Online Supplement
Delayed Purchase Options in Single-Leg Revenue Management

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PROPOSITION 4.1 *Suppose the probability of a request being a commitment is class independent; that is, $\nu_1 = \nu_2 = \cdots = \nu_m$. Then, given the fare ordering $f_1 \geq f_2 \geq ... \geq f_m$, and hence, the ordering of the expected commitment revenues, $\phi_1 \geq \phi_2 \geq ... \geq \phi_m$, we have $u^*_{1t} \geq u^*_{2t} \geq ... \geq u^*_{mt}, t = 1, ..., T$.*

Proof. For given x and y, to maximize $V_t(x_t, y_t)$ we accept a booking or commitment request ($u^*_{it} = 1$) if

$$\begin{aligned} p_{it}(f_i + &\mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))) + q_{it}(\phi_i + \mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)) \\ &\geq p_{it}\mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) + q_{it}\mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)). \end{aligned}$$

Let $\nu := \nu_1 = \nu_2 = \cdots = \nu_m$. Then, by using $p_{it} = (1-\nu)\alpha_{it}$ and $q_{it} = \nu\alpha_{it}$, we obtain

$$\begin{aligned} (1-\nu)(f_i + &\mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))) + \nu(\phi_i + \mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)) \\ &\geq \mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)). \end{aligned} \tag{17}$$

Since $f_{i-1} \geq f_i$ and $\phi_{i-1} \geq \phi_i$, if relation (17) holds for a fare class i request ($u^*_{it} = 1$), then it also holds for the fare class $i-1$ request $u^*_{(i-1)t} = 1$. Similarly, if relation (17) does not hold for the expensive fare class $i-1$, then it does not hold for the cheaper fare class i either. This means that, if $u^*_{(i-1)t} = 0$ then $u^*_{it} = 0$. Therefore, we obtain the desired result. □

PROPOSITION 5.1 *The optimal objective value of the DLP model gives an upper bound on the dynamic programming model (1a)-(1b). That is, $J_1(0, \mathbf{0}) \leq Z^*_{DLP}$.*

Proof. Suppose the random variables $W_{it}, \forall i, t$ denote the number of reservations accepted over the planning horizon under the optimal policy of the dynamic programming model. Each accepted reservation for fare class i either buys the contingent commitment option with probability ν_i or books the seat with probability $(1-\nu_i)$. Let X_{it} and Z_{it} be the random numbers of bookings and commitments accepted for fare class i at time period t, respectively. Since an accepted commitment request can leave with probability p_l, we also let S_{it} and L_{it} be the binary random numbers denoting the sold (exercised) and not exercised commitments, respectively. That is, S_{it} takes value 1, if there is a commitment reservation for fare class i at time period t and this commitment customer decides to exercise the option, and L_{it} takes value 1 if this commitment reservation leaves. As a result, $X_{it} + Z_{it} = W_{it}$ and $S_{it} + L_{it} = Z_{it}$ for all i, t.

Let now D_{it} be the random number of reservation requests for fare class i at time period t. Then, we have,

$$\mathcal{V}_1 = C, \tag{18}$$

$$\mathcal{V}_t = \mathcal{V}_{t-1} - \sum_{i=1}^{m} X_{i(t-1)} - \sum_{i=1}^{m} Z_{i(t-1)}, \qquad 2 \le t \le s+1, \tag{19}$$

$$\mathcal{V}_t = \mathcal{V}_{t-1} - \sum_{i=1}^{m} X_{i(t-1)} - \sum_{i=1}^{m} Z_{i(t-1)} + \sum_{i=1}^{m} L_{i(t-s-1)}, \quad s+2 \le t \le T, \tag{20}$$

$$\mathcal{V}_{T+1} = \mathcal{V}_T - \sum_{i=1}^{m} X_{iT} - \sum_{i=1}^{m} Z_{iT} + \sum_{k=T-s}^{T} \sum_{i=1}^{m} L_{ik}, \tag{21}$$

$$X_{it} + Z_{it} \le D_{it}, \qquad i = 1, ..., m; t = 1, ..., T, \tag{22}$$

$$\sum_{i=1}^{m} X_{it} + \sum_{i=1}^{m} Z_{it} \le \mathcal{V}_t, \quad t = 1, ..., T, \tag{23}$$

where (18)-(21) ensure that the balance equations in each time period holds, (22) ensures that total number of bookings and commitments that we accept under the optimal policy do not exceed the reservation requests. Similarly, (23) guarantees that the total number of bookings and commitments that we accept do not exceed the available capacity. Consequently, the total revenue under the optimal policy of the dynamic programming is

$$\sum_{t=1}^{T}\sum_{i=1}^{m} f_i X_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} f^c Z_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} f_i S_{it}.$$

