Appendix A: Notation

$\underline{b}_t^k$	lower bound on optimal value in iteration k
$\bar{b}_t^k$	upper bound on optimal value in iteration k
B_j	Value currently stored in the state coordinate with the label j
β_n	amount of energy absorbed by the battery in period n of the charging process
C_t	cost contribution in period t
$\bar{C}_t$	cost contribution in period t of lower bounding model
C^K	infinite horizon cost per charging bay
C^N	infinite horizon cost per battery
D_t	demand for batteries in period t
e_t	price of electric energy in period t
E_t	price of electric energy in period prior to period t
$\mathcal{E}$	space of possible values for E
f_t	infinite horizon operating cost from period t
γ	discount factor
I	Number of stations in the network model
$\mathcal{I}$	Set of stations in the network model
K	number of charging bays
κ	mean reversion parameter
λ_t	demand rate in period t
L	battery charge time
μ_t	mean of energy cost distribution in period t
N	number of batteries
p_t	per period, per unit penalty cost for stockouts
Π	set of all coordinates of state space
$R_{t,i}$	number of charging batteries in period t that become available in i periods
σ_t	standard deviation of energy cost distribution in period t
S_t	state
$\mathcal{S}$	state space
$\mathcal{S}^c$	convex subspace of state space
T	planning horizon, i.e., number of periods
U_t	number of available uncharged batteries in period t
$\mathcal{U}$	space of feasible values for U
V_t	value function
$\underline{V}_t$	global lower bound
$\underline{V}_t^K$	coordinatewise lower bound for K charging bays
$\underline{V}_t^N$	coordinatewise lower bound for N batteries
$\tilde{V}_t$	value function of transformed model
$\hat{V}_t$	value function approximation
$\bar{V}_t$	value function of lower bounding model
$W_{t,i}$	number of charged batteries in period t that become available in i periods
$\bar{W}_t$	inventory position of charged batteries in period t
$\mathcal{W}$	space of feasible values for W
x_t	number of batteries to start charging in period t
$\mathcal{X}$	action space
$\tilde{X}$	action space of transformed model
$\bar{X}$	action space of lower bounding model
$y_{t,i,j}$	number of charged batteries transshipped from station i to j starting in t
$z_{t,i,j}$	number of uncharged batteries transshipped from station i to j starting in t
ζ_t	normally distributed noise term of energy price in period t

Appendix B: Proofs of Statements

All properties from Sections 3.2 and 4.2 are proven for finite horizon versions of the dynamic program. In this dynamic program the fixed planning horizon of T periods is cycled through O times, resulting in a total of $\hat{T} = OT$ periods. The corresponding results for the infinite horizon case follow by taking the limit $O \to \infty$ (see, for instance, Bertsekas 2007, Chapter 1).

B.1. Proof of Property 1

We prove the property by a counterexample.

We consider a deterministic model with $L = 1$, $D_t = 10$, $E_t = 3$, and $p_t = 100$ over a horizon of $\bar{T} = 2$ periods, i.e., all cost is zero from the third period on. Note that we assume $T > \bar{T}$ and all costs between T and $\bar{T}$ are zero. Without loss of generality, we set $K = M$, where M is a big number. We omit dependence of V_t on K.

In $t = \bar{T} = 2$ it is always optimal not to start charging batteries, because the costs from $\bar{T}$ onwards are zero and energy prices $E_t > 0$. Then $V_2(S_2, 100) = 100[10 - W_{2,0}]^+ + 2([W_{2,0}]^+ + U_2)$ and V_2 is convex in S_2. Function V_2 takes its minimum at $(W_{2,0}, U_2) = (10, 0)$.

For $L = 1$, S_t is a function of S_{t-1}, x_{t-1}, and D_{t-1}. It is defined by $W_{t,0} = W_{t-1,0} + x_{t-1} - D_{t-1} = W_{t-1,0} + x_{t-1} - 10$ and $U_t = U_{t-1} - x_{t-1} + \min([-W_{t-1,0}]^+ + D_{t-1}, [W_{t-1,0}]^+ + x_{t-1}) = U_{t-1} - x_{t-1} + \min([-W_{t-1,0}]^+ + 10, [W_{t-1,0}]^+ + x_{t-1})$.

