1 Appendix

Proof of Theorem 1: Let t^A be such that $t_r = \sum_{a \in r} t_a^A, r \in \mathscr{R}$ and set

$$t_a'^A = t_a^A/(1+\delta)$$

for all $a \in \mathscr{A}_{\text{toll}}$, where $t_r' = \sum_{a \in r} t_a'^A$ whenever $a \in \mathscr{A}$, $r \in \mathscr{R}$ and $\delta > 0$. Under this perturbation, the paths with the largest tolls are the most penalized, and it follows that, for any positive δ, all shortest paths under t' must have the same toll, which can be made arbitrarily close to the toll on the most profitable shortest path under t.

Proof of Theorem 2: First, we show that $\lim_{\theta \to \infty} \varphi^*_{\text{logit}}(\theta) \geq \varphi^*_{\text{det}}$. Let t^* be such that $f^{\text{det}}(t^*) = \varphi^*_{\text{det}}$. By Theorem 1, for any $\varepsilon > 0$ there exists a perturbation t' of t^* such that (i) $f^{\text{det}}(t') \geq \varphi^*_{\text{det}} - \varepsilon$ and (ii) $r, r' \in \mathscr{R}^q \Rightarrow t_r = t_{r'}, q \in \mathscr{Q}$, where r and r' are shortest paths on OD pair q. As logit probabilities, for fixed t, concentrate on shortest paths with increasing values of θ, and on account of the fact that under t' all shortest paths have identical toll values, we obtain that $\lim_{\theta \to \infty} f^{\text{logit}}(t'|\theta) \geq \varphi^*_{\text{det}} - \varepsilon$, which immediately yields

$$\lim_{\theta \to \infty} \varphi^*_{\text{logit}}(\theta) \geq \varphi^*_{\text{det}} - \varepsilon.$$

Next, let us show that $\lim_{\theta \to \infty} \varphi^*_{\text{logit}}(\theta) \leq \varphi^*_{\text{det}}$. To this aim, let us consider sequences $\theta_k \to \infty$ and $t_k \in \arg\max_t f^{\text{logit}}(t|\theta_k)$ with $x_k := \text{logit}(t_k|\theta_k)$. By compactness of T, there exists a subsequence K of indices and a toll policy $\bar{t} \in T$ such that $\lim_{k \in K} t_k = \bar{t}$. By continuity of the logit probabilities there also exists an assignment $\bar{x}$ such that $\lim_{k \in K} \text{logit}(t_k|\theta_k) = \bar{x}$. For some $q \in \mathscr{Q}$, consider paths $r, r' \in \mathscr{R}^q$ and a positive number δ such that

$$u_r(\bar{t}) \leq u_{r'}(\bar{t}) - \delta.$$

Then there exists an index $\bar{k}$ such that $k > \bar{k}$ yields (i) $\delta + u_r(t_k) \leq u_{r'}(t_k)$ and (ii)

$\lim_{k \geq \bar{k}} \text{logit}_{r'}(t_k, \theta_k) = 0$, where the last equality follows from

$$\begin{aligned} \text{logit}_{r'}(\bar{t}, \theta) &\leq \frac{\exp[-\theta u_{r'}(\bar{t})]}{\exp[-\theta u_{r'}(\bar{t})] + \exp[-\theta u_r(\bar{t})]} \\ &\leq \frac{\exp[-\theta(u_r(\bar{t}) + \delta)]}{\exp[-\theta(u_r(\bar{t}) + \delta)] + \exp[-\theta u_r(\bar{t})]} \\ &= \frac{\exp(-\theta\delta)}{\exp(-\theta\delta) + 1}, \end{aligned}$$

which goes to zero as θ goes to infinity. Since only shortest paths carry positive flow, the solution $(\bar{t}, \bar{x})$ is feasible for Program 1 and $\lim_{\theta \to \infty} \varphi^*_{\text{logit}}(\theta) \leq \varphi^*_{\text{det}}$, as desired.

