APPENDIX A.

The detailed procedure of the modified label-setting algorithm for solving the NASP problem is shown as below.

<u>Step0</u>: For O-D pair (o,d), set the node o's label as $(o,0,0,0,0)$, where the first element represents the current node, the second element is the precedent node (zero means that there is no precedent node for the origin), the third element corresponds to the cumulative travel time, the fourth element represents the cumulative attribute value (i.e., the sum of $g_a(v_a)$ along the path), and the fifth element stands for the cumulative travel cost. Set the temporary label set $\Pi = \{(o,0,0,0,0)\}$ and the permanent label set $\Lambda = \phi$.

<u>Step1</u>: Find the label vector with the minimum travel cost in the set Π. Suppose that the corresponding label is $(n_c, n_p, t_c, g_c, l_c)$. For each subsequent node of n_c, i.e., n_s, on the network, add a new label vector $(n_s, n_c, t_c + t_a, g_c + g_a, l_s)$ to Π, where a is the link from node n_c to node n_s, and $l_s = t_c + t_a + G(g_c + g_a)$. Set $\Lambda = \Lambda \cup (n_c, n_p, t_c, g_c, l_c)$, and remove $(n_c, n_p, t_c, g_c, l_c)$ from Π.

<u>Step2</u>: Perform the label-dominance test for all the nodes associated with the newly generated labels. Specifically, for a node n_c and a particular label $\pi = (n_c, n_p, t_c, g_c, l_c)$, if there exists another label $\pi' = (n_c, n'_p, t'_c, g'_c, l'_c)$ such that $t'_c < t_c$ and $g'_c \leq g_c$, then π is dominated by π'. We delete all the dominated labels from Π.

<u>Step3</u>: If $\Pi \neq \phi$, go to Step 1. Otherwise, identify the shortest path from the label associated with the destination d and with the minimum travel cost in Λ.

In fact, each time we run the procedure above, it provides the shortest paths from the node o to all the other nodes on the network. The validity of the above algorithm is stated in the following proposition.

Proposition A1. *For each O-D pair* $w \in W$, *the proposed label-setting algorithm can: i) find the shortest path with the path cost defined as* $c_r^w = \sum_{a\in A} \delta_{ar} t_a(v_a) + G[\sum_{a\in A} \delta_{ar} g_a(v_a)]$; *ii) find all eligible paths.*

Proof. We only need to prove that the proposed label-setting algorithm can identify all eligible paths between the O-D pair $w \in W$ because the shortest path is also an eligible path. To prove the above claim, we need to show that i) all the eligible paths are tracked in the permanent label set Λ; ii) all the paths tracked in the permanent label set are eligible paths.

We first show that all the eligible paths are tracked in the permanent label set Λ. To prove it, we first prove that at any node $n \in N$ along an eligible path $r \in \bar{R}^w$, the sub-path of path r from the source node o to the node n, denoted by r_o^n, is also an eligible path. We show this by contradiction. If there is another sub-path $\tilde{r}_o^n$ that can dominate r_o^n, then it means that $t_{\tilde{r}_o^n} < t_{r_o^n}$, $g_{\tilde{r}_o^n} \leq g_{r_o^n}$, where $t_{r_o^n}$ and $g_{r_o^n}$ represent the travel time and tolled attribute of the sub-path r_o^n, respectively. Suppose that the path r is composed of the sub-paths r_o^n and r_n^d. Construct a new path $\tilde{r}$ through connecting the sub-paths $\tilde{r}_o^n$ and r_n^d. Then, we can derive that $t_{\tilde{r}} = t_{\tilde{r}_o^n} + t_{r_n^d} < t_{r_o^n} + t_{r_n^d} = t_r$, $g_{\tilde{r}} = g_{\tilde{r}_o^n} + g_{r_n^d} \leq g_{r_o^n} + g_{r_n^d} = g_r$. This contradicts the fact that the path r is an eligible path. To this end, we have shown that any sub-path of an eligible path should also be eligible. Then, if a eligible path is not tracked in the permanent set Λ, then either it is deleted or its sub-path is deleted after a dominance test. However, none of these cases will happen because it is an eligible path. Therefore, all the eligible paths are tracked in the permanent label set Λ.

Next, we show that all the paths tracked in the permanent label set are eligible paths. Suppose that the label corresponding to path r is selected as the label with the minimum travel cost in Step 1 of the modified label-setting algorithm. Then, it is not dominated by any label that is already in the permanent set. Otherwise, it will be deleted in the previous dominant tests. Furthermore, it will not be dominated by any label that is generated later in the algorithm process, either, because the minimum travel cost of labels in the set Π increases with the iteration number. Therefore, all the paths tracked in the permanent label set are eligible paths. □

APPENDIX B.

