)}(Z^{(r)}(s)) 1_{\{W^{(r)}(s) > 0\}} ds \tag{61}$$

Now, introduce the notations

$$\bar{K}^{(r)}(t,x) = \frac{1}{r^2} \sum_{i=1}^{\lceil r^2 t \rceil} 1_{\{v_i^{(r)} \le x\}}, \tag{62}$$

$$\hat{K}^{(r)}(t,x) = r[\bar{K}^{(r)}(t,x) - tG(x)]. \tag{63}$$

The second term on the right-hand side of (61) can be written as

$$r \int_0^t \int_0^\infty x d\bar{K}^{(r)}(\frac{1}{r^2} \Lambda^{(r)}(r^2 s), x)$$

$$
\begin{aligned}
&= \int_0^t \int_0^\infty x d\hat{K}^{(r)}(\frac{1}{r^2}\Lambda^{(r)}(r^2 s), x) + \frac{1}{r}\int_0^t \int_0^\infty x dG(x) d\Lambda^{(r)}(r^2 s) \\
&= \int_0^t \int_0^\infty x d\hat{K}^{(r)}(\frac{1}{r^2}\Lambda^{(r)}(r^2 s), x) + m\int_0^t d\frac{1}{r}[\Lambda^{(r)}(r^2 s) - \lambda r^2 s] + \lambda r m t.
\end{aligned}
$$

The last term on the right-hand side of (61) can be written as

$$
\begin{aligned}
&-r\int_0^t \mu^{(r)}(r\hat{Z}^{(r)}(s)) 1_{\{W^{(r)}(r^2 s)>0\}} ds \\
&= \int_0^t r[\lambda m - \mu^{(r)}(r\hat{Z}^{(r)}(s))]ds + r\int_0^t 1_{\{\hat{W}^{(r)}(s)=0\}} ds - \lambda r m t \\
&= \int_0^t r[\lambda m - \mu^{(r)}(r\Delta(\hat{W}^{(r)}(s)) \wedge k^{(r)})]ds + \int_0^t r[\mu^{(r)}(r\hat{Z}^{(r)}(s)) - \mu^{(r)}(r\Delta(\hat{W}^{(r)}(s)) \wedge k^{(r)})]ds \\
&\quad + r\int_0^t 1_{\{\hat{W}^{(r)}(s)=0\}} ds - \lambda r m t \\
&= \int_0^t \theta^{(r)}\left(\Delta(\hat{W}^{(r)}(s)) \wedge \frac{k^{(r)}}{r}\right) - \theta\left(\Delta(\hat{W}^{(r)}(s)) \wedge \frac{k^{(r)}}{r}\right) ds + r\int_0^t 1_{\{\hat{W}^{(r)}(s)=0\}} ds - \lambda r m t \\
&\quad + \int_0^t r[\mu^{(r)}(r\hat{Z}^{(r)}(s)) - \mu^{(r)}(r\Delta(\hat{W}^{(r)}(s)) \wedge k^{(r)})]ds + \int_0^t \theta\left(\Delta(\hat{W}^{(r)}(s)) \wedge \frac{k^{(r)}}{r}\right) ds
\end{aligned}
$$

In summary, we can write the workload process as

$$
\begin{aligned}
\hat{W}^{(r)}(t) = \hat{W}^{(r)}(0) + \hat{M}_s^{(r)}(t) + \hat{M}_a^{(r)}(t) + \hat{G}_1^{(r)}(t) + \hat{G}_2^{(r)}(t) \\
+ \int_0^t \theta\left(\Delta(\hat{W}^{(r)}(s)) \wedge \frac{k^{(r)}}{r}\right) ds + r\int_0^t 1_{\{\hat{W}^{(r)}(s)=0\}} ds,
\end{aligned}
\tag{64}
$$

where

$$
\hat{M}_s^{(r)}(t) = \int_0^t \int_0^\infty x d\hat{K}^{(r)}(\frac{1}{r^2}\Lambda^{(r)}(r^2 s), x), \tag{65}
$$

$$
\hat{M}_a^{(r)}(t) = m\int_0^t d\frac{1}{r}[\Lambda^{(r)}(r^2 s) - \lambda r^2 s], \tag{66}
$$

$$
\hat{G}_1^{(r)}(t) = \int_0^t r[\mu^{(r)}(r\hat{Z}^{(r)}(s)) - \mu^{(r)}(r\Delta(\hat{W}^{(r)}(s)) \wedge k^{(r)})]ds, \tag{67}
$$

$$
\hat{G}_2^{(r)}(t) = \int_0^t \theta^{(r)}\left(\Delta(\hat{W}^{(r)}(s) \wedge \frac{k^{(r)}}{r})\right) - \theta\left(\Delta(\hat{W}^{(r)}(s)) \wedge \frac{k^{(r)}}{r}\right) ds. \tag{68}
$$

The following lemma is an extension of the classical one-dimensional Skorohod problem. The proof can be found in Lee and Weerasinghe (2011).

Lemma 2. *Suppose g is a Lipschitz continuous function. For any $x \in \mathbf{D}(\mathbb{R}^+)$, there exists a unique pair $(y, z) \in \mathbf{D}^2(\mathbb{R}^+)$ satisfying*

$$
z(t) = \int_0^t g(z(s))ds + x(t) + y(t), \tag{69}
$$

$$
z(t) \geq 0, \quad \textit{for all } t \geq 0, \tag{70}
$$

$$
y(0) = 0 \textit{ and } y \textit{ is non-decreasing}, \tag{71}
$$

$$
\int_0^t z(s)dy(s) = 0. \tag{72}
$$

More over, denote $z = \psi(x)$. The mapping $\psi : \mathbf{D}(\mathbb{R}^+) \to \mathbf{D}(\mathbb{R}^+)$ is continuous in the uniform topology on compact set.

Proof of Theorem 1: We first study the first four terms on the right-hand side of equation (64). For the initial condition $\hat{W}^{(r)}(0)$, its convergence to some random variable w_0 is part of the assumption (20) on the initial state.

According to Lemma 3.8 in Krichagina and Puhalskii (1997),

$$\int_0^t \int_0^\infty x d\hat{K}^{(r)}(\frac{1}{r^2}\Lambda^{(r)}(r^2 s), x) \Rightarrow \sqrt{\lambda} m c_s M_s(t), \quad \text{as } r \to \infty,$$

where $M_s(t)$ is a standard Brownian motion (with zero drift and variance 1).

It follows from the assumption (14) that

$$\hat{M}_a^{(r)}(t) = m\hat{\Lambda}^{(r)}(t) \Rightarrow \sqrt{\lambda} m c_a M_a(t), \quad \text{as } r \to \infty.$$

We now study the terms $\hat{G}_1^{(r)}$ and $\hat{G}_2^{(r)}$. By the stochastic bound (Lemma 3) proved in Section C, for any $\epsilon > 0$, there exists C such that $\mathbb{P}(\Omega_r) \geq 1 - \epsilon$, where $\Omega_r = \left\{ \sup_{t\in[0,T]} \max\left(\hat{Z}^{(r)}(s), \Delta\hat{W}^{(r)}(s)\right) \leq C \right\}$ (noting that we naturally have $\hat{Z}^{(r)}(\cdot) \leq k^{(r)}/r$). According to condition (16), for any sample path in the event Ω_r, we have

$$\hat{G}_1^{(r)}(t) \Rightarrow 0, \qquad \hat{G}_2^{(r)}(t) \Rightarrow 0, \quad \text{as } r \to \infty.$$

Let $\hat{Y}^{(r)}(t) = r\int_0^t 1_{\{\hat{W}^{(r)}(s)=0\}} ds$. It is easy to see that

$$\int_0^t \hat{W}^{(r)}(s) d\hat{Y}^{(r)}(s) = 0. \tag{73}$$

Thus $(\hat{W}^{(r)}, \hat{Y}^{(r)})$ is the solution to the reflection mapping in Lemma 2. So

$$\hat{W}^{(r)} = \psi\left(\hat{W}^{(r)}(0) + \hat{M}_s^{(r)} + \hat{M}_a^{(r)} + \hat{G}_1^{(r)} + \hat{G}_2^{(r)}\right).$$

By the continuous mapping theorem, $\hat{W}^{(r)} \Rightarrow W^*$, where $W^* = \psi(w_0 + \sqrt{\lambda} m c_s M_s(t) + \sqrt{\lambda} m c_a M_a(t))$. In other words, the limit W^* satisfies

$$W^*(t) = w_0 + \sqrt{\lambda} m c_s M_s(t) + \sqrt{\lambda} m c_a M_a(t) - \theta(\Delta(W^*))(t) + Y^*(t), \tag{74}$$

with $Y^*(0) = 0$ and being non-decreasing and

$$\int_0^t W^*(s) dY^*(s) = 0. \tag{75}$$

Thus, we have shown that the diffusion limit of the workload process is an RBM with state-dependent drift $-\theta(\Delta_K(W^*(t)) \wedge K)$ and variance $\lambda m^2(c_s^2 + c_a^2)$. The proof of (25) follows immediately from the continuous mapping theorem.

Proof of Lemma 1: (Karlin and Taylor 1981, Chapter 15) prescribed an approach based on the Kolmogorov equation to compute the stationary distribution for general diffusion processes. Here we provide an alternate derivation using the basic adjoint relationship for an RBM with state-dependent drift and variance.

