Appendix A: An Example of Double-Queue Model

The parameters of the link are described as follows. Free flow travel time $\tau^0 = 1$; shockwave travel time $\tau^\omega = 2$; inflow and exit flow capacities $\overline{C}^p = \overline{C} = 5$; queue capacity $\overline{Q} = (\tau^0 + \tau^\omega)\,\overline{C} = 15$. Without losing generality, the length of the link is 1 unit length. The critical density (when the flow reaches its capacity) is $k_c = 5$, and the jam density is $k_{jam} = 15$. Figure 1 depicts the fundamental diagram (FD) of the link assuming it is triangular.

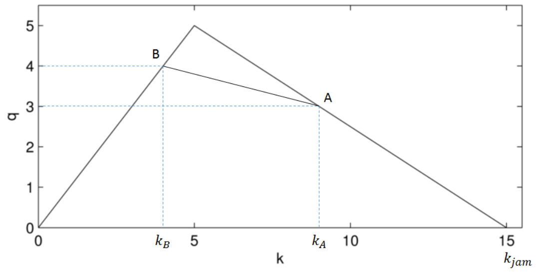

Figure 1 Fundamental diagram of the single link

The desired inflow profile is pre-given as a constant. However, the exit flow is restricted to 3 from the very beginning (time $t = 0$) till time 11 minutes due to the downstream blockage. In this example, we show that the downstream blockage can spill back from the exit to the entrance of this link, and the actual inflow rate is reduced from the desired inflow rate for a time period.

Figure 2 shows the desired inflow rate $p'(t)$, the actual inflow rate $p(t)$, and exit flow rate $v(t)$, the number of vehicles on the link $x(t)$, the upstream queue $q^u(t)$ and the downstream queue $q^d(t)$, respectively. The desired inflow rate is constantly 4 from time 0 to 15 minutes. The exit flow rate is restricted to be no more than 3 from time 0 to 11 minutes, in order to illustrate a downstream blockage. Notice that the downstream blockage is actually effective to reduce the exit flow rate at time 1 minute, since before this time instant, there is no exit flow from this link. The number of vehicles, upstream and downstream queues are zero at the very beginning ($t = 0$).

We first discuss the queue dynamics, particularly the upstream queue. Here we define the (right-side) slope of $q^u(t)$ at time $t+$ as $\dot{q}^u(t^+) = q^u(t+1) - q^u(t)$. Also, according the double queue dynamics, $\dot{q}^u(t) = p(t) - v(t-2)$, as $\tau^w = 2$. Before time $t = 6$, q^u has experienced two different slopes. According to Figure 2, we here list $p(t)$ and $v(t-2)$, and calculate the corresponding slope of $q^u(t)$ for $t = 0, 1, \cdots, 5$ in Table 1.

Table 1 Flow and queue dynamics from time 0 to 5

Time t	0	1	2	3	4	5
Inflow $p(t)$	4	4	4	4	4	4
Exit flow $v(t-2)$	0	0	0	3	3	3
$\dot{q}^u(t^+)$	4	4	4	1	1	1