Proof of Theorem 3: Consider the network of Figure 3, and let

$$c_1 < D_2 c_2 / (D_1 + D_2), \tag{1}$$

where D_i is the demand on OD-pair i (in Figure 3, we have $D_1 = 30$ and $D_2 = 1$). Let $f^{\text{det}(t)}$ be the deterministic revenue associated with toll t and let H a slightly modified Heaviside step function:

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then:

$$\begin{aligned} \max_t f^{\text{det}}(t) &= \max_t D_1 H(t - c_1) t + D_2 H(t - c_2) \\ &= \max\{(D_1 + D_2)c_1, D_2 c_2\} \\ &= D_2 c_2. \end{aligned}$$

Let $f^{\text{logit}}(t)$ denote the logit revenue. We have:

$$\begin{aligned} f^{\text{logit}}(t) &= \frac{D_1 \exp(-\theta t) t}{\exp(-\theta t) + \exp(-\theta c_1)} + \frac{D_2 \exp(-\theta t) t}{\exp(-\theta t) + \exp(-\theta c_2)} \\ &> \frac{D_1 \exp(-\theta t) t}{\exp(-\theta t) + \exp(-\theta c_1)}. \end{aligned}$$

This lower bound on the logit revenue can be made arbitrarily large by way of D_1. Furthermore, for any D_1, bounded values of D_2 and c_2 can be found to satisfy inequality (1). The conclusion follows.

The proof of Theorem 4 will be based on two lemmas.

Lemma 1. *Let w and x be feasible for Program 3. Then w is a piecewise linear function of x such that*

$$w = \sum_{n=1}^{N} m^{x\log x}(x|\alpha_n^{x\log x})\mathbf{1}\{x \in (\beta_{n-1}^{x\log x}, \beta_n^{x\log x}]\},$$

where $\mathbf{1}\{\cdot\}$ is the indicator function.

Proof. Constraints (13)–(15), together with $x \in X$, are the KKT conditions of the mathematical program:

$$\begin{aligned} \min_{w,x} \quad & (c+t)\cdot x + \theta^{-1} 1_{\mathscr{R}} \cdot w \\ \text{s.t.} \quad & w \geq m^{x\cdot\log x}(x \mid \alpha_n^{x\log x}) \quad n \in \mathscr{N} \quad (\varphi) \\ & x \in X \qquad\qquad\qquad\qquad\qquad (\pi). \end{aligned} \tag{2}$$

The convexity of $x\log x$ ensures that $\beta^{x\log x}$ is an increasing sequence such that, for all r, there exists n that satisfies

$$x_r \in (\beta_{n-1}, \beta_n] \Rightarrow m_r^{x\log x}(x \mid \alpha_n^{x\log x}) \geq m_r^{x\log x}(x \mid \alpha_\ell^{x\log x})$$

for all $\ell = 1, \ldots, N$. The problem is bounded and for each path $r \in \mathscr{R}$, at least one of the inequalities in (2) is tight. More precisely $w_r = m_r^{x\log x}(x \mid \alpha_n^{x\log x})$, and the result follows. □

Lemma 2. *In Program 2b, let $F(t,x) = \nabla_x g(t,x)$ and $SOL(F(t,\cdot),X)$, the solution set of the variational inequality* $\mathrm{VI}(F(t,\cdot),X)$*:*

$$SOL(F(t,\cdot),X) = \{y \in X | F(t,y)\cdot(x-y) \geq 0, \forall x \in X\}.$$

Let (t,x) be feasible for Program 2b and $\bar{x} \in \mathrm{SOL}(\overline{F}(t,\cdot),X_0)$ where $\overline{F}$ is an approximation of F. Then setting $d^{\max} = \max\{d_q \mid q \in \mathscr{Q}\}$ we have

$$|f(t,x) - f(t,\bar{x})| \leq \|F(t,\bar{x}) - \overline{F}(t,\bar{x})\|\, d^{\max}\, \mathrm{diam}\mathscr{T}/\theta.$$

