Online Supplement for
The One-Dimensional Dynamic Dispatch Waves Problem

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Proof of Proposition 5.1

Proof. Applying restriction (6) to (2) yields the LP

$$C^{ALP} = \max_{\{\mathbf{a},\mathbf{b},\mathbf{v}\}} \mathbb{E}_{R_T}\left[\sum_{i\in\hat{R}} a_i^T + \sum_{i\in N\setminus\hat{R}} b_i^T\right] - \sum_{t=1}^{T} v_t \tag{1a}$$

$$\text{s.t.} \sum_{i\in R} a_i^0 + \sum_{i\in P} b_i^0 = \sum_{i\in R} p_i, \qquad \forall (R,P)\in\Xi \tag{1b}$$

$$\sum_{i\in R} a_i^t + \sum_{i\in P} b_i^t - \mathbb{E}_{F_1^t}\left[\sum_{i\in R\cup F_1^t} a_i^{t-1} + \sum_{i\in P\setminus F_1^t} b_i^{t-1}\right] \le v_t, \qquad \forall t\in\mathscr{T}, (R,P)\in\Xi \tag{1c}$$

$$\sum_{i\in R} a_i^t + \sum_{i\in P} b_i^t - \mathbb{E}_{F_d^t}\left[\sum_{i\in R_d\cup F_d^t} a_i^{t-d} + \sum_{i\in P\setminus F_d^t} b_i^{t-d}\right] \le \alpha d + \sum_{k=t-d+1}^{t} v_k, \quad \forall t\in\mathscr{T}, (R,P)\in\Xi, d\in\mathscr{A}_R^t. \tag{1d}$$

Model (1) has a polynomial number of variables for a given n and T, but it has exponentially many terms within the expectations and constraints. We prove Proposition 5.1 in two steps. First, we compute a closed form for the expectations in model (1). Then we show a one to one equivalence between both domains.

The expectations in (1) are given by

$$\mathbb{E}_{R_T}\left[\sum_{i\in\hat{R}} a_i^T + \sum_{i\in N\setminus\hat{R}} b_i^T\right] = \sum_{i\in N}\left(\mathbb{P}(\tau_i = T)a_i^T + \mathbb{P}(\tau_i < T)b_i^T\right)$$

$$\mathbb{E}_{F_1^t}\left[\sum_{i\in R\cup F_1^t} a_i^{t-1} + \sum_{i\in P\setminus F_1^t} b_i^{t-1}\right] = \sum_{i\in R} a_i^{t-1} + \sum_{i\in P} f_{it}a_i^{t-1} + \bar{f}_{it}b_i^{t-1}$$

$$\mathbb{E}_{F_d^t}\left[\sum_{i\in R_d\cup F_d^t} a_i^{t-d} + \sum_{i\in P\setminus F_d^t} b_i^{t-d}\right] = \sum_{i\in R_d} a_i^{t-d} + \sum_{i\in P} g_{it}^d a_i^{t-d} + \bar{g}_{it}^d b_i^{t-d}.$$

Replacing them in (1) yields

$$\max_{\{\mathbf{a},\mathbf{b},\mathbf{v}\geq 0\}} \sum_{i\in N}\left(\mathbb{P}(\tau_i = T)a_i^T + \mathbb{P}(\tau_i < T)b_i^T\right) - \sum_{t=1}^{T} v_t \text{ s.t.} \tag{2a}$$

$$\sum_{i\in R} a_i^0 + \sum_{i\in P} b_i^0 = \sum_{i\in R} p_i \qquad \forall (R,P)\in\Xi \tag{2b}$$

$$\sum_{i\in R}\left(a_i^t - a_i^{t-1}\right) + \sum_{i\in P}\left(b_i^t - f_{it}a_t^{t-1} - \bar{f}_{it}a_i^{t-1}\right) \leq v_t \qquad \forall t\in\mathscr{T}, \forall (R,P)\in\Xi \tag{2c}$$

$$\sum_{i\in\bar{R}_d} a_i^t + \sum_{i\in R_d}\left(a_i^t - a_i^{t-d}\right) + \sum_{i\in P}\left(b_i^t - g_{it}^d a_t^{t-d} - \bar{g}_{it}^d a_i^{t-d}\right) \leq \sum_{k=t-d+1}^{t} v_k + \alpha d \quad \forall t\in\mathscr{T}, \forall (R,P)\in\Xi, \forall d\in\mathscr{A}_R^t, \tag{2d}$$

where we still have an exponential number of constraints. We prove that (7) is equivalent to (2) by showing equality between both domains.

1. $(7b) \iff (2b)$: Suppose that $(\mathbf{a},\mathbf{b},\mathbf{v})$ satisfies $(2b)$. If $R=\{i\}$ and $P=\emptyset$ we get $a_i^t = p_i$, and if $R=\emptyset$ and $P=\{i\}$ we get $b_i^t = 0$. Now, suppose that $(\mathbf{a},\mathbf{b},\mathbf{v})$ satisfies $(7b)$ and add $a_i = p_i$ and $b_j = 0$ over any feasible pair of sets $(R,P)\in\Xi$ to get $(2b)$.

2. $(7c),(7d),(7e) \iff (2c)$: Suppose that $(\mathbf{a},\mathbf{b},\mathbf{v})$ satisfies $(2c)$. For each $t\in\mathscr{T}$, choose a particular $(R,P)\in\Xi$ as follows: put $i\in R$ if $a_i^t - a_i^{t-1} > \max\{0, b_i^t - f_{it}a_i^{t-1} - \bar{f}_{it}b_i^{t-1}\}$, and put $i\in P$ if $b_i^t - f_{it}a_i^{t-1} - \bar{f}_{it}b_i^{t-1} > \max\{0, a_i^t - a_i^{t-1}\}$.
   Then, for (i,t) set $s_{it} = \max\left\{0, a_i^t - a_i^{t-1}, b_i^t - f_{it}a_i^{t-1} - \bar{f}_{it}b_i^{t-1}\right\}$ and we get
   $$v_t \geq \sum_{i\in R} a_i^t - a_i^{t-1} + \sum_{i\in P} b_i^t - f_{it}a_t^{t-1} - \bar{f}_{it}a_i^{t-1} = \sum_{i\in N}\max\left\{0, a_i^t - a_i^{t-1}, b_i^t - f_{it}a_i^{t-1} - \bar{f}_{it}b_i^{t-1}\right\} = \sum_{i\in N} s_{it}.$$
   Now suppose that $(\mathbf{a},\mathbf{b},\mathbf{v},\mathbf{s})$ satisfies $(7c),(7d),(7e)$, select any pair $(R,P)\in\Xi$ and we have
   $$v_t \geq \sum_{i\in N} s_{it} \geq \sum_{i\in R} s_{it} + \sum_{i\in P} s_{it} \geq \sum_{i\in R}\left(a_i^t - a_i^{t-1}\right) + \sum_{i\in P}\left(b_i^t - f_{it}a_t^{t-1} - \bar{f}_{it}a_i^{t-1}\right).$$

