Appendix A: Proof of Proposition 1

Proof of Proposition 1

Proposition 1 can be proven by induction using inductive Hypothesis 1:

HYPOTHESIS 1 ($P(n)$). *If $y_l \leq z_l, \forall l \in sites(r)$, then for any service time s_{r_k} such that $k+n=m$ and $y_{r_k} \leq s_{r_k} \leq z_{r_k}$, there is a start of service time s_j for each $j = r_{k+1}, \ldots, r_{m+1}$ such that $y_j \leq s_j \leq z_j$ and such that $[s_{r_k}, s_{r_{k+1}}, \ldots, s_{r_{m+1}}]$ respects constraints (4) to (6) with $i = r_k, i = r_{k+1}, \ldots, i = r_m$ and constraint (5) with $i = r_{m+1}$.*

Base case: n=0

This is the case where $k+0=m$, thus customer r_k is the last customer in the route. After serving customer r_k, the vehicle will return to the depot r_0 (remember we consider the successor of the last customer r_m the depot, thus $r_{m+1} = r_0$). Hypothesis $P(0)$ becomes:
If $y_l \leq z_l, \forall l \in sites(r)$, then for any service time s_{r_k} such that $k=m$ and $y_{r_k} \leq s_{r_k} \leq z_{r_k}$, there is a start of service time s_{r_0} such that $y_{r_0} \leq s_{r_0} \leq z_{r_0}$ and such that $\{s_{r_k}, s_{r_{m+1}}\}$ respects constraints (4) to (6) with $i = r_k = r_m$ and constraint (5) with $i = r_{m+1} = r_0$.

Suppose $s_{r_0} = s_{r_k} + c_{r_k,r_0} + d_{r_k}$. First we need to verify whether s_{r_0} respects $y_{r_0} \leq s_{r_0} \leq z_{r_k}$. It is known that $y_{r_0} = e_{r_0}$ and $z_{r_0} = l_{r_0}$ from (8) and (9), thus it is sufficient to verify $e_{r_0} \leq s_{r_0} \leq l_{r_0}$. Since $s_{r_0} \geq s_{r_k}$ and $s_{r_k} \geq y_{r_k}$ (by hypothesis P(n)) and $y_{r_k} \geq e_{r_k}$ (by definition of y_{r_k}) and $e_{r_k} \geq e_{r_0}$ (by problem definition) we have $s_{r_0} \geq e_{r_0}$. By (9) we know that

$$
\begin{array}{lll}
z_{r_k} & \leq z_{r_0} - c_{r_k,r_0} - d_{r_k} & \\
\Leftrightarrow z_{r_k} & \leq l_{r_0} - c_{r_k,r_0} - d_{r_k} & \text{from (9)} \\
\Leftrightarrow s_{r_k} & \leq l_{r_0} - c_{r_k,r_0} - d_{r_k} & \text{since } s_{r_k} \leq z_{r_k} \text{ by hypothesis P(n)} \\
\Leftrightarrow s_{r_k} + c_{r_k,r_0} + d_{r_k} & \leq l_{r_0} & \\
\Leftrightarrow s_{r_0} & \leq l_{r_0} &
\end{array}
$$

Thus we have $e_{r_0} \leq s_{r_0} \leq l_{r_0}$.

Next we need to show that s_{r_k} respects Eq. (4). This is verified by definition of s_{r_0}:

$$
\begin{array}{lll}
\delta_{r_k} + c_{r_k,r_0} & \leq s_{r_0} & \\
\Leftrightarrow s_{r_k} + d_{r_k} + c_{r_k,r_0} & \leq s_{r_0} & \text{(by Eq.(2))} \\
\Leftrightarrow s_{r_k} + d_{r_k} + c_{r_k,r_0} & \leq s_{r_k} + d_{r_k} + c_{r_k,r_0} &
\end{array}
$$

Both s_{r_k} and s_{r_0} respect Eq. (5) since $y_{r_k} \leq s_{r_k} \leq z_{r_k}$ and $e_{r_k} \leq y_{r_k} \leq z_{r_k} \leq l_{r_k}$ and $y_{r_0} \leq s_{r_0} \leq z_{r_0}$ and $e_{r_0} \leq y_{r_0} \leq z_{r_0} \leq l_{r_0}$.