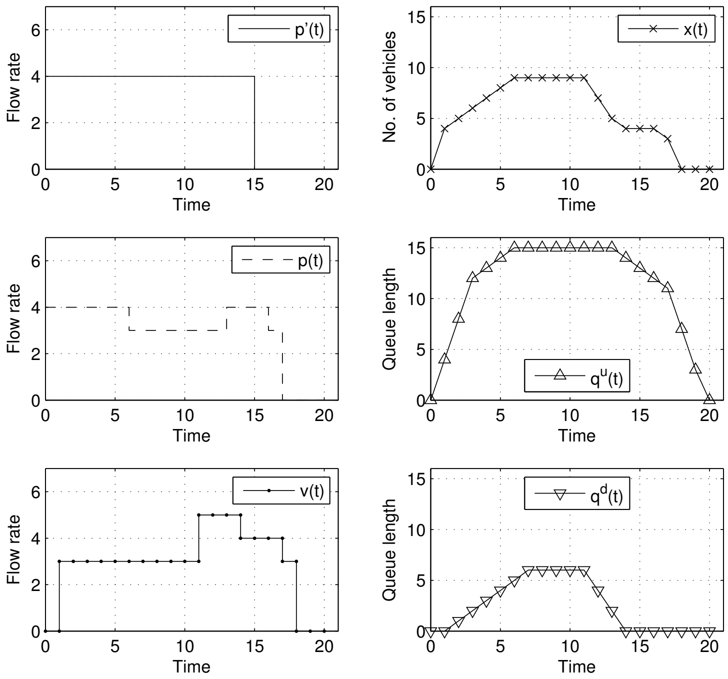

Figure 2 **Flows and queues for a single link with downstream blockage**

For $t = 0, 1$ and 2, the exit flow with time delay τ^w is still zero, while the inflow is 4, so that the slope of $q^u(t)$ is 4 during these times. After time $t = 3$, the exit flow with time delay τ^w becomes to be 3, so that the slope of $q^u(t)$ changes to 1.

After time $t = 13$, $q^u(t)$ has experienced three different slopes. We then list the $p(t)$ and $v(t-2)$, and calculate the corresponding slope of $q^u(t)$ for $t = 13, 14, \cdots, 19$ in Table 2.

Table 2 **Flow and queue dynamics from time 13 to 19**

Time t	13	14	15	16	17	18	19
Inflow $p(t)$	4	4	4	3	0	0	0
Exit flow $v(t-2)$	5	5	5	4	4	4	3
$\dot{q}^u(t^+)$	-1	-1	-1	-1	-4	-4	-3

As we can observe, from time $t = 13$ to 16, the slope is -1; from time $t = 17$ to 18, the slope is -4; for time $t = 19$, the slope is -3. The last one at time $t = 19$ is -3 because it clears the residual $q^u(t)$ at the end; after this time $t = 19$, $q^u(t)$ becomes zero.

We can then observe the shockwave occurs following the analysis for the triangular fundamental diagram. At time 1 minute, the exit flow rate is restricted to 3, and the inflow rate keeps at 4 until time 6 minutes. It takes 5 minutes for the downstream blockage to spill back from the exit to the entrance of the link. We further observe that when the inflow rate starts to be restricted to 3, at the same time ($t = 6$), the upstream queue reaches the queue capacity $\overline{Q} = 15$. This indicates that when the upstream queue of a link reaches its capacity, queue spillback will occur and the inflow to the link is subject to reduction (i.e., flow withholding may happen) due to the spillback.

Such observations from Figure 2 are in fact consistent with the shockwave analysis for the triangular fundamental diagram. When the spillback occurs, the flow rate of the downstream traffic is $C_A = 3$ and the traffic is in congestion state, which is noted as Point A in Figure 2; The corresponding traffic density is $k_A = 9$. The flow rate of the upstream traffic is $C_B = 4$ and the traffic is in free flow state, noted as Point B; The corresponding traffic density is $k_B = 4$. According to the shockwave analysis, the spillback shockwave speed is calculated as

$$\frac{C_A - C_B}{k_A - k_B} = \frac{3-4}{4-9} = -\frac{1}{5}.$$

The negative sign means the spillback shockwave propagates against the traffic flow direction. The absolute value of the shockwave speed is 1/5, which means the shockwave needs 5 minutes to travel from the exit back to the entrance of the link (as the link length is 1 unit length). Thus the analysis from the fundamental diagram is consistent with the double-queue link model for this example.

The above example shows that for the specific link, the double-queue model is consistent with the triangular fundamental diagram when downstream blockage spills back to the entrance. In fact, such consistency also holds for a general link in a network. This is true since (i) LTM is equivalent to the LWR model if triangular fundamental diagram is used (Yperman 2007); and (ii) the double queue model is equivalent to the LTM model (Ma et al. 2014).

