# A Proofs

## A.1 Uniqueness of M-MDM Solution

To see the uniqueness of the solution to the M-MDM system (17)-(18), note that the right-hand-side of the M-MDM recursion (17), and hence, $w_i^d$, is componentwise nondecreasing in $w_j^d, \forall j \in N_d^+(i)$. Under Assumption A3, it is componentwise increasing. Suppose that $\mathbf{w^{d(1)}}$ and $\mathbf{w^{d(2)}}$ are two solutions. Let $\delta_i^d = w_i^{d(1)} - w_i^{d(2)}$, and let $\delta = \max_{i \in N_d}\{\delta_i^d\} \geq 0$. Let $i \in N_d$ be a node where the maximum difference is attained, i.e., $w_i^{d(1)} - w_i^{d(2)} = \delta$. By definition of $\delta$, we have $\delta_j^d \leq \delta, \forall j \in N_d^+(i)$. Strict monotonicity of $w_i^d$ in $w_j^d$ implies that $\delta_j^d = \delta, \forall j \in N_d^+(i)$. Moving forward in the same manner, we can show that $\delta_j^d = \delta$ for all $j$ reachable from node $i$ including the destination. Then, the boundary condition implies that $\delta = 0$. Thus we have $\delta_i^d = 0, \forall i \in N_d$, implying that there is at most one solution.

## A.2 Proof of Proposition 1

*Solution.* The Lagrangian function of the formulation (20)-(22) is given below:

$$
\begin{aligned}
L = & \sum_{i \in N_d} h_i^d \cdot w_i^d - \sum_{i \in N_d} w_i^d \cdot n_i^d - \sum_{i \in N_d} \lambda_i^d \cdot n_i^d \\
& - \sum_{(i,j) \in A_d} n_i^d \int_{\lambda_i^d + t_{ij} + w_j^d}^{\infty} \left[1 - F_{ij}^d(\omega)\right] d\omega,
\end{aligned}
$$

where $n_i^d$ is the dual variable corresponding to constraint (21). The KKT optimality conditions are as follows:

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i^d} = 0 \Longrightarrow n_i^d = n_i^d \sum_{j \in N_d^+(i)} \left[1 - F_{ij}^d\left(\lambda_i^d + t_{ij} + w_j^d\right)\right], \qquad & \forall i \in N_d, \\
\frac{\partial L}{\partial w_i^d} = 0 \Longrightarrow n_i^d = h_i^d + \sum_{k \in N_d^-(i)} n_k^d \left[1 - F_{ki}^d\left(\lambda_k^d + t_{ki} + w_i^d\right)\right], \quad & \forall i \in N_d.
\end{aligned}
$$

For any origin node $o$, i.e., $(o, d) \in W$, we have $h_o^d > 0$ and, therefore, the second KKT condition implies that $n_o^d > 0$. Then, the summation in the right-hand-side of the first KKT condition (for node $o$) must be equal to one. In other words, $\lambda_o^d$ solves the MDM optimality condition (15). Since MDM choice probabilities are strictly positive, there will be a positive flow into each node that is adjacent to $o$. The same reasoning can be used to show that $n_i^d > 0, \forall i \in N_d$ and to conclude that all $\lambda_i^d$'s solve the MDM optimality conditions (15) for all $i \in N_d$. The right-hand-side of constraint (21) is componentwise increasing in $w_j^d$. Monotonicity and the fact that we are maximizing the objective function imply that the constraint holds at equality for the optimal solution. Then, the optimal values of $w_i^d$ and $\lambda_i^d$ solve the M-MDM system of equations (17)-(19). □

### A.3 Proof of Lemma 1

*Solution.* Consider problem (20)-(22) where $F_{ij}^d(t) = 0.5 + 0.5t(2+t^2)^{-0.5}$. In this special case, constraint (21) can be reformulated as follows:

$$\begin{aligned} 2w_i^d &\leq -2\lambda_i^d + \sum_{j \in N_d^+(i)} \left(a_{ij}^d - b_{ij}^d\right), & \forall i \in N_d, \\ a_{ij}^d &= \lambda_i^d + t_{ij} + w_j^d, & \forall (i,j) \in A_d, \\ \sqrt{a_{ij}^d + 2} &\leq b_{ij}^d, & \forall (i,j) \in A_d, \end{aligned}$$

and hence, the problem can be represented as a second-order cone programming model. □

