Modeling and Engineering Constrained Shortest Path Algorithms for Battery Electric Vehicles

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We study the problem of computing constrained shortest paths for battery electric vehicles. Since battery capacities are limited, fastest routes are often infeasible. Instead, users are interested in fast routes on which the energy consumption does not exceed the battery capacity. For that, drivers can deliberately reduce speed to save energy. Hence, route planning should provide both path *and* speed recommendations. To tackle the resulting $\mathcal{NP}$-hard optimization problem, previous work trades correctness or accuracy of the underlying model for practical running times. We present a novel framework to compute *optimal* constrained shortest paths (without charging stops) for electric vehicles that uses more realistic physical models, while taking speed adaptation into account. Careful algorithm engineering makes the approach practical even on large, realistic road networks: We compute optimal solutions in less than a second for typical battery capacities, matching the performance of previous inexact methods. For even faster query times, the approach can easily be extended with heuristics that provide high quality solutions within milliseconds.

Key words: route planning; road networks; speedup techniques; algorithm engineering; constrained shortest paths; electric vehicles; energy consumption

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Appendix A: Omitted Details and Proofs from Section 3

In what follows, we provide technical details for linking consumption functions in the special case considered in Section 3.2, in which both functions represent arcs (Appendix A.1), give a formal proof of Lemma 2 (Appendix A.2), and show how general consumption functions can be linked in linear time (Appendix A.3). Finally, Appendix A.4 mentions technical details omitted from Section 3.3.

A.1. Linking Functions Defined by Single Tradeoff Functions

We show how $c = \operatorname{link}(c_1, c_2)$ can be computed in constant time in the case where each of the two given consumption functions c_1 and c_2 is defined by a *single* tradeoff subfunction, rather than multiple ones (c. f. Section 3.2). Recall Equation (6) from Section 3.2 restated below:

$$c(x) = \min_{\substack{\Delta \in [\underline{\tau}_1, \bar{\tau}_1] \\ \Delta \in [x - \bar{\tau}_2, x - \underline{\tau}_2]}} c_1(\Delta) + c_2(x - \Delta). \tag{6}$$

As argued in Section 3.2, the best choice of Δ depends on the value $x \in [\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2]$. Therefore, we consider the Δ-*function* $\Delta_{\text{opt}} \colon [\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2] \to \mathbb{R}_{\geq 0}$ that maps every admissible value of x to the optimal choice of Δ. Then, we immediately get for arbitrary $x \in \mathbb{R}_{\geq 0}$ that

$$c(x) = \begin{cases} \infty & \text{if } x < \underline{\tau}_1 + \underline{\tau}_2, \\ c_1(\bar{\tau}_1) + c_2(\bar{\tau}_2) & \text{if } x > \bar{\tau}_1 + \bar{\tau}_2, \\ c_1(\Delta_{\text{opt}}(x)) + c_2(x - \Delta_{\text{opt}}(x)) & \text{otherwise.} \end{cases} \tag{7}$$

Hence, we essentially need to compute Δ_{opt} to obtain the desired function c. To this end, consider an arbitrary fixed driving time $x \in [\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2]$. To identify the optimal value $\Delta_{\text{opt}}(x)$, we examine the derivative c'_x of the term $c_x(\Delta) := c_1(\Delta) + c_2(x - \Delta)$. It evaluates to

$$c'_x(\Delta) = \frac{2\alpha_1}{(\beta_1 - \Delta)^3} + \frac{2\alpha_2}{(x - \Delta - \beta_2)^3},$$