3. $(2d) \iff (7f),(7g),(7h)$. Consider that $(\mathbf{a},\mathbf{b},\mathbf{v})$ satisfies $(2d)$. For each $t\in\mathscr{T}$ and $d\in\mathscr{A}_N^t$, choose $(R,P)\in\Xi$ as follows: put $i\in R$ if $a_i^t - \mathbb{I}_{d_i>d}a_i^{t-d} > \max\{0, b_i^t - g_{it}^d a_i^{t-d} - \bar{f}_{it}^d b_i^{t-d}\}$, and put $i\in P$ if $b_i^t - g_{it}^d a_i^{t-d} - \bar{f}_{it}^d b_i^{t-d} > \max\{0, a_i^t - \mathbb{I}_{d_i>d}a_i^{t-d}\}$.
   Then, for each (i,t,d) define $u_{it}^d = \max\left\{0, a_i^t - \mathbb{I}_{d_i>d}a_i^{t-d}, b_i^t - g_{it}^d a_i^{t-d} - \bar{g}_{it}^d b_i^{t-d}\right\}$. By (2d), we get
   $$\begin{aligned}\alpha d + \sum_{k=t-d+1}^{t} v_k &\geq \sum_{i\in R}\left(a_i^t - \mathbb{I}_{d_i>d}a_i^{t-d}\right) + \sum_{i\in P}\left(b_i^t - g_{it}^d a_i^{t-d} - \bar{g}_{it}^d b_i^{t-d}\right) \\ &= \sum_{i\in N}\max\left\{0, a_i^t - \mathbb{I}_{d_i>d}a_i^{t-d}, b_i^t - g_{it}^d a_i^{t-d} - \bar{g}_{it}^d b_i^{t-d}\right\} = \sum_{i\in N} u_{it}^d.\end{aligned}$$

Now, suppose that $(\mathbf{a},\mathbf{b},\mathbf{v},\mathbf{u})$ satisfies $(7f),(7g),(7h)$, select any pair $(R,P)\in\Xi$ and get

$$\begin{aligned}\alpha d+\sum_{k=t-d+1}^{t} v_k &\geq \sum_{i\in N} u_{it}^d \\ &\geq \sum_{i\in R} u_{it}^d+\sum_{i\in P} u_{it}^d \\ &\geq \sum_{i\in R}\left(a_i^t-\mathbb{I}_{d_i>d}a_i^{t-d}\right)+\sum_{i\in P}\left(b_i^t-g_{it}^d a_i^{t-d}-\bar{g}_{it}^d b_i^{t-d}\right) \\ &= \sum_{i\in R_d} a_i^t+\sum_{i\in \bar{R}_d}\left(a_i^t-a_i^{t-d}\right)+\sum_{i\in P}\left(b_i^t-g_{it}^d a_i^{t-d}-\bar{g}_{it}^d b_i^{t-d}\right).\end{aligned}$$

□

0.1 Proof of Property 5.2

Proof. We start proving that there exists at least one optimal solution for (7) satisfying $a_i^t\leq p_i$ and $b_i^t\leq g_{it}^t p_i$ for all $i\in N$ and $t\in\mathscr{T}_0$. Choose any $i\in N$ and do forward induction on t.

- $t=0$ is given by constraints (7b).
- Inductive step:
  Assume that $a_i^k\leq p_i$ and that $b_i^k\leq g_{ik}^k p_i$ for all $k<t$. We prove the statement for step t.

  Suppose that $a_i^t=p_i+\delta_a$ and $b_i^t=g_{it}^t p_i+\delta_b$, with $\varepsilon=\max\{\delta_a,\delta_b\}>0$. By the inductive hypothesis, (7c),(7d), (7f) and (7g) it implies that $s_{it}\geq\varepsilon$, $u_{it}^d\geq\varepsilon$, for all $d\in\mathscr{A}_N^t$ and by (7e) we have $v_t\geq\varepsilon$.

  So, update the variables for time t as follows: $a_i^t\leftarrow p_i$; $b_i^t\leftarrow b_i^t-\varepsilon$; $s_{it}\leftarrow s_{it}-\varepsilon$; $u_{it}^d\leftarrow u_{it}^d-\varepsilon$, $\forall d\in\mathscr{A}_N^t$ and $v_t\leftarrow v_t-\varepsilon$. Also, update the variables for time $v>t$: $a_i^v\leftarrow a_i^v-\varepsilon$ and $b_i^v\leftarrow b_i^v-\varepsilon$. These changes keep (7) feasible and the objective value does not changes (the reduction in v_t increases the objective by ε, but the change in $\mathbb{P}(\tau_i=T)a_i^T+\mathbb{P}(\tau_i<T)b_i^T$ reduces it by ε).

Now, let us show that there exists at least one optimal solution for (7) satisfying $a_i^t\geq 0$, $b_i^t\geq 0$ for all $i\in N$ and $t\in\mathscr{T}_0$. Choose any $i\in N$ and do forward induction on t.

- $t=0$ is given by constraints (7b).
- Inductive step:
  Assume that $a_i^k\geq 0, b_i^k\geq 0$ for all $k<t$. We prove the statement for step t.

  Suppose that: $a_i^t<0$ and/or $b_i^t<0$. We can set these variables equal to 0 without losing feasibility. If $t<T$, then the objective remains unaltered. Else, it improves when $t=T$.

□

0.2 Proof of Property 5.3

Proof. Choose any $i\in N$. We prove by induction on t that there exists an optimal solution satisfying $a_i^t=p_i$ and $b_i^t=g_{it}^t p_i$ for all $i\in N, t\in\mathscr{T}_0: d_i>t$.

- $t=0$ is given by (7b).

