Appendix. Proofs of Propositions in Section 2.2

In this appendix, we prove Propositions 1 through 6 presented in Section 2.2. Recall that $\mathcal{X}$ is the feasible set of the natural formulation given by (2)-(5). Let e_i the i^{th} unit vector of size $|N|$ and h_q the q^{th} unit vector of size $|Q|$. In the following, we refer to a solution in $\mathcal{X}$ as (x, y).

Proof of Proposition 1 Consider the following solutions of $\mathcal{X}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_i \\ 0 \end{pmatrix} i \in N, \begin{pmatrix} x^q \\ h_q \end{pmatrix} q \in Q,$$

where $x^q = \sum_{i \in N(\pi)} e_i$ and π is a path in $\mathcal{P}'_q$. These constitute a set of $|N|+|Q|+1$ solutions of the RSLP-R, which are affinely independent. Hence, the dimension of conv$(\mathcal{X})$ is $|N|+|Q|$. $\square$

Proof of Proposition 2 Consider the $|N|+|Q|$ solutions given in the proof of Proposition 1 that satisfy $y_q = 0$. Obviously, these solutions are affinely independent and they lie on the face of conv$(\mathcal{X})$ defined by inequality $y_q \geq 0$. $\square$

Proof of Proposition 3 Suppose that the condition is satisfied. Consider the following $|N|+|Q|$ solutions of $\mathcal{X}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_j \\ 0 \end{pmatrix} j \in N \setminus \{i\}, \begin{pmatrix} x^q \\ h_q \end{pmatrix} q \in Q,$$

where $x^q = \sum_{j \in N(\pi)} e_j$ and π is a path in $\mathcal{P}'_q$ with $i \notin N(\pi)$ (such a path exists since $\{i\}$ is not a q-node-cut for $\mathcal{P}'_q$). Observe that these solutions are in $\mathcal{X}$, satisfy $x_i = 0$ and are affinely independent.

If there exists a $q \in Q$ such that $\{i\}$ is a q-node-cut for $\mathcal{P}'_q$, then $x_i \geq y_q$ is valid for $\mathcal{X}$ and dominates $x_i \geq 0$. Hence the latter cannot define a facet. $\square$

Proof of Proposition 4 Assume that the conditions are satisfied and consider the following $|N|+|Q|$ solutions:

$$\begin{pmatrix} e_i \\ 0 \end{pmatrix}, \begin{pmatrix} e_i + e_j \\ 0 \end{pmatrix} j \in N \setminus \{i\}, \begin{pmatrix} x^q \\ h_q \end{pmatrix} q \in Q,$$

where $x^q = \sum_{j \in N(\pi^q) \cup \{i\}} e_j$ and π^q is a path in $\mathcal{P}'_q$ such that $|N(\pi^q) \cup \{i\}| \leq p$. So, these vectors are solutions of the RSLP-R, satisfy $x_i = 1$ and are affinely independent.

If $p = 1$, then $x(N) \leq 1$ dominates $x_i \leq 1$. If there exists $q \in Q$ such that $|N(\pi^q) \cup \{i\}| > p$ for all $\pi^q \in \mathcal{P}'_q$, then $x_i + y_q \leq 1$ is a valid inequality and it dominates $x_i \leq 1$. $\square$

Proof of Proposition 5 It suffices to exhibit $|N|+|Q|$ affinely independent solutions of $\mathcal{X}$ satisfying $x(N) \leq p$ with equality. Suppose that $p < |N|$. For each $q \in Q$, we can find a solution

$$\begin{pmatrix} x^q \\ h_q \end{pmatrix}$$

where x^q is such that $x^q(N) = p$, $x^q(S) \geq 1$ for all $S \in \Gamma_q$. Consider additional $|N|$ solutions:

$$\begin{pmatrix} w_{p+1} - e_i \\ 0 \end{pmatrix} i = 1, \ldots, p-1$$

and

$$\begin{pmatrix} w_{p-1}+e_i \\ 0 \end{pmatrix} i=p,\ldots,n,$$

where $w_j=\sum_{l=1}^{j} e_l$. Observe that all these $|N|+|Q|$ solutions are in $conv(\mathcal{X})$, satisfy $x(N)=p$ and are affinely independent. Thus the face of $conv(\mathcal{X})$ defined by inequality $x(N)\le p$ has dimension $|N|+|Q|-1$ and is a facet.

If $p=|N|$, then all solutions in $conv(\mathcal{X})$ that satisfy $x(N)=p$ also satisfy $x_i=1$ for $i\in N$. As $conv(\mathcal{X})$ is full dimensional and $x(N)\le p$ is not a positive multiple of $x_i\le 1$, $x(N)\le p$ cannot be facet defining. □

Proof of Proposition 6 Suppose that the conditions are satisfied. Consider the following $|N|+1$ solutions:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_i \\ 0 \end{pmatrix} i\in N\setminus S, \begin{pmatrix} x^i \\ h_q \end{pmatrix} i\in S,$$

where $x^i=\sum_{j\in N(\pi_i)} e_j$ and π_i is a path in $\mathcal{P}'_q$ with $N(\pi_i)\cap S=\{i\}$. In other words, we construct a path for demand q that contains only node i from set S. Note that S being a minimal q-node-cut for $\mathcal{P}'_q$ ensures that we have such a path $\pi_i\in\mathcal{P}'_q$. Next, we describe additional $|Q|-1$ solutions for every $\hat{q}\in Q\setminus\{q\}$ depending on the two cases. If there exists a path $\pi^{\hat{q}}\in\mathcal{P}'_{\hat{q}}$ with $N(\pi^{\hat{q}})\cap S=\emptyset$, then we construct the solution

$$\begin{pmatrix} x^{\hat{q}} \\ h_{\hat{q}} \end{pmatrix}$$

where $x^{\hat{q}}=\sum_{j\in N(\pi^{\hat{q}})} e_j$. Note that $x^{\hat{q}}(S)=0$ in this case. If such a path does not exist, then by assumption, there exists a node $i\in S$, a path $\pi^q\in\mathcal{P}'_q$ and a path $\pi^{\hat{q}}\in\mathcal{P}'_{\hat{q}}$ such that $N(\pi^q)\cap S=N(\pi^{\hat{q}})\cap S=\{i\}$ and $|N(\pi^q)\cup N(\pi^{\hat{q}})|\le p$. In this case we construct the solution

$$\begin{pmatrix} x^{\hat{q}} \\ h_q+h_{\hat{q}} \end{pmatrix}$$

where $x^{\hat{q}}=\sum_{j\in N(\pi^q)\cup N(\pi^{\hat{q}})} e_j$. Observe that these $|Q|-1$ solutions together with the previous $|N|+1$ solutions are affinely independent, are in $conv(\mathcal{X})$ and satisfy $y_q=x(S)$. Hence, the inequality $y_q\le x(S)$ is facet defining for $conv(\mathcal{X})$.

If the conditions are not satisfied, then $y_q+y_{\hat{q}}\le x(S)$ is a valid inequality for $conv(\mathcal{X})$. As this new inequality dominates $y_q\le x(S)$, the latter cannot be facet defining for $conv(\mathcal{X})$. □