The detailed procedure of adopting the modified Fukushima algorithm (Zhu and Marcotte, 1993) to solve R-VI-FD is shown as below.

Step0: Given a tolling function $G(\cdot)$, a restricted path set $\mathcal{P}$, and an initial point $(\boldsymbol{v}_s^0, \boldsymbol{f}_s^0) \in \Omega_F^{\mathcal{P}}(\boldsymbol{d})$, let ϵ be a predetermined tolerance factor, and define α^0, $\Delta\alpha$, and γ as positive parameters ($\gamma < 1$). Set iteration number $n = 0$. Define the merit function $\hbar_{\alpha_n}(\boldsymbol{v}, \boldsymbol{f}) = \max_{\boldsymbol{q} \geq \mathbf{0}, \sum_{r \in \mathcal{P}^w} q_r^w = d^w, \forall w} \boldsymbol{c}(\boldsymbol{v})^T(\boldsymbol{f} - \boldsymbol{q}) - \frac{1}{2\alpha_n}\|\boldsymbol{f} - \boldsymbol{q}\|^2$, where $\boldsymbol{c}(\boldsymbol{v})$ is the path cost vector.

Step1: Let $\boldsymbol{q}^* \in \operatorname*{argmax}_{\boldsymbol{q} \geq \mathbf{0}, \sum_{r \in \mathcal{P}^w} q_r^w = d^w, \forall w} \boldsymbol{c}(\boldsymbol{v}_s^n)^T(\boldsymbol{f}_s^n - \boldsymbol{q}) - \frac{1}{2\alpha_n}\|\boldsymbol{f}_s^n - \boldsymbol{q}\|^2$. If $\hbar_{\alpha_n}(\boldsymbol{v}_s^n, \boldsymbol{f}_s^n) \leq \frac{1}{2(1-\gamma)\alpha_n}\|\boldsymbol{f}_s^n - \boldsymbol{q}^*\|^2$, let $\alpha_{n+1} \leftarrow \alpha_0 + n\Delta\alpha$, and $(\boldsymbol{v}_s^{n+1}, \boldsymbol{f}_s^{n+1}) \leftarrow (\boldsymbol{v}_s^n, \boldsymbol{f}_s^n)$. Otherwise, let $\alpha_{n+1} \leftarrow \alpha_n$, and $\boldsymbol{f}_s^{n+1} \leftarrow \boldsymbol{f}_s^n + \beta^n(\boldsymbol{q}^* - \boldsymbol{f}_s^n)$, where β^n obeys the limited minimization rule. Calculate $\boldsymbol{v}_s^{n+1}$ through $\boldsymbol{f}_s^{n+1}$. Go to Step 2.

Step2: If $\hbar_{\alpha_{n+1}}(\boldsymbol{v}_s^{n+1}, \boldsymbol{f}_s^{n+1}) < \epsilon$, then terminate the algorithm. Otherwise, set $n \leftarrow n + 1$, and go to Step 1.

APPENDIX C.

In this appendix, we provide a geometric explanation for the fact that $\bar{R}_2^w(\boldsymbol{T}) \subseteq \bar{R}_2^w(\boldsymbol{T}^*), \forall w \in W$, where $\boldsymbol{T}$ satisfies Constraints (14)-(17); $\bar{R}_2^w(\boldsymbol{T}) = \{r \in \bar{R}^w | t_r^{SO} + T_r^w - B^w(d^{w,SO}) = 0\}$ and $\boldsymbol{T}^*$ is calculated by Equations (18)-(21). For brevity, we assume that $\Psi = \emptyset$. We also intuitively explain why determining the existence results in the fixed demand case is more difficult than in the elastic demand case.

We plot eligible paths and tolling functions in one diagram, as shown in Figure C1. For any eligible path, we use its corresponding $g_r^{w,SO}$ and $B^w(d^{w,SO}) - t_r^{w,SO}$ as its x and y coordinates, respectively, whereas a tolling function's x and y coordinates correspond to its $g_r^{w,SO}$ and T_r^w, respectively. In Figure C1, each point corresponds to an eligible path and the curve represents a tolling function. Obviously, the y coordinates of the eligible paths and tolling function share the same axis.