Appendix B: Discussions on Link Travel Time

The link travel time definition in equations (4)and (5) are based on the cumulative link inflow and cumulative link exit flow, which can be defined as $P_{ij}(t) = \int_0^t p_{ij}(w)dw$ and $V(t) = \int_0^t v_{ij}(w)dw$, respectively, for a link (i, j). Following these two equations, Figure 3 shows three possible cases for calculating link travel time. Figure 3 (a) indicates that if both the cumulative inflow and cumulative exit flow curves are strictly increasing, the link travel time is continuous and well-defined.

If the cumulative exit flow curve is strictly increasing, while the cumulative inflow curve is not strictly increasing, the link travel time is also continuous, as shown in Figure 3 (c). Notice that for the time period when the cumulative inflow does not increase, the link travel time is still defined as the horizontal distance of the two curves. Since at any time t, there is a unique point on the cumulative inflow curve, one can easily find a unique definition of the link travel time. In particular, the slope of the link travel time is -1, when the cumulative inflow curve does not increase. The link travel time in this case is continuous and well-defined; the derivative of link travel time $\dot{\tau}(t)$ is not (the slopes of $\tau(t)$ are not well-defined at the kinky points). However, the link travel time function has a slope that is larger than or equal to -1. This means that FIFO still holds.

If the cumulative exit curve is not strictly increasing, i.e., there are periods of times when the exit flow stops and the cumulative exit curve does not change, the link travel time will be discontinuous at the time when the exit flow first stops. This is shown in Figure 3 (b). In this case, the link travel time is not well-defined. In the figure, at time t when the exit flow first stops, there is a "jump" of the travel time at t and

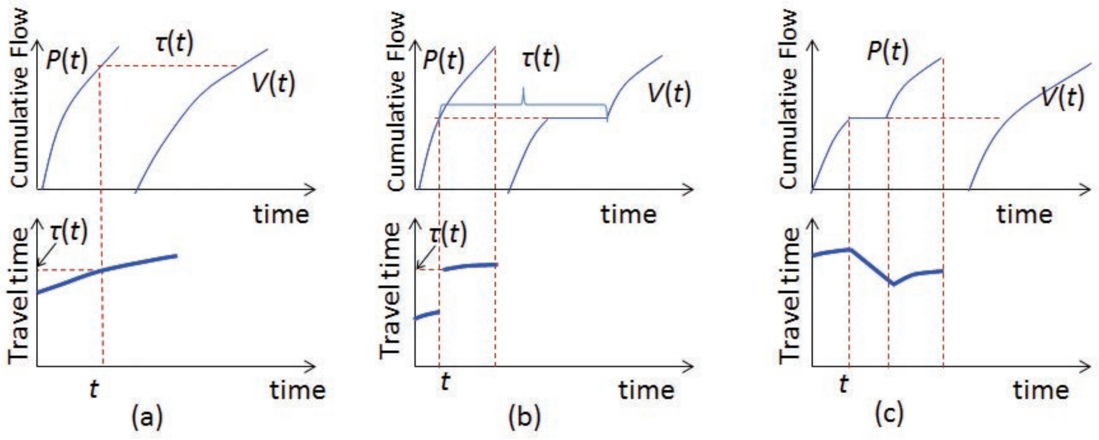

Figure 3 **Nonunique link travel times**

the travel time definition is not unique at this time instant. To remedy this issue, we may find the travel time in this case as the maximum of the possible link travel times, i.e.,

$$\tau_{ij}(t) = \sup\{t' > t | P_{ij}(t) = V_{ij}(t')\} - t. \tag{1}$$

The derivative of link travel time $\dot{\tau}(t)$ at the "jump" time t is not well-defined either. However, $\dot{\tau}(t) \to +\inf > -1$, so the FIFO condition holds as well at the "jump" time. Therefore, the FIFO condition holds in all three cases if link travel time is calculated using equation (1) for the case in Figure 3 (b). Notice that (1) here is a very intuitive way to resolve the nonunique travel time problem. How to deal with this issue more rigorously needs further investigation.