### A.4 Derivation of Primal M-MDM Objective Function

Given the link choice probabilities, and using (23) and (24), we can express the optimal objective function value of the dual M-MDM formulation (20)-(22) as follows:

$$\begin{aligned} \sum_{i \in N_d} h_i^d \cdot w_i^d &= \sum_{i \in N_d} h_i^d \sum_{j \in N_d} \mathbf{M_{ij}^d} \cdot \underline{w}_j^d \\ &= \sum_{j \in N_d} \underline{w}_j^d \sum_{i \in N_d} h_i^d \cdot \mathbf{M_{ij}^d} \\ &= \sum_{j \in N_d} n_j^d \cdot \underline{w}_j^d \\ &= \sum_{i \in N_d} n_i^d \sum_{j \in N_d^+(i)} \left( t_{ij} \cdot p_{ij}^d - \int_{1-p_{ij}^d}^{1} F_{ij}^{d\,(-1)}(\omega)\, d\omega \right) \\ &= \sum_{i \in N_d} \sum_{j \in N_d^+(i)} \left( t_{ij} \cdot x_{ij}^d - n_i^d \int_{1-\frac{x_{ij}^d}{n_i^d}}^{1} F_{ij}^{d\,(-1)}(\omega)\, d\omega \right). \end{aligned}$$

### A.5 Proof of Proposition 2

*Solution.* Let $\varphi(\mathbf{x^d}, \mathbf{n^d}) = \sum_{(i,j)\in A_d} \varphi_{ij}(x_{ij}^d, n_i^d)$ represent the objective function where:

$$\varphi_{ij}(x_{ij}^d, n_i^d) = \left( t_{ij} \cdot x_{ij}^d - n_i^d \int_{1-\frac{x_{ij}^d}{n_i^d}}^{1} {F_{ij}^d}^{(-1)}(\omega) d\omega \right).$$

This is the perspective of the function $\varphi_{ij}^0(x)$ defined as:

$$\varphi_{ij}^0(x) = \left( t_{ij} \cdot x - \int_{1-x}^{1} {F_{ij}^d}^{(-1)}(\omega) d\omega \right),$$

such that $\varphi_{ij}(x_{ij}^d, n_i^d) = n_i^d \varphi_{ij}^0(x_{ij}^d / n_i^d)$. Since the function $\varphi_{ij}^0(x)$ is strictly convex under the assumptions on the marginal distribution, the perspective of the function denoted by $\varphi_{ij}(x_{ij}^d, n_i^d)$ is also strictly convex. This implies that $\varphi(\mathbf{x^d}, \mathbf{n^d})$ is strictly convex. Furthermore the constraints are convex and hence it is a convex optimization problem with a unique optimal solution. The KKT optimality conditions of the formulation are given below:

$$\frac{x_{ij}^d}{n_i^d} = 1 - F_{ij}^d\left(\lambda_i^d + t_{ij} + w_j^d\right), \qquad \forall (i,j) \in A_d,$$

$$\begin{aligned} w_i^d &= -\lambda_i^d - \sum_{j\in N_d^+(i)} \left( \int_{1-\frac{x_{ij}^d}{n_i^d}}^{1} {F_{ij}^d}^{(-1)}(\omega) d\omega - \frac{x_{ij}^d}{n_i^d} {F_{ij}^d}^{(-1)}\left(1 - \frac{x_{ij}^d}{n_i^d}\right)\right) \\ &= -\lambda_i^d - \sum_{j\in N_d^+(i)} \int_{\lambda_i^d + t_{ij} + w_j^d}^{\infty} \left[1 - F_{ij}^d(\omega)\right] d\omega, \qquad \forall i \in N_d, \end{aligned}$$

where $\lambda_i^d$ and $w_i^d$ are the dual variables corresponding to constraints (26) and (27), respectively. By constraint (26), $\lambda_i^d$ satisfies the normalization condition (15). Therefore, the KKT conditions are equivalent to the M-MDM system of equations. □