Based on Constraints (14)-(17), an SO-inducing tolling function must be nonnegative, monotonically non-decreasing, and beyond all the points in the diagram. Now, we focus on those points surrounded by circles , denoted by $r \in \mathcal{R}^p$. Those paths satisfy the condition that $\forall r'$ with $g_{r'}^{SO} < g_r^{SO}$, we have $B^w(d^{w,SO}) - t_r^{SO} > B^{w'}\left(d^{w',SO}\right) - t_{r'}^{SO}$. Then, for any toll vector $\boldsymbol{T}$ satisfying Constraints (14)-(17), it is straightforward to verify that $\forall r \notin \mathcal{R}^p$, $T_r > B^w(d^{w,SO}) - t_r^{SO}$. Therefore, we have $\bar{R}_2^w(\boldsymbol{T}) \subseteq \mathcal{R}^{w,p}$, where $\mathcal{R}^{w,p} = \mathcal{R}^p \cap \bar{R}^w$. Subsequently, to prove that $\bar{R}_2^w(\boldsymbol{T}) \subseteq \bar{R}_2^w(\boldsymbol{T}^*)$, $\forall w \in \mathcal{W}$, it is sufficient to show that $\bar{R}_2^w(\boldsymbol{T}^*) = \mathcal{R}^{w,p}$.

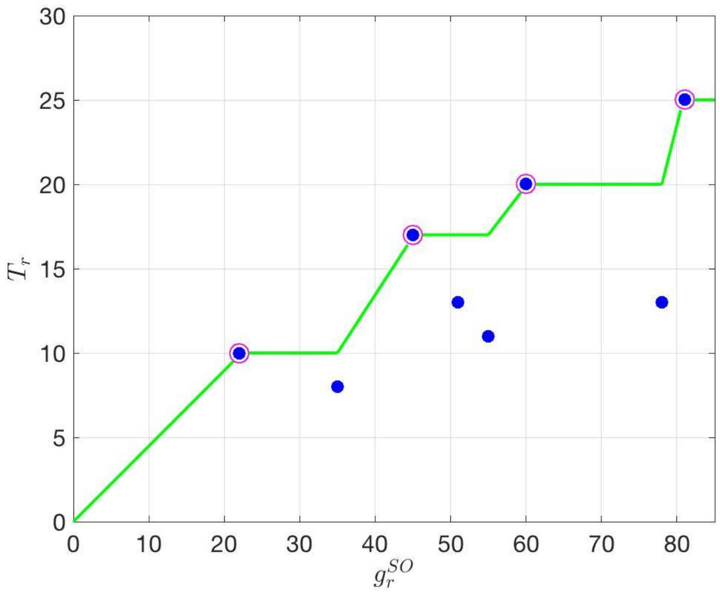
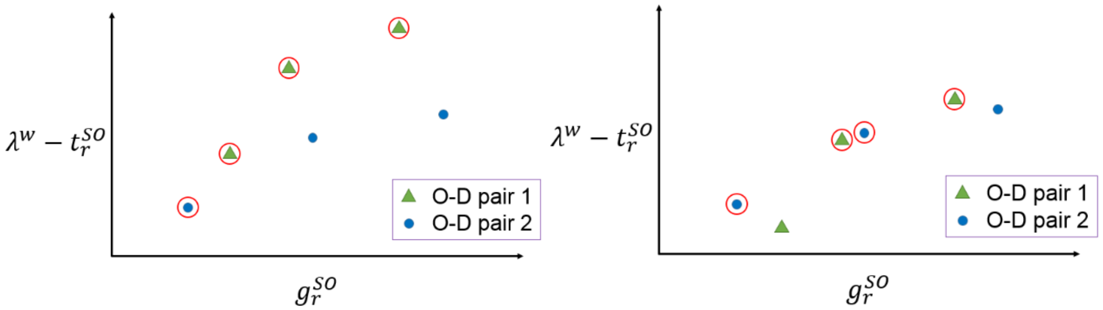


Figure C1. Illustration of toll vector $\boldsymbol{T}^*$ satisfying (18)-(21)

It can be verified that the curve in Figure C1 satisfies Equations (18)-(21). Clearly, we see that $T_r^* = B^w(d^{w,SO}) - t_r^{SO}$, if and only if $r \in \mathcal{R}^{w,p}$, implying that $\bar{R}_2^w(\boldsymbol{T}^*) = \mathcal{R}^{w,p}, \forall w \in W$, which, together with $\bar{R}_2^w(\boldsymbol{T}) \subseteq \mathcal{R}^{w,p}$, implies that $\bar{R}_2^w(\boldsymbol{T}) \subseteq \bar{R}_2^w(\boldsymbol{T}^*)$, $\forall w \in W$.