Appendix C: Proofs of Properties of the DCS-based Traffic Flow Model in Section 3.4

C.1. Proof of Property 2 of the DCS-based traffic flow model

Proof We first prove that for $\widehat{m} \in \{1, \cdots, M_i\}$, $\sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) \geq \sum_{1\leq\widehat{m}\leq M_i} \delta_{n_i^{\widehat{m}},i}(t)$ by an inductive augment.

Assume that $\sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t) \geq 0$ for some $m \in \{1, \cdots, M_i\}$. This is clearly true for $m = 1$ because the second sum is zero.

- If $0 \leq \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) \leq \sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t)$, $v_{n_i^m,i}(t) = 0$ by Eq(12)This leads to $\underline{\delta}_{n_i^m,i} = \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t)$ from Eq(11). From Eq(11), $\underline{\delta}_{n_i^m,i} \geq 0$, which implies $\delta_{n_i^m,i} = \underline{\delta}_{n_i^m,i}$. Then $\delta_{n_i^m,i} = \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t)$ so that

$$\sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m} \delta_{n_i^{\widehat{m}},i}(t) = \sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t) - \left(\overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t)\right) \geq 0.$$

- If $\overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) \leq 0 \leq \sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t)$, then using the same argument as shown above, $v_{n_i^m,i}(t) = 0$ and $\underline{\delta}_{n_i^m,i} = \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) \leq 0$. This leads to $\delta_{n_i^m,i} = 0$ so that

$$\sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m} \delta_{n_i^{\widehat{m}},i}(t) = \sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t) \geq 0 \text{ by induction.}$$

- If $0 \leq \sum\limits_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum\limits_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t) \leq \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t)$, then $\underline{\delta}_{n_i^m,i} > 0$ and $\delta_{n_i^m,i} = \underline{\delta}_{n_i^m,i} = \sum\limits_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum\limits_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t)$ based on Eq(11).

This leads to

$$\sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m} \delta_{n_i^{\widehat{m}},i}(t) = 0$$

Hence, $\sum\limits_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum\limits_{1\leq\widehat{m}\leq m} \delta_{n_i^{\widehat{m}},i}(t) \geq 0$ for $m = 1, \cdots, M_i$.

By setting $m = M_i$, it follows that $\sum\limits_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum\limits_{1\leq\widehat{m}\leq M_i} \delta_{n_i^{\widehat{m}},i}(t) \geq 0$.

C.2. Proof of Property 3 of the DCS-based traffic flow model

Proof From the definition of exit flow in (12), it follows easily that $v_{n_i^m,i}(t) \geq 0$.

On one hand, if $\overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) \leq \sum\limits_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum\limits_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t)$, then $v_{n_i^m,i}(t) = 0$.

On the other hand, if $\overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) \geq \sum\limits_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum\limits_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t)$, by Property 1

$$\underline{\delta}_{n_i^m,i}(t) = \sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\leq\widehat{m}\leq m-1} \delta_{n_i^{\widehat{m}},i}(t) \geq 0.$$

We also have $\mu_{n_i^m,i}(t) \geq 0$, hence

$$v_{n_i^m,i}(t) = \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) - \underline{\delta}_{n_i^m,i}(t) \leq \overline{C}_{n_i^m,i}.$$

C.3. Proof of Property 7 of the DCS-based traffic flow model

When $\dot{\tau}_{ij}(t)$ is well-defined, $\tau_{ij}(t)$ is differentiable for almost all time t. Differentiating (5) and we have