where α_1, β_1, α_2, and β_2 are the corresponding coefficients in the tradeoff functions of c_1 and c_2. We assume that $\alpha_1 > 0$ and $\alpha_2 > 0$ are positive (the other cases $\alpha_1 = 0$ or $\alpha_2 = 0$ are trivial: at least one of the consumption functions is constant in this case and hence, has a unique admissible driving time). Then, we obtain a unique zero Δ^*_x for this derivative under the assumption that $\beta_1 < \Delta$ and $\Delta < x - \beta_2$. This holds true for valid choices of Δ, because $\Delta \geq \underline{\tau}_1 > \beta_1$ and $\Delta \leq x - \underline{\tau}_2 < x - \beta_2$ always holds; see Equation (6). Therefore, solving the equation $c'_x(\Delta) = 0$ for Δ yields

$$\Delta^*_x = \frac{x - \beta_2 + \beta_1 \sqrt[3]{\frac{\alpha_2}{\alpha_1}}}{1 + \sqrt[3]{\frac{\alpha_2}{\alpha_1}}}.$$

The value Δ^*_x minimizes energy consumption for an unrestricted distribution of driving times that sum up to x. From Equation (6) we get the additional constraints $\Delta_{\text{opt}}(x) \geq \max\{\underline{\tau}_1, x - \bar{\tau}_2\}$ and $\Delta_{\text{opt}}(x) \leq \min\{\bar{\tau}_1, x - \underline{\tau}_2\}$. Since Δ^*_x is the unique zero of c'_x in the open interval $(\beta_1, x - \beta_2)$, monotonicity of c_x in the intervals $(\beta_1, \Delta^*_x]$ and $[\Delta^*_x, x - \beta_2)$ follows. Thus, we get

$$\Delta_{\text{opt}}(x) = \begin{cases} \max\{\underline{\tau}_1, x - \bar{\tau}_2\} & \text{if } \Delta^*_x < \max\{\underline{\tau}_1, x - \bar{\tau}_2\}, \\ \min\{\bar{\tau}_1, x - \underline{\tau}_2\} & \text{if } \Delta^*_x > \min\{\bar{\tau}_1, x - \underline{\tau}_2\}, \\ \Delta^*_x & \text{otherwise.} \end{cases} \tag{8}$$

We already argued in Section 3.2 that Equation (7) and Equation (8) together are sufficient to specify the desired function c. Since we want to explicitly represent c using tradeoff functions, we now derive the actual subfunctions that define c, depending on the value Δ_x^*.

First, solving the conditions $\Delta_x^* < \max\{\underline{\tau}_1, x - \bar{\tau}_2\}$ and $\Delta_x^* > \min\{\bar{\tau}_2, x - \underline{\tau}_2\}$ in Equation (8) for x yields four equivalences in total, namely

$$\Delta_x^* < \underline{\tau}_1 \quad \Leftrightarrow \quad x < \underline{\tau}_1^* := \underline{\tau}_1 + \beta_2 + (\underline{\tau}_1 - \beta_1)\sqrt[3]{\frac{\alpha_2}{\alpha_1}}, \tag{1}$$

$$\Delta_x^* < x - \bar{\tau}_2 \quad \Leftrightarrow \quad x > \bar{\tau}_2^* := \bar{\tau}_2 + \beta_1 + (\bar{\tau}_2 - \beta_2)\sqrt[3]{\frac{\alpha_1}{\alpha_2}}, \tag{2}$$

$$\Delta_x^* > \bar{\tau}_1 \quad \Leftrightarrow \quad x > \bar{\tau}_1^* := \bar{\tau}_1 + \beta_2 + (\bar{\tau}_1 - \beta_1)\sqrt[3]{\frac{\alpha_2}{\alpha_1}}, \tag{3}$$

$$\Delta_x^* > x - \underline{\tau}_2 \quad \Leftrightarrow \quad x < \underline{\tau}_2^* := \underline{\tau}_2 + \beta_1 + (\underline{\tau}_2 - \beta_2)\sqrt[3]{\frac{\alpha_1}{\alpha_2}}. \tag{4}$$

Note that we obtain similar statements when solving for equality, e. g., we have $\Delta_x^* = \underline{\tau}_1$ if and only if $x = \underline{\tau}_1^*$. Consequently, we also get $\Delta_x^* > \underline{\tau}_1$ if and only if $x > \underline{\tau}_1^*$ (and analogous results for Equations (2)–(4)). To obtain the actual function c and its subfunctions, we use the following Lemma A-1.