Proof. For ease of exposition, we only consider the case of unit demand and a single OD pair. The generalization to the case with multiple OD pairs is straightforward. Program 2b is equivalently expressed

$$\begin{aligned} \max_{t,x} \quad & f\,(t,x) \\ \text{s.t.} \quad & t \ \in \mathscr{T} \\ & x \in \mathrm{SOL}(F(t,\cdot),X). \end{aligned} \tag{3}$$

Using (10) we have $x \in \mathrm{SOL}(F(t,\cdot),X) \Rightarrow x \in \mathrm{SOL}(F(t,\cdot),X_0)$ where $X_0 = \{x | x \geq \delta_{\min} 1_{\mathscr{R}}\}$. Then

$$F(t,x) = \nabla_x g(t,x) = c + t + \theta^{-1}(\log x + 1_{\mathscr{R}})$$

and so the associated Jacobian matrix F' is uniformly positive definite over X_0 with largest eigenvalue $(\theta\delta_{\min})^{-1}$ and smallest eigenvalue θ^{-1}. As X_0 is compact and F is strongly monotone with modulus θ, we have that

$$\|x - \bar{x}\| \leq \theta^{-1}\|F(t,\bar{x}) - \overline{F}(t,\bar{x})\|,$$

and thus

$$\begin{aligned} |f(t,\bar{x}) - f(t,x)| &= \sum_{q\mathscr{Q}} d_q t^q \cdot (\bar{x}^q - x^q) \\ &\leq d^{\max}\|t\|\|\bar{x} - x\| \\ &\leq d^{\max} \mathrm{diam}\, T\, \|F(t,\bar{x}) - \overline{F}(t,\bar{x})\|/\theta. \end{aligned}$$

□

Proof of Theorem 4: Program 3 is equivalently expressed

$$\begin{aligned} \max_{x,t} \quad & f(x,t) \\ \text{s.t.} \quad & t \in \mathscr{T} \\ & x \in \mathrm{SOL}(F^0(t,\cdot|\alpha^{x\log x}),X), \end{aligned} \tag{4}$$

where

$$F_r^0(t,x|\alpha^{x\log x}) = \begin{cases} F_r(t,\alpha_n^{x\log x} 1_{\mathscr{R}}) & \text{if } x_r \in (\beta_{n-1}^{x\log x}, \beta_n^{x\log x}) \\ k \in [F_r(t,\alpha_n^{x\log x} 1_{\mathscr{R}}), F_r(t,\alpha_{n+1}^{x\log x} 1_{\mathscr{R}})] & \text{if } x_r = \beta_n^{x\log x}. \end{cases}$$

Indeed, for $n(r)$ such that $x_r^0 \in (\beta_{n(r)-1}^{x\log x}, \beta_{n(r)}^{x\log x}]$, it follows from Lemma 1 that the vector function $F^0(\cdot,\cdot|\alpha^{x\log x})$ is constant over $(\beta_{n(r)-1}^{x\log x}, \beta_{n(r)}^{x\log x})$. Thus

$$|F_r(t,x^0) - F_r^0(t,x^0|\alpha)| \leq (\theta\delta_{\min})^{-1}|x_r - \alpha_{n(r)}^{x\log x}| \leq (\theta\delta_{\min})^{-1}(\beta_{n(r)}^{x\log x} - \beta_{n(r)-1}^{x\log x}) \leq (N\theta\delta_{\min})^{-1},$$

where $(\theta\delta_{\min})^{-1}$ is the largest eigenvalue value of F's Jacobian. Thus

$$|F(t,x^0) - F^0(t,x^0|\alpha^{x\log x})| = \sqrt{\sum_{r\in\mathscr{R}} (F_r(t,x^0) - F_r^0(t,x^0|\alpha^{x\log x}))^2} \leq \sqrt{R}/(N\theta\delta_{\min})$$

and the conclusion follows from Lemma 2.

The proof of Theorem 5 is based on the following lemma.