- Inductive step:
  Assume that $a_i^k = p_i$, $b_i^k = g_{ik}^k p_i, \forall k \in \mathscr{T}_0 : d_i > k$ with $k < t$ and suppose that the optimal solution is such that $a_i^t = p_i - \delta_a, b_i^t = g_{it}^t p_i - \delta_b$, where $\max\{\delta_a, \delta_b\} > 0$. We can reassign these two variables, *i.e.*, $a_i^t \leftarrow p_i$ and $b_i^k \leftarrow g_{ik}^k p_i$, keeping feasibility and without reducing the objective value. Just note that for constraints (7f) we have $d_i > d$ (given by $d_i > t$ and $d \in \mathscr{A}_N^t$). Thus, all constraints involving the reassigned variables are

$$\begin{aligned}
s_{it} &\geq a_i^t - a_i^{t-1} = -\delta_a \\
s_{it} &\geq b_i^t - f_{it} a_i^{t-1} - \bar{f}_{it} b_i^{t-1} = -\delta_b \\
s_{it+1} &\geq a_i^{t+1} - a_i^t = a_i^{t+1} - p_i + \delta_a \\
s_{it+1} &\geq b_i^{t+1} - f_{it+1} a_i^t - \bar{f}_{it+1} b_i^t = b_i^{t+1} - g_{it+1}^{t+1} p_i + f_{it+1}\delta_a + \bar{f}_{it+1}\delta_b \\
u_{it}^d &\geq a_i^t - a_i^{t-d} = -\delta_a \\
u_{it}^d &\geq b_i^t - g_{it}^d a_i^{t-d} - \bar{g}_{it}^d b_i^{t-d} = -\delta_b \\
u_{it+d}^d &\geq a_i^{t+d} - a_i^t = a_i^{t+d} - p_i + \delta_a \\
u_{it+d}^d &\geq b_i^{t+d} - g_{it+d}^d a_i^t - \bar{g}_{it+d}^d b_i^t = b_i^{t+d} - g_{it+d}^{t+d} + g_{it+d}^d \delta_a + \bar{g}_{it+d}^d \delta_b,
\end{aligned}$$

  and when $\delta_a, \delta_b \to 0$ the lower bounds for $\mathbf{u}$ and $\mathbf{s}$ do not increase, since $\mathbf{u}$ and $\mathbf{s}$ are nonnegative. The missing case, i.e. $b_i^t = g_{it}^t p_i$ when $d_i = t$ follows a similar proof.

$\square$

0.3 Proof of Theorem 5.4

For this proof we simplify our formulation to keep the intuition as simple as possible. The action set $\mathscr{A}_R^t$ in state (t, R, P) will be $\{d \in \mathbb{Z}_+ : d \leq t\}$, and so, will include possibly suboptimal actions. So, consider the stochastic DDWP

$$\begin{aligned}
C^* = &\max_{C} \ \mathbb{E}_{\hat{R}}\left[C_T\left(\hat{R}, N \setminus \hat{R}\right)\right] \\
\text{s.t. } & C_0(R,P) \leq \textstyle\sum_{i \in R} p_i, && (R,P) \in \Xi \\
& C_t(R,P) \leq \mathbb{E}_{F_1^t}\left[C_{t-1}\left(R \cup F_1^t, P \setminus F_1^t\right)\right], && t \in \mathscr{T},\ (R,P) \in \Xi \\
& C_t(R,P) \leq \alpha d + \mathbb{E}_{F_d^t}\left[C_{t-d}\left(R_d \cup F_d^t, P \setminus F_d^t\right)\right], && t \in \mathscr{T},\ d \in \mathbb{Z}_+ : d \leq t,\ (R,P) \in \Xi,
\end{aligned}$$

and its ALP bound

$$\begin{aligned}
C' = &\max_{a,v,s,u} \sum_{i \in N} \left(\mathbb{P}(\tau_i = T) a_i^T + \mathbb{P}(\tau_i < T) b_i^T\right) - \sum_{k=1}^{T} v_k \\
\text{s.t. } & a_i^0 \leq p_i,\ b_i^0 \leq 0, && i \in N \\
& a_i^t - a_t^{t-1} - s_{it} \leq 0, && i \in N, t \in \mathscr{T} \\
& b_i^t - f_{it} a_i^{t-1} - \bar{f}_{it} b_i^{t-1} - s_{it} \leq 0, && i \in N, t \in \mathscr{T} \\
& \textstyle\sum_{i \in N} s_{it} - v_t \leq 0, && t \in \mathscr{T} \\
& a_i^t - u_{it}^d \leq 0, && i \in N, t \in \mathscr{T},, d \in \{d_i, \ldots, t\} \\
& a_i^t - a_i^{t-d} - u_{it}^d \leq 0, && i \in N, t \in \mathscr{T},, d \in \{1, \ldots, \min(d_i - 1, t)\}
\end{aligned}$$

$$b_i^t - g_{it}^d a_i^{t-d} - \bar{g}_{it}^d b_i^{t-d} - u_{it}^d \leq 0, \qquad i \in N, t \in \mathscr{T}, d \in \{1,\dots,t\}$$
$$\sum_{i\in N} u_{it}^d - \sum_{k=t-d+1}^t v_k \leq \alpha d, \qquad t \in \mathscr{T},, d \in \{1,\dots,t\}$$
$$\mathbf{s},\mathbf{u} \geq 0.$$

For the deterministic case we get $\mathbb{P}(\tau_i = T) = \mathbb{I}_{(\tau_i = T)}$, $f_{it} = \mathbb{I}_{(\tau_i = t-1)}$ and $g_{it}^d = \mathbb{I}_{(t-d \leq \tau_i < t)}$. The ALP collapses to

$$C^{'} = \max_{a,v,s,u} \sum_{i\in N} \left(a_i^T \mathbb{I}_{(\tau_i=T)} + b_i^T \mathbb{I}_{(\tau_i<T)}\right) - \sum_{k=1}^T v_k$$
$$\text{s.t. } a_i^0 \leq 0,\ b_i^0 \leq 0, \qquad i \in N$$
$$a_i^t - a_t^{t-1} - s_{it} \leq 0, \qquad i \in N, t \in \mathscr{T}$$
$$b_i^t - b_i^{t-1} - \mathbb{I}_{(t-1\leq\tau_i<t)}(a_i^{t-1} - b_i^{t-1}) - s_{it} \leq 0, \qquad i \in N, t \in \mathscr{T}$$
$$\sum_{i\in N} s_{it} - v_t \leq 0, \qquad t \in \mathscr{T}$$
$$a_i^t - u_{it}^d \leq 0, \qquad i \in N, t \in \mathscr{T}, d \in \{d_i,\dots,t\}$$
$$a_i^t - a_i^{t-d} - u_{it}^d \leq 0, \qquad i \in N, t \in \mathscr{T},, d \in \{1,\dots,\min(d_i-1,t)\}$$
$$b_i^t - b_i^{t-d} - \mathbb{I}_{(t-d\leq\tau_i<t)}(a_i^{t-d} - b_i^{t-d}) - u_{it}^d \leq 0, \qquad i \in N, t \in \mathscr{T}, d \in \{1,\dots,t\}$$
$$\sum_{i\in N} u_{it}^d - \sum_{k=t-d+1}^t v_k \leq \alpha d, \qquad t \in \mathscr{T}, d \in \{1,\dots,t\}$$
$$\mathbf{s},\mathbf{u} \geq 0.$$

From this point we assume without loss of generality that $d_i \leq \tau_i$. Otherwise, we can transform the model to an equivalent one satisfying this requirement. If request i does not arrive ($\tau_i < 0$), the optimal ALP value does not get altered by removing it, since at optimality $a_i^t = b_i^t = 0$, $\forall t \in \mathscr{T}_0$. In case that $0 < \tau_i < d_i$, i.e. the order arrives but cannot be served, one optimal solution is $a_i^t = p_i$, $\forall t \in \mathscr{T}_0$ and may be removed from the analysis by adding a constant p_i to the objective.