We can write $W(t)$ as

$$W(t) = W(0) - \int_0^t \beta(W(\tau)) ds + \int_0^t \sqrt{s(W(\tau))} dB(\tau) + Y(t), \tag{76}$$

where B is a standard Brownian motion and Y is the regulator process that prevents W from becoming negative. The process Y is non-decreasing and satisfies

$$\int_0^t W(\tau) dY(\tau) = 0. \tag{77}$$

Let f be a twice differentiable function. By Ito's formula,

$$\begin{aligned} f(W(t)) - f(W(0)) = &\int_0^t \sqrt{s(W(\tau))} f'(W(\tau)) dB(\tau) \\ &+ \int_0^t \left[\frac{1}{2} s(W(\tau)) f''(W(\tau)) - \beta(W(\tau)) f'(W(\tau))\right] d\tau \\ &+ \int_0^t f'(W(\tau)) dY(\tau). \end{aligned}$$

Note that $\int_0^t f'(W(\tau)) dB(\tau)$ is a martingale, and that

$$\int_0^t f'(W(\tau)) dY(\tau) = f'(0) Y(t),$$

due to regulation (77). Taking the conditional expectation with respect to the stationary distribution π on both sides of the above formula, we have

$$0 = \mathbb{E}_\pi \int_0^t \left[\frac{1}{2} s(W(\tau)) f''(W(\tau)) - \beta(W(\tau)) f'(W(\tau))\right] d\tau + f'(0) \mathbb{E}_\pi[Y(t)].$$

So

$$\int_0^\infty \left[\frac{1}{2} s^2(W(\tau)) f''(w) - \beta(w) f'(w)\right] d\pi(w) + f'(0) \frac{\mathbb{E}_\pi[Y(t)]}{t} = 0. \tag{78}$$

This is known as the *basic adjoint relation* (BAR) in the literature. Our goal is to guess a functional form for π so that the integral in the above expression can be decomposed as $f'(0)$ times a term independent of f.

Consider the following derivative:

$$\begin{aligned} &\left[c(w) h(w) e^{\int_0^w g(u) du}\right]' \\ &= \left[c(w) h'(w) + c(w) h(w) g(w) + c'(w) h(w)\right] e^{\int_0^w g(u) du} \\ &= \left[c(w) h'(w) + \left[c(w) g(w) + c'(w)\right] h(w)\right] e^{\int_0^w g(u) du} \end{aligned}$$

Now substituting $h(w) = f'(w)$ and $c(w) = \frac{1}{2} s(w)$, we obtain the expression inside the square brackets in the integral term of BAR if $c'(w) + c(w) g(w) = -\mu(w)$. Equivalently, $g(w) = \frac{-\mu(w) - \frac{1}{2} s'(w)}{\frac{1}{2} s(w)}$. Therefore, letting $d\pi(w) = \alpha \cdot e^{-\int_0^w \frac{\mu(v) + \frac{1}{2} s'(v)}{\frac{1}{2} s(v)} dv}$,

$$\begin{aligned} &\int_0^\infty \left[\frac{1}{2} s(w) f''(w) - \mu(w) f'(w)\right] d\pi(w) \\ &= \int_0^\infty \left[\frac{1}{2} s(w) f''(w) - \mu(w) f'(w)\right] \alpha \cdot e^{-\int_0^w \frac{\mu(v) + \frac{1}{2} s'(v)}{\frac{1}{2} s'(v)} dv} \\ &= \alpha \int_0^\infty d\left[\frac{1}{2} s(w) f'(w) e^{-\int_0^w \frac{\mu(v) + 0.5 s'(v)}{0.5 s(v)} dv}\right] \\ &= \alpha \left[\frac{1}{2} s(\infty) f'(\infty) e^{-\int_0^\infty \frac{\mu(v) + 0.5 s'(v)}{0.5 s(v)} dv} - \frac{1}{2} s(0) f'(0)\right] \\ &= -\frac{1}{2} s(0) \alpha f'(0). \end{aligned}$$

If we plug in $f(w) = w$ into the BAR (78), we will get $\frac{\mathbb{E}_\pi[Y(t)]}{t} = \frac{1}{2} s(0) \alpha$. This proves the lemma.

Proof of Proposition 2: We start with Lemma 1 and substitute state-dependent variance and drift as

$$\begin{aligned} s(w) &= \lambda m^2 (c_a^2 + c_s^2) \\ \beta(w) &= \begin{cases} \theta(w/m_e) = -\lambda m \left.\frac{d\log f(x)}{dx}\right|_{x=\frac{w}{m_e}} & w \le K \cdot m_e \\ \theta(K) = -\lambda m \left.\frac{d\log f(x)}{dx}\right|_{x=K} & w > K \cdot m_e \end{cases} \end{aligned}$$

To obtain a further simplification, we use our assumption that $\frac{d\log f(x)}{dx}$ is a constant for $x \ge K$, and therefore

$$\beta(w) = -\lambda m \left.\frac{d\log f(x)}{dx}\right|_{x=\frac{w}{m_e}}, \quad \forall\ w \in [0, \infty)$$

We then get

$$\begin{aligned} \mathbf{Pr}[W^*(\infty) \le w] &= \frac{\alpha'}{\lambda m^2 (c_a^2 + c_s^2)} \int_0^w e^{\frac{2}{\lambda m^2 (c_a^2 + c_s^2)} \int_0^u \lambda m d\log f(v/m_e)} du \\ &= \frac{\alpha'}{\lambda m^2 (c_a^2 + c_s^2)} \int_0^w e^{\frac{2\lambda m m_e}{\lambda m^2 (c_a^2 + c_s^2)} \int_0^{u/m_e} d\log f(z)} du \\ &= \frac{\alpha''}{\lambda m^2 (c_a^2 + c_s^2)} \int_0^w e^{\frac{1+c_s^2}{c_a^2+c_s^2} \log f(u/m_e)} du \\ &= \frac{\alpha''}{\lambda m^2 (c_a^2 + c_s^2)} \int_0^w f\left(\frac{u}{m_e}\right)^{\frac{1+c_s^2}{c_a^2+c_s^2}} du \\ &= \alpha \int_0^{\frac{w}{m_e}} f(u)^{\frac{1+c_s^2}{c_a^2+c_s^2}} du \end{aligned}$$

which proves (29).

From (25) and the continuous mapping theorem

$$X^*(\infty) = \frac{W^*(\infty) \wedge K m_e}{m_e} + \frac{(W^*(\infty) - K m_e)^+}{m}. \tag{79}$$

It now follows that

$$\mathbf{Pr}[X^*(\infty) \le x] = \begin{cases} \mathbf{Pr}[W^*(\infty) \le x m_e] & x \le K \\ \mathbf{Pr}[W^*(\infty) \le K m_e + (x - K) m] & x > K \end{cases}$$

which, together with (29), gives (30).

To find $\mathbf{E}[X^*(\infty)]$, we will find it convenient to start with (29) and rewrite it as

$$\mathbf{Pr}\left[\frac{W^*(\infty)}{m_e} \le z\right] = \alpha \int_0^z f(x)^{\frac{c_s^2+1}{c_s^2+c_a^2}} dx. \tag{80}$$

Therefore, $f(x)^{\frac{c_s^2+1}{c_s^2+c_a^2}}$ is the density of $\frac{W^*(\infty)}{m_e}$. Now we again use the map (79) to write

$$\begin{aligned} \mathbf{E}[X^*(\infty)] &= \mathbf{E}\left[\frac{W^*(\infty)}{m_e} \wedge K\right] + \frac{m_e}{m} \mathbf{E}\left[\left(\frac{W^*(\infty)}{m_e} - K\right)^+\right] \\ &= \frac{\int_0^\infty (x \wedge K) f(x)^{\frac{c_s^2+1}{c_s^2+c_a^2}} dx}{\int_0^\infty f(x)^{\frac{c_s^2+1}{c_s^2+c_a^2}} dx} + \frac{c_s^2 + 1}{2} \frac{\int_0^\infty (x - K)^+ f(x)^{\frac{c_s^2+1}{c_s^2+c_a^2}} dx}{\int_0^\infty f(x)^{\frac{c_s^2+1}{c_s^2+c_a^2}} dx}, \end{aligned}$$

which proves (31).

Proof of Theorem 2: This theorem essentially establishes the interchange of the steady state and heavy traffic limits for the constructed sequence of Sd-LPS models. Proving such an interchange usually involves quite a complicated analysis of a well-constructed Lyapunov function (see, for example, Gamarnik and Zeevi (2006) and Lee and Weerasinghe (2011)). Taking advantage of the existing studies, we use a coupling argument to prove the interchange for our model. The proofs for both the workload and queue length essentially follow the same argument. We only focus on the queue length in this proof.

For each r, we construct an auxiliary system which takes exactly the same arrival stream as the rth Sd-LPS system and the same initial condition. Denote

$$\mu_{\dagger}^{(r)} = \mu^{(r)}(k^{(r)}).$$

When the number of jobs in the auxiliary system is more than $k^{(r)}$, the server works at rate $\mu_{\dagger}^{(r)}$. When the number of jobs drops below $k^{(r)}$, the server works at speed 0 (in other words it completely shuts down). Without loss of generality, we assume that the initial number of jobs is larger than $k^{(r)}$. Let $Q^{(r)}(t)$ and $Q_{\dagger}^{(r)}(t)$ denote the number of jobs in the queue in the Sd-LPS and auxiliary systems, respectively. It is clear that

$$Q^{(r)}(t) < Q_{\dagger}^{(r)}(t). \tag{81}$$

Due to parallel processing, overtaking can happen in each system, i.e., the jth arriving job may leave the system earlier than the ith arriving job even if $j > i$. However, due to the coupling, the ith arriving job in the auxiliary system can never enter service earlier than the corresponding job in the Sd-LPS system.

