Robust Hazmat Network Design Problems Considering Risk Uncertainty

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Appendices

A Proof of Projection Method

The procedure introduced in Section 5.2.3 obtains $\bar{\mu}$ as the projection of μ onto $\{\mu : 0 \leqslant \mu_{ij} \leqslant 1 \ \forall (i,j) \in \mathcal{A}_k, \Gamma - \sum_{(i,j)\in\mathcal{A}_k} \mu_{ij} \geqslant 0\}$.

Proof. The projection of μ onto $\{\mu : 0 \leqslant \mu_{ij} \leqslant 1 \ \forall (i,j) \in \mathcal{A}_k, \Gamma - \sum_{(i,j)\in\mathcal{A}_k} \mu_{ij} \geqslant 0\}$ can be formulated as:

$$\min \sum_{(i,j)\in\mathcal{A}_k} \frac{1}{2}(\mu_{ij} - \bar{\mu}_{ij})^2 \tag{1}$$

subject to

$$\sum_{(i,j)\in\mathcal{A}_k} \bar{\mu}_{ij} \leqslant \Gamma \qquad : \lambda$$

$$\bar{\mu}_{ij} \leqslant 1 \qquad \forall (i,j) \in \mathcal{A}_k \qquad : v_{ij}$$

$$\bar{\mu}_{ij} \geqslant 0 \qquad \forall (i,j) \in \mathcal{A}_k \qquad : \rho_{ij}$$

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The Karush-Kuhn-Tucker (KKT) conditions for problem (1) are

$$\bar{\mu}_{ij} - \mu_{ij} + \lambda + v_{ij} - \rho_{ij} = 0 \tag{2}$$

$$\lambda\Bigg(\sum_{(i,j)\in\mathcal{A}_k} \bar{\mu}_{ij} - \Gamma \Bigg) = 0 \tag{3}$$

$$v_{ij}(\bar{\mu}_{ij} - 1) = 0 \tag{4}$$

$$\rho_{ij}\bar{\mu}_{ij} = 0 \tag{5}$$

$$\lambda, v_{ij}, \rho_{ij} \geqslant 0 \tag{6}$$

$$\sum_{(i,j)\in\mathcal{A}_k} \bar{\mu}_{ij} \leqslant \Gamma \tag{7}$$

$$\bar{\mu}_{ij} \leqslant 1 \tag{8}$$

$$\bar{\mu}_{ij} \geqslant 0 \tag{9}$$

Problem (1) is strictly convex over a nonempty bounded region, so there must exist multipliers λ, v_{ij} and ρ_{ij} that satisfy the KKT conditions. We show that we can find a solution to the above KKT conditions by following the projection procedure in Section 5.2.3.

1. For any λ, we set

$$\begin{aligned} \bar{\mu}_{ij} &= \max\{0, \min\{1, \mu_{ij} - \lambda\}\} \\ v_{ij} &= \max\{0, \mu_{ij} - \lambda - 1\} \\ \rho_{ij} &= \max\{0, -(\mu_{ij} - \lambda)\} \end{aligned} \tag{10}$$

   for each $(i, j) \in \mathcal{A}_k$. We consider the following three cases:

   - When $\mu_{ij} - \lambda \geqslant 1$, we have $\bar{\mu}_{ij} = 1$, $\rho_{ij} = 0$, and $v_{ij} = \mu_{ij} - \lambda - 1$.
   - When $0 \leqslant \mu_{ij} - \lambda < 1$, we have $\bar{\mu}_{ij} = \mu_{ij} - \lambda$, $\rho_{ij} = 0$, and $v_{ij} = 0$.
   - When $\mu_{ij} - \lambda < 0$, we have $\bar{\mu}_{ij} = 0$, $\rho_{ij} = -(\mu_{ij} - \lambda)$, and $v_{ij} = 0$.

   The above $\bar{\mu}_{ij}$, ρ_{ij}, and v_{ij} satisfy conditions (2), (4), (5), (6), (8), and (9) for any given λ. Remaining conditions are (3), (7), and the nonnegativity of λ.

2. When $\lambda = 0$, if the resulting $\bar{\mu}_{ij} = \max\{0, \min\{1, \mu_{ij}\}\}$ satisfies (7), then we are done as described in Step 0 of the projection procedure. If not, we can show that we can obtain $\lambda > 0$ such that (3) and (7) hold, by following Step 1 of the projection procedure. That is, we can find $\lambda > 0$ such

that

$$\sum_{(i,j)\in\mathcal{A}_k} \bar{\mu}_{ij} = \sum_{(i,j)\in\mathcal{A}_k} \max\{0, \min\{1, \mu_{ij} - \lambda\}\} = \Gamma. \tag{11}$$

Equation (11) can be rewritten as

$$\sum_{(i,j)\in\mathcal{A}_k} \bar{\mu}_{ij} = \sum_{\{(i,j)\in\mathcal{A}_k : \mu_{ij}-1 \leqslant \lambda < \mu_{ij}\}} (\mu_{ij} - \lambda) + \sum_{\{(i,j)\in\mathcal{A}_k : \lambda < \mu_{ij}-1\}} (1) + \sum_{\{(i,j)\in\mathcal{A}_k : \lambda \geqslant \mu_{ij}\}} (0) = \Gamma. \tag{12}$$

In equation (12), μ_{ij} and $\mu_{ij} - 1$ are critical values. Thus we can find the value of λ by searching the interval of these values as used in Step 1.

This completes the proof. $\square$