# Online Appendix for Relay Network Design with Capacity and Link-Imbalance Considerations: A Lagrangean Decomposition Algorithm and Analysis

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## Proof of Proposition 1

Let $\mathcal{P}$ be the set of RPs, $k, l \in \mathcal{P}$ and $G_{kl}$ be the total flow from RP $k$ to RP $l$. Let $(k', l')$ be the RP-RP link with the maximum link-imbalance value in the solution of **CRND-LI**. Assuming, without loss of generalization, that $G_{k'l'} \geq G_{l'k'} \geq 0$, we have

$$\frac{G_{k'l'} - G_{l'k'}}{2\, G_{k'l'}} \geq \frac{G_{kl} - G_{lk}}{2\, G_{kl}}, \qquad \forall\, (k, l).$$

Then,

$$\frac{G_{k'l'} - G_{l'k'}}{G_{k'l'}} - \frac{G_{kl} - G_{lk}}{G_{kl}} = \frac{G_{kl}\big(G_{k'l'} - G_{l'k'}\big) \quad - \quad G_{k'l'}\big(G_{kl} - G_{lk}\big)}{G_{k'l'} G_{kl}} \geq 0, \qquad \forall\, (k, l)$$

and, thus,

$$G_{kl}\big(G_{k'l'} - G_{l'k'}\big) - G_{k'l'}\big(G_{kl} - G_{lk}\big) \geq 0, \qquad \forall\, (k, l). \tag{1}$$

The difference, $\Delta^k_{Imb}$, between the twice the maximum link-imbalance in the network and node-imbalance at any node $k \in \mathcal{R}$ is written as

$$\Delta^k_{Imb} = \frac{G_{k'l'} - G_{l'k'}}{G_{k'l'}} - \frac{\sum_{j \in \mathcal{R}_k} G_{kj} - \sum_{j \in \mathcal{R}_k} G_{jk}}{\sum_{j \in \mathcal{R}_k} G_{kj}}$$

Rearranging, we obtain

$$\Delta^k_{Imb} \quad = \quad \frac{\sum_{j \in \mathcal{R}_k} \Big( G_{kj}\big(G_{k'l'} - G_{l'k'}\big) \quad - \quad G_{k'l'}\big(G_{kj} - G_{jk}\big) \Big)}{G_{k'l'} \sum_{j \in \mathcal{R}_k} G_{kj}}.$$

Clearly, (1) holds for each RP $k$ and $j \in \mathcal{R}_k$, thus, $\Delta^k_{Imb} \geq 0,\ \ \forall\, k \in \mathcal{P}$, and the result follows. $\square$

## Proof of Proposition 2

Before we show the optimality of Algorithm 2 for $\mathbf{LSP}^{kl}_y$, we state the following remarks:

**Remark 1** *If the resulting link-imbalance value* $\bar{\Phi}$ *is less than or equal to* $\Phi$*, the solution obtained by Algorithm 1 is an optimal solution to* $\mathbf{LSP}^{kl}_{\mathbf{y}}$*.*

Recall that, without loss of generality, we assume that the total flow in $(k \to l)$ direction is greater than the total flow in $(l \to k)$ direction. Let these total flows in opposing directions be denoted by $G_{kl}$ and $G_{lk}$, respectively. If the resulting link-imbalance $\bar{\Phi}$ is greater than $\Phi$ upon termination of Algorithm 1 in the first part of the Algorithm 2, least objective improving portion of the load in $k \to l$ direction is off-loaded in such a way that the link-imbalance constraint is satisfied as equality. Let $\mathcal{R}$ be the set of these off-loaded commodities, $\mathcal{R} = \{r_1, \ldots, r_R\}$ such that $B^{r_1}_{kl} \leq B^{r_2}_{kl} \leq \ldots \leq B^{r_R}_{kl} < 0$. On the other hand, let $\mathcal{S}$ be the set of commodities that are not loaded in the $(l \to k)$ direction when

Algorithm 1 is terminated, $\mathcal{S} = \{s_1, \ldots, s_S\}$ such that $B_{lk}^{s_1} \leq B_{lk}^{s_2} \leq \ldots \leq B_{lk}^{s_S}$. Furthermore, let $\mathcal{R}' = \{r'_1, \ldots, r'_{R'}\}$ and $\mathcal{S}' = \{s'_1, \ldots, s'_{S'}\}$ be the set of commodities loaded in $(k \to l)$ and $(l \to k)$ directions, respectively, in the imbalance-feasible solution obtained by the completion of the first stage of Algorithm 2

**Remark 2** *We must have* $B_{kl}^{r_R} < B_{lk}^{s_1}$ *since Algorithm 1 is a greedy algorithm.*

**Remark 3** *The greedy solution by Algorithm 1 implies that* $B_{kl}^{r'_1} \leq \ldots \leq B_{kl}^{r'_{R'}} \leq \ldots \leq B_{kl}^{r_1} \leq \ldots \leq B_{kl}^{r_R} < 0$ *and* $B_{lk}^{s'_1} \leq \ldots \leq B_{lk}^{s'_{S'}} < 0$, $B_{lk}^{s'_{S'}} \leq B_{lk}^{s_1} \leq \ldots \leq B_{lk}^{s_S}$.