Since $U_0 = 0$, we have $x_0 = 0$. As a result, $V_0(-11, 0) = 2100 + V_1(-21, 0)$ and $V_0(-10, 0) = 2000 + V_1(-20, 0)$, $V_0(0, 0) = 1000 + \gamma V_1(-10, 0)$, $V_0(1, 0) = 902 + \gamma V_1(-9, 1)$, $V_0(20, 0) = 40 + V_1(10, 10)$ and $V_0(21, 0) = 42 + V_1(11, 10)$.

Consider $V_1(S_1)$. It is given by

$$V_1(S_1) = 100[10 - W_{1,0}]^+ + 2([W_{1,0}]^+ + U_1) + \min_{x \le U_1} \left\{3x + \gamma(100[10 - W_{2,0}]^+ + 2([W_{2,0}]^+ + U_2))\right\}.$$

We obtain $V_1(-10, 0) = 2000 + \gamma \cdot 3000$, $V_1(-9, 1) = 1902 + 3 + \gamma \cdot 2802$, $V_1(-21, 0) = 3100 + \gamma \cdot 4100$, $V_1(-20, 0) = 3000 + \gamma \cdot 4000$, $V_1(10, 10) = 40 + 30 + \gamma \cdot 40$ and $V_1(11, 10) = 42 + 27 + \gamma \cdot 42$.

Let us assume $\gamma = 0.9$. The first difference of V_0 is then not increasing for all S_0, because

$$V_0(-10, 0) - V_0(-11, 0) = 2000 - 2100 + \gamma(3000 - 3100) + \gamma^2(4000 - 4100) = -100(1 + \gamma + \gamma^2) = -271$$

$$V_0(1, 0) - V_0(0, 0) = 902 - 1000 + \gamma(1905 - 2000) + \gamma^2(2802 - 3000) = -98 - 95\gamma - 198\gamma^2 = -343.88$$

and

$$V_0(21, 0) - V_0(20, 0) = 42 - 40 + \gamma(69 - 70) + \gamma^2(42 - 40) = 2 - \gamma + 2\gamma^2 = 2.72.$$

This completes the counterexample and shows that $V_t(S_t, K)$ is not coordinatewise convex in $S_t \in \mathcal{S}$.

B.2. Proof of Property 2

We prove coordinatewise convexity of $\tilde{V}_t$ in W, K, and N by analyzing a continuous extension $\bar{V}_t$. We first show that $\bar{V}_t$ is convex in W, K, and N. We then show that $\tilde{V}_t$ is the restriction of $\bar{V}_t$ to integer values for W, K, and N. Coordinatewise convexity then follows from the fact that $\bar{V}_t$ is convex.

We define:

- $\bar{\mathcal{X}}(W, K, N) = \mathbf{conv}\ \tilde{\mathcal{X}}(W, K, N)$
- $\bar{\mathcal{W}}(K) = \mathbf{conv}\ \mathcal{W}(K)$.

- $J_t(x, E_t, W_t, E_{t+1}, D_t, K, N) = C_t(W_t, x, E_t, D_t) + \gamma \bar{V}_{t+1}(W_{t+1}, E_{t+1}, K, N)$
- $x_t^*(E_t, W_t, E_{t+1}, D_t, K, N) = \arg\min_{x \in \bar{\mathcal{X}}} J_t(x, E_t, W_t, E_{t+1}, D_t, K, N)$
- $\bar{J}_t(x, E_t, W_t, E_{t+1}, D_t, K, N) = J_t(\lfloor x \rfloor, E_t, W_t, E_{t+1}, D_t, K, N)(\lceil x \rceil - x) + J_t(\lceil x \rceil, E_t, W_t, E_{t+1}, D_t, K, N)(x - \lfloor x \rfloor)$
- $\bar{V}_t(W_t, E_t, K, N) = \mathbb{E}_{E_{t+1}, D_t}\left[\min_{x \in \bar{\mathcal{X}}(W_t, K, N)} \left\{\bar{J}_t(x, E_t, W_t, E_{t+1}, D_t, K, N)\right\} \middle| E_t\right]$

Note that:

1. J_t is convex, if C_t and $\bar{V}_{t+1}$ are convex and W_{t+1} is an affine function of W_t and x.
2. $\bar{J}_t$ is convex, if J_t is convex, because it is the linear interpolation of J_t between integer points in x.