By conditioning on W_{it} we trivially obtain $\mathbb{E}(Z_{it}) = \nu\mathbb{E}(W_{it})$. Since $X_{it} = W_{it} - Z_{it}$, we have $\mathbb{E}(X_{it}) = (1-\nu_i)\mathbb{E}(W_{it})$. Similarly, conditioning on Z_{it} leads to $\mathbb{E}(S_{it}) = p_b\nu_i\mathbb{E}(W_{it})$. Therefore, the total expected revenue is given by

$$J_1(0, \mathbf{0}) = \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)\mathbb{E}(W_{it}) + \sum_{t=1}^{T}\sum_{i=1}^{m} f^c\nu_i\mathbb{E}(W_{it}) + \sum_{t=1}^{T}\sum_{i=1}^{m} f_i p_b \nu_i \mathbb{E}(W_{it}).$$

Taking the expectations (18)-(23) and noting that $\mathbb{E}(D_{it}) = \alpha_{it}$, the solution given by $w_{it} = \mathbb{E}(W_{it})$ and $\vartheta_t = \mathbb{E}(\mathcal{V}_t)$ is feasible for the DLP model (4)-(12). Therefore, we have

$$Z^*_{DLP} \ge J_1(0, \mathbf{0}) = \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)\mathbb{E}(W_{it}) + \sum_{t=1}^{T}\sum_{i=1}^{m} \phi_i\nu_i\mathbb{E}(W_{it}),$$

and the desired result holds. $\square$

To prove the asymptotic bound result in Proposition 5.2, we first define a lower bound on the rate of convergence. Let d_{ib} and d_{ic} denote the random numbers of total fare class i requests for bookings and commitments, respectively. Then, the expected demands are computed as $\mu_i^b := \mathbb{E}(d_{ib}) = (1-\nu_i)\sum_{t=1}^{T}\alpha_{it}$ and $\mu_i^c := \mathbb{E}(d_{ic}) = \nu_i\sum_{t=1}^{T}\alpha_{it}$. Likewise, σ_{ib} and σ_{ic} denote the corresponding standard deviations. Then, the coefficient of variation of the number of requests for bookings and commitments are given as

$$CV_i^b = \frac{\sqrt{\sigma_{ib}^2}}{\mu_i^b} \text{ and } CV_i^c = \frac{\sqrt{\sigma_{ic}^2}}{\mu_i^c}, \text{ for } i = 1, ..., m.$$

We also define

$$CV = \max_{1\le i\le m}\left\{CV_i^b, CV_i^c\right\},$$

as the maximum coefficient of variation.

Proposition A *Let CV denote the maximum coefficient of variation over bookings and commitments for all fare classes. Then for $\epsilon \in [1-p_b, 1]$, we have $J_1(0, \mathbf{0}) \geq \left(1-\epsilon-\frac{CV^2}{\epsilon^2}\right) Z^*_{DLP-UB}$.*

Proof. Let $\{w^*_{it} : \forall i, t\}$ be the optimal value of the decision variables in problem (13)-(16). We consider a policy π that accepts at most $(1-\epsilon)(1-\nu_i)\sum_{t=1}^{T} w^*_{it}$ booking requests and $(1-\epsilon)\nu_i \sum_{t=1}^{T} w^*_{it}$ contingent commitment requests for fare class i for $\epsilon \in (0, 1)$. Due to the capacity constraint (14) in DLP-UB model, the policy π is feasible if $(1-\epsilon) \leq p_b$. The expected revenue $\mathcal{P}^\pi$ is given by

$$\mathcal{P}^\pi = \mathbb{E}[\sum_{i=1}^{m} f_i \min(d_{ib}, (1-\epsilon)\sum_{t=1}^{T}(1-\nu_i)w^*_{it}) + \sum_{i=1}^{m} f^c \min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}) + \sum_{i=1}^{m} f_i \mathcal{S}(\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}))],$$

where $\mathcal{S}(k)$ is a binomial random variable with k independent trials with success probability p_b and it gives the number of purchased committed seats. A lower bound to the generic term in the expression for $\mathcal{P}^\pi$ is then given by

$$\mathbb{E}[\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it})] \geq (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\mathbf{P}(d_{ic} \geq (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}) \tag{24}$$

$$\geq (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\mathbf{P}(d_{ic} \geq (1-\epsilon)\sum_{t=1}^{T}\nu_i \alpha_{it}) \tag{25}$$

$$\geq (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\left(1-\frac{CV_i^{c2}}{CV_i^{c2}+\epsilon^2}\right) \tag{26}$$

$$\geq (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\left(1-\frac{CV^2}{\epsilon^2}\right). \tag{27}$$