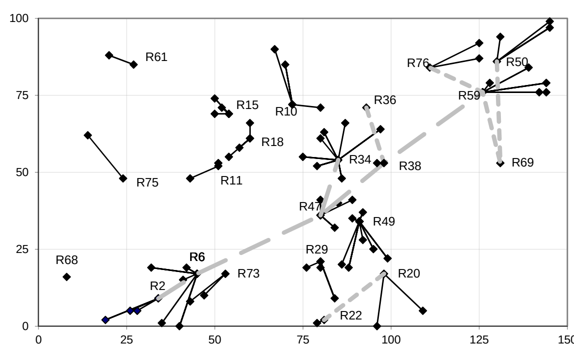
**Remark 4** *None of the commodities that is not loaded in the* $(k \to l)$ *direction upon termination of Algorithm 1 is to appear in the solution by Algorithm 2 since either the link capacity is completely used up before the commodities in set* $\mathcal{R}$ *are off-loaded or they have nonnegative* $B_{kl}^{-}$ *values.*

**Proof** In the first stage of the Algorithm 2 , the solution is converted to a feasible solution (lines 2-11) by excluding least objective improving commodities in the $(k \to l)$ direction. It follows from the first part of Remark 3 (related to $\mathcal{R}$ and $\mathcal{R}'$) and fractional knapsack *optimal substructure property* that the solution at the end of this first stage of Algorithm 2 is optimal for the used level of link capacity at that point. Let $\tilde{G}_{kl}$ and $\tilde{G}_{lk}$ be the total flows in corresponding directions and $\tilde{Z}_{kl}$ be the objective value upon completion of the first stage of Algorithm 2. To further improve the objective value $\tilde{Z}_{kl}$ optimally by utilizing the residual capacity, we have four possible operations as follows:

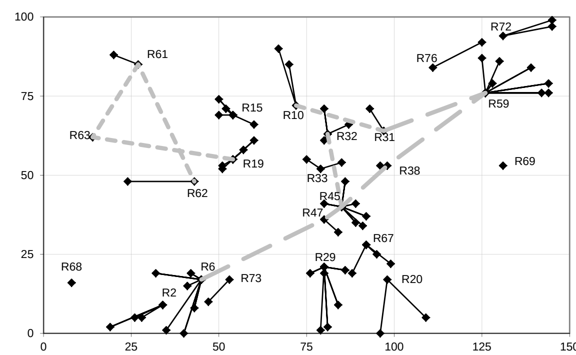
1. Decrease $\tilde{G}_{kl}$; Decrease or maintain $\tilde{G}_{lk}$: It follows from Remark 3 that this operation only worsens the objective value.
2. Decrease $\tilde{G}_{kl}$; Increase $\tilde{G}_{lk}$: It follows from Remark 2 that this operation also only worsens the objective value.
3. Increase $\tilde{G}_{kl}$; Decrease or maintain $\tilde{G}_{lk}$: This operation clearly increases the link-imbalance and, thus, leads to infeasibility as it reverses the first stage of Algorithm 2
4. Increase $\tilde{G}_{kl}$; Increase $\tilde{G}_{lk}$: To achieve this optimally, in Algorithm 2, we essentially solve a specially structured fractional knapsack problem in which commodities are to be added in both $(k \to l)$ and $(l \to k)$ directions using a greedy algorithm in such a way that the link-imbalance feasibility is maintained. Specifically, we add flow in both directions by maintaining a $1/(1-\mathbf{\Phi})$ ratio for $\tilde{G}_{kl}/\tilde{G}_{lk}$ by implicitly defining new entries entering into knapsack as pairs of commodities $[m, n]$ and $[o, p]$ in $(k \to l)$ and $(l \to k)$ directions, respectively, in such a way that their weighted combined improvement ratio (specifically, $(B_{kl}^{mn} + (1-\Phi)B_{lk}^{op})$) is the most negative. □

## Illustration of RP-networks and RP-RP Link Utilizations

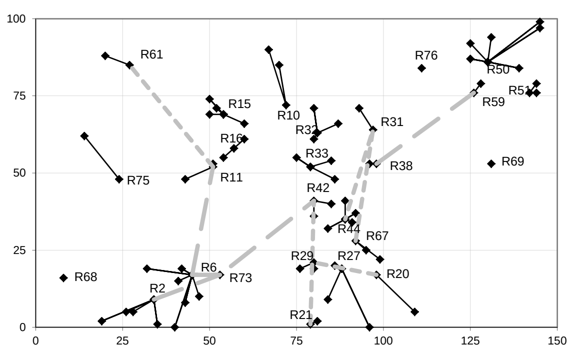
Figure 1 provides the RP-networks for one of the instances in class C14 with all four settings of commodity o-d distance distributions. In Figure 1, nodes are represented by dots and are labeled if they are RPs. A solid black line indicates the assignment of non-RP node to RP. Five most heavily used links are indicated using dashed grey lines and, similarly, five links with the highest link-imbalance are indicated by dotted grey lines. In a group of nodes, RPs are usually located on the nodes that are towards the center of the regions to avoid excessive transportation. Moreover, RPs are also located on the non-group nodes in order to connect them to the RP-network. We also observe that the most utilized links are those that connect the south-west and the noth-east territories (groups of regions). However, with the increased number of shorter range commodities, links connecting other regions can also be used more heavily. Unlike the heavily utilized links, links that have high link-imbalance level are typically those that connect regions with few nodes.



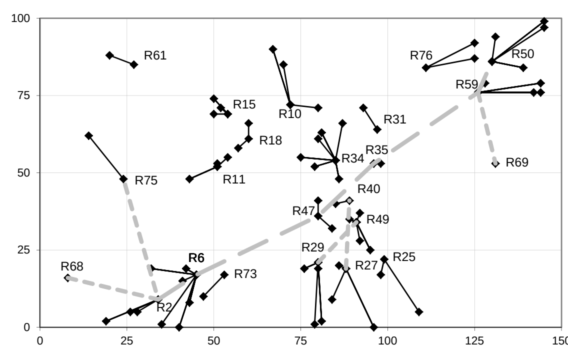

(a) 60-20-20




(b) 20-60-20




(c) 20-20-60




(d) 40-30-30

Figure 1: RP-Network for Varying Long-Medium-Short Distance Commodity Distributions