The state transition function is given by $W_{t+1}(W_t, D_t, x) = [W_{t,0} + W_{t,1} - D_t, W_{t,2}, \ldots, W_{t,L}, x]$, and it is clearly an affine function of W_t and x. We also use the standard trick to turn a constrained convex problem into an unconstrained convex problem via the set characteristic function.

We denote the domain of J_t for given E_t, E_{t+1}, and D_t by $\mathcal{J}(E_t, E_{t+1}, D_t)$ and the domain of $\bar{V}_{t,i}$ for given E_t by $\mathcal{V}(E_t)$. We have

$$\mathcal{J}(E_t, E_{t+1}, D_t) = \{(x, E_t, W_t, E_{t+1}, D_t, K, N) : K \in \mathbb{R}_0^+, N \in \mathbb{R}_0^+, W_t \in \bar{\mathcal{W}}(K), x \in \bar{\mathcal{X}}(W_t, K, N)\}$$

and

$$\mathcal{V}(E_t) = \{(W_t, E_t, K, N) : K \in \mathbb{R}_0^+, N \in \mathbb{R}_0^+, W_t \in \bar{\mathcal{W}}(K), x \in \bar{\mathcal{X}}(W_t, K, N)\}.$$

Sets $\mathcal{J}(E_t, E_{t+1}, D_t)$ and $\mathcal{V}(E_t)$ are polyhedra and therefore convex.

We show convexity of $\bar{V}_t(W_t, E_t, K, N)$ by induction. First note that $C_t(W_t, x, E_t, D_t)$ is convex in x and W_t for all E_t, D_t, and t when $p_t \geq 0$.

Base Case.

Let us assume $\bar{V}_{\bar{T}+1} = 0$. Then $J_{\bar{T}}(x, E_{\bar{T}}, W_{\bar{T}}, E_{\bar{T}+1}, D_{\bar{T}}, K, N) = C_{\bar{T}}(W_{\bar{T}}, x, E_{\bar{T}}, D_{\bar{T}})$. Set $\mathcal{J}(E_{\bar{T}}, E_{\bar{T}+1}, D_{\bar{T}})$ is convex and nonempty. Additionally, $J_{\bar{T}}$ and $\bar{J}_{\bar{T}}$ are convex. Then the following lemma holds:

LEMMA 1. $\min_{x \in \bar{\mathcal{X}}(W_{\bar{T}}, K, N)} \left\{\bar{J}_{\bar{T}}(x, E_{\bar{T}}, W_{\bar{T}}, E_{\bar{T}+1}, D_{\bar{T}}, K, N)\right\}$ *is convex in* $W_{\bar{T}}$, K, *and* N.

The proof for Lemma 1 can be found in Appendix B.9. Taking the expected value with respect to e preserves convexity and we conclude that $\bar{V}_{\bar{T}}(W_{\bar{T}}, E, K, N)$ is convex in $W_{\bar{T}}$, K, and N.

Induction Step.

Now assume that $\bar{V}_{t+1}(W_{t+1}, E_{t+1}, K, N)$ is convex in W_{t+1}, K, and N on $\mathcal{V}(E_{t+1})$.

Then, J_t and $\bar{J}_t$ are convex in W_t, K, and N, because C_t is convex and W_{t+1} is an affine mapping of W_t and x. Then, because $\mathcal{V}(E)$ is convex and nonempty, and because taking the expected value preserves convexity, we conclude that $\bar{V}_t(W, E, K, N)$ is convex in W, K, and N on $\mathcal{V}(E)$.

This concludes the first part of the proof.

It remains to show, that $\tilde{V}_t(W, E, K, N) = \bar{V}_t(W, E, K, N)$ for all $(E, W, K, N) \in \mathcal{E} \times \{W \in \mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\} \subseteq \mathcal{E} \times \mathcal{V}(E)$. We show this by induction. First, note that for $(E, W, K, N) \in \mathcal{E} \times \{W \in$

$\mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$ we have $\max\{x : x \in \bar{\mathcal{X}}(W,K,N)\} = \max\{x : x \in \tilde{\mathcal{X}}(W,K,N)\} \in \mathbb{Z}_0^+$ and $\min\{x : x \in \bar{\mathcal{X}}(W,K,N)\} = \min\{x : x \in \tilde{\mathcal{X}}(W,K,N)\} \in \mathbb{Z}_0^+$.

