Appendix A: Detailed computational process for Table 1

Problem 1:

Firstly, it is easy to verify the satisfiability of Condition A and C. The equilibrium total flow can be obtained from the flow conservation and credit conservation conditions:

$$\begin{cases} X_1 + X_2 = 18, \\ 5X_1 + 2X_2 = 60. \end{cases} \Rightarrow \begin{cases} X_1^* = 8, \\ X_2^* = 10. \end{cases}$$

So X^* is uniquely determined, and there exists one unique break point $\hat{\alpha} = \frac{14}{9} \in (1, 2)$ such that

$$d \int_{\hat{\alpha}}^{2} h(\alpha) d\alpha = X_1^*, \quad d \int_{1}^{\hat{\alpha}} h(\alpha) d\alpha = X_2^*.$$

Apparently, both path 1 and path 2 are strongly undominated, so Condition B is satisfied. From the equality $16\hat{\alpha} + 5p^* = 20\hat{\alpha} + 2p^*$, we can obtain the unique equilibrium credit price $p^* = \frac{56}{27}$.

Problem 2:

The same data described in Problem 1 is utilized, except the density function

$$h(\alpha) = \begin{cases} \frac{100}{18}, & \alpha \in [1,\ 1.1]; \\ \frac{80}{18}, & \alpha \in [1.9,\ 2]; \\ 0 & \text{otherwise.} \end{cases}$$

For any $\alpha' \in (1.1, 1.9)$, $h(\alpha') = 0$. So when α=1.1, it is impossible to find δ, such that $\rho * h(\rho * \alpha) > h(\alpha)$ is always fulfilled for any $\rho \in [1, 1 + \delta]$. Condition C thus fails.

The total flows satisfy the same flow and credit conservation condition as in Problem 1, therefore we still have $X_1^* = 8$ and $X_2^* = 10$. And it is easy to verify that users on path 1 have a high VOT with $\alpha \in [1.9, 2]$, and users on path 2 have a low VOT with $\alpha \in [1, 1.1]$. From the following user equilibrium conditions: $16\alpha + 5p^* > 20\alpha + 2p^*$, for $\alpha \in [1, 1.1]$; $16\alpha + 5p^* < 20\alpha + 2p^*$, for $\alpha \in [1.9, 2]$, we obtain the equilibrium credit price p^* belonging to [22/15,38/15].

At maximum equilibrium credit price $p^*_{\max} = 38/15$, the breakpoint $\hat{\alpha} = 1.9$, so both path 1 and path 2 are strongly undominated, Condition B thus is satisfied at $p^*_{\max} = 38/15$.

Problem 3:

Similarly as in Problem 1, Condition A and C are easy to verify. The equilibrium total flow can be obtained from the following equation:

$$\begin{cases} X_1 + X_2 = 18, \\ 5X_1 + 2X_2 = 90. \end{cases} \Rightarrow \begin{cases} X_1^* = 18, \\ X_2^* = 0. \end{cases}$$

If $\hat{\alpha}$ is the breakpoint, then it satisfies the following equations:

$$d \int_{\hat{\alpha}}^{2} h(\alpha) d\alpha = 18, \quad d \int_{1}^{\hat{\alpha}} h(\alpha) d\alpha = 0.$$

From the following traffic equilibrium condition: $18\alpha + 5p^* < 27\alpha + 2p^*$, for any $\alpha \in [1, 2]$, we obtain equilibrium credit price $p^* \in [0, 3]$. At maximum equilibrium credit price $p^*_{\max} = 3$, the breakpoint $\hat{\alpha} = 1$, path 2 is weakly undominated, Condition B is thus violated.

Problem 4:

$$K = \min_{x\in\Omega_x} \langle \kappa, x\rangle = 36.$$

So Condition A is violated. The equilibrium total flow can be obtained from the following equation:

$$\begin{cases} X_1 + X_2 = 18, \\ 5X_1 + 2X_2 = 36. \end{cases} \Rightarrow \begin{cases} X_1^* = 0, \\ X_2^* = 18. \end{cases}$$

Apparently, path 1 is not strongly undominated, so Condition B is violated. From the traffic equilibrium condition, $36\alpha + 2p^* < 40\alpha + 5p^*$ for any $\alpha > 0$, we have $p^* > 0$.

Appendix B: Sensitivity analysis of the MNEPTCS

To establish the stability of optimal credit scheme design, we provide a sensitivity analysis of the MNEPTCS with credit scheme parametric (κ, K).