Now, let us preset some variables in the ALP:

- $a_i^t = 0$, for all $i \in N, t > \tau_i$; the open order cost before arrival is zero.
- $b_i^t = a_i^{\tau_i}$, for all $i \in N, t > \tau_i$; the potential order cost before arrival is equal to the open order cost upon arrival.
- $b_i^t = 0$, for all $i \in N, t \leq \tau_i$, i.e., the potential order cost after arrival is zero.

We have restricted the feasible space, and thus the remaining model is still an underestimate of C^* given by

$$C^{''} = \max_{a,v,s,u} \sum_{i\in N} a_i^{\tau_i} - \sum_{k=1}^T v_k$$
$$\text{s.t. } (w)\quad a_i^t - a_t^{t-1} - s_{it} \leq 0, \qquad i \in N, t \in \{1,\dots,\tau_i\}$$
$$(m)\quad a_i^t - u_{it}^d \leq 0, \qquad i \in N, t \in \{d_i,\dots,\tau_i\}, d \in \{d_i,\dots,t\}$$
$$(\alpha)\quad a_i^t - a_i^{t-d} - u_{it}^d \leq 0, \qquad i \in N, d \in \{1,\dots,d_i-1\}, t \in \{d,..,\tau_i\},$$
$$(\beta)\quad a_i^{\tau_i} - a_i^{t-d} - u_{it}^d \leq 0, \qquad i \in N, t \in \{\tau_i+1,\dots,T\}, d \in \{t-\tau_i,\dots,t\}$$
$$(\gamma)\quad a_i^0 \leq p_i, \qquad i \in N$$

$$
\begin{aligned}
&(Z) \quad \sum_{i\in N} s_{it} - v_t \le 0, && t \in \mathscr{T}\\
&(Y) \quad \sum_{i\in N} u_{it}^d - \sum_{k=t-d+1}^{t} v_k \le d, && t \in \mathscr{T}, d \in \{1,\dots,t\}\\
&\quad\ \mathbf{s},\mathbf{u} \ge 0,
\end{aligned}
$$

and its dual problem is

$$
C'' = \min_{Z,Y,\alpha,\beta,\gamma,m,w\ge 0} \sum_{i\in N} p_i\gamma_i + \sum_{t=1}^{T}\sum_{d=1}^{t} dY_{t,d} \tag{9a}
$$

$$
\text{s.t.} \ (v)\ Z_t + \sum_{t'=t}^{T}\sum_{d=t'-t+1}^{t'} Y_{t',d} = 1, \qquad t \in \mathscr{T} \tag{9b}
$$

$$
(s)\ w_i^t \le Z_t, \qquad i \in N, t \in \{1,\dots,\tau_i\} \tag{9c}
$$

$$
(u)\ m_{i,t}^d \le Y_{t,d}, \qquad i \in N, t \in \{d_i,\dots,\tau_i\}, \ d \in \{d_i,\dots,t\} \tag{9d}
$$

$$
\alpha_{i,t}^d \le Y_{t,d}, \qquad i \in N, d \in \{1,..,d_i-1\}, \ t \in \{d,\dots,\tau_i\} \tag{9e}
$$

$$
\beta_{i,t}^d \le Y_{t,d}, \qquad i \in N, t \in \{\tau_i+1,\dots,T\}, \ d \in \{t-\tau_i,\dots,t\} \tag{9f}
$$

$$
(a)\ \gamma_i = \left(w_i^1 + \textstyle\sum_{k=1+\tau_i}^{T} \beta_{i,k}^k + \sum_{k=1}^{d_i-1} \alpha_{i,k}^k\right), \qquad i \in N, \ t = 0 \tag{9g}
$$

$$
\begin{aligned}
&\left(w_i^t + \textstyle\sum_{d=1}^{t} \alpha_{i,t}^d\right)\\
&= \left(w_i^{t+1} + \textstyle\sum_{k=1+\tau_i}^{T} \beta_{i,k}^{k-t} + \sum_{k=1}^{d_i-1} \mathbb{I}_{(k+t\le\tau_i)}\alpha_{i,k+t}^k\right), \qquad i \in N, \ t \in \{1,\dots,d_i-1\}
\end{aligned} \tag{9h}
$$

$$
\begin{aligned}
&\left(w_i^t + \textstyle\sum_{d=d_i}^{t} m_{i,t}^d + \sum_{d=1}^{d_i-1} \alpha_{i,t}^d\right)\\
&= \left(w_i^{t+1} + \textstyle\sum_{k=1+\tau_i}^{T} \beta_{i,k}^{k-t} + \sum_{k=1}^{d_i-1} \mathbb{I}_{(k+t\le\tau_i)}\alpha_{i,k+t}^k\right), \qquad i \in N, \ t \in \{d_i,\dots,\tau_i-1\}
\end{aligned} \tag{9i}
$$

$$
\begin{aligned}
&\left(w_i^{\tau_i} + \textstyle\sum_{d=d_i}^{\tau_i} m_{i,\tau_i}^d + \sum_{d=1}^{d_i-1} \alpha_{i,t}^d\right)\\
&+ \left(\textstyle\sum_{k=\tau_i+1}^{T}\sum_{d=k-\tau_i+1}^{k} \beta_{i,k}^d\right) = 1, \qquad i \in N, \ t = \tau_i.
\end{aligned} \tag{9j}
$$

Consider (9):

1. First note that (9b) are equivalent to the following network flow balance constraints

$$
1 = Z_T + \textstyle\sum_{d=1}^{T} Y_{T,d} \tag{10a}
$$

$$
Z_{t+1} + \textstyle\sum_{t'=t+1}^{T} Y_{t',t'-t} = Z_t + \sum_{d=1}^{t} Y_{t,d}, \qquad t \in \{1,..,T-1\} \tag{10b}
$$