By condition (17), $\mu_{\dagger}^{(r)} > \lambda m$ for all large enough r. So both $Q^{(r)}$ and $Q_{\dagger}^{(r)}$ are stationary. Let $\pi^{(r)}$ denote the stationary probability measure of the diffusion-scaled process $\hat{Q}^{(r)}$. Similarly, Let $\pi_{\dagger}^{(r)}$ denote the stationary probability measure of the diffusion-scaled queue length $\hat{Q}_{\dagger}^{(r)}$ in the coupled system. The key step to showing that $X^{(r)}(\infty) \Rightarrow X^{*}(\infty)$ as $r \to \infty$ is to show that the family of probability measures $\{\pi^{(r)}\}_{r\in\mathbb{N}}$ is tight. (Since $\hat{X}^{(r)}(t) \le \hat{Q}^{(r)}(t) + k^{(r)}/r$, studying only the queue length suffices.) Readers can refer to the proof of Theorem 8 in Gamarnik and Zeevi (2006) for a standard argument of how to prove the convergence using tightness. We now focus on proving the tightness of probability measures $\{\pi^{(r)}\}_{r\in\mathbb{N}}$.

We can model the rth auxiliary system as if it has $k^{(r)}$ identical servers. All the servers either work or stop in perfect synchronization. Denote by $S_{\dagger,i}^{(r)}(\cdot)$, $i = 1, \ldots, k^{(r)}$, independent renewal processes with inter-renewal time following distribution $G(\cdot/k^{(r)})$, where G is the distribution of job sizes. In other words, the inter-renewal time has mean $mk^{(r)}$ and SCV c_s^2. The queueing dynamics of the rth auxiliary system can be written as

$$Q_{\dagger}^{(r)}(t) = Q_{\dagger}^{(r)}(0) + \Lambda^{(r)}(t) - \sum_{i=1}^{k^{(r)}} S_{\dagger,i}^{(r)}(B_{\dagger}^{(r)}(t)),$$

where $B_{\dagger}^{(r)}(t)$ is the cumulative busy time for each of the servers. Applying the diffusion scaling, we have

$$\hat{Q}_{\dagger}^{(r)}(t) = \hat{Q}_{\dagger}^{(r)}(0) + \hat{\Lambda}^{(r)}(t) - \sum_{i=1}^{k^{(r)}} \hat{S}_{\dagger,i}^{(r)}(\frac{1}{r^2} B_{\dagger}^{(r)}(r^2 t)) + r(\lambda - \mu_{\dagger}^{(r)}/m)t + \frac{\mu_{\dagger}^{(r)}}{rm}(r^2 t - B^{(r)}(r^2 t)) \tag{82}$$

where

$$\hat{\Lambda}^{(r)}(t) = \frac{1}{r}\left(\Lambda^{(r)}(r^2 t) - \lambda r^2 t\right), \quad \hat{S}^{(r)}_{\dagger,i} = \frac{1}{r}\left(S^{(r)}_{\dagger,i}(r^2 t) - \frac{\mu^{(r)}_{\dagger} r^2}{mk^{(r)}} t\right).$$

Note that $r^2 t - B^{(r)}(r^2 t)$ increases only when $\hat{Q}^{(r)}_{\dagger}(t) = 0$, so (82) is the same as the Skorohod mapping for the $G/G/1$ queue except that the service process is the superposition of $k^{(r)}$ renewal processes with a much lower speed (roughly $1/k^{(r)} \approx 1/r$) rather than a single renewal process. We now take advantage of the tools developed in Lee and Weerasinghe (2011) by verifying that the processes $\hat{\Lambda}^{(r)}(t)$ and $\hat{S}^{(r)}_{\dagger,i}$ satisfy condition (A8.p) there. That is, we want to show

$$\mathbb{E}\left[\sup_{0\le s\le t} |\hat{\Lambda}^{(r)}(s)|^2\right] < C(1+t), \tag{83}$$

$$\mathbb{E}\left[\sup_{0\le s\le t} |\hat{S}^{(r)}_{\dagger,i}(s)|^2\right] < \frac{C}{r}(1+t). \tag{84}$$

Condition (83) directly follows from assumption (14), following the same argument as in Lee and Weerasinghe (2011) (essentially using Lemma 3.5 in Budhiraja and Ghosh (2006)). To verify (84), we need to further investigate the proof of Lemma 3.5 in Budhiraja and Ghosh (2006). It follows from (3.31) and (3.32) in the proof that the result of Lemma 3.5 can be enhanced as follows: The right-hand side of the second inequality in (3.20), which is $C^*(1+t)$, can be replaced by $\frac{2}{r} + \frac{C_2}{r^2} + 3C_2 t$. Since our renewal process $S^{(r)}_{\dagger,i}$ has speed $1/k^{(r)}$ rather than 1, by time change, we can replace the time t by $\frac{t}{k^{(r)}}$. So $\mathbb{E}\left[\sup_{0\le s\le t} |\hat{S}^{(r)}_{\dagger,i}(s)|^2\right] < \frac{2}{r} + \frac{C_2}{r^2} + 3C_2 \frac{t}{k^{(r)}} \le \frac{3}{r} + 6KC_2 \frac{t}{r}$ for all large enough r. This proves (84). Then following exactly the same argument, Theorem 3.3 in Lee and Weerasinghe (2011) holds for our problem. This implies Theorem 3.2 in Lee and Weerasinghe (2011), i.e., $\sup_r \int_0^\infty w\pi^{(r)}_{\dagger}(dw) < \infty$. By the coupling construction (81), we have $\int_0^\infty x\pi^{(r)}(dx) < \int_0^\infty x\pi^{(r)}_{\dagger}(dx)$. This implies tightness of $\{\pi^{(r)}\}_{r\in\mathbb{N}}$.

Appendix C: State Space Collapse for the Sd-LPS system

We introduce a strengthened version of the mapping Δ_K as the follows. Let $\Delta_{K,\nu} : \mathbb{R}_+ \to \mathbf{M} \times \mathbf{M}$ be the lifting map associated with the probability measure ν and constant K given by

$$\Delta_{K,\nu} w = \Big(\frac{(w - K\beta_e)^+}{\beta}\nu, \frac{w \wedge K\beta_e}{\beta_e}\nu_e\Big) \quad \text{for } w \in \mathbb{R}_+.$$

We aim to prove the following full version of the SSC

Theorem 3 **(Full State Space Collapse)**. *Under the conditions* (14)–(16) *and* (19)–(21)*, for any* $T > 0$*,*

$$\sup_{t\in[0,T]} \mathbf{d}[(\hat{\mathcal{Q}}^{(r)}(t), \hat{\mathcal{Z}}^{(r)}(t)), \Delta_{K,\nu}\hat{W}^{(r)}(t)] \Rightarrow 0 \qquad \text{as } r \to \infty.$$

It is clear that Theorem 3 implies Proposition 1. The rest of this section is devoted to the proof of the full SSC.

C.1. Tightness of Shifted Fluid-Scaled Processes

The key to proving SSC, which was originally developed by Bramson (1998), is to "chop" the diffusion-scaled processes into pieces.

Shifted Fluid Scaling Introduce,

$$\bar{\mathcal{Q}}^{(r,l)}(t) = \frac{1}{r}\mathcal{Q}^{(r)}(rl+rt), \quad \bar{\mathcal{Z}}^{(r,l)}(t) = \frac{1}{r}\mathcal{Z}^{(r)}(rl+rt), \tag{85}$$

for all $m \in \mathbb{N}$ and $t \geq 0$. To see the relationship between these two scalings, consider the diffusion-scaled process on the interval $[0,T]$, which corresponds to the interval $[0, r^2T]$ for the unscaled process. Fix a constant $L > 1$, the interval will be covered by $\lfloor rT \rfloor + 1$ overlapping intervals

$$[rl, rl+rL] \qquad l = 0, 1, \cdots, \lfloor rT \rfloor.$$

For each $t \in [0,T]$, there exists an $l \in \{0, \cdots, \lfloor rT \rfloor\}$ and $s \in [0, L]$ (which may not be unique) such that $r^2 t = rl + rs$. Thus

$$\hat{\mathcal{Q}}^{(r)}(t) = \bar{\mathcal{Q}}^{(r,l)}(s), \quad \hat{\mathcal{Z}}^{(r)}(t) = \bar{\mathcal{Z}}^{(r,l)}(s). \tag{86}$$

This will serve as a key relationship between fluid and diffusion-scaled processes.