Now consider the multiclass equilibrium problem with credit scheme parameters: Find $x(\kappa, K) \in \Gamma_x(\kappa, K)$, such that

$$\mathrm{VI}_x(\kappa, K): \qquad \langle \alpha F(X(\kappa, K)), x - x(\kappa, K)\rangle \geq 0, \quad \forall\ x \in \Gamma_x(\kappa, K).$$

where $\Gamma_x(\kappa, K) = \{x | x \in \Omega_x, \langle \kappa, X\rangle \leq K\}$. Using the same technique described in Section 2, the parametric variational inequality $\mathrm{VI}_x(\kappa, K)$ can be formulated as the following parametric variational inequality systems:

Find $x(\kappa, K) \in \Omega_x$, $p^*(\kappa, K) \geq 0$ such that

$$\mathrm{VIS}_x(\kappa, K): \qquad \langle \alpha F(X(\kappa, K)) + p^*(\kappa, K)\kappa, x - x(\kappa, K)\rangle \geq 0, \quad \forall\ x \in \Omega_x, \tag{1}$$

$$\langle \kappa^t X(\kappa, K) - K, p - p^*(\kappa, K)\rangle \leq 0, \qquad \forall\ p \geq 0, \tag{2}$$

where $x(\kappa, K)$ is the user equilibrium flow pattern, $X(\kappa, K)$ is the aggregate user equilibrium link flow pattern, $X(\kappa, K) = \int_{\alpha_{\min}}^{\alpha_{\max}} x(\kappa, K)(\alpha) d\alpha$, $p^*(\kappa, K)$ is the equilibrium credit price. From Theorem 2, the uniqueness of the user equilibrium flow $(x(\kappa, K), X(\kappa, K))$ and credit price $p^*(\kappa, K)$ are provided.

Lemma 1. For given (κ^0, K^0), suppose the following conditions hold:

(1) For all credit schemes in the neighborhood $B(\kappa^0, K^0)$ of (κ^0, K^0), F is average strongly monotone on $S_x(p)$, $\forall p \geq 0$.

(2) The credit scheme satisfies Condition A at (κ^0, K^0), i.e., $K^0 > K_l(\kappa^0)$.

(3) For credit scheme (κ^0, K^0), all paths have distinct credit charges, i.e.,

$$T_1 > T_2 > \cdots > T_{|R|-1} \ \mathit{or}\ (\Delta^t \kappa^0)_j - (\Delta^t \kappa^0)_{j+1} \geq \delta > 0, \ \forall\ j \in < 1, |R| - 1 > .$$

Then the equilibrium price $p^*(\kappa, K)$ is uniformly bounded on $B(\kappa^0, K^0)$.

Proof. Obviously, Assumptions (2) and (3) hold for the credit scheme (κ, K) in the sufficiently small neighborhood $B(\kappa^0, K^0)$ of (κ^0, K^0). Together with the assumption of average strong monotonicity for F, $X^*(\kappa, K)$ is unique and the equilibrium price $p^*(\kappa, K)$ is bounded for all $(\kappa, K) \in B(\kappa^0, K^0)$. Furthermore, for the maximum equilibrium price $p^*_{\max}(\kappa, K)$, there exists at least one breakpoint $\hat{\alpha}_i(\kappa, K)$, $i = 1, 2, \cdots, |R| - 1$ such that

$$\hat{\alpha}_i(\kappa, K) = \frac{(T_i - T_{i+1})p^*_{\max}(\kappa, K)}{C_{i+1}(V(\kappa, K)) - C_i(V(\kappa, K))} \leq \alpha_{\max}, \tag{3}$$

or

$$\hat{\alpha}_i(\kappa, K) = \frac{((\Delta^t \kappa^0)_i - (\Delta^t \kappa^0)_{i+1})p^*_{\max}(\kappa, K)}{(\Delta^t F(X(\kappa, K))_{i+1} - (\Delta^t F(X(\kappa, K))_i} \leq \alpha_{\max}. \tag{4}$$

Otherwise, the minimum credit-charged equilibrium path is uniquely strongly undominated and all $p \geq p^*_{\max}$ become equilibrium price which contradicts the boundedness of credit price.