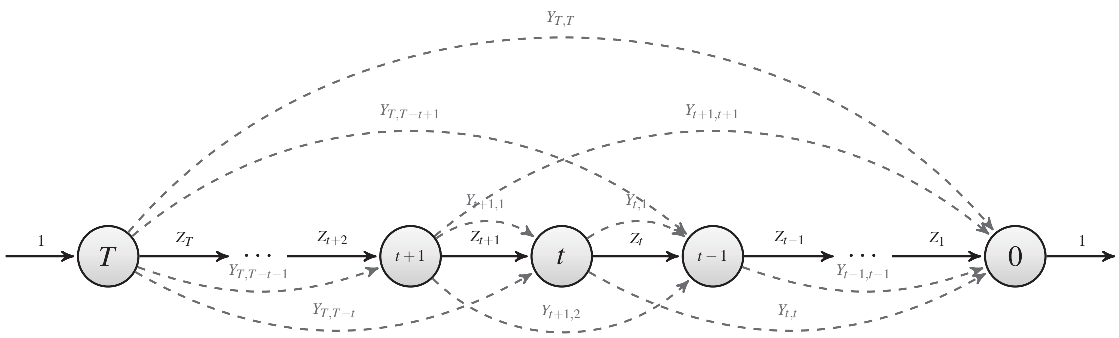

Figure 1: Network Structure in (Z,Y)-domain

$$Z_1 + \sum_{t'=2}^{T} Y_{t',t'-1} = 1 \tag{10c}$$
$$Z,Y \geq 0, \tag{10d}$$

represented in Figure 1.

Proof. Equivalence is obtained by subtracting constraint t from constraint $t+1$ in (9b) for all $t \in \{T-1,\ldots,2\}$. The flow balance constraint at node $t=T$ comes explicitly, and the flow balance constraint at node $t=1$ is obtained by adding the previously derived equations. □

Therefore, substructure (10) has integral extreme points.

2. Now, let us study the remaining constraints. Note that for a given (Z,Y) the resulting problem in variables $(\alpha,\beta,\gamma,w,m)$ collapses to n independent capacitated minimum cost network flow problems (CMCNF) for each order $i \in N$ defined in (11)

$$\gamma_i(Z,Y) = \min_{\alpha_i,\beta_i,\gamma_i,m_i,w_i \geq 0} \gamma_i \tag{11a}$$
$$\text{s.t. } (9c),(9d),(9e),(9f),(9g),(9h),(9i),(9j). \tag{11b}$$

In this network there is a set of nodes given by $\{0,\ldots,\tau_i\}$ and a sink node S^i defined by the (redundant) flow balance constraint $\gamma_i + \sum_{t=d_i}^{\tau_i} \sum_{x=d_i}^{t} m_{i,t}^d = 1$ obtained when adding (9g),(9h),(9i) and (9j). We would like to minimize the cost of moving one unit of flow from node τ_i to the sink node. There are five arc types available in (11) given by

- Type 1 arc (γ_i) going from node 0 to S^i. Our objective is to minimize the value of this flow, since it is the only one with non-zero cost.
- Type 2 arcs ($m_{i,t}^d$) going from node $t \in \{d_i,\ldots,\tau_i\}$ to S^i. We want to maximize these flows, but these arc flows are bounded by $Y_{t,d}$.
- Type 3 arcs (w_i^t) going from t to $t-1$ for each $t \in \{1,\ldots,\tau_i\}$. These flows are bounded by Z_t.
- Type 4 arcs ($\alpha_{i,t}^d$) going from a node $t \in \{1,\ldots,\tau_i\}$ to any node $t-d$ for each $d < d_i$ and $d \leq t$; also bounded by $Y_{t,d}$.

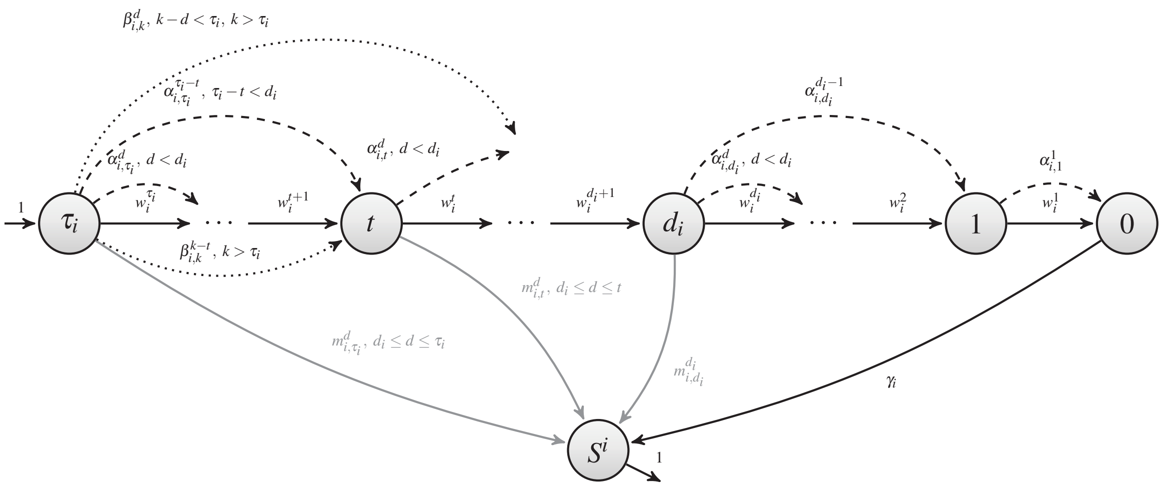

Figure 2: Network for i^{th} order subproblem.

- Type 5 arcs ($\beta^d_{i,k}$) going from node τ_i to any node $t\in\{0,\dots,\tau_i-1\}$ for each $k\in\mathbb{Z}_+$ and $d\in\mathbb{Z}_+$ satisfying $\tau_i<k\le T$ and $k-d=t$; also bounded by $Y_{k,d}$.

Note that problem (11) is feasible for any value $(Z,Y)\in(10)$. Its network is graphically represented in Figure 2.

If we put these two comments together, the dual ALP in (9) is equal to

$$\min_{(Z,Y)\in(10)} C(Z,Y) := \underbrace{\sum_{i\in N} p_i\gamma_i(Z,Y)}_{P(Z,Y)} + \underbrace{\sum_{t=1}^{T}\sum_{d=1}^{t} dY_{t,d}}_{C_{OP}(Y)}. \tag{12}$$

We show in two parts that (12) has an optimal value equal to the optimal cost of the deterministic DDWP in (4). First, we prove that any feasible dispatch for the deterministic DDWP has a one-to-one mapping with integer feasible solutions (Z,Y) to (12). Then, we show that without loss of optimality a solution of (12) can be assumed integral.