Base Case.

Assume $\bar{V}_{\bar{T}+1} = 0$. Function $\bar{J}_{\bar{T}}$ is piece-wise linear in x and has integer breakpoints. Then it holds

$$\begin{aligned}
&\min_{x \in \bar{\mathcal{X}}(W_{\bar{T}},K,N)} \bar{J}_{\bar{T}}(x, E_{\bar{T}}, W_{\bar{T}}, E_{\bar{T}+1}, D_{\bar{T}}, K, N) \\
&= \min\{\bar{J}_{\bar{T}}(\lfloor x^*_{\bar{T}} \rfloor, E_{\bar{T}}, W_{\bar{T}}, E_{\bar{T}+1}, D_{\bar{T}}, K, N), \bar{J}_{\bar{T}}(\lceil x^*_{\bar{T}} \rceil, E_{\bar{T}}, W_{\bar{T}}, E_{\bar{T}+1}, D_{\bar{T}}, K, N)\} \\
&= \min_{x \in \tilde{\mathcal{X}}(W_{\bar{T}},K,N)} C_{\bar{T}}(W_{\bar{T}}, x, E_{\bar{T}}, D_{\bar{T}}).
\end{aligned}$$

It follows that $\tilde{V}_{\bar{T}}(W_{\bar{T}}, E_{\bar{T}}, K, N) = \bar{V}_{\bar{T}}(W_{\bar{T}}, E_{\bar{T}}, K, N)$ for all $(E_{\bar{T}}, W_{\bar{T}}, K, N) \in \mathcal{E} \times \{W_{\bar{T}} \in \mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$.

Induction Step.

The induction assumption is $\tilde{V}_{t+1}(W_{t+1}, E_{t+1}, K, N) = \bar{V}_{t+1}(W_{t+1}, E_{t+1}, K, N)$ for all $(E_{t+1}, W_{t+1}, K, N) \in \mathcal{E} \times \{W_{t+1} \in \mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$.

From the induction assumption, it follows that $C_t(W_t, x, E_t, D_t) + \gamma \tilde{V}_{t+1}(W_{t+1}, E_{t+1}, K, N) = C_t(W_t, x, E_t, D_t) + \gamma \bar{V}_{t+1}(W_{t+1}, E_{t+1}, K, N) = \bar{J}_t(x, E_t, W_t, E_{t+1}, D_t, K, N)$ for all $(E_{t+1}, W_{t+1}, K, N) \in \mathcal{E} \times \{W_{t+1} \in \mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$.

We have that $(E_{t+1}, W_{t+1}, K, N) \in \mathcal{E} \times \{\mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$, where W_{t+1} is a function of W_t, x_t, and D_t, if the following conditions hold:

1. $x_t \in \tilde{\mathcal{X}}(W_t, K, N)$
2. $D_t \in \mathbb{Z}_0^+$
3. $(E_t, W_t, K, N) \in \mathcal{E} \times \{\mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$.

Condition 2 holds by definition and condition 3 holds, if $W_0 \in \mathbb{Z}_+$ and $x_t \in \mathbb{Z}_+$ for all $t = 0, \dots, t-1$. This is true since $W_0 = (N, 0, \dots, 0)$ and $N \in \mathbb{Z}_+$ and all x^*_t and D_t are integer.

We show that $x^*_t \in \tilde{\mathcal{X}}(W_t, K, N)$. Since $\bar{J}_t$ is piece-wise linear in x with integer breakpoints, $\arg\min_{x \in \bar{\mathcal{X}}(W_t,K,N)} \bar{J}_t(x, E_t, W_t, E_{t+1}, D_t, K, N)$ is integer, if condition 3 holds (for the case $\bar{J}_t(\lfloor x^*_t \rfloor, E_t, W_t, E_{t+1}, D_t, K, N) = \bar{J}_t(\lceil x^*_t \rceil, E_t, W_t, E_{t+1}, D_t, K, N)$, we use the convention $\arg\min_x \bar{J}_t(x, E_t, W_t, E_{t+1}, D_t, K, N) = \lfloor x^*_t \rfloor$). This means that $x^*_t \in \tilde{\mathcal{X}}(W_t, K, N)$.