### A.6 Proof of Proposition 3

*Solution.* The exponential distribution satisfies the assumption on marginals, A3. Substituting the inverse cumulative distribution function ${F_{ij}^d}^{(-1)}(\omega) = -(1 + \ln(1-\omega))/\beta$ in the objective function (25), the integral is calculated as follows:

$$\int_{1-\frac{x_{ij}^d}{n_i^d}}^{1} F_{ij}^{d\,(-1)}(\omega)d\omega = \frac{1}{\beta n_i^d}\left(-x_{ij}^d \cdot \ln x_{ij}^d + x_{ij}^d \cdot \ln n_i^d\right),$$

and the objective function simplifies to the following:

$$\sum_{(i,j)\in A_d}\left(t_{ij}\cdot x_{ij}^d + \frac{x_{ij}^d \cdot \ln x_{ij}^d}{\beta} - \frac{x_{ij}^d \cdot \ln n_i^d}{\beta}\right).$$

Constraints (26) and (27) can be expressed as a single constraint by substituting $n_i^d$ in the objective function. Using substitution $n_i^d = \sum_{j\in N_d^+(i)} x_{ij}^d$, the last term in the objective function can be expressed as follows:

$$\begin{aligned}
-\frac{1}{\beta}\sum_{(i,j)\in A_d} x_{ij}^d \cdot \ln n_i^d &= -\frac{1}{\beta}\sum_{i\in N_d}\sum_{j\in N_d^+(i)} x_{ij}^d \cdot \ln\left(\sum_{j\in N_d^+(i)} x_{ij}^d\right) \\
&= -\frac{1}{\beta}\sum_{i\in N_d}\left(\sum_{j\in N_d^+(i)} x_{ij}^d\right)\cdot \ln\left(\sum_{j\in N_d^+(i)} x_{ij}^d\right).
\end{aligned}$$

Then, the M-MDM formulation reduces to the Markovian logit model given in (29)-(32). □

### A.7 Proof of Proposition 4

*Solution.* Let $\lambda_i^d$, $w_i^d$ and $t_{ij}$ be the dual variables corresponding to constraints (38), (39) and (40), respectively. Then, the KKT optimality conditions of the convex optimization problem are as follows:

$$\begin{aligned}
& t_{ij} = \tau_{ij}\left(f_{ij}\right), && \forall (i,j)\in A, \\
& \frac{x_{ij}^d}{n_i^d} = 1 - F_{ij}^d\left(\lambda_i^d + t_{ij} + w_j^d\right), && \forall (i,j)\in A_d, d\in D, \\
& w_i^d = -\lambda_i^d - \sum_{j\in N_d^+(i)} \int_{\lambda_i^d + t_{ij} + w_j^d}^{\infty}\left[1 - F_{ij}^d(\omega)\right] d\omega, && \forall i \in N_d, d\in D.
\end{aligned}$$

The last two optimality conditions solve the M-MDM system of equations for a given link cost vector. Then, conditions (35) and (36) are satisfied since $x_{ij}^d/n_i^d = p_{ij}(\mathbf{t})$, where $t_{ij} = \tau_{ij}\left(f_{ij}\right)$ by the first KKT condition. Conditions

(33) and (34) are satisfied by the constraints of the model. Therefore, the optimal solution of the formulation (37)-(40) solves the MTE-MDM fixed point problem. The constraints of the model are linear, and the objective function is strictly convex and hence, the solution of the model is unique. □