Part 1: Consider any feasible dispatch with lengths $\{d^1,...,d^K\}$ and dispatch times $\{t^1,..,t^K\}$. Then, there is a unique integer solution of (Z,Y) representing this operation. Just set to zero all components of Y except for $Y_{t^k,d^k}=1, \forall k\in\{1,\dots,K\}$ and set Z to satisfy (9b). Thus, $Y_{t,d}$ represents a dispatch at t with distance length d and Z_t represents waiting at the depot between t and $t-1$. Its corresponding operational dispatch cost matches the second term in the objective of (12), i.e. $C_{OP}(Y) := \sum_{t=1}^{T}\sum_{d=1}^{t} dY_{t,d} = \sum_{k=1}^{K} d^k$. Also, let

$$\eta_i = \begin{cases} 1 & \text{if } d^k\ge d_i \text{ and } t^k\le\tau_i, \text{ for some } k\in\{1,\dots,K\} \\ 0 & \text{otherwise} \end{cases}$$

indicate whether order i is covered by any dispatch or not. If $\eta_i=0$, then all type 2 arcs for subproblem (11) cannot be used, i.e. $m^d_{i,t}\le 0$ for $d_i\le t\le\tau_i$ and $d_i\le d\le t$, so there is a unique path from τ_i to S^i with $\gamma_i=1$.

If $\eta_i = 1$, then a new $(\tau_i - S^i)$-path arises with capacity one. The idea is to move the unit flow horizontally using type 3 arcs ($w_i^t = 1$) at each node $t : 1 \leq t \leq \tau_i$ when $Z_t = 1$. Otherwise, if $Z_t = 0$ there are three potential scenarios:

- A dispatch at t covers i, *i.e.*, $d \geq d_i$. Then we can use the corresponding type 2 arc $m_{i,t}^d = Y_{t,d} = 1$ and reach the sink node S^i at zero cost ($\gamma_i = 0$).
- A dispatch at t does not cover i, i.e. $d < d_i$. Then we can use the corresponding type 4 arc $\alpha_{i,t}^d = Y_{t,d} = 1$ and reach node $t-d$ at zero cost. Since $\eta_i = 1$, we proceed until we find the type 2 arc associated with the earliest dispatch that covers i.
- We have $t = \tau_i$ and there is a dispatch at time $k > \tau_i$ with distance d such that $k - d < \tau_i$. Then we can send one unit of flow in a type 5 arc to node $k-d$, *i.e.*, $b_{i,k}^d = Y_{k,d} = 1$. Again, we proceed until we find the earliest type 2 arc.

The first cost term in (12) will be exactly equal to the penalties paid for orders left unattended: $P(Z,Y) := \sum_{i \in N} p_i \gamma_i = \sum_{\substack{i \in N: \\ n_i = 0}} p_i$.

Part 2: Now we prove that without loss of optimality Z, Y is binary, and hence an optimal solution is an optimal dispatch for the deterministic DDWP. Assume by contradiction that Y has fractional components and that $C(Z,Y) < C(\bar{Z},\bar{Y})$ for any integral solution $(\bar{Z},\bar{Y}) \in (10)$. We can express (Z,Y) as a convex combination of the extreme points $(Z^1,Y^1),\ldots,(Z^p,Y^p)$ of (10) which are binary. Thus, we have $(Z,Y) = \sum_{l=1}^p \lambda_l (Z^l, Y^l)$ for a given nonnegative vector $\lambda \geq 0$ such that $\sum_{l=1}^p \lambda_l = 1$. The operational cost term $C_{OP}(Y)$ in (12) is additive in Y, since

$$C_{OP}(Y) = \sum_{t=1}^{T} \sum_{d=1}^{t} dY_{t,d} = \sum_{t=1}^{T} \sum_{d=1}^{t} d \left(\sum_{l=1}^{p} \lambda_l Y_{t,d}^l \right) = \sum_{l=1}^{p} \lambda_l \left(\sum_{t=1}^{T} \sum_{d=1}^{t} dY_{t,d}^l \right) = \sum_{l=1}^{p} \lambda_l C_{OP}(Y^l).$$

So, if (Z,Y) satisfies for each $i \in N$ that

$$\gamma_i(Z,Y) = \sum_{l=1}^{p} \lambda_l \gamma_i(Z^l, Y^l), \tag{13}$$

then the additive relation follows for the penalty cost term $P(Z,Y)$ in (12), because

$$P(Z,Y) = \sum_{i \in N} \gamma_i(Z,Y) p_i = \sum_{i \in N} \left(\sum_{l=1}^{p} \lambda_l \gamma_i(Z^l,Y^l) p_i \right) = \sum_{l=1}^{p} \lambda_l \left(\sum_{i \in N} \gamma_i(Z^l,Y^l) p_i \right) = \sum_{l=1}^{p} \lambda_l P(Z^l,Y^l),$$

and the total cost is additive in (Z,Y), i.e. $C(Z,Y) = \sum_{l=1}^p \lambda_l C(Z^l,Y^l)$. So, if condition (13) is true, the optimal cost is a convex combination of binary extreme point costs and it directly implies that there should be an integer extreme point l^* satisfying $C(Z^{l^*}, Y^{l^*}) \leq C(Z,Y)$. This is our desired contradiction.

Proof of condition (13): Note that $\gamma_i(Z,Y) \leq \sum_{l=1}^p \lambda_l \gamma_i(Z^l,Y^l)$ is trivial, since the optimal value of (11) is a convex function of the right-hand-side argument (Z,Y). Also, we have that $\gamma_i(Z^l,Y^l) = 1$ when the operation encoded in Y^l covers order i, else it is equal to 0. So, the right-hand-side of (13) yields $\sum_{l=1}^p \lambda_l \gamma_i(Z^l,Y^l) = 1 - \sum_{l:Y^l \text{ covers } i} \lambda_l$.