This establishes that $(E_{t+1}, W_{t+1}, K, N) \in \mathcal{E} \times \{\mathcal{W}(K), K \in \mathbb{Z}_{0,+}, N \in \mathbb{Z}_{0,+}\}$ and this in turn yields, by the induction assumption, that $\min_{x \in \bar{X}} \bar{J}_t = \min_{x \in \tilde{X}} J_t$.

It directly follows that $\tilde{V}_t(W_t, E_t, K, N) = \bar{V}_t(W_t, E_t, K, N)$ and we conclude that $\tilde{V}_{t,i}(W, E, K, N)$ is coordinatewise convex in W, K, and N.

B.3. Proof of Property 3

By Property 2, we know that $\tilde{V}_t$ is coordinatewise convex in W for W, K, and N integer. Additionally, for any W by Proposition 1 equation $V_t((W_t, U_t, E_t), K) = \tilde{V}_t(W_t, E_t, K, U_t + [W_{t,0}]^+ + \sum_{n=1}^L W_{t,n})$ holds. The statements then follows by noting that $\tilde{V}_t(W_t, E_t, K, U_t + [W_{t,0}]^+ + \sum_{n=1}^L W_{t,n})$ is coordinatewise convex on $\{(W, U) : W \in \mathcal{W}(K), [W_0]^+ + \sum_{n=1}^L W_n + U = N\}$.

B.4. Proof of Property 4

It is easy to construct counterexamples showing that f_t does not inherit any of the mentioned properties in general.

B.5. Proof of Property 5

We show coordinatewise convexity of $f_t(N,K)$ in N and K first.

Because $f_t(N,K) = \tilde{V}_t(W_0(N), E_0, K, N)$ and $W_0(N) = (N, 0, 0, \ldots, 0)$ for all N and K, we can rely on the proof of Property 2 to show coordinatewise convexity of f_t. From the proof we know that $\tilde{V}_t(W,E,K,N) = \bar{V}_t(W,E,K,N)$ for integer values of W, K, and N. Because $W_0(N)$ is integer, $\tilde{V}_t(W_0(N),E,K,N) = \bar{V}_t(W_0(N),E,K,N)$. Additionally, $\bar{V}_t(W_0(N),E,K,N)$ is convex and coordinatewise convexity of $\tilde{V}_t(W_0(N),E,K,N)$ in N follows. Coordinatewise convexity of $f_t(N,K)$ in K, follows from similar arguments.

We show that $f_t(N,K)$ is non-increasing in N and K next.

The following relations are easily verified:

1. $\mathcal{W}(K_1,N) \subseteq \mathcal{W}(K_2,N)$ for $K_2 > K_1$
2. $\mathcal{W}(K,N_1) \subseteq \mathcal{W}(K,N_2)$ for $N_2 > N_1$
3. $\tilde{\mathcal{X}}(W,K,N_1) \subseteq \tilde{\mathcal{X}}(W,K,N_2)$ for $N_2 > N_1$ and $W \in \mathcal{W}(K,N_1)$
4. $\tilde{\mathcal{X}}(W,K_1,N) \subseteq \tilde{\mathcal{X}}(W,K_2,N)$ for $K_2 > K_1$ and $W \in \mathcal{W}(K_1,N)$

We show that $f_t(N,K) = \tilde{V}_t(W_0(N),E,K,N)$ is non-increasing in K by induction on t first. Let $K_2 > K_1 \geq 0$.

Base Case.

We show $\tilde{V}_{\bar{T}}(W_{\bar{T}},E_{\bar{T}},K_1,N) \geq \tilde{V}_{\bar{T}}(W_{\bar{T}},E_{\bar{T}},K_2,N)$ for all $W_{\bar{T}} \in \mathcal{W}(K_1,N)$. By relation 4,

$$\min_{x \in \tilde{\mathcal{X}}(W_{\bar{T}},K_1,N)} C_{\bar{T}}(W_{\bar{T}},x,E_{\bar{T}},D_{\bar{T}}) \geq \min_{x \in \tilde{\mathcal{X}}(W_{\bar{T}},K_2,N)} C_{\bar{T}}(W_{\bar{T}},x,E_{\bar{T}},D_{\bar{T}})$$

and, as a result,

$$\tilde{V}_{\bar{T}}(W_{\bar{T}},E_{\bar{T}},K_1,N) \geq \tilde{V}_{\bar{T}}(W_{\bar{T}},E_{\bar{T}},K_2,N).$$

The statement of the property follows for the base case by noting that $W = W_0(N) \in \mathcal{W}(K_1,N)$.