Now, we intuitively explain why the problem of determining the existence of an SO-inducing tolling function in fixed demand case cannot be reformulated as a LP problem. Consider a network with two O-D pairs, and Figure C2 plots their eligible paths, where the x and y coordinates correspond to an eligible path's $g_r^{w,SO}$ and $\lambda^w - t_r^{w,SO}$, respectively. Different from the elastic demand case, $\lambda^w - t_r^{w,SO}$ in the y axis is not a fixed value but can vary with the value of λ^w. Figures C2(a) and C2(b) illustrate two scenarios with different values of λ^w. Subsequently, for each scenario, we can identify the set $R_2(\boldsymbol{T}^*)$ (the paths are surrounded by circles). Comparing $R_2(\boldsymbol{T}^*)$ of two scenarios helps us understand that for a general network, we cannot always guarantee that there exists a vector $\boldsymbol{\lambda}$ such that the corresponding $R_2(\boldsymbol{T}^*)$ is the superset of any other possible $R_2(\boldsymbol{T}^*)$. This is exactly the reason why the problem in the fixed demand case is more complicated, and cannot be reduced to a linear programming problem.



(a) Scenario 1 **(b) Scenario 2**

Figure C2. Illustration of non-uniqueness of $R_2(\boldsymbol{T}^*)$ in fixed demand case

APPENDIX D.

Proof of Proposition 6

We first prove that there must exist a solution to LPCC-FD. Let $\mathcal{U} = \{(\cdots, U_r^w, \cdots) | (15)-(17), (28)-(29)\}$ denote the feasible domain for $\boldsymbol{U}$, i.e., $\forall \boldsymbol{U} \in \mathcal{U}$, we can at least find one $\boldsymbol{T}$ and $\boldsymbol{\lambda}$ such that Constraints (15)-(17) and (28)-(29) are satisfied. Let $\Xi(\boldsymbol{U}) = \{r \in \cup_{w \in W} \bar{R}^w | U_r^w = 0, \boldsymbol{U} \in \mathcal{U}\}$ be a possible set of paths whose U_r^w equal zero. Define Θ through $\Theta = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{|\Theta|}\}$, where $\mathcal{P}_i$ is a possible set of paths whose U_r^w equal zero, i.e., $\boldsymbol{U} \in \mathcal{U}$ and $\Xi(\boldsymbol{U}) = \mathcal{P}_i$. Note that Θ is a finite set since the number of all the possible combinations of paths in $\cup_{w \in W} \bar{R}^w$ is finite. Subsequently, LPCC-FD can be reformulated as $\min\limits_{\mathcal{P} \in \Theta} Z(\mathcal{P})$, where $Z(\mathcal{P})$ is defined by the following linear programming problem.

LP-2

$$Z(\mathcal{P}) = \min_{\boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-, \boldsymbol{f}} \sum_{a \in A} (\gamma_a^+ + \gamma_a^-)$$

s.t. (25)

$$v_a^{SO} - \sum_{w \in W} \sum_{r \in \mathcal{P}} \delta_{ar} f_r^w = \gamma_a^+ - \gamma_a^- \qquad \forall a \in A$$

$$d^{w,SO} = \sum_{r \in \mathcal{P}} f_r^w \qquad \forall w \in W$$

$$f_r^w \geq 0 \qquad \forall r \in \mathcal{P}$$

We note that LP-2 may not be feasible because $\mathcal{P}$ may not include any path for some O-D pairs, and in this case we set $Z(\mathcal{P}) = +\infty$.

We now prove that $-\infty < \min\limits_{\mathcal{P} \in \Theta} Z(\mathcal{P}) < +\infty$, i.e., there must exist a solution to LPCC-FD. Since $-\infty < \min\limits_{\mathcal{P} \in \Theta} Z(\mathcal{P})$ is guaranteed by Constraint (25), then we only need to show that there exists a $\mathcal{P} \in \Theta$ such that $Z(\mathcal{P}) < +\infty$. Denote $\hat{r}_w$ as the eligible path between O-D pair $w \in \mathcal{W}$ with the minimum value of g_r^{SO}, i.e., $g_{\hat{r}_w}^{SO} \leq g_r^{SO}, \forall r \in \bar{R}^w$, and then we index all the O-D pairs based on their corresponding $g_{\hat{r}_w}^{SO}$. The indexed O-D pairs in the ascending order of $g_{\hat{r}_w}^{SO}$ is denoted by $w^{[1]}, w^{[2]}, \dots, w^{[|W|]}$ where

$g^{SO}_{\hat{r}_{w^{[1]}}} \leq g^{SO}_{\hat{r}_{w^{[2]}}} \leq \cdots \leq g^{SO}_{\hat{r}_{w^{[|W|]}}}$. It can be verified that we can recursively set the values of $\lambda^{w^{[k]}}$ and $T_{\hat{r}_{w^{[k]}}}$ such that the corresponding $\boldsymbol{U}$ belongs to set $\mathcal{U}$ and $\forall w \in W$, there exists at least one path whose U_r^w equals zero. Then, the corresponding path set $\mathcal{P}$ will make LP-2 feasible, i.e., $Z(\mathcal{P}) < +\infty$, and consequently $\min\limits_{\mathcal{P} \in \Theta} Z(\mathcal{P}) < +\infty$.