Finally, since $F(.)$ is bounded and Assumption (3) holds, the right-hand side term of the following inequality is bounded for any (κ, K) on $B(\kappa^0, K^0)$:

$$p^*_{\max}(\kappa, K) \leq \frac{(\Delta^t F(X(\kappa, K))_{i+1} - (\Delta^t F(X(\kappa, K))_i}{((\Delta^t \kappa^0)_i - (\Delta^t \kappa^0)_{i+1})} \alpha_{\max}. \tag{5}$$

which gives rise to the desired uniform boundedness of credit price. □

Based on Lemma 1 in Appendix B and the uniqueness result of Section 3, the following sensitivity theorem establishes that a small change in credit scheme (κ, K) entering the parametric variational inequality VI(κ, K) induces a small change in the resulting solution.

THEOREM 1 **(Sensitivity results)**. Suppose the assumptions of Lemma 1 hold and the density function $h(\alpha)$ satisfies Condition C with $\delta > 0$. Then,

(i) the aggregate user equilibrium flow pattern $X(\kappa, K)$ is continuous in (κ, K) on $B(\kappa^0, K^0)$ and

$$\|X(\kappa_1, K_1) - X(\kappa_2, K_2)\|^2 \leq \frac{\overline{M}}{\beta}(|K_1 - K_2| + \|\kappa_1 - \kappa_2\|). \tag{6}$$

where $\overline{M}$ is a positive constant;

(ii) credit price $p^*(\kappa, K)$ and user equilibrium solution $x(\kappa, K)$ of the MNEPTCS with credit scheme parameters are also continuous in (κ, K).

Proof. For simplicity, denote $x(\kappa_i, K_i)$ and $p^*(\kappa_i, K_i)$ by x_i, p^*_i for $i = 1, 2, ...$ respectively.

(i) Rewriting (1) with $\kappa = \kappa_1, K = K_1$, $x = x_2$ and $\kappa = \kappa_2, K = K_2$, $x = x_1$, then adding the resulting inequalities, one obtains

$$\langle \alpha(F(X(\kappa_1, K_1)) - F(X(\kappa_2, K_2))) + p^*_1\kappa_1 - p^*_2\kappa_2, x_2 - x_1 \rangle \geq 0 \tag{7}$$

Again rewriting (2) with $\kappa = \kappa_1$, $K = K_1$, $p^* = p^*_2$ and $\kappa = \kappa_2$, $K = K_2$, $p^* = p^*_1$, then adding the resulting inequalities, one obtains

$$\langle (\kappa^t_1 X(\kappa_1, K_1) - K_1) - (\kappa^t_2 X(\kappa_2, K_2) - K_2), p^*_1 - p^*_2 \rangle \geq 0. \tag{8}$$

Using the average strong monotonicity condition, (7) yields

$$\begin{aligned}\langle p_1^*\kappa_1 - p_2^*\kappa_2, X(\kappa_2,K_2) - X(\kappa_1,K_1)\rangle &\geq \langle \alpha(F(X(\kappa_2,K_2)) - F(X(\kappa_1,K_1))), x_2 - x_1\rangle \\ &\geq \beta\|X(\kappa_2,K_2) - X(\kappa_1,K_1)\|^2. \end{aligned} \tag{9}$$

By virtue of (7)-(9), we have

$$\begin{aligned}&\langle p_2^* - p_1^*, K_1 - K_2\rangle + \langle p_1^* X(\kappa_2,K_2) + p_2^* X(\kappa_1,K_1), \kappa_1 - \kappa_2\rangle \\ &\geq \langle p_1^*\kappa_1 - p_2^*\kappa_2,, X(\kappa_2,K_2) - X(\kappa_1,K_1)\rangle \\ &\geq \beta\|X(\kappa_2,K_2) - X(\kappa_1,K_1)\|^2. \end{aligned} \tag{10}$$

From Lemma 1, the equilibrium price $p^*(\kappa,K)$ is uniformly bounded on $B(\kappa^0,K^0)$, (10) yields (6) and the continuity of $X(\kappa,K)$.

(ii) Suppose sequence $\{(\kappa_n,K_n)\}$ converges to (κ,K). Therefore, the following variational inequality system holds:

$$\text{VIS}_x(\kappa_n,K_n): \qquad \langle \alpha F(X(\kappa_n,K_n)) + p_n^*\kappa_n, x - x_n\rangle \geq 0, \quad \forall\, x \in \Omega_x, \tag{11}$$

$$\langle \kappa_n^t X(\kappa_n,K_n) - K_n, p - p_n^*\rangle \leq 0, \qquad \forall\, p \geq 0. \tag{12}$$

Since x_n and p_n^* are bounded, then there exists the subsequence $\{(x_{n'},p_{n'}^*)\} \to (x_0,p_0^*)$. Taking the limit in $\text{VIS}_x(\kappa_{n'},K_{n'})$, we get

$$\text{VIS}_x(\kappa,K): \qquad \langle \alpha F(X(\kappa,K)) + p_0^*\kappa, x - x_0\rangle \geq 0, \quad \forall\, x \in \Omega_x, \tag{13}$$

$$\langle \kappa^t X(\kappa,K) - K, p - p_0^*\rangle \leq 0, \qquad \forall\, p \geq 0 \tag{14}$$

which states that $(X(\kappa,K),x_0)$ is the solution of the MNEPTCS with credit scheme (κ,K), p_0 is the credit price.