The inequality (25) holds since $\nu_i w^*_{it} \leq \nu_i \alpha_{it}$, (26) follows from the Marshall's inequality and (27) holds due to the definition of CV. Since $\mathbb{E}[\mathcal{S}(\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}))] = p_b\mathbb{E}[\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it})]$, we can give a lower bound to $\mathcal{P}^\pi$ by using the inequality (27) as follows:

$$\begin{aligned}
\mathcal{P}^\pi &\geq (1-\epsilon)\left(1-\frac{CV^2}{\epsilon^2}\right)(\sum_{i=1}^{m}\sum_{t=1}^{T} f_i(1-\nu_i)w^*_{it} + \sum_{i=1}^{m}\sum_{t=1}^{T} f^c\nu_i w^*_{it} + \sum_{i=1}^{m}\sum_{t=1}^{T} f_i p_b \nu_i w^*_{it}) \\
&\geq (1-\epsilon)\left(1-\frac{CV^2}{\epsilon^2}\right) Z^*_{DLP-UB} \\
&\geq \left(1-\epsilon-\frac{CV^2}{\epsilon^2}\right) Z^*_{DLP-UB}
\end{aligned}$$

This implies

$$J_1(0, \mathbf{0}) \geq \mathcal{P}^\pi \geq \left(1-\epsilon-\frac{CV^2}{\epsilon^2}\right) Z^*_{DLP-UB}.$$

□

To tighten the lower bound in the above inequality, we maximize it over ϵ and obtain $\epsilon^* = \max\{(2CV^2)^{1/3}, 1-p_b\}$. Since $\epsilon \in [1-p_b, 1]$, this tighter bound is only obtained when $2CV^2 < 1$. Consequently we have,

$$J_1(0, \mathbf{0}) \geq \mathcal{P}^\pi \geq \left(1-\epsilon^*-\frac{CV^2}{\epsilon^{*2}}\right) Z^*_{DLP-UB}.$$

Next we examine the structure of the lower bound as the problem size gets large.

PROPOSITION 5.2 *Given $\epsilon \in [1-p_b, 1]$ and $\kappa > 0$, we have*

$$Z^{\kappa}_{DLP-UB} \geq Z^{\kappa}_{DLP} \geq J^{\kappa}_{1}(0,\mathbf{0}) \geq \left(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\right) Z^{\kappa}_{DLP-UB},$$

where CV denotes the maximum coefficient of variation over bookings and commitments for all fare classes. Therefore,

$$p_b \leq \lim_{\kappa\to\infty} \frac{J^{\kappa}_{1}(0,\mathbf{0})}{Z^{\kappa}_{DLP}} \leq 1.$$

Proof. We observe that if $\{w^*_{it} : \forall i, t\}$ is an optimal solution to problem (13)-(16). Then $\{w^*_{i\lceil t/\kappa\rceil} : \forall i, t\}$ is an optimal solution for the scaled problem. Thus, it follows that $Z^{\kappa}_{DLP-UB} = \kappa Z_{DLP-UB}$. For the scaled problems, the expected demand and the variance are scaled with κ. If μ and σ^2 denote the mean demand and variance for problem $\mathcal{P}^1$, then the mean demand is $\kappa\mu$ and the variance is $\kappa\sigma^2$ for the problem $\mathcal{P}^{\kappa}$. Therefore the maximum coefficient of variation of the scaled problem is

$$CV^{\kappa} = \max_{1\leq i\leq m}\left\{\frac{\sqrt{\kappa\sigma^2_{ib}}}{\kappa\mu^b_i}, \frac{\sqrt{\kappa\sigma^2_{ic}}}{\kappa\mu^c_i}\right\} = \frac{CV\sqrt{\kappa}}{\kappa}$$

By following the result of Proposition A above and replacing CV^{κ} with $\frac{CV\sqrt{\kappa}}{\kappa}$, we have

$$J^{\kappa}_{1}(0,\mathbf{0}) \geq \left(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\right)Z^{\kappa}_{DLP-UB}.$$

When κ goes to infinity, the expression $\left(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\right)$ approaches to $(1-\epsilon)$. Since $\epsilon \in [1-p_b, 1]$, this bound is tighter when $\epsilon = 1-p_b$. Therefore, as p_b goes to 1, the upper bound obtained from Z^*_{DLP-UB} becomes asymptotically tight. Following the result of Proposition A, we obtain the following convergence rate

$$\kappa Z_{DLP-UB} \geq Z^{\kappa}_{DLP} \geq J^{\kappa}_{1}(0,\mathbf{0}) \geq \left(1-\epsilon-\frac{CV^2}{\kappa\epsilon^2}\right)\kappa Z_{DLP-UB},$$

Dividing the chain of inequalities by κZ_{DLP-UB} and taking the limit as κ goes to infinity, we get

$$p_b \leq \lim_{\kappa\to\infty}\frac{J^{\kappa}_{1}(0,\mathbf{0})}{Z^{\kappa}_{DLP-UB}} \leq \lim_{\kappa\to\infty}\frac{Z^{\kappa}_{DLP}}{Z^{\kappa}_{DLP-UB}} \leq 1$$

which implies,

$$p_b \leq \lim_{\kappa\to\infty} \frac{J^{\kappa}_{1}(0,\mathbf{0})}{Z^{\kappa}_{DLP}} \leq 1.$$

□