Now, we are ready to prove that setting $M_U = \sum_{w \in W} \bar{t}^w$ can guarantee that MILP-FD and LPCC-FD own the same optimal objective function value. We first show the following definitions and lemmas that will be useful in the proof.

Definition D-1. *Given the minimum generalized user cost vector* $\boldsymbol{\lambda}$, *a path* $r \in \bar{R}^w$ *is defined to be "active" if and only if* $\forall w' \in W, r' \in \bar{R}^{w'}$, $g_{r'}^{w,SO} \leq g_r^{w,SO}$ and $\lambda^w - t_r^{w,SO} < \lambda^{w'} - t_{r'}^{w,SO}$ will not simultaneously hold.

The set of active paths under vector $\boldsymbol{\lambda}$ is denoted by $\mathcal{T}(\boldsymbol{\lambda}) \subseteq (\cup_w \bar{R}^w)$. It is straightforward to verify that a set of active paths is an element in the set $\Theta = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{|\Theta|}\}$. In other words, the active path set $\mathcal{T}(\boldsymbol{\lambda})$ is the set of paths capable of supporting positive flows under $\boldsymbol{\lambda}$ and a tolling function satisfying Assumption 1.

Definition D-2. *Consider two subsets of* W, W_1 *and* W_2 *where* $W_1 \cap W_2 = \emptyset$. *We define that* W_1 *is "upon"* W_2 *under* $\boldsymbol{\lambda}$ *if and only if* $\min\limits_{r \in \bar{R}^w, w \in W_1} \left(\lambda^w - t_r^{w,SO}\right) \geq \max\limits_{r' \in \bar{R}^{w'}, w' \in W_2} \left(\lambda^{w'} - t_{r'}^{w',SO}\right)$. *And,* W_1 *is "strictly upon"* W_2 *under* $\boldsymbol{\lambda}$ *if and only if* $\min\limits_{r \in \bar{R}^w, w \in W_1} \left(\lambda^w - t_r^{w,SO}\right) > \max\limits_{r' \in \bar{R}^{w'}, w' \in W_2} \left(\lambda^{w'} - t_{r'}^{w',SO}\right)$.

Consider the following two lemmas.

Lemma D-1. *Consider two subsets of* W*,* W_1 *and* W_2*, where* $W_1 \cap W_2 = \emptyset$*,* $W_1 \cup W_2 = W$*, and* W_1 *is upon* W_2 *under* $\tilde{\boldsymbol{\lambda}}$ *and strictly upon* W_2 *under* $\boldsymbol{\lambda}$*. If* $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \hat{\boldsymbol{\lambda}}$*, where* $\hat{\lambda}^w = \begin{cases} \lambda_0, & w \in W_1 \\ 0, & w \in W_2 \end{cases}$ *and* $\lambda_0 \in \mathbb{R}$*, then* $\mathcal{T}(\boldsymbol{\lambda}) \subseteq \mathcal{T}(\tilde{\boldsymbol{\lambda}})$*.*