The quantities $Q^{(r)}(\cdot)$, $Z^{(r)}(\cdot)$, $X^{(r)}(\cdot)$, $W^{(r)}(\cdot)$ are essentially functions of $(\mathcal{Q}^{(r)}(\cdot), \mathcal{Z}^{(r)}(\cdot))$, so the scaling for these quantities is defined as the functions of the corresponding scaling for $(\mathcal{Q}^{(r)}(\cdot), \mathcal{Z}^{(r)}(\cdot))$. For example

$$\bar{W}^{(r,l)}(t) = \langle \chi, \bar{\mathcal{Q}}^{(r,l)}(t) + \bar{\mathcal{Z}}^{(r,l)}(t) \rangle = \frac{1}{r} W^{(r)}(rl+rt).$$

We define the shifted fluid scaling for the arrival process as

$$\bar{\Lambda}^{(r,l)}(t) = \frac{1}{r}\Lambda^{(r)}(rl+rt),$$

for all $t \geq 0$. By (6), the shifted fluid scaling for $B^{(r)}(\cdot)$ is

$$\bar{B}^{(r,l)}(t) = \bar{E}^{(r,l)}(t) - \bar{Q}^{(r,l)}(t),$$

for all $t \geq 0$. A shifted fluid-scaled version of the stochastic dynamic equations (4) and (5) can be written as, for any $A \subset (0, \infty)$, $0 \leq s \leq t$,

$$\begin{aligned} \bar{\mathcal{Q}}^{(r,l)}(t)(A) &= \bar{\mathcal{Q}}^{(r,l)}(s)(A) + \frac{1}{r} \sum_{i = r\bar{E}^{(r,l)}(s)+1}^{r\bar{E}^{(r,l)}(t)} \delta_{v_i}(A) \\ &\quad - \frac{1}{r} \sum_{i = r\bar{B}^{(r,l)}(s)+1}^{r\bar{B}^{(r,l)}(t)} \delta_{v_i}(A), \end{aligned} \tag{87}$$

$$\begin{aligned} \bar{\mathcal{Z}}^{(r,l)}(t)(A) &= \bar{\mathcal{Z}}^{(r,l)}(s)(A + S^{(r)}(rl+rs, rl+rt)) \\ &\quad + \frac{1}{r} \sum_{i = r\bar{B}^{(r,l)}(s)+1}^{r\bar{B}^{(r,l)}(t)} \delta_{v_i^{(r)}}(A + S^{(r)}(\tau_i^{(r)}, rl+rt)). \end{aligned} \tag{88}$$

We point out that the cumulative service process $S^{(r)}$ is never scaled because it tracks the amount of service received by each individual customer. However, via some algebra we can see that

$$S^{(r)}(rl+rs, rl+rt) = \int_{rl+rs}^{rl+rt} \frac{\mu^{(r)}(Z^{(r)}(\tau))}{Z^{(r)}(\tau)} d\tau = \int_s^t \frac{\mu^{(r)}(r\bar{Z}^{(r,l)}(\tau))}{\bar{Z}^{(r,l)}(\tau)} d\tau. \tag{89}$$

This gives two interesting observations. First, the shifted fluid scaling is essentially fluid scaling, meaning the shifted fluid-scaled processes should be close to some fluid model solutions. Second, the corresponding fluid model is essentially the same as the fluid model in Zhang et al. (2011) since by (16),

$$\mu^{(r)}(r\bar{Z}^{(r,l)}(\tau)) = 1 + O^+(\frac{1}{r}),$$

where $O^+(1/r)$ means the quantity is positive and of the same order as $1/r$ when $r \to \infty$. So

$$S^{(r)}(rl + rs, rl + rt) = \int_s^t \frac{1}{\bar{Z}^{(r,l)}(\tau)} d\tau + O^+(\frac{1}{r}). \tag{90}$$

Intuitively, $\bar{Z}^{(r,l)}$ is close to some fluid limit denoted by $\tilde{Z}$ as r becomes very large (in the mathematical sense of convergence in probability), then

$$S^{(r)}(rl + rs, rl + rt) \Rightarrow \int_s^t \frac{1}{\tilde{Z}(\tau)} d\tau. \tag{91}$$

So we can conclude that the underlying fluid is the same as the one for the regular LPS system. Thus, we can use existing properties developed in Zhang et al. (2011). We hope to make the argument rigorous and concise in the follows.

Some Bound Estimation

The tightness property, which guarantees that the shifted fluid-scaled process $\{\bar{\mathcal{Q}}^{(r,l)}, \bar{\mathcal{Z}}^{(r,l)}\}$ has a convergent subsequence, can be proved in a similar way as in Zhang et al. (2011). There are two key differences. First is the service process as pointed out before. Second is that Zhang et al. (2011) heavily relies on the known result on the diffusion of the workload (see Proposition 2.1). However, we do not have such a diffusion limit of workload a priori. Instead, we try to prove such a diffusion limit by SSC. Looking into the details of the machinery in Zhang et al. (2011), what essentially is needed for the workload process is some kind of stochastic bound, which we prove in the following lemma.

Lemma 3 **(An Upper Bound of the Workload)**. *For any $\eta > 0$ there exists a constant M such that*

$$\mathbb{P}\left(\max_{l \le rT} \sup_{t\in[0,L]} \bar{W}^{(r,l)}(t) < M\right) > 1 - \eta. \tag{92}$$

Proof: Using the relationship between the shifted fluid scaling and diffusion scaling, we essentially need to prove that

$$\mathbb{P}\left(\sup_{t\in[0,L]} \hat{W}^{(r)}(t) < M\right) > 1 - \eta.$$

Recall the representation (64) for the diffusion-scaled workload processes. Let $\underline{\theta} = \inf_{x\in[0,K]} \theta(x)$, which is finite due to condition (16), so the process $\hat{W}_1^{(r)}$ satisfying

$$\hat{W}_1^{(r)}(t) = \hat{W}^{(r)}(0) - \underline{\theta}t + \hat{M}_s^{(r)}(t) + \hat{M}_a^{(r)}(t) + \hat{G}_1^{(r)}(t) + \hat{G}_2^{(r)}(t) + r\int_0^t 1_{\{\hat{W}_1^{(r)}(s)=0\}} ds$$

is an upperbound of $\hat{W}^{(r)}$ due to the definition of $\underline{\theta}$ and condition (17). By Lemma 2, $\hat{W}_1^{(r)}$ converges to a driftless RBM, which is stochastically bounded. This implies the result.

Such a stochastic bound of the workload process helps to establish some useful bound estimates for the stochastic processes underlying the Sd-LPS model.

Lemma 4 **(Further Bound Estimations)**. *For any $\eta > 0$, there exists a constant $M > 0$ and a probability event $\Omega^r_B(M)$ for each index r such that*

$$\liminf_{r\to\infty} \mathbb{P}\left(\Omega^r_B(M)\right) \geq 1-\eta, \tag{93}$$

and on the event $\Omega^r_B(M)$, we have

$$\max_{l\leq \lfloor rT\rfloor} \sup_{t\in[0,L]} \bar{Q}^{(r,l)}(t) \leq M, \tag{94}$$

$$\max_{l\leq \lfloor rT\rfloor} \sup_{t\in[0,L]} \langle \chi^{1+p}, \bar{\mathcal{Q}}^{(r,l)}(t) + \bar{\mathcal{Z}}^{(r,l)}(t)\rangle \leq M. \tag{95}$$

Proof: The result (94) holds due to Lemma 4.2 in Zhang et al. (2011), which only utilizes the regularity of the arrival process (14) and the stochastic bound (92) for the workload process proved in Lemma 3. For (95), the first half, $\max_{l\leq\lfloor rT\rfloor}\sup_{t\in[0,L]}\langle\chi^{1+p}, \bar{\mathcal{Q}}^{(r,l)}(t)\rangle \leq M$, also follows the same reasoning as Lemma 4.3 in Zhang et al. (2011). Essentially, any results for the "queue" part follows the same argument in Zhang et al. (2011).

The challenge with the state-dependent service rate lies in the analysis of the server. It follows from the shifted fluid-scaled dynamic equation (88) that for any Borel set $A\subset(0,\infty)$,

$$\begin{aligned}\frac{1}{r}\mathcal{Z}^{(r)}(rl+rt)(A) &= \frac{1}{r}\mathcal{Z}^{(r)}(0)(A+S^{(r)}(0,rl+rt)) \\ &\quad + \sum_{j=0}^{m-1}\frac{1}{r}\sum_{i=B^{(r)}(r(l-j-1))+1}^{B^{(r)}(r(l-j))} \delta_{v_i}(A+S^{(r)}(\tau_i^{(r)},rl+rt)) \\ &\quad + \frac{1}{r}\sum_{i=B^{(r)}(rl)+1}^{B^{(r)}(rl+rt)} \delta_{v_i}(A+S^{(r)}(\tau_i^{(r)},rl+rt)).\end{aligned}$$

Given $0\leq j\leq m-1$, for those i's with $B^{(r)}(r(l-j-1)) < i \leq B^{(r)}(r(l-j))$ we have

$$\tau_i^{(r)} \in [r(l-j-1), r(l-j)].$$

For the sake of simplicity, let us assume that $Z^{(r)}(s) > 0$ for all $s\in[0,rl+rt]$. If this does not hold, we can use a technical trick presented in the proof of Lemma 4.3 in Zhang et al. (2011) to deal with it. Here we show the main difference coming from the state-dependent service rate. By (90) and the fact that $Z^{(r)}\leq k^{(r)}$, we have a lower bound on the cumulative service amount

$$S^{(r)}(rs,rt) \geq \int_{rs}^{rt}\frac{1}{Z^{(r)}(s)}ds \geq \frac{r(t-s)}{k^{(r)}}. \tag{96}$$

Thus,

$$S^{(r)}(\tau_i^{(r)}, rl+rt) \geq S^{(r)}(r(l-j),rl) \geq \frac{rj}{k^{(r)}} \geq \frac{j}{2K},$$

for all large r where the last inequality is due to (9). For those i's such that $\tau_i^{(r)}$ is larger than $B^{(r)}(rl)$, we use the trivial lower bound $S^{(r)}(\tau_i^{(r)}, rl+rt)\geq 0$. Also take the trivial lower bound that $S^{(r)}(0,rl+rt)\geq 0$. Then we have the following inequality on the $(1+p)$th moment:

$$\begin{aligned}\langle\chi^{1+p}, \frac{1}{r}\mathcal{Z}^{(r)}(rl+rt)\rangle &\leq \langle\chi^{1+p}, \frac{1}{r}\mathcal{Z}^{(r)}(0)\rangle \\ &\quad + \sum_{j=0}^{m-1}\langle\big((\chi-\frac{j}{2K})^+\big)^{1+p}, \frac{1}{r}\sum_{i=B^{(r)}(r(l-j-1))+1}^{B^{(r)}(r(l-j))}\delta_{v_i}\rangle \\ &\quad + \langle\chi^{1+p}, \frac{1}{r}\sum_{i=B^{(r)}(rl)+1}^{B^{(r)}(rl+rt)}\delta_{v_i}\rangle.\end{aligned} \tag{97}$$

This is the same as (4.22) in Zhang et al. (2011). The estimation of the first term on the right-hand side in the above follows directly from the initial condition (20). The analysis of the second and third terms follows the same way as in Zhang et al. (2011).

