Appendix A

Convergence of the NIFE's Solution Algorithm

In Section (3), we have shown that the NIFE's solution algorithm adjusts the vector of LMs sequentially along unit directions $e_1, e_2, \ldots, e_{n_w}$. In the nth iteration, W_n is adjusted n_w times. Let $w_{n,k}$ denote the vector of LMs after the kth adjustment, and $\tau_{n,k}$ be the adjustment factor along the unit direction e_k. That is, we have

$$\begin{aligned} w_{n,0} &= W_n, && (31)\\ w_{n,k} &= w_{n,k-1} + \tau_{n,k} e_k, k = 1, 2, \ldots, n_w, && (32)\\ W_{n+1} &= w_{n,n_w}. && (33) \end{aligned}$$

Proposition A.1. *Let Ψ be one iteration of the solution algorithm, that is,*

$$W_{n+1} = \Psi(W_n).$$

If $W_n \notin \Omega$, then

$$\theta(W_{n+1}) > \theta(W_n),$$

where Ω is defined by Eq (29).

Proof. Since $W_n \notin \Omega$, there must exist an adjustment $\tau_{n,k_0} \neq 0, 1 \leq k_0 \leq n_w$, such that

$$w_{n,k_0} = w_{n,k_0-1} + \tau_{n,k_0} e_{k_0}. \tag{34}$$

Note that, by Eq (20), we have

$$\theta(w_{n,k_0}) \quad = \quad \inf\{H(w_{n,k_0})(F),\ F \in R_+^{n_f}\}.$$

By Proposition (3.4) and Eq (34), we have

$$\theta(w_{n,k_0}) = H(w_{n,k_0})(F_{w_{n,k_0}}) = H(w_{n,k_0-1})(F_{w_{n,k_0}}) + \tau_{n,k_0} e_{k_0}^t \times \nabla\theta(w_{n,k_0}).$$

By Proposition (3.6)-(3.9), we have

$$\theta(w_{n,k_0}) \quad \geq \quad H(w_{n,k_0-1})(F_{w_{n,k_0}}).$$

Since τ_{n,k_0} is made to only one component of w_{n,k_0-1}, we have

$$F_{w_{n,k_0-1}} \neq F_{w_{n,k_0}}.$$

By Proposition (3.4), $X(w_{n,k_0-1}) = \{F_{w_{n,k_0-1}}\}$ is a singleton. Therefore, we have

$$\theta(w_{n,k_0}) \geq H(w_{n,k_0-1})(F_{w_{n,k_0}}) > H(w_{n,k_0-1})(F_{w_{n,k_0-1}}) = \theta(w_{n,k_0-1}).$$

Thus, we have

$$\theta(W_{n+1}) \geq \theta(w_{n,k_0}) > \theta(w_{n,k_0-1}) \geq \theta(W_n).$$

$\square$

Adding Eq (31), Eq (32) and Eq (33) together, we have,

$$W_{n+1} \quad = \quad W_n + \tau_{n,1}e_1 + \tau_{n,2}e_2 + \ldots, \tau_{n,n_w}e_{n_w}.$$

For simplicity, the above equation is written in the following matrix form:

$$W_{n+1} \quad = \quad W_n + T_n I, \tag{35}$$

where $T_n = (\tau_{n,1}, \tau_{n,2}, \ldots, \tau_{n,n_w})$ and I is the identity matrix.

Proposition A.2. *Let $W_n \to \overline{W}$, and $V_n \to \overline{V}$, as $n \to +\infty$, where $V_n = \Psi(W_n)$. Since D_w is closed, we have $\overline{V}, \overline{W} \in D_w$. Further, we have*

$$\overline{V} = \Psi(\overline{W}). \tag{36}$$

Proof. By Eq (35), we have

$$V_n \quad = \quad W_n + T_n I. \tag{37}$$

The $\{W_n\}_n$ and $\{V_n\}_n$ are convergent sequences. Therefore, by Eq (37), sequence $\{T_n\}_n$ is also convergent. That is, we have

$$T_n \quad \to \quad T, \text{as } n \to +\infty,$$

where $T = (\tau_1, \tau_2, \ldots, \tau_{n_w})$. In other words, we have

$$\tau_{n,k} \quad \to \quad \tau_k, \text{ as } n \to +\infty, \text{ for } k = 1, \ldots, n_w.$$