Proof of Lemma D-1. Consider the paths r and r' in the set $\cup_w \bar{R}^w$, where $r \in \mathcal{T}(\boldsymbol{\lambda})$ and $g_{r'}^{SO} \le g_r^{SO}$. If $r \in \bar{R}^w, r' \in \bar{R}^{w'}$, where $w, w' \in W_1$, then by Definition D-1, we have $\tilde{\lambda}^w - t_r^{w,SO} = \lambda^w + \lambda_0 - t_r^{w,SO} \ge \lambda^{w'} + \lambda_0 - t_{r'}^{w',SO} = \tilde{\lambda}^{w'} - t_{r'}^{w',SO}$. If $r \in \bar{R}^w, r' \in \bar{R}^{w'}$, where $w \in W_1, w' \in W_2$, then by Definition D-2 we know $\tilde{\lambda}^w - t_r^{w,SO} \ge \tilde{\lambda}^{w'} - t_{r'}^{w',SO}$. If $r \in \bar{R}^w, r' \in \bar{R}^{w'}$, where $w, w' \in W_2$, then by Definition D-1, $\tilde{\lambda}^w - t_r^{w,SO} = \lambda^w - t_r^{w,SO} \ge \lambda^{w'} - t_{r'}^{w',SO} = \tilde{\lambda}^{w'} - t_{r'}^{w',SO}$. Lastly, it is impossible that $r \in \bar{R}^w, r' \in \bar{R}^{w'}$ and $w \in W_2, w' \in W_1$. Otherwise, by Definition D-2 we know $\lambda^w - t_r^{w,SO} < \lambda^{w'} - t_{r'}^{w',SO}$, contradicting $r \in \mathcal{T}(\boldsymbol{\lambda})$. With above derivations we then conclude that $r \in \mathcal{T}(\boldsymbol{\lambda}) \Rightarrow r \in \mathcal{T}(\tilde{\boldsymbol{\lambda}})$. Thus, $\mathcal{T}(\boldsymbol{\lambda}) \subseteq \mathcal{T}(\tilde{\boldsymbol{\lambda}})$. □

The next lemma provides one condition that set W can be divided into two subsets W_1 and W_2 such that W_1 is strictly upon W_2.

Lemma D-2. *Recall that* $\bar{t}^w = \max_{r \in \bar{R}^w} t_r^{SO} - \min_{r \in \bar{R}^w} t_r^{SO}$ *. If* $\max_{w \in W, r \in \bar{R}^w} \left(\lambda^w - t_r^{w,SO}\right) - \min_{w \in W, r \in \bar{R}^w} \left(\lambda^w - t_r^{w,SO}\right) > \sum_{w \in W} \bar{t}^w$*, then we can divide* W *into two subsets* W_1 *and* W_2 *where* $W_1 \cap W_2 = \emptyset$*,* $W_1 \cup W_2 = W$ *and* W_1 *is strictly upon* W_2 *under* $\boldsymbol{\lambda}$*.*

Proof of Lemma D-2. Set $\min_{w \in W, r \in \bar{R}^w} (\lambda^w - t_r^{SO}) = x$, $\min_{r \in \bar{R}^w} (\lambda^w - t_r^{SO}) = x^w$ and $\max_{w \in W, r \in \bar{R}^w} (\lambda^w - t_r^{SO}) = x + y$. Then, because $y > \sum_{w \in W} \bar{t}^w$ and $\max_{r \in \bar{R}^w} (\lambda^w - t_r^{SO}) = x^w + \bar{t}^w$, there always exists $\hat{x} \in [x, x + y]$ such that $\hat{x} \notin [x^w, x^w + \bar{t}^w], \forall w \in W$. Now, define $W_1 = \{w \in W | x^w > \hat{x}\}$ and $W_2 = \{w \in W | x^w + \bar{t}^w < \hat{x}\}$. Then, we have $W_1 \cap W_2 = \emptyset$, $W_1 \cup W_2 = W$, and W_1 is strictly upon W_2. Proof completes. □

We henceforth refer to the set of $\hat{x}$ as "uncovered set". With above definitions and lemmas, we are now ready to prove Proposition 6. To prove Proposition 6, we only need to prove that for any active path set $\mathcal{P} \subseteq \cup_w \bar{R}^w$, we can always find a vector $\boldsymbol{\lambda}$ and a toll vector $\boldsymbol{T}$ such that $\boldsymbol{U} \in \mathcal{U}, \mathcal{P} \subseteq \mathcal{T}(\boldsymbol{\lambda})$ and $U_r^w \le \sum_{w \in W} \bar{t}^w,\ \forall w \in W, r \in \bar{R}^w$.