Finally s_{r_k} respects Eq. (6) since:

$$\begin{aligned} s_{r_k} + d_{r_k} + c_{r_k,r_0} &\geq s_{r_0} - w_{max} \\ \Leftrightarrow s_{r_k} + d_{r_k} + c_{r_k,r_0} + w_{max} &\geq s_{r_k} + d_{r_k} + c_{r_k,r_0} \end{aligned}$$

Thus for any s_{r_k} s.t. $y_{r_k} \leq s_{r_k} \leq z_{r_k}$ with $k+0=m$ there is a s_{r_0} such that $\{s_{r_k}, s_{r_0}\}$ respects constraints (4) to (6) with $i = r_k = r_m$ and constraint (5) with $i = r_{m+1} = r_0$ and we have proven $P(0)$.

Inductive step: if P(n) holds then P(n+1) holds as well

Assume that the following holds: If $y_l \leq z_l, \forall l \in sites(r)$, then for any service time s_{r_k} such that $k+n=m$ and $y_{r_k} \leq s_{r_k} \leq z_{r_k}$, there is a start of service time s_j for each $j = r_{k+1}, \ldots, r_{m+1}$ such that $y_j \leq s_j \leq z_j$ and such that $\{s_{r_k}, s_{r_{k+1}}, \ldots, s_{r_{m+1}}\}$ respects constraints (4) to (6) with $i = r_k, i = r_{k+1}, \ldots, i = r_m$ and constraint (5) with $i = r_{m+1} = r_0$.

We need to show that if $y_l \leq z_l, \forall l \in sites(r)$, then for any service time $s_{r_{k-1}}$ such that $k-1+n+1=m$ and $y_{r_{k-1}} \leq s_{r_{k-1}} \leq z_{r_{k-1}}$, there is a start of service time s_j for each $j = r_k, \ldots, r_{m+1}$ such that $y_j \leq s_j \leq z_j$ and such that $\{s_{r_{k-1}}, s_{r_k}, \ldots, s_{r_{m+1}}\}$ respects constraints (4) to (6) with $i = r_{k-1}, i = r_k, \ldots, i = r_m$ and constraint (5) with $i = r_{m+1} = r_0$..

Given $P(n)$ it is sufficient to show that for any $s_{r_{k-1}}$ such that $y_{r_{k-1}} \leq s_{r_{k-1}} \leq z_{r_{k-1}}$ there is a s_{r_k} such that $y_{r_k} \leq s_{r_k} \leq z_{r_k}$ and such that $s_{r_{k-1}}$ and s_{r_k} respect Constraints (4) to (6) with $i = k-1$ and $i = k$.

Assume some s_{r_k-1} such that $y_{r_{k-1}} \leq s_{r_{k-1}} \leq z_{r_{k-1}}$ and some s_{r_k} such that $y_{r_k} \leq s_{r_k} \leq z_{r_k}$. Then the following hold:

$$\begin{aligned} y_{r_{k-1}} &\geq y_{r_k} - c_{r_{k-1},r_k} - d_{r_{k-1}} - w_{max} && \text{by Eq. (8)} \\ \Leftrightarrow y_{r_k} &\leq y_{r_{k-1}} + c_{r_{k-1},r_k} + d_{r_{k-1}} + w_{max} \\ \Leftrightarrow y_{r_k} &\leq s_{r_{k-1}} + c_{r_{k-1},r_k} + d_{r_{k-1}} + w_{max} && \text{since } s_{r_{k-1}} \geq y_{r_{k-1}} \end{aligned}$$

and

$$\begin{aligned} z_{r_{k-1}} &\leq z_{r_k} - c_{r_{k-1},r_k} - d_{r_{k-1}} && \text{by Eq. (9)} \\ \Leftrightarrow z_{r_k} &\geq z_{r_{k-1}} + c_{r_{k-1},r_k} + d_{r_{k-1}} \\ \Leftrightarrow z_{r_k} &\geq s_{r_{k-1}} + c_{r_{k-1},r_k} + d_{r_{k-1}} && \text{since } s_{r_{k-1}} \leq z_{r_{k-1}} \end{aligned}$$