Since sequences $\{W_n\}_n$, $\{V_n\}_n$, and $\{\tau_{n,k}\}_n, k = 1, \ldots, n_w$, are convergent, by Eq (31) - (33), we have

$$\{w_{n,k}\}_n, \text{ for } k = 0, 1, \ldots, n_w,$$

are also convergent. We assume that

$$\lim_{n \to +\infty} w_{n,k} = w_k, \text{ for } k = 0, 1, \ldots, n_w. \tag{38}$$

Then, by letting $n \to +\infty$ in Eq (31) - (33), we have

$$\begin{aligned} w_0 &= \overline{W}, \\ w_k &= w_{k-1} + \tau_k e_k, \ k = 1, \ldots, n_w, \\ \overline{V} &= w_{n_w}. \end{aligned} \tag{39}$$

To show Eq (36) holds, we need to show that

$$\theta(w_k) = \theta(w_{k-1} + \tau_k e_k) \geq \theta(w_{k-1} + s e_k), s \in D_k, \text{ for } k = 1, 2, \ldots, n_w,$$

where D_k is the feasible region of s.

For generality, we assume that the adjustment τ_k is made to LM λ which is the kth component of w_{k-1}. Consider the following two cases of λ:

Case 1: λ is associated with constraints (1) or (2). In this case, $D_k = D_{n,k} = R$. By Proposition (3.6), (3.7), and Eq (39), for any $s \in D_{n,k}$, we have

$$\theta(w_{n,k}) = \theta(w_{n,k-1}, +\tau_{n,k} e_k) \geq \theta(w_{n,k-1}, +s e_k), \text{ for } k = 1, 2, \ldots, n_w.$$

By the continuity of θ, as $n \to +\infty$, we have

$$\theta(w_k) = \theta(w_{k-1}, +\tau_k e_k) \geq \theta(w_{k-1}, +s e_k), s \in D_k, \text{ for } k = 1, 2, \ldots, n_w$$

Case 2: λ is associated with constraints (3). Let λ_n be the kth component of $w_{n,k-1}$, then, by Eq (38), we have

$$\lim_{n \to +\infty} \lambda_n = \lambda. \tag{40}$$

Further, we have

$$D_k = \{s : s \geq -\lambda\}, \text{and } D_{n,k} = \{s : s \geq -\lambda_n\}, k = 1, 2, \ldots, n_w.$$

Given k, for any $s \in (-\lambda, +\infty)$, by Eq (40), there exists a $N > 0$ such that $-\lambda_n < s$. This means $s \in D_{n,k}$ for $n > N$. Therefore, for the s, we have

$$\theta(w_{n,k}) = \theta(w_{n,k-1} + \tau_{n,k} e_k) \geq \theta(w_{n,k-1} + s e_k).$$

Let $n \to +\infty$, we have

$$\theta(w_k) = \theta(w_{k-1} + \tau_k e_k) \geq \theta(w_{k-1} + s e_k).$$

We have shown that the above inequality holds for every $s \in (-\lambda, +\infty)$. By the continuity of θ, we have

$$\theta(w_k) = \theta(w_{k-1} + \tau_k e_k) \geq \theta(w_{k-1} + s e_k), \text{for } s \in D_k.$$

□

Proposition A.3. *Let $\{W_n\}_{n=1}^{+\infty}$ be the sequence produced by the solution algorithm. If problem (P) is feasible, then we have*

1)

$$\lim_{n \to +\infty} ||W_{n+1} - W_n|| = 0. \tag{41}$$

2) The solution algorithm is convergent. That is, every convergent subsequence of $\{W_n\}_{n=1}^{+\infty}$ *has a limit in* Ω*. In other words, all accumulation points of* $\{W_n\}_{n=1}^{+\infty}$ *belong to* Ω*, where* Ω *is the set defined by (29).*

3) $\theta(W_n) \to \theta(\overline{W})$*, for some* $\overline{W} \in \Omega$*.*