To prove that a family of measure-valued processes is tight, there are three properties to verify, namely *Compact Containment*, *Asymptotic Regularity* and *Oscillation Bound*. For brevity, we will not repeat the exact mathematical statements and their proofs. For the LPS system, these three properties were proved in Lemmas 4.4–4.6 in Zhang et al. (2011). We just point out that the proof for the above mentioned three properties for the Sd-LPS system relies on (*a*) the bound estimate in Lemma 4; and (*b*) the fact that (90) implies the lower bound of the cumulative service process (96). The proof of Lemma 4 has demonstrated point (b) clearly, we therefore omit a repeat of the argument used in Zhang et al. (2011). So we reach the conclusion:

PROPOSITION 7 **(Tightness of Shifted Fluid-scaled Processes)**. *The family of shifted fluid-scaled processes* $\{(\bar{\mathcal{Q}}^{(r,l)}, \bar{\mathcal{Z}}^{(r,l)}\}_{l\le rT, r\in\mathbb{N}}$ *is tight.*

Loosely speaking, tightness means that any subsequence from the family of shifted fluid-scaled processes has a convergent subsequence. This is formally stated in Theorem 4.1 in Zhang et al. (2011).

C.2. Bramson's Framework for SSC

Sd-LPS and LPS essentially use the same measure-valued framework. The difference lies in the cumulative service process as we explained when deriving (90) and the workload process as we studied in Lemma 3. After obtaining the tightness, we can apply the framework invented by Bramson Bramson (1998) in the same way as how Section 5 in Zhang et al. (2011) applies it to the measure-valued process. The high level-logic is as the follows: the shifted fluid-scaled processes are "close" to the fluid model solution, and the fluid model solution converges to some invariant which exhibits SSC (Theorem 3.1 in Zhang et al. (2011)). Thus SSC, which happens on the diffusion scaling, can be proved based on the relationship (86) between diffusion scaling and shifted fluid scaling. We thus refer to Section 5 in Zhang et al. (2011) for the proof of Theorem 3.

Appendix D: Analysis of Algorithms for Finding Optimal Control

Recall some notation and definitions used in this section.

$$\begin{aligned}
\hat{k} &= \arg\max_{k} \theta(k)\\
\hat{\theta} &= \theta(\hat{k})\\
\Delta_k(w) &= \frac{w}{m} + k\left(1 - \frac{m_e}{m}\right)\\
d_\theta &= \sup_k \theta(k) - \inf_k \theta(k)\\
k_f(w) &= \arg\max_{k\in[0,w/m_e]} \theta(k_f)
\end{aligned}$$

Throughout this section, we assume that d_θ is finite, and therefore $\theta(k)$ is bounded from above and below.

D.1. Some Auxiliary Results

We first provide some auxiliary results (Lemmas 5 and 6) which will be useful in proving the results in Section 4.

Lemma 5. *Consider the solution of the following ODE, parameterized by v and W:*

Terminal condition:

$$G_{v,W}(w) = \alpha w + \beta v + \gamma \qquad \ldots w \geq \max\{W, \hat{k} m_e\}$$

ODE:

$$\begin{aligned} v &= \frac{w}{m} + k_f(w)\left(1 - \frac{m_e}{m}\right) - \theta(k_f(w)) G_{v,W}(w) + \frac{\sigma^2}{2} G'_{v,W}(w) && \ldots w \in [W, \hat{k} m_e] \\ v &= \min_{k \in [0, w/m_e]} \left\{ \frac{w}{m} + k\left(1 - \frac{m_e}{m}\right) - \theta(k) G_{v,W}(w) + \frac{\sigma^2}{2} G'_{v,W}(w) \right\} && \ldots w \in [0, W] \end{aligned}$$

Then $G_{v,W}(w)$ is continuous in both v and W for all w.

Let (v_a, W_a) and (v_b, W_b) denote two parameter settings, and for succinctness, denote the corresponding solutions to the ODE as G_a and G_b, respectively. We will consider the case $W_a, W_b \geq \hat{k} m_e$ as other cases are analogous.

Let $W_a \leq W_b$.

At $w = W_b$, we have

$$|G_a(W_b) - G_b(W_b)| = \beta |v_a - v_b|. \tag{98}$$

For $w \in [W_a, W_b]$, we have

$$\begin{aligned} G_a(w) &= \alpha w + \beta v_a + \gamma, && (99) \\ G'_b(w) &= \frac{2}{\sigma^2}\left(v_b + \theta(k_b(w)) G_b(w) - \frac{w}{m} + k_b\left(1 - \frac{m_e}{m}\right)\right), && (100) \end{aligned}$$

which gives

$$\frac{2}{\sigma^2}\left(v_b - \frac{W_b}{m \wedge m_e} - d_\theta G_b(w)\right) \leq G'_b(w) \leq \frac{2}{\sigma^2}\left(v_b - \frac{W_a}{m \vee m_e} + d_\theta G_b(w)\right). \tag{101}$$

Since the derivatives are bounded, $G_b(w)$ is bounded in the interval $[W_a, W_b]$. Let $D = \sup_{w \in [W_a, W_b]} |G_b(w)|$. Then,

$$\begin{aligned} |G_a(w) - G_b(w)| &\leq |G_a(w) - G_a(W_a)| + |G_a(W_b) - G_b(W_b)| + |G_b(w) - G_b(W_b)| && (102) \\ &\leq \alpha |W_b - w| + \beta |v_a - v_b| + (W_b - w)\frac{2}{\sigma^2}\left(v_b + \frac{W_b}{m \wedge m_e} + d_\theta D\right) && (103) \\ &\leq \alpha |W_b - W_a| + \beta |v_a - v_b| + (W_b - W_a)\frac{2}{\sigma^2}\left(v_b + \frac{W_b}{m \wedge m_e} + d_\theta D\right), && (104) \end{aligned}$$

which goes to 0 as $|v_a - v_b| + |W_a - W_b| \to 0$.

For $w \in [0, W_a]$, by Lemma 7,

$$|G_a'(w) - G_b'(w)| \leq \frac{2}{\sigma^2}|v_a - v_b| + \frac{2}{\sigma^2}\left| \min_{k_a \in [0, w/m_e]} (k_b(1 - m_e/m) - \theta(k_a)G_a(w)) \right.$$
$$\left. - \min_{k_b \in [0, w/m_e]} (k_b(1 - m_e/m) - \theta(k_b)G_2(w)) \right|$$
$$\leq \frac{2}{\sigma^2}|v_a - v_b| + \frac{2d_\theta}{\sigma^2}|G_a(w) - G_b(w)|. \tag{105}$$

Applying Gronwall's inequality, for all $w \in [0, W_a]$

$$|G_a(w) - G_b(w)| \leq |G_a(W_a) - G_b(W_a)| e^{\frac{2d_\theta}{\sigma^2}(W_a - w)} + \frac{|v_a - v_b|}{d_\theta}\left(e^{\frac{2d_\theta}{\sigma^2}(W_a - w)} - 1\right),$$

which, together with (104), implies that for all $w \in [0, W_a]$

$$|G_a(w) - G_b(w)| \leq |v_a - v_b| \left(|\beta| e^{\frac{2d_\theta}{\sigma^2}(W_a - w)} + \frac{e^{\frac{2d_\theta}{\sigma^2}(W_a - w)} - 1}{d_\theta} \right)$$
$$+ |W_b - W_a| \left(\alpha + \frac{2}{\sigma^2}\left(v_b + \frac{W_b}{m \wedge m_e} + d_\theta D \right) \right) e^{\frac{2d_\theta}{\sigma^2}(W_a - w)},$$

which goes to 0 as $|v_a - v_b| + |W_a - W_b| \to 0$.