Suppose that $\mathcal{T}(\tilde{\boldsymbol{\lambda}}) = \mathcal{P}$ and $\max\limits_{w \in W, r \in \bar{R}^w}\left(\tilde{\lambda}^w - t_r^{w,SO}\right) - \min\limits_{w \in W, r \in \bar{R}^w}\left(\tilde{\lambda}^w - t_r^{w,SO}\right) > \sum_{w \in W} \bar{t}^w$. Then, by Lemma D-2, we can divide W into two subsets W_1 and W_2 such that $W_1 \cap W_2 = \emptyset$, $W_1 \cup W_2 = W$ and W_1 is strictly upon W_2, i.e., $\min\limits_{r \in \bar{R}^w, w \in W_1}\left(\lambda^w - t_r^{w,SO}\right) = \max\limits_{r' \in \bar{R}^{w'}, w' \in W_2}\left(\lambda^{w'} - t_{r'}^{w',SO}\right) + \delta, \delta > 0$. Now define $\tilde{\boldsymbol{\lambda}}_n = \tilde{\boldsymbol{\lambda}} + \hat{\boldsymbol{\lambda}}$, where $\hat{\lambda}^w = \begin{cases} -\delta, & w \in W_1 \\ 0, & w \in W_2 \end{cases}$. Then W_1 is upon W_2, and by Lemma D-1 we know that $\mathcal{P} = \mathcal{T}(\tilde{\boldsymbol{\lambda}}) \subseteq \mathcal{T}(\tilde{\boldsymbol{\lambda}}_n)$. In this way, we reduce the length of this uncovered piece into 0 through using $\tilde{\boldsymbol{\lambda}}_n$ to replace $\tilde{\boldsymbol{\lambda}}$ and meanwhile expand the set of active paths. Moreover, if there exists other uncovered pieces with length greater than 0, we can repeat the above procedure and iteratively set $\tilde{\boldsymbol{\lambda}} = \tilde{\boldsymbol{\lambda}}_n$ until the uncovered pieces are all with the length of 0. Since the whole uncovered set contains at most $|W| - 1$ separate pieces, then we have $\mathcal{P} \subseteq \mathcal{T}(\tilde{\boldsymbol{\lambda}})$ and $\max\limits_{w \in W, r \in \bar{R}^w}\left(\tilde{\lambda}^w - t_r^{w,SO}\right) - \min\limits_{w \in W, r \in \bar{R}^w}\left(\tilde{\lambda}^w - t_r^{w,SO}\right) \le \sum_{w \in W} \bar{t}^w$.

With the above results, for any active path set $\mathcal{P} \subseteq \bar{R}$, we can always find another active path set $\mathcal{P}'$ such that $\mathcal{P} \subseteq \mathcal{P}' \subseteq \bar{R}$ and its associated $\boldsymbol{\lambda}$ satisfies $\max\limits_{w \in W, r \in \bar{R}^w}\left(\lambda^w - t_r^{w,SO}\right) - \min\limits_{w \in W, r \in \bar{R}^w}\left(\lambda^w - t_r^{w,SO}\right) \le \sum_{w \in W} \bar{t}^w$; we define a toll vector $\boldsymbol{T}$ through $T_r^w = \max\left\{\lambda^w - t_r^{w,SO}, \max\limits_{r' \in \cup_{w \in W} g_{r'}^{SO} \le g_r^{SO}} \lambda^{w'} - t_{r'}^{w',SO}\right\}$, then Constraints (15)-(17) and (24)-(25) are satisfied, and

$$
\begin{aligned}
U_r^w &= T_r^w + t_r^{w,SO} - \lambda^w \\
&= \max\left\{0, \max_{r' \in \cup_{w \in W} g_{r'}^{SO} \le g_r^{SO}} t_r^{w,SO} - \lambda^w + \lambda^{w'} - t_{r'}^{w',SO}\right\}
\end{aligned}
$$

$$= \max\left\{0, \max_{r' \in \cup_{w \in W} g_{r'}^{SO} \leq g_r^{SO}} \lambda^{w'} - t_{r'}^{w',SO} - (\lambda^w - t_r^{w,SO})\right\}$$

$$\leq \sum_{w \in W} \bar{t}^w$$

The proof is then completed. □

APPENDIX E

If the existence of the SO-inducing tolling functions is confirmed in the fixed demand case, in order to alleviate drivers' financial burden, we investigate the problem of finding the SO-inducing tolling function with the minimum revenue. Replacing $B^w(d^{w,SO})$ with λ^w in EC-2, the minimum revenue problem (MRP) can be formulated as below.

MRP1

$$\min_{\boldsymbol{f},\boldsymbol{T},\boldsymbol{\lambda}} \sum_{w\in W} \sum_{r\in \bar{R}^w} f_r^w T_r^{SO}$$

s.t. (10)-(17)

As formulated, MRP1 is a mathematical program with complementarity constraints (MPCC) and its objective function is bilinear. MPCC is difficult to solve because Mangasarian-Fromovitz constraint qualification is violated, and its feasible region is non-convex (He et al., 2013b). Summing up all of the Constraint (13), we have $\sum_{w\in W}\sum_{r\in\bar{R}^w} f_r^w T_r^{SO} = \sum_{w\in W}\sum_{r\in\bar{R}^w} f_r^w \lambda^w - \sum_{w\in W}\sum_{r\in\bar{R}^w} f_r^w \sum_{a\in A}\delta_{ar} t_a(v_a^{SO}) = \sum_{w\in W} d^w\lambda^w - \sum_{a\in A} v_a^{SO} t_a(v_a^{SO})$, thus linearizing the objective function of MRP1. If we further introduce binary variables $z_r^w \in \{0,1\}$, then MRP1 can be reformulated as the following MILP problem.