Also, in order for $s_{r_{k-1}}$ to respect Eqs. (4) and (6) the following must hold:

$$s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k} \leq s_{r_k} \leq s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k} + w_{max}$$

Therefore a start of service time s_{r_k} such that $y_{r_k} \leq s_{r_k} \leq z_{r_k}$ and such that Eqs. (4) and (6) are respected must lie in the interval:

$$[max(y_{r_k}, s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k}); min(z_{r_k}, s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k} + w_{max})]$$

Since $y_{r_k} \leq s_{r_{k-1}} + c_{r_{k-1},r_k} + d_{r_{k-1}} + w_{max}$ and $z_{r_k} \geq s_{r_{k-1}} + c_{r_{k-1},r_k} + d_{r_{k-1}}$ as shown previously and since $y_{r_k} \leq z_{r_k}$ this interval is non-empty. Since $e_{r_k} \leq y_{r_k} \leq z_{r_k} \leq l_{r_k}$ a start of service time s_{r_k} in this interval also respects Eq. (5).

Finally, if s_{r_k} lies in this interval then $s_{r_{k-1}}$ respects Eqs. (4) to (6) since:

- $s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k} \leq max(y_{r_k}, s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k})$
  (Eq. (4) holds with $i = r_{k-1}$)
- $s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k} + w_{max} \geq min(z_{r_k}, s_{r_{k-1}} + d_{r_{k-1}} + c_{r_{k-1},r_k} + w_{max})$
  (Eq. (6) holds with $i = r_{k-1}$)
- $e_{r_{k-1}} \leq y_{r_{k-1}} \leq s_{r_{k-1}} \leq z_{r_{k-1}} \leq l_{r_{k-1}}$
  (Eq. (5) holds with $i = r_{k-1}$)

This shows that if $y_l \leq z_l, \forall l \in sites(r)$, then for any service time s_{r_k} such that $y_{r_k} \leq s_{r_k} \leq z_{r_k}$, there is a start of service time s_j for each $j = r_{k+1}, \ldots, r_m$ such that $y_j \leq s_j \leq z_j$ and such that $\{s_{r_k}, s_{r_{k+1}}, \ldots, s_{r_{m+1}}\}$ respects constraints (4) to (6) with $i = r_k, i = r_{k+1}, \ldots, i = r_m$ and constraint (5) with $i = r_{m+1} = r_0$. ■

Appendix B: Additional Results

Table 4 Type 3, problem set C1

$p_{\max}$	$w_{\max}$	Instance	*K/Dist.*	*Non-viol. TW (%)*	*sec_{tt} (sec)*
5%	5%	C101	11/867	99	487
		C102	10/1076	100	776
		C103	10/1092	100	909
		C104	10/1009	100	1103
		C105	10/871	100	577
		C106	10/880	100	568
		C107	10/865	100	632
		C108	11/957	100	719
		C109	10/897	100	897
5%	10%	C101	11/901	100	520
		C102	10/922	100	732
		C103	10/1036	100	944
		C104	10/971	100	1089
		C105	10/829	100	568
		C106	10/851	100	572
		C107	10/869	100	638
		C108	11/895	100	735
		C109	10/868	100	890
10%	5%	C101	10/831	98	591
		C102	10/1131	100	862
		C103	10/1010	100	1082
		C104	10/985	100	1235
		C105	10/871	100	714
		C106	11/972	100	733
		C107	10/869	98	803
		C108	11/902	100	850
		C109	10/888	100	1041
10%	10%	C101	10/935	99	631
		C102	10/1019	100	871
		C103	10/865	100	1159
		C104	10/972	100	1270
		C105	11/907	100	704
		C106	11/926	100	747
		C107	10/842	97	785
		C108	11/917	100	888
		C109	10/936	100	1079