Induction Step.

We assume that $\tilde{V}_{t+1}(W_{t+1},E_{t+1},K_1,N) \geq \tilde{V}_{t+1}(W_{t+1},E_{t+1},K_2,N)$ for all $W_{t+1} \in \mathcal{W}(K_1,N)$, E_{t+1}, and N. Note that from $W_t \in \mathcal{W}(K_1,N)$ it follows $W_{t+1} \in \mathcal{W}(K_1,N)$, if $x \in \tilde{\mathcal{X}}(W_t,K_1,N)$. It immediately follows that

$$\begin{aligned}
&\min_{x \in \tilde{\mathcal{X}}(W_t,K_1,N)} C_t(W_t,x,E_t,D_t) + \tilde{V}_{t+1}(W_{t+1},E_{t+1},K_1,N) \\
&\qquad \geq \min_{x \in \tilde{\mathcal{X}}(W_t,K_1,N)} C_t(W_t,x,E_t,D_t) + \tilde{V}_{t+1}(W_{t+1},E_{t+1},K_2,N) \\
&\qquad \geq \min_{x \in \tilde{\mathcal{X}}(W_t,K_2,N)} C_t(W_t,x,E_t,D_t) + \tilde{V}_{t+1}(W_{t+1},E_{t+1},K_2,N),
\end{aligned}$$

where the first inequality is by the induction assumption and the second inequality is by relation 4. This implies $\tilde{V}_t(W_t, E_t, K_1, N) \geq \tilde{V}_t(W_t, E_t, K_2, N)$ for $W_t \in \mathcal{W}(K_1, N)$, E_t, and N.

The statement of the property then follows, because $W_0(N) \in \mathcal{W}(K_1, N)$.

The proof that $\tilde{V}_t(W_0(N), E_0, K, N)$ is non-increasing in N follows similar arguments for two number $N_2 > N_1$ and is omitted.

B.6. Proof of Property 6

The proof is essentially identical to the proof of Property 2 and thus omitted.

B.7. Proof of Proposition 2

The proof of Proposition 2 is a byproduct of the proof of Proposition 3.

B.8. Proof of Proposition 3

We prove the statement by induction.

Base Case. We want to show that $\underline{V}_{\bar{T}} \leq \underline{V}^N_{\bar{T}} \leq f_{\bar{T}}(N, K)$ and $\underline{V}_{\bar{T}} \leq \underline{V}^K_{\bar{T}} \leq f_{\bar{T}}(N, K)$. Because for all $t > \hat{\bar{T}}$, $V_t = \underline{V}_t = 0$ and $p_t = E_t = D_t = 0$, it suffices to show that

$$\min_{x_{\bar{T}} \geq 0} \{x_{\bar{T}} E_{\bar{T}}\} \leq \min_{0 \leq x_{\bar{T}} \leq N - \bar{W}_{\bar{T}}} \{x_{\bar{T}} E_{\bar{T}}\} \leq \min_{x_{\bar{T}} \in \mathcal{X}(S_{\bar{T}}, K)} \{x_{\bar{T}} E_{\bar{T}}\}$$

and

$$\min_{x_{\bar{T}} \geq 0} \{x_{\bar{T}} E_{\bar{T}}\} \leq \min_{0 \leq x_{\bar{T}} \leq K} \{x_{\bar{T}} E_{\bar{T}}\} \leq \min_{x_{\bar{T}} \in \mathcal{X}(S_{\bar{T}}, K)} \{x_{\bar{T}} E_{\bar{T}}\}$$

for all $\bar{W}_{\bar{T}} = \sum_{n=0}^{L} W_{\bar{T}}$. The relations hold obviously by definition of $\mathcal{X}(S_{\bar{T}}, K)$.