Proof. By Proposition (A.1), the sequence $\{\theta(W_n)\}_{n=1}^{+\infty}$ is monotonically increasing. Since problem (P) is feasible, the sequence is bounded and hence convergent. Therefore, we have

$$\lim_{n\to+\infty} \theta(W_{n+1}) - \theta(W_n) = 0, \tag{42}$$

For a given $k,\ 1 \le k \le n_w$, for any n, by the proof of Proposition (A.1), we have

$$\theta(W_{n+1}) \ge \theta(w_{n,k}) \ge \theta(w_{n,k-1}) \ge \theta(W_n).$$

That is

$$\theta(W_{n+1}) - \theta(W_n) \ge \theta(w_{n,k}) - \theta(w_{n,k-1}) \ge 0$$

By Eq (42), we have

$$\lim_{n\to+\infty} \theta(w_{n,k}) - \theta(w_{n,k-1}) = 0. \tag{43}$$

Using Eq (8), Eq (9), Eq (19), and Proposition (3.4), we have

$$\begin{aligned}\theta(w_{n,k-1}) \quad = \quad & \sum_i \lambda_i M_i - \sum_{d\in D, a\in B^d, k\in\kappa_a^d} \eta_{k,a}^d C_{k,a}^d + \sum_{d\in D, a\in B^d, k\in\kappa_a^d} \omega_{k,a}^d O_{k,a}^d - \sum_{d\in D, a\in B^d} (1-\mu_a^d) q_a^d \\ & - \sum_{d\in D, a\in B^d, k\in\kappa_a^d} \mu_a^d f_{k,a}^d, \end{aligned} \tag{44}$$

where $f_{k,a}^d$s and q_a^ds are the components of $F_{w_{n,k-1}}$. By Eq (32), $w_{n,k}$ and $w_{n,k-1}$ have at most one different component (i.e., the kth component). By Eq (44), we have

$$\theta(w_{n,k}) - \theta(w_{n,k-1}) = \begin{cases} \tau_{n,k} M_i + \sum\limits_{D, B^d, \kappa_a^d} \mu_a^d f_{k,a}^d \left\{1 - \exp\left(\dfrac{\tau_{n,k}\delta_{k,i}^d}{\mu_a^d}\right)\right\}, \text{if } \tau_{n,k} \text{ is made to } \lambda_i, \\ (1-\mu_a^d) q_a^d \left\{1 - \exp\left(\dfrac{-\tau_{n,k}}{1-\mu_a^d}\right)\right\} + \sum\limits_{k\in\kappa_a^d} \mu_a^d f_{k,a}^d \left\{1 - \exp\left(\dfrac{\tau_{n,k}}{\mu_a^d}\right)\right\}, \text{if } \tau_{n,k} \text{ is made to } \nu_a^d, \\ -\tau_{n,k} C_{k,a}^d + \mu_a^d f_{k,a}^d \left\{1 - \exp\left(\dfrac{-\tau_{n,k}}{\mu_a^d}\right)\right\}, \text{if } \tau_{n,k} \text{ is made to } \eta_{k,a}^d, \\ \tau_{n,k} O_{k,a}^d + \mu_a^d f_{k,a}^d \left\{1 - \exp\left(\dfrac{\tau_{n,k}}{\mu_a^d}\right)\right\}, \text{if } \tau_{n,k} \text{ is made to } \omega_{k,a}^d. \end{cases}$$

Without loss of generality, we assume that $\tau_{n,k}$ is made to LM λ_i. Then, we have

$$\theta(w_{n,k}) - \theta(w_{n,k-1}) = \tau_{n,k} M_i + \sum_{d\in D, a\in B^d, k\in\kappa_a^d} \mu_a^d f_{k,a}^d \left\{1 - \exp\left(\frac{\tau_{n,k}}{\mu_a^d}\right)\right\} \delta_{k,i}^d.$$

Note that, we always have

$$1 - \exp\left(\frac{\tau_{n,k}}{\mu_a^d}\right) \leq -\frac{\tau_{n,k}}{\mu_a^d}.$$

Therefore, it is easy to verify that

$$\theta(w_{n,k}) - \theta(w_{n,k-1}) \leq \tau_{n,k} M_i - \tau_{n,k}(\sum_{d\in D, a\in B^d, k\in\kappa_a^d} f_{k,a}^d \delta_{k,i}^d) = \tau_{n,k}(M_i - M) = \tau_{n,k} M_i \left\{1 - \exp\left(-\frac{\tau_{n,k}}{\tau'}\right)\right\} \leq 0,$$

where M is defined in the step 2 of the solution algorithm. Note that M_i and τ' are constants for airport i. Therefore, by Eq (43), we have

$$\lim_{n\to+\infty} \tau_{n,k} \left\{1 - \exp\left(-\frac{\tau_{n,k}}{\tau'}\right)\right\} = 0. \tag{45}$$