Lemma 6. *Consider $G_{v,W}$ defined in Lemma 5 for a given $W \geq \hat{k}m_e$. Then $G_{v,W}(w)$ is monotonic and Lipschitz continuous in v for all w.*

Fix $W \geq \hat{k}m_e$, and consider $v_a > v_b$. Let G_a and G_b denote the solutions of the ODE defined in Lemma 5 for v_a and v_b, respectively. We will show that $G_a(w) < G_b(w)$ for all $w \geq 0$. We rely on the following two facts:

1. Terminal condition:

$$G_b(w) - G_a(w) = -\beta(v_a - v_b) \quad w \geq W$$

2. Bounds on $G_b'(w) - G_a'(w)$ for $w \in [0, W]$:

$$G_b'(w) - G_a'(w) = -\frac{2}{\sigma^2}\left[(v_a - v_b) - \min_{k \in [0, w/m_e]} (\Delta_k(w) - \theta(k)G_a(w)) - \min_{k \in [0, w/m_e]} (\Delta_k(w) - \theta(k)G_b(w))\right]$$

where recall that $\Delta_k(w) = \frac{w}{m} + k\left(1 - \frac{m_e}{m}\right)$. Under the assumption $G_a(w) \leq G_b(w)$, from Lemma 7:

$$-\frac{2}{\sigma}\left[(v_a - v_b) + d_\theta(G_b(w) - G_a(w))\right] \leq G_b'(w) - G_a'(w) \leq -\frac{2}{\sigma^2}\left[(v_a - v_b) - d_\theta\left(G_b(w) - G_a(w)\right)\right] \tag{106}$$

Combining these two facts, we get for any $w \in [0, W]$

$$(v_a - v_b)\left[-\beta + \frac{1}{d_\theta}\left(1 - e^{-\frac{2d_\theta W}{\sigma^2}}\right)\right] \leq G_b(w) - G_a(w) \leq (v_a - v_b)\left[-\beta + \frac{1}{d_\theta}\left(e^{\frac{2d_\theta W}{\sigma^2}} - 1\right)\right] \tag{107}$$

Lemma 7. *Let $x_1 = \arg\min_{x \in [u,v]} f_1(x)$ and $x_2 = \arg\min_{x \in [u,v]} f_2(x)$. Then,*

$$|f_1(x_1) - f_2(x_2)| \leq \sup_{x \in [u,v]} |f_1(x) - f_2(x)|$$

Proof Since $f_1(x_1) \leq f_1(x_2)$ and $f_2(x_2) \leq f_2(x_1)$,

$$f_1(x_1) - f_2(x_1) \leq f_1(x_1) - f_2(x_2) \leq f_1(x_2) - f_2(x_2)$$

and therefore, $|f_1(x_1) - f_2(x_2)| \leq \sup_{x \in [u,v]} |f_1(x) - f_2(x)|$.

D.2. Proofs of Results in Section 4

Proof of Proposition 3: We should point out that the monotonicity of the value function is not immediate because under the optimal policy $k^*(\cdot)$, the state-dependent cost function $\Delta_{k^*}(w)$ need not be monotonic in w. If it were, a simple sample path coupling argument could be used to deduce the monotonicity of the discounted value function by considering initial workloads $w_1 \le w_2$.

Let $k^*_\gamma(\cdot)$ be the optimal policy minimizing expected discounted cost, and $V_\gamma(w)$ be the corresponding value function. Consider $w_1 \le w_2$. We will create an alternate control policy π_1 when the initial workload is w_1, and denote the corresponding expected discounted cost by $\tilde{v}_1$. We will then show that $\tilde{v}_1 \le V_\gamma(w_2)$ (in fact, our construction involves stochastic coupling and implies that the discounted reward starting with w_1 and using π_1 is stochastically smaller than the discounted reward starting with w_2 and using $k^*_\gamma(\cdot)$).

Construction of π_1: We simulate two independent systems in parallel: system 1 with initial workload $W_1(0) = w_1$ under control policy π_1 (which we will describe shortly); and system 2 with initial workload $W_2(0) = w_2$ under the optimal control policy $k^*_\gamma(w)$. The control at time t under π_1 is chosen to be

$$k_{\pi_1}(t) = \operatorname*{arg\,min}_{k\in[0,W_1(t)/m_e]} \frac{W_1(t)}{m} + k(1 - m_e/m)$$

for $t \in [0,\tau]$, where $\tau \doteq \min\{s \ge 0 : W_1(s) = W_2(s)\}$ is the coupling time of the two systems. That is, τ is the first time the workloads of the two coupled processes W_1 and W_2 coincide. For $t \ge \tau$, $k_{\pi_1}(t) = k^*_\gamma(W_1(t))$.

It is easy to see that since W_1 and W_2 have continuous sample paths, $W_1(t) \le W_2(t)$ for $t \le \tau$. Due to the choice of k_{π_1}, this further implies that

$$\begin{aligned}\min_{k\in[0,W_1(t)/m_e]} \left(\frac{W_1(t)}{m} + k\left(1 - \frac{m_e}{m}\right)\right) &= \min\left\{\frac{W_1(t)}{m}, \frac{W_1(t)}{m_e}\right\}\\ &\le \min\left\{\frac{W_2(t)}{m}, \frac{W_2(t)}{m_e}\right\}\\ &\le \frac{W_2(t)}{m} + k^*_\gamma(W_2(t))\left(1 - \frac{m_e}{m}\right).\end{aligned}$$

For $t \ge \tau$, $W_1(t)$ is stochastically equal to $W_2(t)$. Therefore, the discounted cost of π_1 (with initial workload w_1) is stochastically smaller than the discounted cost of k^*_γ (with initial workload w_2). This implies $\tilde{v}_1 \le V_\gamma(w_2)$, but $V_\gamma(w_1) \le \tilde{v}_1$ (since $V_\gamma(w_1)$ is the optimal expected discounted cost). Therefore, $V_\gamma(w_1) \le V_\gamma(w_2)$ when $w_1 \le w_2$.

Since $(V_\gamma(w_2) - V_\gamma(w_1)) \ge 0$ for all γ, this also holds as $\gamma \downarrow 0$.

Note : The only facts we relied on to argue monotonicity were (i) continuity of sample paths, and (ii) the cost of the cheapest action available in each state is monotonic in w. These appear to be weaker than the conditions typically used in the literature where the set of available actions is assumed to be independent of the state. Further, the cost is assumed to be non-decreasing in the state variable for each action. ■

We now provide the proofs of Proposition 4 and 6 for the analysis of our algorithms. We omit the proof of Proposition 5 as it mirrors the proof of Proposition 4.

Proof of Proposition 4: Consider the diffusion control formulation for the Sd-LPS system but with a finite workload buffer of W. For the diffusion corresponding to this loss system, we have reflections at both

$w = 0$ and $w = W$. Therefore, for any policy for this loss system, the value function gradient is 0 at both these values Mandl (1968):

$$G(0) = G(W) = 0.$$

Therefore, (46) defines the HJB equation for the value function gradient of the finite buffer system with workload buffer W, together with $G_v(0) = 0$ and the additional boundary condition $G_v(W) = 0$. Lemma 5 guarantees that ODE (46) has a unique solution (by choosing the terminal condition $G_{v,W}(W) = 0$).

We first show that for all $v < v^*$, there is a *unique* value of W such that v is the average cost of the optimal finite buffer policy with workload buffer W.

Consider an arbitrary pair W, v and solve the following ODE

$$v = \min_{k\in[0,w/m_e]} \left\{ \frac{w}{m} + k\left(1 - \frac{m_e}{m}\right) - \theta(k) G_{v,W}(w) \right\} + \frac{\sigma^2}{2} G'_{v,W}(w) \tag{108}$$

backwards with terminal condition $G_{v,W}(W) = 0$ (note that this is the same ODE as (46) but we do not enforce $G_{v,W}(0) = 0$). Lemma 6 then shows that $G_{v,W}(0)$ is monotonic in v. Therefore, for each W, there exists a unique $v^*(W)$ such that $G_{v^*(W),W}(0) = 0$ for the ODE above, with terminal condition $G_{v^*(W),W}(W) = 0$. Further, Lipschitz continuity and Lemma 5 imply that the map $v^*(W)$ is continuous. From the foregoing discussion, we see that $v^*(W)$ denotes the cost of the optimal finite buffer policy with finite buffer W.

We next show that $v = v^*(W_1) \neq v^*(W_2)$ if $W_1 \neq W_2$. This would imply that two different workload buffer sizes must yield different optimal costs. Assume the contrapositive, and further $W_1 < W_2$. This implies that $G_{v,W_1}(w) = G_{v,W_2}(w)$ for $w \in [0, W_1]$ when the $G_{v,W}$ ODEs are evolved forward with initial condition $G_{v,W_1} = G_{v,W_2} = 0$. Then by (108),

$$\begin{aligned} G'_{v,W_2}(w)\big|_{w=W_1} = G'_{v,W_1}(w)\big|_{w=W_1} &= \frac{2}{\sigma^2}\left[v - \min_{k\in[0,W_1/m_e]} \Big(\Delta_k(W_1) - \theta(k) G_{v,W_1}(W_1)\Big)\right] \\ &= \frac{2}{\sigma^2}\left[v - \min_{k\in[0,W_1/m_e]} \Delta_k(W_1)\right] \\ &= \frac{2}{\sigma^2}\left[v - \min\left\{\frac{W_1}{m}, \frac{W_1}{m_e}\right\}\right] < 0. \end{aligned}$$

The last inequality is true because $\theta()$ is bounded from below, and hence for the optimal policy with buffer W_1, the average cost is strictly smaller than $\min\{W_1/m, W_1/m_e\}$. This implies $G_{v,W_2}(W_1 + \epsilon) < 0$ for any $\epsilon > 0$. A similar argument as in Proposition 3 shows that the optimal value function for the finite buffer system is monotonic and hence $G_{v,W_2}(w) \geq 0,\ w \in [0, W_2]$, which contradicts $G_{v,W_2}(W_1) = 0$ and $G'_{v,W_2}(W_1) < 0$.

Therefore, $\underline{W}(v)$ as defined in (47) (if it exists) is the unique buffer size corresponding to the optimal finite buffer policy with average cost v. To get a control on how $\underline{W}(v)$ grows as $v \uparrow v^*$ (where v^* denotes the average cost of the infinite buffer control), we will next argue that $\underline{W}(v) = O\left(\log \frac{1}{v^*-v}\right)$ (and hence also finite). We will instead prove the following equivalent result: let v^*_W denote the average cost of the optimal finite buffer control with workload buffer limit W, then $(v^* - v^*_W) = O(e^{-\beta W})$ as $W \to \infty$ for some constant $\beta > 0$.