We need to show that the left-hand-side of (13) is also equal to the above value. There is two cases:

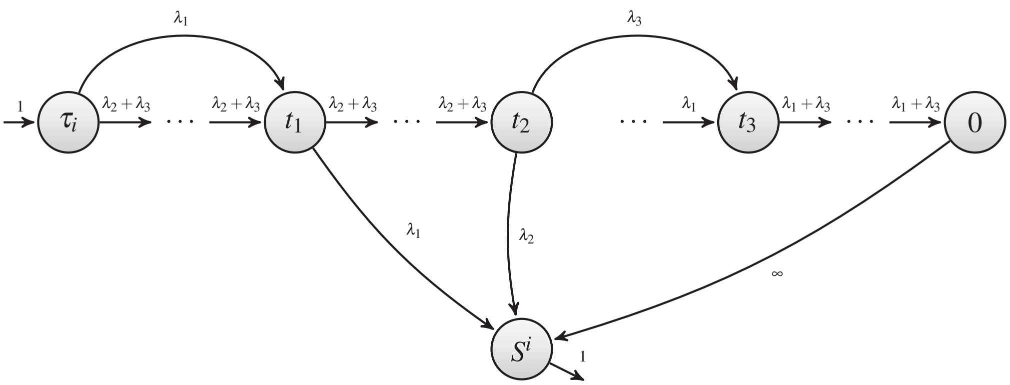

Figure 3: Example of a convex combination of three operations in i^{th} order subproblem.

1. Suppose that for each $l \in \{1,\dots,p\}$ with $0<\lambda_l<1$, the operation encoded in Y^l covers order $i \in N$ at most in one dispatch. In case that Y^l covers i exactly once, then $(\lambda_l Y^l, \lambda_l Z^l)$ will add in (11) exactly one type 2 arc $m^d_{i,t}$ with capacity $\lambda_l > 0$, where $Y^l_{t,d}$ is such that $d_i \le d$ and $\tau_i \ge t$. Also, $(\lambda_l Y^l, \lambda_l Z^l)$ will produce a zero cost path from τ_i to S^i with capacity λ_l that uses arc $m^d_{i,t}$. On the other hand, if Y^l does not cover i there will be no additional paths to S^i. If we put all these solutions Y^l together for each $l \in \{1,\dots,p\}$ with $0<\lambda_l<1$ and form $Y=\sum_{l=1}^{p}\lambda_l Y^l$, the binding cut between τ_i and S^i with zero-cost flows will be defined by $U=\{1,\dots,\tau_i\}$ with capacity $\sum_{l:Y^l \text{ covers } i}\lambda_l$. So, given that the cut is always binding, if we put these paths together in one single network it does not affect the output and $\gamma_i(Z,Y)=1-\sum_{l:Y^l \text{ covers } i}\lambda_l$. Figure 3 provides an example of this network showing the arc capacities of subproblem (11) for order i. This case has three integer extreme points Y: Y^1, Y^2 and Y^3 defining $Y=\lambda_1 Y^1+\lambda_2 Y^2+\lambda_2 Y^3$ and $1=\lambda_1+\lambda_2+\lambda_2$ for $\lambda \ge 0$. Y^1 and Y^2 cover order i, but Y^3 does not. It is clear that the maximum zero-cost flow from τ_i to S^i is equal to the capacity of the cut U equal to $\lambda_1+\lambda_2<1$. So, $\gamma_i = 1-\lambda_1-\lambda_2$.

2. A potential problem could occur if an operation covers an order more than once in multiple dispatches. For example, suppose that there exists an operation l^1 with $0<\lambda^{l^1}<1$ such that Y^{l^1} covers order i twice and that there exists another operation l^2 not covering i such that the vehicle is at the depot when operation l^1 dispatches the latest dispatch covering i. Then, an "artificial" coverage is created for order i. Figure 4 illustrates this problem. In this example, operation $l=1$ with weight $\lambda_1=0.5$ waits at the depot until t_1, covers order i at t_1, returns at t_2 and covers order i again at t_2. Operation $l=2$ with weight $\lambda_2=0.5$ waits at the depot all the time (between τ_i and 0). We have that $0.5\gamma_i(Z^1,Y^1)+0.5\gamma_i(Z^2,Y^2)=0.5$, but $\gamma_i\left(0.5(Z^1,Y^1)+0.5(Z^2,Y^2)\right)=1-0.5-\min\{0.5,0.5\}=0$. So condition (13) does not hold. Fortunately, we can prove that there exists an alternative set of operations $l \in E$ such that Y can also be written as $Y=\sum_{l\in E}\lambda_l Y^l$ and such that condition (13) holds.

   Let us solve this problem for the example in Figure 4 first. Define Y^3 and Y^4 as follows. Let

$$Y^3_{t,d} := \begin{cases} Y^1_{t,d} & t>t_2, 1\le d\le t \\ Y^2_{t,d} & t\le t_2, 1\le d\le t \end{cases} \quad \text{and} \quad Y^4_{t,d} := \begin{cases} Y^2_{t,d} & t>t_2, 1\le d\le t \\ Y^1_{t,d} & t\le t_2, 1\le d\le t \end{cases}.$$

   Note that $Y=0.5Y^3+0.5Y^4$ and, thus, this new decomposition does not affect operational costs. Also, it covers the same amount of orders plus the "artificial" coverage which is now valid. So

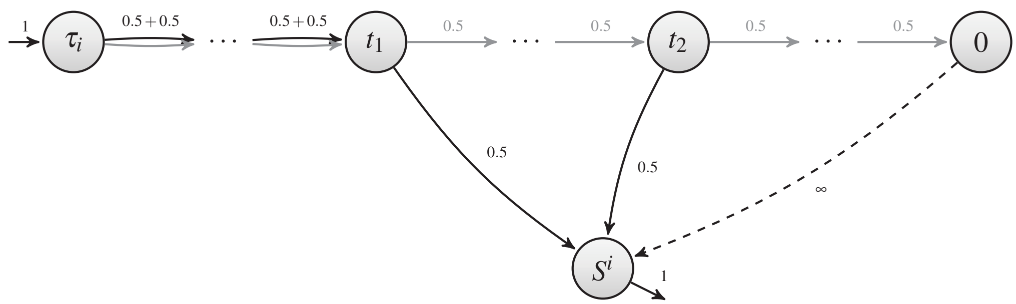

Figure 4: Example of a convex combination of two operations where subproblem for order i is not additive in the argument (Z,Y).

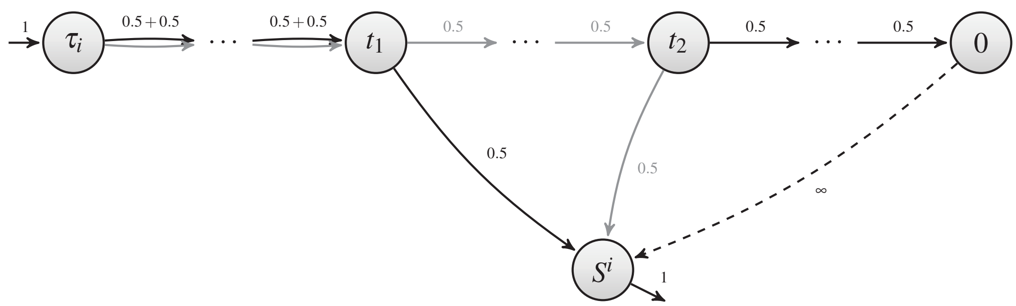

Figure 5: Same example with two operations where subproblem for order i is additive in the argument (Z,Y).