From Theorem 2, we get $x_0 = x(\kappa,K)$, $p_0^* = p^*(\kappa,K)$. The continuity of $x(\kappa,K)$ and $p^*(\kappa,K)$ is provided by the aforementioned process. □

Corollary 1. If all assumptions of Theorem 1 hold, then for the MNEPTCS we have

(i) The first order moment of link flow $\bar{X}(\kappa,K)$ is continuous in (κ,K).

(ii) The total cost function $\langle \alpha F(X), x\rangle$ is also continuous in (κ,K).

Proof. For any x, $y \in \Omega_x$, we have

$$\begin{aligned}\|\bar{X} - \bar{Y}\|^2 = \|\int_{\alpha_{\min}}^{\alpha_{\max}} \alpha(x(\alpha) - y(\alpha))d\alpha\|^2 &\leq 2\int_{\alpha_{\min}}^{\alpha_{\max}} \langle \alpha(x(\alpha) - y(\alpha)), \alpha(x(\alpha) - y(\alpha))\rangle d\alpha \\ &\leq 2\alpha_{\max}^2 \int_{\alpha_{\min}}^{\alpha_{\max}} \|x - y\|_2^2 d\alpha \\ &= 2\alpha_{\max}^2\|x - y\|^2. \end{aligned}$$

Thus, the continuity of $x(\kappa, K)$ ensures the continuity of $\bar{X}$. Combining the continuity of X and $F(.)$, we conclude that the total cost function $\langle \alpha F(X), x\rangle = \langle F(X), \bar{X}\rangle$ is also continuous in (κ, K). $\square$

Finally, we provide the sensitivity analysis results with respect to the total number of credits distributed K. Fixing credit charge κ, take $K_l = \min_{x\in\Omega_x}\langle \kappa, X\rangle$ and $K_u = \langle \kappa, X(0)\rangle$ where $X(0) \in \text{Sol}(\text{VI}_x(0))$. We call (K_l, K_u) the effective credit supply interval. The following proposition is useful for analysis and design of the credit scheme for the MNEPTCS.

Proposition 1 **(Relation between credit supply and credit market price equilibrium)**. For fixed κ, suppose the assumptions of Theorem 2 hold for credit scheme (κ, K) with $K \in (K_l, +\infty)$.

(1) If $K \in [K_u, +\infty)$, then we have unique zero credit price market equilibrium and unique user equilibrium $X(0) \in \text{Sol }(\text{VI}_x(0))$.

(2) If $K \in (K_l, K_u)$, then we have unique positive credit price market equilibrium. Moreover, the equilibrium credit price is decreasing with credit supply within the effective credit supply interval.

Proof.

(1) Let $\Phi_K(p) = \langle \kappa, X(p)\rangle - K$ where $X(p) \in \text{Sol}(\text{VI}_x(p))$. Note that $\Phi_{K_1}(p) < \Phi_{K_2}(p)$, when $K_1 > K_2$. If $K \in [K_u, +\infty)$, we have $\Phi_K(0) \le \Phi_{K_u}(0) = \langle \kappa, X(0)\rangle - K_u = 0$ and we have unique credit market equilibrium $p^*(\kappa, K) = 0$ and user equilibrium $X(\kappa, K) = X(0)$.

(2) If $K \in (K_l, K_u)$, we have $\Phi_K(0) > \Phi_{K_u}(0) = 0$, therefore $p^*(\kappa, K) > 0$. For $K_1 < K_2,\ K_i \in (K_l, K_u),\ i = 1, 2$, we obtain

$$0 = \Phi_{K_2}(p^*(\kappa, K_2)) = \Phi_{K_1}(p^*(\kappa, K_1)) > \Phi_{K_2}(p^*(\kappa, K_1)).$$

From Proposition 7, $\Phi_{K_2}(p)$ is nonincreasing, thus we get $p^*(\kappa, K_2) < p^*(\kappa, K_1)$. $\square$