Intuitively, the service rate of the optimal control must asymptotically approach $\hat{\theta}$ as the backlog builds up, and hence the distribution of the workload (and therefore number of jobs in the system) should decay at an exponential rate. Therefore, the effect of truncation at workload W for a finite buffer system, that is $(v^* - v^*_W)$, should also be $O(e^{-\beta W})$ for some constant $\beta > 0$.

The proof will proceed in several steps:

Step 1: For the optimal infinite buffer control, there exists a constant α such that the optimal value function gradient $G^*(w)$ satisfies

$$G^*(w) \leq \alpha + \frac{w}{m\hat{\theta}}. \tag{109}$$

Proof: Consider the upper envelope function $\overline{G}_{v^*}(\cdot)$ given by

$$\overline{G}_{v^*}(w) = \frac{w}{m\hat{\theta}} + \left(\hat{k}\left(1 - \frac{m_e}{m}\right) + \frac{\sigma^2}{2m\hat{\theta}} - v^*\right)\frac{1}{\hat{\theta}}, \quad w \geq \hat{k}m_e. \tag{110}$$

For $w \in [0, \hat{k}m_e]$, $\overline{G}_{v^*}$ is obtained by solving ODE (51) backwards starting with the terminal condition

$$\overline{G}_{v^*}(\hat{k}m_e) = \left(\hat{k} - v^* + \frac{\sigma^2}{2m\hat{\theta}}\right)\frac{1}{\hat{\theta}} \ .$$

Note that this is the same upper envelope function we use for detecting feasibility in the binary search algorithm. Now $v^* \leq v_f(0)$ implies $\overline{G}_{v^*}(0) \geq 0$ since $\overline{G}_v(0)$ is monotonically decreasing and linear in v, and $\overline{G}_{v_f(0)}(0) = 0$.

Assume that (109) does not hold. Then $G^*(\cdot)$ must cross $\overline{G}_{v^*}(\cdot)$ from below at some $\hat{w} \geq 0$. But then, by following the fluid control for $w \geq \hat{w}$ we get a feasible control with average cost v^*. Therefore, $G^*(w) = \overline{G}_{v^*}(w)$ for $w \geq \hat{w}$, and hence $G^*(w) \leq \overline{G}_{v^*}(w)$ for $w \geq 0$ contradicting our assumption.

Step 2: Let $G^*_W(w)$ denote the value function gradient for the optimal finite buffer control with workload buffer W and average cost v^*_W. Then

$$G^*_W(w) \leq G^*(w) \qquad \text{for } w \in [0, W]$$

and hence $G^*_W(w) \leq \alpha + \frac{w}{m\hat{\theta}}$.

Proof: Compare the HJB equation for $G^*(w)$ and $G^*_W(w)$:

$$v^* = \min_{k \in [0, w/m_e]} (\Delta_k(w) - \theta(k)G^*(w)) + \frac{\sigma^2}{2}(G^*)'(w)$$
$$v^*_W = \min_{k \in [0, w/m_e]} (\Delta_k(w) - \theta(k)G^*_W(w)) + \frac{\sigma^2}{2}(G^*_W)'(w)$$

Now, for any $w \geq 0$, if $G^*(w) = G^*_W(w)$, then $(G^*)'(w) \geq (G^*_W)'(w)$ since the terms in the parentheses are equal and $v^* \geq v^*_W$. That is, $G^*_W(\cdot)$ can never cross $G^*(\cdot)$ from below. Since $G^*(0) = G^*_W(0)$, $G^*(w) \geq G^*_W(w)$ for all $w \geq 0$.

Step 3: Define

$$T_W(w) \doteq \int_0^w \theta(k^*_W(u))du,$$

where $k^*_W(w)$ is the optimal finite buffer control with workload buffer W. Then as $W \to \infty$, $T_W(W) = \Theta(W)$. That is, the integral of the drift over the interval $[0, W]$ for the family of controls $k^*_W(\cdot)$ parameterized by W must asymptotically grow linearly in the buffer limit.

Proof: We begin by rewriting the HJB equation for $G^*_W(w)$

$$\begin{aligned}\frac{\sigma^2}{2}(G^*_W)'(w) &= v^*_W - \min_{k \in [0, w/m_e]} (\Delta_k(w) - \theta(k)G^*_W(w)) \\ &\leq v^*_W - \frac{w}{m \vee m_e} + \theta(k^*_W(w))G^*_W(w).\end{aligned}$$

Take the integration

$$\int_{w=0}^{W} \frac{\sigma^2}{2}(G_W^*)'(w)dw \le W \cdot v_W^* - \frac{W^2}{2(m \vee m_e)} + \left(\alpha + \frac{W}{m\hat{\theta}}\right) \int_0^W \theta(k_W^*(w))dw.$$

This implies

$$\frac{\sigma^2}{2}\left(G_W^*(W) - G_W^*(0)\right) \le W \cdot v_W^* - \frac{W^2}{2(m \vee m_e)} + \left(\alpha + \frac{W}{m\hat{\theta}}\right) T_W(W).$$

The left-hand side of the above inequality is 0 (since the workload reflects at $w = 0$ and $w = W$). The first term on the right-hand side grows linearly in W since v_W^* is bounded by v^*. The second term grows as $\Theta(W^2)$. If $T_W(W) = o(W)$ then the right-hand side becomes negative for W large enough – a contradiction. Therefore, $T_W(W)$ must grow at least linearly. By our assumptions, $\theta(w) \le \hat{\theta} < \infty$. Therefore, $T_W(W) = \Theta(W)$.

Step 4: Denote the density function of the workload under control k_W^* by f_W. The preceding step implies

$$f_W(W) = O(e^{-\beta W})$$

for some positive constant $\beta > 0$. That is, the density function for the workload under optimal finite buffer controls falls exponentially.

Proof: The density function is given by

$$\begin{aligned} f_W(w) &= \kappa_W e^{-\int_0^w \frac{\theta(k_W^*(u))}{\sigma^2/2}} du \\ &= \kappa_W e^{-\frac{T_W(w)}{\sigma^2/2}}, \end{aligned}$$

where κ_W is the normalization constant. Since $T_W(W) = \Theta(W)$ by the preceding step, $f_W(W) = O(e^{-\beta W})$ for some $\beta > 0$.

Step 5: Finally consider the following infinite buffer control, parameterized by workload W:

$$\tilde{k}_W(w) = \begin{cases} k_W^*(w) & w \le W, \\ \hat{k} & w > W. \end{cases}$$

That is, we create a fluid continuation control with prefix $k_W^*(w)$. This results in a suboptimal infinite buffer control with average cost $\tilde{v}_W \ge v^* \ge v_W^*$. However, since the workload density decays exponentially under control $k_W^*(\cdot)$, and $\theta(\hat{k}) > 0$, the cost of $\tilde{k}_W(\cdot)$ is at most $O(e^{-\beta W})$ higher than v_W^*. That is

$$(v^* - v_W^*) \le (\tilde{v}_W - v_W^*) = O(e^{-\beta W}).$$

■

PROOF OF PROPOSITION 6: Recall the Newton-Raphson algorithm from Algorithm 2: We first pick a large enough value of workload $W \ge \hat{k} m_e$ (which is not changed during subsequent iterations). The goal of the Newton-Raphson algorithm then is to find the average cost of the optimal dynamic policy under the restriction that the control for $w \ge W$ is the fluid control $\hat{k}$. With v_n as our guess in the nth iteration, we backwards evolve the ODEs:

$$v_n = \min_{k \in [0, w/m_e]} \left[\frac{w}{m} + k\left(1 - \frac{m_e}{m}\right) - \theta(k) G_{v_n}(w)\right] + \frac{\sigma^2}{2} G'_{v_n}(w) \tag{111}$$

$$1 = -\theta(k_{v_n}(w)) g_{v_n}(w) + \frac{\sigma^2}{2} g'_{v_n}(w) \tag{112}$$

for $w \in [0, W]$ with (terminal) boundary conditions:

$$G_{v_n}(W) = \left(\hat{k}\left(1 - \frac{m_e}{m}\right) - v_n + \frac{\sigma^2}{2\hat{\theta}}\right)\frac{1}{\hat{\theta}} + \frac{1}{m\hat{\theta}}W \tag{113}$$

$$g_{v_n}(W) = -\frac{1}{\hat{\theta}} \tag{114}$$

Here $k_{v_n}(w)$ denotes the policy obtained while solving the ODE for G_{v_n}.

The updated guess for the $(n+1)$st iteration is

$$v_{n+1} = v_n - \frac{G_{v_n}(0)}{g_{v_n}(0)}.$$

We develop our proof of the proposition in several steps.