$0.5\gamma_i(Z^3,Y^3)+0.5\gamma_i(Z^4,Y^4)=\gamma_i\left(0.5(Z^3,Y^3)+0.5(Z^4,Y^4)\right)=0$. Figure 5 presents this solution.

The general proof can be constructed by induction on r_1+r_2, where r_1 is the total number of additional dispatches covering $i\in N$ in operations inside S, and r_2 is the number of operations not covering i in S with the vehicle at the depot at a time t^* where another operation $l'\in S$ executes a dispatch covering i which is not the earliest such dispatch.

- Case $r_1=0,r_2=0$: This case is trivial, since the set $S:Y=\sum_{l\in S}\lambda_l Y^l$ satisfies (13).
- Case $r_1>0,r_2=0$: This case is also trivial, since the multiple dispatches cannot be used to generate "artificial coverages" and any S such that $Y=\sum_{l\in S}\lambda_l Y^l$ satisfies (13).
- Case $r_1=0,r_2>0$: This case is impossible, by the definition of r_2 ($r_1=0 \implies r_2=0$).
- Case $r_1>0,r_2>0$: Let $l^1\in S$ be the operation with a repeated dispatch to i at time t^* such that there exists another operation $l^2\in S$ not covering i and with the vehicle available at the depot at time t^*. Construct two new operations l^3 and l^4 as follows:

$$Y^{l^3}_{t,d}:=\begin{cases}Y^{l^1}_{t,d} & t>t^*,1\le d\le t\\ Y^{l^2}_{t,d} & t\le t^*,1\le d\le t\end{cases}\quad\text{and}\quad Y^{l^4}_{t,d}:=\begin{cases}Y^{l^2}_{t,d} & t>t^*,1\le d\le t\\ Y^{l^1}_{t,d} & t\le t^*,1\le d\le t\end{cases}.$$

  We have three cases:

  - If $\lambda_{l^1}<\lambda_{l^2}$, we have $Y=\sum_{l\in S\setminus\{l^1,l^2\}}\lambda_{l^1}Y^l+\lambda_{l^1}(Y^{l^3}+Y^{l^4})+(\lambda_{l^2}-\lambda_{l^1})Y^{l^2}$ and r_1 decreases by one. Use induction with $S'=S\setminus\{l^1\}\cup\{l^3,l^4\}$.

– If $\lambda_{l^2} < \lambda_{l^1}$, we have $Y = \sum_{l \in S \setminus \{l^1, l^2\}} \lambda_{l^1} Y^l + \lambda_{l^2}(Y^{l^3} + Y^{l^4}) + (\lambda_{l^1} - \lambda_{l^2}) Y^{l^1}$ and r_2 decreases by one. Use induction with $S' = S \setminus \{l^2\} \cup \{l^3, l^4\}$.
– If $\lambda_{l^2} = \lambda_{l^1}$, set $Y = \sum_{l \in S \setminus \{l^1, l^2\}} \lambda_{l^1} Y^l + \lambda_{l^1}(Y^{l^3} + Y^{l^4})$ and r_1 and r_2 each decrease by one. Use induction with $S' = S \setminus \{l^1, l^2\} \cup \{l^3, l^4\}$.

□

0.4 ALP solution pruning

We can reduce the computational effort involved in getting the ALP optimal policy defined by (9) with the following proposition:

Proposition 0.1 (ALP solution pruning)**.** *Suppose $\delta \in \mathscr{A}_R^t$ is a feasible dispatch distance at state (t, R, P) and its related ALP solution to* (9) *is $\{\mathbf{a}(\delta), \mathbf{b}(\delta), \mathbf{v}(\delta)\}$. Let $\mu \in \mathscr{A}_R^t$ be a different feasible dispatch distance. If*

$$\begin{aligned} &\alpha\delta + \textstyle\sum_{i \in R_\delta} a_i(\delta)^{t-\delta} + \sum_{i \in P} \left(g_{it}^{\delta} a_i(\delta)^{t-\delta} + \bar{g}_{it}^{\delta} b_i(\delta)^{t-\delta} \right) - \sum_{k=1}^{t-\delta} v_k(\delta) \\ &< \alpha\mu + \textstyle\sum_{i \in R_\mu} a_i(\delta)^{t-\mu} + \sum_{i \in P} \left(g_{it}^{\mu} a_i(\delta)^{t-\mu} + \bar{g}_{it}^{\mu} b_i(\delta)^{t-\mu} \right) - \sum_{k=1}^{t-\mu} v_k(\delta), \end{aligned}$$

then μ is suboptimal for (9) *and can be discarded before solving its related ALP.*

Proof. The proof is based on the fact that $\{\mathbf{a}(\delta), \mathbf{b}(\delta), \mathbf{v}(\delta)\}$ is also a feasible solution for the ALP problem related to μ. By proposition (0.1) and the feasibility of $\mathbf{a}(\delta), \mathbf{b}(\delta), \mathbf{v}(\delta)$ in any ALP problem we get

$$\begin{aligned} &\alpha\delta + \sum_{i \in R_\delta} a_i(\delta)^{t-\delta} + \sum_{i \in P} \left(g_{it}^{\delta} a_i(\delta)^{t-\delta} + \bar{g}_{it}^{\delta} b_i(\delta)^{t-\delta} \right) - \sum_{k=1}^{t-\delta} v_k(\delta) \\ &< \alpha\mu + \sum_{i \in R_\mu} a_i(\delta)^{t-\mu} + \sum_{i \in P} \left(g_{it}^{\mu} a_i(\delta)^{t-\mu} + \bar{g}_{it}^{\mu} b_i(\delta)^{t-\mu} \right) - \sum_{k=1}^{t-\mu} v_k(\delta) \\ &\leq \alpha\mu + \max_{\{(\mathbf{a},\mathbf{b},\mathbf{v}) \in (7\mathrm{b})-7i\}} \sum_{i \in R_\mu} a_i^{t-\mu} + \sum_{i \in P} \left(g_{it}^{d} a_i^{t-\mu} + \bar{g}_{it}^{d} b_i^{t-\mu} \right) - \sum_{k=1}^{t-\mu} v_k, \end{aligned}$$

and this proves that the dispatch distance δ yields a lower approximate expected cost than μ for the ALP policy. □