Table 5 **Type 4,** $p_{max} = 5\%$

Instance	*K/Dist.*	*Non-viol. TW (%)*	sec_{tt} *(sec)*
R101	15/1599	84	346
R102	13/1530	86	406
R103	11/1246	95	445
R104	10/1097	100	484
R105	12/1434	83	358
R106	11/1323	94	392
R107	10/1191	97	415
R108	9/1037	98	490
R109	11/1248	95	378
R110	10/1203	91	404
R111	10/1206	97	406
R112	10/1036	100	461
C101	10/829	100	467
C102	10/842	100	702
C103	11/882	100	866
C104	10/917	100	954
C105	10/869	100	522
C106	10/831	100	536
C107	10/879	100	574
C108	10/868	100	680
C109	10/904	100	789
RC101	13/1668	86	323
RC102	12/1593	96	380
RC103	11/1344	98	393
RC104	10/1229	100	428
RC105	12/1611	91	344
RC106	11/1466	96	351
RC107	11/1373	100	386
RC108	10/1267	99	396
R201	4/1390	100	2702
R202	3/1296	98	4729
R203	3/1079	100	7889
R204	3/841	100	9218
R205	3/1170	100	4427
R206	3/1033	100	5745
R207	3/980	100	7445
R208	2/928	100	8392
R209	3/1078	100	4880
R210	3/1038	99	5792
R211	3/884	100	6306
C201	3/591	100	1957
C202	3/591	100	3976
C203	4/673	100	5528
C204	4/731	100	6854
C205	3/589	100	1789
C206	3/589	100	1934
C207	3/589	100	2116
C208	3/591	100	2441
RC201	4/1600	97	2527
RC202	4/1384	99	3083
RC203	3/1269	100	5794
RC204	3/951	100	6867
RC205	4/1391	97	2913
RC206	4/1303	100	3229
RC207	4/1233	100	3478
RC208	3/1014	100	4255

Table 6 Type 4, $p_{max} = 10\%$

Instance	*K/Dist.*	*Non-viol. TW (%)*	sec_{tt} *(sec)*
R101	13/1485	62	342
R102	12/1349	81	393
R103	10/1184	84	421
R104	10/1078	100	476
R105	12/1377	83	366
R106	11/1262	92	405
R107	10/1132	93	427
R108	9/1018	99	434
R109	11/1254	97	392
R110	10/1187	94	397
R111	10/1154	95	416
R112	10/1031	100	457
C101	10/910	97	525
C102	10/1133	100	706
C103	10/1065	100	925
C104	10/939	100	1150
C105	10/869	100	607
C106	10/878	100	621
C107	10/913	99	646
C108	10/871	100	750
C109	10/870	100	899
RC101	12/1635	75	337
RC102	11/1473	85	363
RC103	10/1316	89	385
RC104	10/1269	100	443
RC105	12/1585	89	355
RC106	11/1441	91	347
RC107	10/1322	85	411
RC108	10/1245	100	384
R201	4/1333	96	3328
R202	3/1244	95	4745
R203	3/1055	98	7582
R204	3/916	100	10398
R205	3/1132	99	5407
R206	3/1035	100	6842
R207	3/959	100	8711
R208	2/832	100	9950
R209	3/1001	99	5945
R210	3/1076	100	6825
R211	3/865	100	7937
C201	3/591	100	2116
C202	4/687	99	5421
C203	4/752	100	6674
C204	4/725	100	7241
C205	3/590	100	2563
C206	3/589	100	2156
C207	3/592	100	2029
C208	3/592	100	2313
RC201	4/1536	94	2523
RC202	3/1375	94	3467
RC203	3/1165	98	5097
RC204	3/979	100	7146
RC205	4/1507	96	3192
RC206	3/1291	98	3338
RC207	4/1201	99	4043
RC208	3/1069	100	5469

