

E-Companion: Asymptotically Optimal Scheduling of Multiple Parallelizable Job Classes

EC.1. Proposition 3 from Glynn and Zeevi (2008)

The high-level idea of this drift argument is to leverage the fact that the *expected change* in steady-state stochastic quantities is zero. Defining a well-chosen Lyapunov function (sometimes referred to as a *test function* in this context) and setting its expected steady-state change to zero yields useful bounds and identities in many cases Eryilmaz and Srikant (2012). We outline this argument here, but see Glynn and Zeevi (2008) for a complete treatment of this technique.

Consider a CTMC with states $x \in \mathbb{Z}_{\geq 0}^\ell$ and generator matrix Q with entries q_{xy} defining the transition rates from any state x to any other state y and appropriate negative entries q_{xx} along the diagonal. We first choose a Lyapunov function $V : \mathbb{Z}_{\geq 0}^\ell \rightarrow \mathbb{R}$. We define the drift of the Lyapunov function, $\Delta V(x)$, to be

$$\Delta V(x) = \lim_{\delta \rightarrow 0^+} \frac{\mathbb{E}[V(X(\delta)) - V(X(0)) \mid X(0) = x]}{\delta}.$$

Rewriting this definition in terms of the generator Q gives the equivalent definition

$$\Delta V(x) = \sum_{x' \in \mathcal{X}: x \neq x'} q_{xx'} (V(x') - V(x)). \quad (\text{EC.1})$$

Intuitively, when the CTMC has reached stationarity, we should have $\mathbb{E}[\Delta V(X)] = 0$. This is shown via the following lemma.

Lemma EC.1 (Adapted from Proposition 3 of Glynn and Zeevi (2008)). *If there exists $C < \infty$ such that $|q_{xy}| < C$ for all states x, y , and a Lyapunov function $V(\cdot)$ such that $\mathbb{E}[V(X)^2] < \infty$, then $\mathbb{E}[\Delta V(X)] = 0$.*

Proof. To show this, we define $\pi(x)$ to be the stationary probability that the system is in state x . We note that

$$\mathbb{E}[\Delta V(X)] = \sum_x \pi(x) \Delta V(x) = \sum_x \pi(x) \left[\sum_{x' \in \mathcal{X}: x \neq x'} q_{xx'} V(x') - V(x) \sum_{x' \in \mathcal{X}: x \neq x'} q_{xx'} \right] = \sum_x \pi(x) \sum_y q_{xy} V(y).$$

We then have

$$\sum_x \pi(x) \sum_y q_{xy} V(y) \leq \sum_x \pi(x) \sum_y |q_{xy}| \cdot |V(y)| = 2 \sum_x \pi(x) |q_{xx}| \cdot |V(x)|.$$

Because $|q_{xx}|$ is bounded, this last sum is bounded by $2C \cdot \mathbb{E}[|V(X)|]$. Our assumption that $\mathbb{E}[V(X)^2] < \infty$ then implies that this sum is finite. This allows us to interchange the sums as follows:

$$\mathbb{E}[\Delta V(X)] = \sum_x \pi(x) \sum_y q_{xy} V(y) = \sum_y V(y) \sum_x \pi(x) q_{xy} = 0,$$

where we exploit the fact that $\sum_x \pi(x) q_{xy} = 0$ because π is the stationary distribution. $\square$

We leverage Lemma EC.1 by finding alternate ways of writing $\Delta V(X)$ and setting the expectation of these alternate forms to 0 to obtain bounds. Specifically, we generally start with (EC.1), observe that

$$\mathbb{E}[\Delta V(X)] = \sum_{X \neq x'} \mathbb{E}[q_{Xx'} (V(x') - V(X))] = 0, \quad (\text{EC.2})$$

and plug in various steady-state quantities to obtain new bounds on these quantities.

EC.2. Lemma 10 from Wang et al. (2022)

Lemma EC.2 (Lemma 10 from Wang et al. (2022)). *Let $\{X(t)\}$ for $t \geq 0$ be an irreducible, nonexplosive, positive-recurrent continuous-time Markov chain over a countable state space χ . Let X be the stationary distribution of this Markov chain. Consider a Lyapunov function $V : \chi \rightarrow \mathbb{R}_+$. Let $\Delta V(x)$ be the drift of the Lyapunov function, defined as*

$$\Delta V(x) = \sum_{x' \in \chi: x \neq x'} q_{xx'} (V(x') - V(x))$$

where $q_{xx'}$ is the corresponding entry of the infinitesimal generator of $\{X(t)\}$. If V satisfies the following conditions

1. *There exist constants $\gamma > 0$ and $B > 0$ such that $V(x) > B \implies \Delta V(x) \leq -\gamma$ for any $x \in \chi$.*
2. *There exists a constant $\nu_{max} < \infty$ such that $\sup_{x, x' \in \chi: q_{xx'} > 0} |V(x') - V(x)| = \nu_{max}$.*
3. *There exists a constant $\bar{q} < \infty$ such that $\sup_{x \in \chi} -q_{xx} = \bar{q}$.*

Then for any $j > 0$, we have

$$\Pr(V(X) > B + 2\nu_{max}j) \leq \left(\frac{q_{max}\nu_{max}}{q_{max}\nu_{max} + \gamma} \right)^{j+1} \quad (\text{EC.3})$$

where

$$q_{max} = \sup_{x \in \chi} \sum_{x' \in \chi: V(x) < V(x')} q_{xx'}.$$

The bound in (EC.3) can then be used to bound the m th moment of $V(X)$ as follows:

$$\mathbb{E}[V(X)^m] \leq (2B)^m + (4\nu_{max})^m \left(\frac{q_{max}\nu_{max} + \gamma}{\gamma} \right)^m m! \quad (\text{EC.4})$$

EC.3. Lemma A.1 from Weng and Wang (2020)

Lemma EC.3 (Lemma A.1 from Weng and Wang (2020)). *This lemma extends the result from Lemma EC.2. While this lemma was originally proven for finite-state continuous-time Markov chains, it is trivially extended to infinite-state Markov chains obeying the conditions in Lemma EC.2. This lemma adds the condition that some states with $V(x) > B$ are permitted to have bounded positive drift. Specifically, for constants $\delta > 0$, $\gamma > 0$, and $B > 0$, let $\mathcal{E}$ be a set of states such that:*

- $V(x) > B \implies \Delta V(x) \leq -\gamma$ for any $x \in \mathcal{E}$.

- $V(x) > B \implies \Delta V(x) \leq \delta$ for any $x \notin \mathcal{E}$.

In this case, the tail bound in (EC.3) becomes

$$\Pr(V(X) > B + 2\nu_{max}j) \leq \left(\frac{q_{max}\nu_{max}}{q_{max}\nu_{max} + \gamma} \right)^j + \left(\frac{\delta}{\gamma} + 1 \right) \Pr(X \notin \mathcal{E}) \quad (\text{EC.5})$$