Induction Step. Let us assume that $\underline{V}_{t+1} \leq \underline{V}^N_{t+1} \leq f_{t+1}(N, K)$ and $\underline{V}_{t+1} \leq \underline{V}^K_{t+1} \leq f_{t+1}(N, K)$ for all $\bar{W}_{t+1} = \sum_{n=0}^{L} W_{t+1,n}$.

We show relation $\underline{V}_t \leq f_t(N, K)$. The other relations are obtained similarly.

Lower bound $\underline{V}_t$ is defined by

$$\begin{aligned}\underline{V}_t =& \bar{V}_t(\bar{W}_t, E_t) + F_t(W_t, E_t) \\ =& \mathbb{E}\left[\min_{x_t \geq 0} \left\{\bar{C}_t(\bar{W}_t, x_t, E_t) + \gamma \bar{V}_{\mathrm{mod}(t+1,T)}(\bar{W}_{\mathrm{mod}(t+1,T)}, E_{\mathrm{mod}(t+1,T)}) + F_t(W_t, E_t)\right\}\right].\end{aligned}$$

By noting that

$$\begin{aligned}F_t(W_t, E_t) = \gamma F(W_{t+1}, E_{t+1}) + & p_t \mathbb{E}[(D_t - W_{t,0})^+] - \gamma^L p_{t+L} \mathbb{E}[(D_{t+L} - W_{t+L-1,0} + x_t)^+] \\ & + E_t \sum_{n=1}^{L} \beta_n W_{t,n} - x_t \left(\mathbb{E}\left[\sum_{\tau=1}^{L} \gamma^\tau \beta_\tau E_{t+\tau} | E_{t+\tau-1}\right]\right),\end{aligned}$$

this simplifies to

$$\begin{aligned}\underline{V}_t = \mathbb{E}\Bigg[& p_t (D_t - W_{t,0})^+ + E_t \left(\beta_0 x_t + \sum_{n=1}^{L} \beta_n W_{t,n}\right) + \gamma^L p_{t+L}((D_{t,t+L} - \bar{W}_t)^+ - (D_{t+L} - W_{t+L,0})^+) \\ & + \min_{x_t \geq 0} \left\{\gamma \bar{V}_{t+1}(\bar{W}_{t+1}, E_{t+1}) + \gamma F_{t+1}(W_{t+1}, E_{t+1})\right\}\Bigg].\end{aligned}$$

Because $\mathbb{E}[(D_{t,t+L} - \bar{W}_t)^+] = \mathbb{E}[(D_{t+L} - W_{t+L,0})^+]$, this is the same as

$$\underline{V}_t = \mathbb{E}\left[\min_{x_t \geq 0} \left\{C_t(W_t, x_t, E_t, D_t) + \gamma \bar{V}_{t+1}(\bar{W}_{t+1}, E_{t+1}) + \gamma F_{t+1}(W_{t+1}, E_{t+1})\right\}\right]. \tag{1}$$

The relation $\underline{V}_t \leq f_t(N, K)$ then follows from the induction assumption, comparing Equation (1) to the definition of $f_t(N, K)$ and noting that the feasible set of actions $x_t \geq 0$ always includes all feasible $x \in \mathcal{X}(N, K)$.

B.9. Proof of Lemma 1

We need to show that

$$\min_{x\in\bar{\mathcal{X}}(W_{\bar{T}},K,N)} \left\{\bar{J}_{\bar{T}}(x,E_{\bar{T}},W_{\bar{T}},E_{\bar{T}+1},D_{\bar{T}},K,N)\right\} \tag{2}$$

is convex in $W_{\bar{T}}$, K and N if $\bar{J}_t(x,E_t,W_t,E_{t+1},D_t,K,N)$ is convex.