MRP2

$$\min_{\boldsymbol{f},\boldsymbol{T},\boldsymbol{\lambda},\boldsymbol{z}} \sum_{w\in W} d^w \lambda^w$$

s.t. (10)-(12) and (14)-(17)

$$f_r^w \le d^w(1-z_r^w) \qquad \forall w\in W, r\in\bar{R}^w$$

$$T_r^{SO} - \lambda^w + t_r^{SO} \le C z_r^w \qquad \forall w\in W, r\in\bar{R}^w$$

$$z_r^w \in \{0,1\} \qquad \forall w\in W, r\in\bar{R}^w$$

where C is a sufficiently large number. We note that the existence of a finite C such that MRP2 and MRP1 are equivalent can also be proven with a similar procedure to the proof of Proposition 6. It is straightforward to verify that if $z_r^w = 0$, then we have $f_r^w \ge 0$ and $T_r^{SO} - \lambda^w + t_r^{SO} = 0$, and if $z_r^w = 1$, then we have $f_r^w = 0$ and $T_r^{SO} - \lambda^w + t_r^{SO} \ge 0$, which is equivalent to Constraint (13).

APPENDIX F

In this appendix, we show that the emission function $g_a(v_a) = 0.2038 \cdot t_a(v_a) \cdot e^{0.7962(l_a/t_a(v_a))}$ is monotonically increasing and convex, when the speed $s_a = l_a/t_a(v_a)$ is not larger than 56 km/h, and the travel time function adopts the BPR function.

Specifically, calculating the partial derivative of $g_a(v_a)$, we have $g_a'(v_a) = 0.2038 \cdot t_a'(v_a) \cdot e^{0.7962 s_a}[1 - 0.7962 s_a]$. Since $0.2038 \cdot t_a'(v_a) \cdot e^{0.7962 s_a} > 0$, then $g_a'(v_a) > 0$ if and only if $s_a < \frac{1}{0.7962} = 1.256$ km/min = 75.36 km/h. Therefore, $g_a(v_a)$ is monotonically increasing when the speed is less than 56 km/h.

Next, calculating the second-order derivative of $g_a(v_a)$, we have $g_a''(v_a) = 0.2038[t_a'(v_a) \cdot e^{0.7962 s_a}]'[1 - 0.7962 s_a] + 0.2038[t_a'(v_a) \cdot e^{0.7962 s_a}][1 - 0.7962 s_a]'$. Since $s_a \leq 56\,\mathrm{km/h}$, we have $1 - 0.7962 s_a > 0$ and $[1 - 0.7962 s_a]' > 0$. In this context, if we can verify that $[t_a'(v_a) \cdot e^{0.7962 s_a}]' > 0$, then the inequality $g_a''(v_a) > 0$ must hold, and the convexity of $g_a(v_a)$ can be confirmed.

For $[t_a'(v_a) \cdot e^{0.7962 s_a}]'$, we have $[t_a'(v_a) \cdot e^{0.7962 s_a}]' = t_a''(v_a) - \frac{0.7962[t_a'(v_a)]^2 l_a}{t_a^2(v_a)} = \frac{t_a''(v_a) t_a(v_a) - 0.7962[t_a'(v_a)]^2 s_a}{t_a(v_a)}$. In this context, $[t_a'(v_a) \cdot e^{0.7962 s_a}]' > 0$ is equivalent to $s_a < 1.256 \frac{t_a''(v_a) t_a(v_a)}{[t_a'(v_a)]^2}$. Plugging $t_a(v_a) = t_0(1 + \beta v_a^4)$ into $1.256 \frac{t_a''(v_a) t_a(v_a)}{[t_a'(v_a)]^2}$, we can obtain that $1.256 \frac{t_a''(v_a) t_a(v_a)}{[t_a'(v_a)]^2} = 1.256 \frac{12\beta v_a^2(1+\beta v_a^4)}{16\beta^2 v_a^6} > 1.256 \times 0.75 = 0.942$ km/min. In this context, if $s_a \leq$ 0.942 km/h = 56.52 km/h, then the inequality $s_a < 1.256 \frac{t_a''(v_a) t_a(v_a)}{[t_a'(v_a)]^2}$ must hold. Therefore, if $s_a \leq 56$ km/h, then $[t_a'(v_a) \cdot e^{0.7962 s_a}]' > 0$, and $g_a(v_a)$ is convex.