Lemma 3. *Program 4 can be written as*

$$\max_{t\in\mathscr{T},(x,\pi)\in X} \quad f^1(t,x) = \sum_{q\in\mathscr{Q}} d_q \left(-c^q\cdot x^q - \frac{1}{\theta}1_{\mathscr{R}^q}\cdot w^q + \pi_q\right) \tag{5}$$

$$\text{s.t.} \quad t\in T$$

$$w \geq m^{x\log x}(x|\alpha_n^{x\log x}) \qquad n\in\mathscr{N}_{obj}$$

$$x\in \mathrm{SOL}(F^1,X), \tag{6}$$

where

$$F^1(t,x) \quad = \quad \sum_{n\in\mathscr{M}} [\nabla_x F(t,\alpha_n^{\log x}1_{\mathscr{R}})(x-\alpha_n^{\log x}1_{\mathscr{R}}) + F(t,\alpha_n^{\log x}1_{\mathscr{R}})]\mathbf{1}\left\{x\in(\beta_n^{logx},\beta_{n+1}^{logx}]\right\}.$$

Proof. The function F^1 is obtained by replacing the logarithm in $F(t,x) = c+t+\theta^{-1}\log(x+1)$ by the a piecewise linear approximation:

$$\begin{aligned}
F^1(t,x) &= \sum_{n\in\mathscr{N}} [\nabla_x F(t,\alpha_n 1_{\mathscr{R}})(x-\alpha_n 1_{\mathscr{R}}) + F(t,\alpha_n 1_{\mathscr{R}})]1\{x\in(\beta_n,\beta_{n+1}]\} \\
&= \sum_{n\in\mathscr{N}} [(\theta\alpha_n)^{-1}(x-\alpha_n 1_{\mathscr{R}}) + c + t + \theta^{-1}(\log\alpha_n 1_{\mathscr{R}} + 1_{\mathscr{R}})]\mathbf{1}\left\{x\in(\beta_n^{\log x},\beta_{n+1}^{\log x}]\right\} \\
&= \theta^{-1}1_{\mathscr{R}} + c + t + \theta^{-1}\sum_{n\in\mathscr{N}} m^{\log x}(x|\alpha_n)\mathbf{1}\left\{x\in(\beta_n^{\log x},\beta_{n+1}^{\log x}]\right\}
\end{aligned}$$

□

Proof of Theorem 5: To prove Theorem 5 we now show that (i) $F^1(x^1) = F(x^1) + O(M^{-2})$, and then that (ii) $f^1(t,x^1) = f(t,x^1) + O(N^{-2})$, which allows to complete the proof. (i) For $r\in\mathscr{R}$, let $n(r)$ be such that $x_r^1 \in (\beta_{n(r)-1}^{\log x}, \beta_{n(r)}^{\log x}]$. From Lemma 3 it follows that F_r^1 is a first-order approximation of F_r over $(\beta_{n(r)-1}^{\log x}, \beta_{n(r)}^{\log x})$. Thus $|F_r(t,x^1) - F_r^1(t,x^1)| \leq (\theta\delta_{\min})^{-1}(x_r - \alpha_n^{\log x})^2 \leq (\theta\delta_{\min})^{-1}(\beta_{n(r)}^{\log x} - \beta_{n(r)-1}^{\log x})^2 \leq (M^2\theta\delta_{\min})^{-1}$, where $(\theta\delta_{\min})^{-1}$ is the largest eigenvalue value of the Jacobian F', and $\delta_{\min}$ is the smallest logit choice probability (defined as in (10). It follows that

$$|F(t,x^1) - F(t,x^1|\alpha^{\log x})| = \sqrt{\sum_r [F_r(t,x^1) - F_r^1(t,x^1)]^2} \leq \sqrt{R}/(M^2\theta\delta_{\min})$$

and by Lemma 2 it follows that $f(t,x^0)\in O(M^{-2})$. (ii) By Theorem 1 the function f^1 is obtained from a piecewise linear approximation of the entropic term in the exact logit revenue expression of Program 2c. Now by construction of the sequence $\beta^{x\log x}$, where the width of each step is inversely proportional to N, we have that $f(t,x^0)\in O(N^{-2})$. The conclusion follows.