Next, we will show that

$$\lim_{n\to+\infty} \tau_{n,k} = \tau_k = 0, \text{for } \ 1 \leq k \leq n_w. \tag{46}$$

The sequence $\{\tau_{n,k}\}_{n=1}^{+\infty}$ is bounded, otherwise, it must have a subsequence with ∞ as its limit. This contradicts Eq (45). Assume that the sequence $\{\tau_{n,k}\}_{n=1}^{+\infty}$ does not converge to zero, then $\exists\varepsilon > 0$ and a subsequence $\{\tau_{n,k}\}_{n\in N}$ such that

$$|\tau_{n,k} - 0| > \varepsilon, \ n \in N.$$

Since the sequence $\{\tau_{n,k}\}_{n\in N}$ is also bounded, there exists a convergent subsequence of $\{\tau_{n,k}\}_{n\in N}$. Without loss of generality, we assume that it is $\{\tau_{n,k}\}_{n\in N}$ itself, that is, we have

$$\lim_{n\in N} \tau_{n,k} = \tau_k.$$

Note that we have

$$\tau_k \neq 0. \tag{47}$$

Further, by Eq (45), we have

$$\tau_k \left\{1 - \exp\left(-\frac{\tau_k}{\tau'}\right)\right\} = 0.$$

Note that the above equation has only one solution, that is $\tau_k = 0$, which contradicts Eq (47). Therefore, Eq (46) holds.

Further, note that

$$||W_{n+1} - W_n||^2 = \sum_{k=1}^{n_w} \tau_{n,k}^2.$$

By Eq (46), Eq (41) holds. This completes the proof of *1)*

Let $\{W_n\}_{n\in N_1}$ be a convergent subsequent of $\{W_n\}_{n=1}^{+\infty}$, and $\lim_{n\in N_1} W_n = W_1$. We will show that

$$W_1 \in \Omega. \tag{48}$$

Assume that

$$W_1 \notin \Omega. \tag{49}$$

Let

$$W_{n+1} = \Psi(W_n), \text{for } n \in N_1,$$

$\{W_{n+1}\}_{n\in N_1}$ is also a subsequence of $\{W_n\}_{n=1}^{+\infty}$. Next, we will show that $\{W_{n+1}\}_{n\in N_1}$ is bounded.

Since $\{W_n\}_{n\in N_1}$ is convergent, it is bounded. That is, $\exists M_1 > 0$ such that

$$||W_n|| \leq M_1, \text{for } n \in N_1.$$

By Eq (41), for a given $\varepsilon > 0$, $\exists N$ such that

$$||W_{n+1}|| \leq ||W_n|| + \varepsilon \leq M_1 + \varepsilon, \text{for } n \geq N.$$

Therefore, $||W_{n+1}|| \leq \max\{||W_{i+1}||_{1\leq i\leq N-1}, M_1+\varepsilon\}$. That is, $\{W_{n+1}\}_{n\in N_1}$ is bounded. Then, $\{W_{n+1}\}_{n\in N_1}$ contains a convergent subsequence $\{W_{n+1}\}_{n\in N_2}$. Assume that

$$\lim_{n\in N_2} W_{n+1} = W_2.$$

By Proposition (A.2), Ψ is closed. Hence, we have

$$W_2 = \Psi(W_1).$$

Note that in Eq (49) we assume that $W_1 \notin \Omega$. By Proposition (A.1), we have

$$\theta(W_2) > \theta(W_1). \tag{50}$$

However, $\{\theta(W_{n+1})\}_{n\in N_2}$ and $\{\theta(W_n)\}_{n\in N_2}$ are subsequences of $\{\theta(W_n)\}_{n=1}^{+\infty}$. Therefore, we have

$$\lim_{n\in N_2} \theta(W_{n+1}) = \theta(W_2) = \lim_{n\in N_2} \theta(W_n) = \theta(W_1) = \lim_{n\to+\infty} \theta(W_n)$$

This contradicts Eq (50). Therefore, Eq (48) holds, and $\lim_{n\to+\infty} \theta(W_n) = \theta(W_1)$. This completes the proof of *2)* and *3)*. □