Table 7 **Type 5,** $p_{max} = 5\%$

Instance	*K/Dist.*	*Non-viol. TW (%)*	sec_{tt} *(sec)*
R101	14/1574	64	353
R102	12/1432	76	439
R103	10/1241	68	429
R104	9/1025	90	476
R105	12/1411	74	364
R106	11/1312	93	409
R107	10/1166	97	430
R108	9/1043	96	456
R109	11/1261	92	404
R110	10/1197	93	399
R111	10/1179	94	414
R112	10/1047	100	472
C101	10/829	100	498
C102	10/940	100	758
C103	10/954	100	923
C104	10/931	100	1125
C105	10/875	100	544
C106	10/839	100	595
C107	10/868	100	630
C108	10/842	100	727
C109	10/896	100	883
RC101	12/1578	69	332
RC102	11/1507	76	361
RC103	10/1298	88	398
RC104	10/1227	99	523
RC105	11/1483	70	349
RC106	11/1463	86	363
RC107	11/1384	98	394
RC108	10/1229	98	395
R201	4/1376	98	3442
R202	3/1394	94	5059
R203	3/1125	100	8760
R204	3/826	100	11747
R205	3/1140	99	5165
R206	3/1050	100	7053
R207	3/961	100	8660
R208	3/790	100	10771
R209	3/1030	100	6974
R210	3/1212	100	7407
R211	3/928	100	7913
C201	3/591	100	1814
C202	4/626	100	4462
C203	4/667	100	6213
C204	3/683	100	7255
C205	3/591	100	1755
C206	3/594	100	1985
C207	3/598	100	2100
C208	3/605	100	2250
RC201	4/1594	95	2440
RC202	4/1340	98	4510
RC203	4/1162	100	5764
RC204	3/921	100	7018
RC205	4/1374	96	3248
RC206	4/1340	100	3868
RC207	3/1214	99	4099
RC208	3/1006	100	5571

Table 8 Type 5, $p_{max} = 10\%$

Instance	*K/Dist.*	*Non-viol. TW (%)*	sec_{tt} *(sec)*
R101	12/1445	45	377
R102	11/1333	68	413
R103	10/1168	79	409
R104	9/1018	88	443
R105	11/1262	62	374
R106	10/1245	73	392
R107	10/1107	91	428
R108	9/996	98	440
R109	10/1185	78	396
R110	10/1187	91	401
R111	10/1061	93	460
R112	9/990	93	433
C101	10/863	98	626
C102	10/865	100	891
C103	10/912	100	1126
C104	10/938	100	1305
C105	10/829	100	723
C106	10/876	100	786
C107	10/862	100	811
C108	11/876	100	921
C109	10/863	100	1044
RC101	11/1467	52	363
RC102	11/1486	84	367
RC103	10/1293	88	397
RC104	10/1221	97	436
RC105	11/1420	77	354
RC106	11/1385	86	395
RC107	10/1333	88	396
RC108	10/1244	96	407
R201	3/1402	83	3639
R202	3/1301	94	6586
R203	3/1061	97	9683
R204	3/844	100	11660
R205	3/1152	100	6315
R206	3/1040	100	9322
R207	3/918	99	11647
R208	3/771	100	9994
R209	3/1082	100	8240
R210	3/1062	99	10096
R211	3/868	100	11241
C201	3/591	100	2087
C202	4/676	100	6693
C203	4/715	100	6688
C204	3/805	99	6469
C205	3/589	100	2012
C206	3/591	100	2521
C207	3/622	100	3275
C208	3/632	100	4848
RC201	4/1602	91	3286
RC202	3/1357	95	4400
RC203	3/1191	96	6493
RC204	3/1001	99	9705
RC205	3/1379	86	3599
RC206	3/1273	95	4003
RC207	3/1311	95	5299
RC208	3/1091	100	7959