$$p_{ij}(t) = v_{ij}(t + \tau_{ij}(t))\,[1 + \dot{\tau}_{ij}(t)].$$

Since $v_{ij}(t + \tau_{ij}) \geq 0$, we have

$$p_{ij}(t) > 0 \rightarrow 1 + \dot{\tau}_{ij}(t) \geq 0,$$

which is the FIFO condition. Further, it follows that

$$p_{ij}(t) > 0 \Rightarrow v_{ij}(t + \tau_{ij}(t)) > 0.$$

Appendix D: Proofs of Properties of the Continuous-Time DUE Model in Section 5

D.1. Proof of Property 1 of DUE model

Proof Suppose by way of contradiction that $\lambda_i \leq 0$ for some $i \in \widetilde{\mathcal{O}}$. By (28), it follows that $\overline{D}_i = D_i(T)$ since $\overline{\lambda}_i > 0$. If $d_i(t) > 0$ for some t, then $\pi_i(t) = F_i(t, \pi_i(t)) = \lambda_i = 0$ by the departure-time choice. By Wardrop's route-choice principle, $p_{ij}(t) = 0$ for all j such that $(i,j) \in \widetilde{\mathcal{L}}$, since $\tau_{ij}(t) \geq \tau_{ij}^0 > 0$; but this contradicts the flow conservation at node i. Hence $d_i(t) = 0$ for all times $t \in (0, T]$ at which $d_i(t)$ is well defined. But this contradicts

$$0 < \overline{D}_i = D_i(T) = \int_0^T d_i(s)\,ds = 0.$$

Consequently, the claim is proved.

D.2. Proof of Property 2 of DUE model

Proof For each node $i \in \mathcal{N} \cup \widetilde{\mathcal{O}} \setminus \{s\}$ at time t, denote $i_0^t \triangleq i$, $t_0 = t$, and $t_1 = t_0 + \tau_{i_0^t, i_1^t}(t_0)$. Since $\sum_{k:(i,k)\in\overline{\mathcal{L}}} p_{ik}(t) > 0$, we can find a link $(i_0^t, i_1^t) \in \mathcal{L}$ such that $p_{i_0^t,i_1^t}(t) > 0$. By Property 7′ of the capacitated queue dynamics in Section 3.4, if $p_{i_0^t,i_1^t}(t_0) > 0$, then $v_{i_0^t,i_1^t}(t_1) > 0$. At node i_1^t, by the flow conservation complementarity (29) and non-negativity of the demand rate $d_{i_1^t}(t) \geq 0$, if $\sum_{k:(k,i_1^t)\in\overline{\mathcal{L}}} v_{k,i_1^t}(t_1) > 0$, then $\sum_{j:(i_1^t,j)\in\overline{\mathcal{L}}} p_{i_1^t,j}(t_1) > 0$.

Therefore, by following the similar procedure, for any node i_{m-1}^t, denote $t_m = t_{m-1} + \tau_{i_{m-1}^t, i_m^t}(t_{m-1})$, and we can find a link $(i_{m-1}^t, i_m^t) \in \mathcal{L}$ such that $p_{i_{m-1}^t, i_m^t}(t) > 0$, where $m = 1, 2, \cdots, K_i^t + 1$ and $i_{K_i^t+1} \triangleq s$. Let

$$\mathcal{P}_i^t : i \triangleq i_0^t \longrightarrow i_1^t \cdots \longrightarrow i_{K_i^t}^t \longrightarrow i_{K_i^t+1}^t \triangleq s \tag{2}$$

be a dynamic shortest path joining node $i \in \mathcal{N} \cup \widetilde{\mathcal{O}} \setminus \{s\}$ to the destination s, starting at time t. Notice that if multiple dynamic shortest paths exist, at least one of them carries positive flow, which is denoted as $\mathcal{P}_i^t$.

We then have the following inequalities

$$\begin{array}{l} p_{i_m^t, i_{m+1}^t}(t_m) > 0, \\ v_{i_m^t, i_{m+1}^t}(t_{m+1}) > 0, \end{array} \quad m = 0, 1, \cdots, K_i^t. \tag{3}$$

From the route choice complementarities (25) we know that for $m = 0, 1, \cdots, K_i^t$,

$$0 \leq p_{i_m^t, i_{m+1}^t}(t_m) \perp \tau_{i_m^t, i_{m+1}^t}(t_m) + \pi_{i_{m+1}^t}(t_{m+1}) - \pi_{i_m^t}(t_m) \geq 0.$$