Step 1 : $v_n \geq v_f(W)$ for $n \geq 1$

Proof: Let $\widetilde{G}_v(w)$ (parameterized by v, w) be given by the ODE

$$v = \frac{w}{m} + k_{v_n}(w)\left(1 - \frac{m_e}{m}\right) - \theta(k_{v_n}(w))\widetilde{G}_v(w) + \frac{\sigma^2}{2}\widetilde{G}'_v(w)$$

for $w \in [0, W]$ with boundary condition

$$\widetilde{G}_v(W) = \left(\hat{k}\left(1 - \frac{m_e}{m}\right) - v + \frac{\sigma^2}{2\hat{\theta}}\right)\frac{1}{\hat{\theta}} + \frac{1}{m\hat{\theta}}W$$

This is essentially the same ODE as (111) but with the min operator replaced by the fixed policy k_{v_n}. The first observation is that $\widetilde{G}_v(0)$ is a linear function of v, and $\widetilde{G}_{v_n}(w) = G_{v_n}(w)$ for all $w \in [0, W]$. Further, denoting

$$\widetilde{g}_v(w) = \frac{d}{dv}\widetilde{G}_v(w)$$

it is easy to see that $\widetilde{g}_v(w) = g_{v_n}(w)$. Therefore,

$$v_{n+1} = v_n - \frac{G_{v_n}(0)}{g_{v_n}(0)} = v_n - \frac{\widetilde{G}_{v_n}(0)}{\widetilde{g}_{v_n}(0)}$$

Since $\widetilde{G}_v(w)$ is a linear function in v for all w, the Newton-Raphson update for $\widetilde{G}_v(0)$ directly yields that value of v for which $\widetilde{G}_v(0) = 0$. But this must be the average cost of policy k_{v_n}. Therefore, v_{n+1} is in fact the average cost of policy k_{v_n}. Since k_{v_n} is a feasible policy in the set $\mathcal{F}_W$, its average cost must be no less than $v_f(W)$ and hence all the iterates $\{v_1, v_2, \ldots\}$ produced are larger than $v_f(W)$.

Step 2: The iterates for average cost $\{v_1, v_2, \ldots\}$ form a strictly decreasing sequence.

Proof: For this, we will show that $G_v(0)$ is monotonically decreasing and Lipschitz continuous in v with derivative bounded away from 0. This would imply that for $v > v_f(W)$, $G_v(0) < 0$, as well as $g_v(0) < 0$, and hence $v_1 > v_2 > \ldots > v_f(W)$.

Consider $v_a > v_b$, and let G_a, g_a, k_a and G_b, g_b, k_b represent the solution of (111)-(114) and the optimal controls for v_a and v_b, respectively. Our goal is to show $G_a(w) < G_b(w)$ for all $w \geq 0$. We rely on the following two facts:

1. Terminal condition:

$$G_b(w) - G_a(w) = \frac{a - b}{\hat{\theta}} \quad w \geq W$$

2. Bounds on $G'_b(w) - G'_a(w)$ for $w \in [0, W]$: [1]

$$G'_b(w) - G'_a(w) = -\frac{2}{\sigma^2}\left[(v_a - v_b) - \min_{k \in [0, w/m_e]} \left(\Delta_k(w) - \theta(k) G_a(w)\right) - \min_{k \in [0, w/m_e]} \left(\Delta_k(w) - \theta(k) G_b(w)\right)\right]$$

where recall that $\Delta_k(w) = \frac{w}{m} + k\left(1 - \frac{m_e}{m}\right)$. By the assumption $G_a(w) \le G_b(w)$ and Lemma 7,

$$-\frac{2}{\sigma}\left[(v_a - v_b) + d_\theta (G_b(w) - G_a(w))\right] \le G'_b(w) - G'_a(w) \le -\frac{2}{\sigma^2}\left[(v_a - v_b) - d_\theta\left(G_b(w) - G_a(w)\right)\right].$$

Combining these two facts, we get

$$(v_a - v_b)\left[\frac{1}{\hat{\theta}} + \frac{1}{d_\theta}\left(1 - e^{-\frac{2 d_\theta W}{\sigma^2}}\right)\right] \le G_b(0) - G_a(0) \le (v_a - v_b)\left[\frac{1}{\hat{\theta}} + \frac{1}{d_\theta}\left(e^{\frac{2 d_\theta W}{\sigma^2}} - 1\right)\right]. \tag{115}$$

Thus we have proved that $G_v(0)$ is monotonically decreasing in v and therefore the Newton-Raphson iterates will form a monotonically decreasing sequence. Further, $G_v(w)$ is Lipschitz continuous in v for all w, and therefore its derivative with respect to v exists almost everywhere, and according to the first inequality of (115) this derivative is bounded away from 0.

The properties proved so far are sufficient to prove that the Newton-Raphson algorithm converges to the optimal $v_f(W)$, and the convergence rate is at least linear.

Step 3: The second derivative of $G_v(0)$ with respect to v is finite in some neighborhood of $v_f(W)$, and hence the Newton-Raphson iterates converge quadratically to $v_f(W)$.

Proof: A sufficient condition to show a quadratic convergence rate for the Newton-Raphson method is that the first derivative is non-zero and the second derivative is finite in some neighborhood of the root. We have already shown that the first condition holds. That is $g_v(0) > 0$ for all v. What remains to be done is to show that $\frac{\partial^2 G_v(0)}{\partial v^2} = \frac{\partial g_v(0)}{\partial v}$ is finite in a neighborhood of $v_f(W)$.

First consider the case where $\left|1 - \frac{m_e}{m}\right| = 0$, or in other words $m_e = m$. As pointed out before, the optimal policy in this case is the fluid policy for which the optimal average cost is found in the initialization phase itself and the algorithm terminates after one step. Therefore we assume $\left|1 - \frac{m_e}{m}\right| > 0$ in the remainder of the proof.

Abbreviating $\theta_a(w) \doteq \theta(k_a(w))$ and $\theta_b(w) \doteq \theta(k_b(w))$, consider the ODE for $g_a(w)$:

$$g'_a(w) = \frac{2}{\sigma^2}\left(1 + \theta_a(w) g_a(w)\right)$$

with boundary condition $g_a(W) = -\frac{1}{\hat{\theta}}$. Therefore, we have

$$\begin{aligned} |g'_a(w) - g'_b(w)| &= \frac{2}{\sigma^2}|\theta_a(w) g_a(w) - \theta_b(w) g_b(w)| && (116) \\ &\le \frac{2}{\sigma^2}\left(|\theta_a(w) - \theta_b(w)|\left(|g_a(w)| + |g_b(w)|\right) + 2 d_\theta |g_a(w) - g_b(w)|\right). && (117) \end{aligned}$$

As we have shown in the previous step, $|G_b(w) - G_a(w)| \le \kappa(w)|v_a - v_b|$ for some bounded function $\kappa(w)$. Combined with our regularity assumptions on $\theta()$, this implies

$$|\theta_a(w) - \theta_b(w)| \le |G_b(w) - G_a(w)| \frac{S_\theta^3}{D_\theta\left|1 - \frac{m_e}{m}\right|} \le |v_a - v_b| \frac{\kappa(w) S_\theta^3}{D_\theta\left|1 - \frac{m_e}{m}\right|}. \tag{118}$$

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[1] We use primes to denote derivatives with respect to w. Derivatives with respect to v are denoted with $\frac{\partial}{\partial v}$ notation.

A rough justification of the first inequality is the following: Since

$$k_a(w) = \operatorname*{arg\,min}_{k \in [0, w/m_e]} \left(\frac{w}{m_e} - k\left(1 - \frac{m_e}{m}\right) + \theta(k) G_a(w) \right),$$

one of three cases occurs: (1) $k_a(w) = 0$, (2) $k_a(w) = w/m_e$, or (3) $(1 - \frac{m_e}{m})/G_a(w) \in \partial\theta(k_a)$ where $\partial\theta(k_a)$ is the set of subderivatives of $\theta(k)$ at $k = k_a$. Consider the scenario where the third case occurs for $k_a(w)$ as well as $k_b(w)$ (other cases are easier to handle). By (59), the absolute value of the first derivative of $\theta(k)$ is bounded by S_θ. Therefore for the case $0 < k_a(w) < \frac{w}{m_e}$ to occur $|G_a(w)|, |G_b(w)|$ must be bounded away from 0. More precisely, $\min(|G_a(w)|, |G_b(w)|) \geq \frac{\left|1 - \frac{m_e}{m}\right|}{S_\theta}$. Now, by (60) $D_\theta > 0$, we must have

$$\left|1 - \frac{m_e}{m}\right| \left|\frac{1}{G_a(w)} - \frac{1}{G_b(w)}\right| \geq D_\theta |k_a(w) - k_b(w)|,$$

or

$$\begin{aligned} |k_a(w) - k_b(w)| &\leq \frac{\left|1 - \frac{m_e}{m}\right|}{D_\theta} \left|\frac{1}{G_a(w)} - \frac{1}{G_b(w)}\right| \\ &= \frac{\left|1 - \frac{m_e}{m}\right|}{D_\theta} \left|\frac{G_a(w) - G_b(w)}{G_a(w) G_b(w)}\right| \\ &\leq \frac{S_\theta^2}{D_\theta \left|1 - \frac{m_e}{m}\right|} \left|G_a(w) - G_b(w)\right|. \end{aligned}$$

Finally,

$$\begin{aligned} |\theta_a(w) - \theta_b(w)| &= |\theta(k_a(w)) - \theta(k_b(w))| \\ &\leq S_\theta \left|k_a(w) - k_b(w)\right| \\ &\leq \frac{S_\theta^3}{D_\theta \left|1 - \frac{m_e}{m}\right|} \left|G_a(w) - G_b(w)\right|. \end{aligned}$$

Finally substituting (118) into (117) gives

$$|g_a'(w) - g_b'(w)| \leq \frac{2}{\sigma^2} \left(|v_a - v_b| \frac{\kappa(w) S_\theta^3}{D_\theta \left|1 - \frac{m_e}{m}\right|} \left(|g_a(w)| + |g_b(w)|\right) + 2 d_\theta |g_a(w) - g_b(w)| \right). \tag{119}$$

Now, a similar calculation to that in step 2 shows that $g_v(w)$ is Lipschitz continuous in v for all w, and hence for $w = 0$. We believe the lower bound on $\left|\frac{d^2\theta(k)}{dk^2}\right|$ we used to prove quadratic convergence is an artifact of our rather crude proof technique, and that quadratic convergence holds even without this restriction. ■