To simplify the notation we define $y_t=(W_t,K,N)$ and $g_t(x,y_t)=\bar{J}_t(x,E_t,W_t,E_{t+1},D_t,K,N)$. Also let $\tilde{g}_t(x,y_t)=I_{\mathcal{X}(y_t)}(x)+g_t(x,y_t)$ with

$$I_{\bar{\mathcal{X}}(y_t)}(x)=\begin{cases}0, & x\in\bar{\mathcal{X}}(y_t)\\ \infty, & x\notin\bar{\mathcal{X}}(y_t)\end{cases}.$$

So we need to show that $\min_{x\in\mathbb{R}_0^+}\tilde{g}_t(x,y_t)$ is convex in y_t. As $\mathbb{R}_0^+$ is a non-empty convex set this holds if $\tilde{g}_t(x,y_t)$ is convex in x and y_t (Boyd and Vandenberghe 2004, Section 3.2.5). The general property of convexity states that the function $\tilde{g}_t(x,y_t)$ is convex in x and y_t if the following inequality holds for all $\lambda\in(0,1)$, all $x\in\mathbb{R}$ and all $y_t\in\{(W_t,K,N):W_t\in\mathbb{R}^{L+1},K\in\mathbb{R},N\in\mathbb{R}\}$:

$$\tilde{g}_t\left(\lambda(x,y_t)+(1-\lambda)(x',{y'}_t)\right)\le\lambda\tilde{g}_t\left(x,y_t\right)+(1-\lambda)\tilde{g}_t\left(x',{y'}_t\right)$$

which is equivalent to

$$\tilde{g}_t\left(\lambda x+(1-\lambda)x',\lambda y_t+(1-\lambda){y'}_t)\right)\le\lambda\tilde{g}_t\left(x,y_t\right)+(1-\lambda)\tilde{g}_t\left(x',{y'}_t\right).$$

Due to the definition of $\tilde{g}_t(x,y_t)$ and the included definition of $I_{\mathcal{X}(y_t)}(x)$ the right side of the inequality is only $<\infty$ if $x\in\bar{\mathcal{X}}(y_t)$ and $x'\in\bar{\mathcal{X}}({y'}_t)$ simultaneously. If additionally $(\lambda x+(1-\lambda)x')\in\bar{\mathcal{X}}(\lambda y_t+(1-\lambda){y'}_t)$ the inequality reduces to the convexity property of $g_t(x,y_t)$ which holds as the function is convex. It is therefore sufficient to show that

$$(x\in\bar{\mathcal{X}}(y_t)\wedge x'\in\bar{\mathcal{X}}({y'}_t))\Rightarrow((\lambda x+(1-\lambda)x')\in\bar{\mathcal{X}}(\lambda y+(1-\lambda)y')) \tag{3}$$

is finite.

This can be easily seen from the definition of $\bar{x}$ because all of the constraints are linear.

This completes the proof of (3) and thus the proof for the convexity of

$$\min_{x\in\bar{\mathcal{X}}(W_{\bar{T}},K,N)} \left\{\bar{J}_{\bar{T}}(x,E_{\bar{T}},W_{\bar{T}},E_{\bar{T}+1},D_{\bar{T}},K,N)\right\} \tag{4}$$

in $W_{\bar{T}}$, K and N.

Appendix C: Demand scaling factors to account for traffic

The following table shows the assumed location for each swapping station and the respective minimum and maximum traffic measured in vehicles per day. The data was acquired from California Highways (2016). Although the data is quite old (1993), we used it to scale the total daily demand at each station as follows. We calculated the mean between the minimum (Traffic low) and maximum traffic (Traffic high) and divided this value by the sum over all stations' means to obtain the relative scaling factor (Factor).

References

Bertsekas, D. P. 2007. *Dynamic Programming and Optimal Control, Vol. II*. Athena Scientific.

California Highways. 2016. Overall statistics california highways. URL `http://www.cahighways.org/`. Last accessed on 2016-05-21.

Station Location	Highway	Traffic low [veh. per day]	Traffic high [veh. per day]	Factor
San Rafael	US 101	3,900	308,000	1.11
San Francisco	I-80	21,500	250,000	0.97
Walnut Creek	Route 24	107,000	171,000	0.99
Oakland	I-580	14,100	286,000	1.07
Hayward	I-880	95,000	204,000	1.06
Fremont	I-880	95,000	204,000	1.06
San Mateo	US 101	3,900	308,000	1.11
Palo Alto	I-280	11,000	226,000	0.84
San Jose	I-680	46,000	203,000	0.89
Santa Cruz	Route 17	54,000	197,000	0.89

Table 1 Station locations, traffic and scaling factors