## A.8 Proof of Lemma 2

*Solution.* The M-MDM recursion (17) implies that $w_i^d \leq t_{ij} - E[\tilde{\epsilon}_{ij}^d] + w_j^d, \forall j \in N_d^+(i)$. This inequality recursively implies that $s_i^d$ is an upper bound for $w_i^{d*}$. We can make the induction assumption that $w_i^{d(n)} \leq w_i^{d(n-1)}, \forall n \in \{1, \ldots, m-1\}, i \in N_d$, since $w_i^{d(1)} \leq w_i^{d(0)} = s_i^d, \forall i \in N_d$. By this assumption, we have $t_{ij} + w_j^{d(m-1)} \leq t_{ij} + w_j^{d(m-2)}, \forall (i,j) \in A_d$. The monotonicity of $w_i^d$ in $w_j^d, \forall j \in N_d^+(i)$, then implies that $w_i^{d(m)} \leq w_i^{d(m-1)}, \forall i \in N_d$. By induction, the second inequality holds for all $n$. Thus, the sequence is monotonic nonincreasing, moreover bounded from below provided that the M-MDM solution exists. This proves that the sequence converges to the unique fixed point $\mathbf{w^{d*}}$. □

## A.9 Proof of Lemma 3

*Solution.* We start the proof by defining $Y_{ai}^d(\mathbf{t})$, $Q_{ji}^d(\mathbf{t})$ and $L_{ai}^d(\mathbf{t})$ as follows:

$$Y_{ai}^d(\mathbf{t}) = \frac{\partial w_i^d(\mathbf{t})}{\partial t_a}, \qquad \forall a \in A_d, d \in N_d, d \in D,$$

$$Q_{ji}^d(\mathbf{t}) = \begin{cases} p_{ij}^d(\mathbf{t}) & \text{if } (i,j) \in A_d \\ 0 & \text{othwerise} \end{cases}, \qquad \forall a \in A_d, d \in N_d, d \in D,$$

$$L_{ai}^d(\mathbf{t}) = \begin{cases} p_a^d(\mathbf{t}) & \text{if tail of } a \text{ is } i \\ 0 & \text{othwerise} \end{cases}, \qquad \forall a \in A_d, d \in N_d, d \in D,$$

where $p_{ij}^d(\mathbf{t}) = 1 - F_{ij}^d(\lambda_i^d(\mathbf{t}) + t_{ij} + w_j^d(\mathbf{t}))$. Taking the derivative of the expected cost with respect to link cost, we obtain:

$$\begin{aligned} Y_{ai}^d(\mathbf{t}) &= -\frac{\partial \lambda_i^d(\mathbf{t})}{\partial t_a} + \sum_{j \in N_d^+(i)} \left( \frac{\partial \lambda_i^d(\mathbf{t})}{\partial t_a} + \frac{\partial t_{ij}}{\partial t_a} + Y_{aj}^d(\mathbf{t}) \right) \cdot p_{ij}^d(\mathbf{t}) \\ &= \sum_{j \in N_d^+(i)} \left( \frac{\partial t_{ij}}{\partial t_a} + Y_{aj}^d(\mathbf{t}) \right) \cdot p_{ij}^d(\mathbf{t}) \\ &= L_{ai}(\mathbf{t}) + \sum_{j \in N_d^+(i)} Y_{aj}^d(\mathbf{t}) \cdot Q_{ji}^d(\mathbf{t}). \end{aligned}$$

This leads to the linear system $Y^d(\mathbf{t}) = L^d(\mathbf{t}) \cdot (M^d(\mathbf{t}))^\intercal$ where $M^d(\mathbf{t})$ is the fundamental matrix of the absorbing Markov chain. Then, the first derivative of the objective function (45) is given by the following:

$$\begin{aligned}
\frac{\partial Z\left(\mathbf{t}\right)}{\partial t_a} &= \tau_a^{-1}\left(t_a\right) - \sum_{d \in D} \sum_{i \in N_d} h_i^d \cdot Y_{ai}^d\left(\mathbf{t}\right), \\
&= \tau_a^{-1}\left(t_a\right) - \sum_{d \in D} x_a^d\left(\mathbf{t}\right), \\
&= \tau_a^{-1}\left(t_a\right) - f_a\left(\mathbf{t}\right), && \forall a \in A,
\end{aligned}$$

where $x_a^d(\mathbf{t})$ and $f_a(\mathbf{t})$ are the flows obtained by the stochastic network loading method (Algorithm 1) at link cost vector $\mathbf{t}$. This completes the proof. □