From (3), $p_{i_m^t, i_{m+1}^t}(t_m) > 0$, so that we have the equations for $m = 0, 1, \cdots, K_i^t$,

$$\tau_{i_m^t, i_{m+1}^t}(t_m) + \pi_{i_{m+1}^t}(t_{m+1}) - \pi_{i_m^t}(t_m) = 0. \tag{4}$$

By summing (4) with $m = 0, 1, \cdots, K_i^t$,

$$\sum_{m=0}^{m=K_i^t} \tau_{i_m^t, i_{m+1}^t}(t_m) + \pi_{i_{m+1}^t}(t_{m+1}) - \pi_{i_m^t}(t_m) = 0.$$

we then have

$$\pi_{i_0^t}(t_0) - \pi_{i_{K_i^t+1}^t}(t_{K_i^t+1}) = \sum_{m=0}^{K_i^t} \tau_{i_m^t, i_{m+1}^t}(t_m) \geq \sum_{m=0}^{K_i^t} \tau^0_{i_m^t, i_{m+1}^t}$$

The inequality is from the Property 7 of the capacitated queue dynamics.

As defined in (33), π_i^0 is the total free-flow time on the minimum free-flow-time path $\mathcal{P}_i$ joining node i to the destination s. This implies that

$$\sum_{m=0}^{K_i^t} \tau^0_{i_m^t, i_{m+1}^t} \geq \sum_{k=0}^{K_i} \tau^0_{i_k, i_{k+1}} = \pi_i^0.$$

Since $t = t_0$, $i \triangleq i_0^t$, $s \triangleq i_{K_i^t+1}^t$ and $\pi_s(t_{K_i^t+1}) = 0$, we have $\pi_i(t) \geq \pi_i^0$.

D.3. Proof of Property 3 of DUE model

Proof First we look for an upper bound of $\pi_i(t)$. As defined in Property 2 of the DUE model, for each dummy origin node $i \in \widetilde{\mathcal{O}}$ and any time t, $\mathcal{P}_i^t$ is a dynamic shortest path. For link $(i_m^t, i_{m+1}^t) \in \mathcal{P}_i^t$, $m = 0, 1, \cdots, K_i^t$, the Wardrop's route choice principle indicates the following inequalities:

$$\tau_{i_m^t, i_{m+1}^t}(t_m) + \pi_{i_{m+1}^t}(t_{m+1}) - \pi_{i_m^t}(t_m) \geq 0.$$

Summing up the above inequalities over all $m = 0, 1, \cdots, K_i^t$, we then have

$$\pi_i(t) \leq \sum_{m=0}^{K_i^t} \tau_{i_m^t, i_{m+1}^t}(t_m) \leq T \triangleq \overline{\pi}_i \tag{5}$$

Next, recalling the definition of the penalty function F_i, we deduce

$$F_i(t, \pi_i(t)) \leq \max\{\, \alpha R_i, \gamma\, (T + \overline{\pi}_i\,)\,\} \triangleq \overline{F}_i$$

With the upper bounds of $\pi_i(t)$ and $F_i(t, \pi_i(t))$, we define $\lambda_i^{\text{critical}} \triangleq \overline{\pi}_i + \overline{F}_i$. From departure time choice complementarity, we have

$$0 \leq \pi_i(t) + F_i(t, \pi_i(t)) - \lambda_i \leq \overline{\pi}_i + \overline{F}_i - \lambda_i = \lambda_i^{\text{critical}} - \lambda_i.$$

Hence $\lambda_i \leq \lambda_i^{\text{critical}}$. The value $\lambda_i^{\text{critical}}$ is the critical upper bound on disutility λ_i. Since $\overline{\lambda_i}$ can be arbitrarily chosen, we choose $\overline{\lambda_i} > \lambda_i^{\text{critical}}$ so that $\overline{\lambda_i} > \lambda_i$. According to Eq(28), $\overline{D}_i = D_i(T)$.