Table 9 Type 6, $p_{max} = 5\%$

Instance	*K/Dist.*	*Non-viol. TW (%)*	sec_{tt} *(sec)*
R101	14/1555	64	343
R102	12/1399	78	393
R103	10/1252	79	417
R104	10/1069	98	473
R105	12/1412	78	354
R106	11/1324	91	415
R107	10/1202	95	419
R108	9/1027	97	483
R109	11/1231	92	448
R110	10/1206	91	405
R111	10/1179	94	428
R112	10/1047	100	509
C101	10/829	100	506
C102	10/951	100	764
C103	10/999	100	890
C104	10/964	100	1078
C105	10/829	100	563
C106	10/841	100	555
C107	10/898	100	628
C108	10/837	100	766
C109	11/891	100	899
RC101	12/1549	66	333
RC102	11/1509	79	367
RC103	10/1267	90	396
RC104	10/1220	100	425
RC105	11/1508	73	351
RC106	11/1451	93	361
RC107	10/1277	87	385
RC108	10/1256	98	389
R201	4/1391	98	3205
R202	3/1262	93	5346
R203	3/1046	99	8978
R204	3/858	100	10569
R205	4/1081	100	4933
R206	3/1121	100	6960
R207	3/940	100	8901
R208	3/808	100	8506
R209	3/1023	100	6753
R210	3/1092	100	7459
R211	3/883	100	8504
C201	3/591	100	1859
C202	3/618	100	4015
C203	4/679	100	6846
C204	4/769	100	6417
C205	3/598	100	1941
C206	3/600	100	1866
C207	3/594	100	2063
C208	3/600	100	2272
RC201	4/1539	97	2252
RC202	4/1427	98	3838
RC203	4/1162	100	5983
RC204	3/979	99	7027
RC205	4/1495	95	3254
RC206	4/1357	100	3860
RC207	3/1362	100	4230
RC208	3/994	100	5598

Table 10 **Type 6,** $p_{max} = 10\%$

Instance	*K/Dist.*	*Non-viol. TW (%)*	sec_{tt} *(sec)*
R101	12/1446	47	359
R102	11/1290	65	424
R103	10/1158	82	414
R104	9/1014	90	421
R105	11/1328	67	379
R106	10/1182	78	412
R107	10/1126	92	453
R108	9/1003	97	462
R109	10/1188	78	396
R110	10/1122	91	422
R111	9/1008	70	435
R112	9/1023	95	497
C101	10/837	96	633
C102	10/964	99	819
C103	10/1003	100	1110
C104	10/975	100	1229
C105	10/830	98	714
C106	10/834	100	717
C107	10/883	98	771
C108	10/871	99	897
C109	10/917	100	1006
RC101	11/1471	58	353
RC102	11/1446	86	375
RC103	10/1279	90	389
RC104	10/1205	98	456
RC105	11/1498	80	363
RC106	10/1305	73	380
RC107	10/1309	85	380
RC108	10/1239	97	445
R201	3/1439	86	3864
R202	3/1172	92	6476
R203	3/1020	97	10234
R204	3/859	100	11787
R205	3/1151	100	7167
R206	3/1041	98	8291
R207	3/968	99	10296
R208	2/854	97	10898
R209	3/1076	99	7631
R210	3/1060	98	9677
R211	3/866	100	10049
C201	3/591	100	2276
C202	4/681	100	5774
C203	4/758	100	6830
C204	3/805	100	6046
C205	3/589	100	2400
C206	3/612	100	2353
C207	3/667	100	3044
C208	3/721	100	5183
RC201	4/1504	90	3470
RC202	3/1387	89	4174
RC203	3/1251	96	6613
RC204	3/934	100	9229
RC205	3/1362	87	3651
RC206	3/1409	99	4348
RC207	3/1285	98	5690
RC208	3/1047	100	7529