LEMMA A-1. *Let c_1 and c_2 be two consumption functions, such that each is defined by a single tradeoff function g_1 and g_2, respectively. Moreover, let $\underline{\tau}_1$, $\bar{\tau}_1$, $\underline{\tau}_2$, and $\bar{\tau}_2$ denote their respective minimum and maximum driving times. Then the following statements hold for their derivatives g_1' and g_2' (even if we replace all occurrences of the relation "$\leq$" by "$=$").*

1. $g_1'(\underline{\tau}_1) \leq g_2'(\underline{\tau}_2) \Leftrightarrow \underline{\tau}_1^* \leq \underline{\tau}_1 + \underline{\tau}_2 \leq \underline{\tau}_2^*$,
2. $g_1'(\bar{\tau}_1) \leq g_2'(\bar{\tau}_2) \Leftrightarrow \bar{\tau}_1^* \leq \bar{\tau}_1 + \bar{\tau}_2 \leq \bar{\tau}_2^*$,
3. $g_1'(\bar{\tau}_1) \leq g_2'(\underline{\tau}_2) \Leftrightarrow \bar{\tau}_1^* \leq \underline{\tau}_2^*$,
4. $g_1'(\underline{\tau}_1) \leq g_2'(\bar{\tau}_2) \Leftrightarrow \underline{\tau}_1^* \leq \bar{\tau}_2^*$.

Proof. All equivalences follow after simple rearrangements. As an example, we show the first part of the first statement, namely, $g_1'(\underline{\tau}_1) \leq g_2'(\underline{\tau}_2)$ if and only if $\underline{\tau}_1^* \leq \underline{\tau}_1 + \underline{\tau}_2$. For $i \in \{1,2\}$, let α_i and β_i denote the coefficients of the tradeoff function g_i as in Equation (5). We exploit that $\alpha_1 > 0$, $\alpha_2 > 0$, $\underline{\tau}_1 > \beta_1$, and $\underline{\tau}_2 > \beta_2$ must hold to get

$$\begin{aligned}
& g_1'(\underline{\tau}_1) \leq g_2'(\underline{\tau}_2) \\
\Leftrightarrow \quad & -\frac{2\alpha_1}{(\underline{\tau}_1 - \beta_1)^3} \leq -\frac{2\alpha_2}{(\underline{\tau}_2 - \beta_2)^3} \\
\Leftrightarrow \quad & (\underline{\tau}_2 - \beta_2)^3 \geq \frac{2\alpha_2}{2\alpha_1}(\underline{\tau}_1 - \beta_1)^3 \\
\Leftrightarrow \quad & \underline{\tau}_2 \geq \beta_2 + \sqrt[3]{\frac{\alpha_2}{\alpha_1}}(\underline{\tau}_1 - \beta_1) \\
\Leftrightarrow \quad & \underline{\tau}_1 + \underline{\tau}_2 \geq \underline{\tau}_1 + \beta_2 + \sqrt[3]{\frac{\alpha_2}{\alpha_1}}(\underline{\tau}_1 - \beta_1) = \underline{\tau}_1^*.
\end{aligned}$$

All other statements follow from similar rearrangements. Q.E.D.

Together with Equations (1)–(4), Lemma A-1 enables us to construct the desired function $c = \mathrm{link}(c_1, c_2)$, depending on the slopes (i. e., the derivatives) of c_1 and c_2 at their respective subdomain borders. Exploiting that $\underline{\tau}_1 \le \bar{\tau}_1$ and $\underline{\tau}_2 \le \bar{\tau}_2$ hold by definition, we obtain the function c after the following case distinction. As in Lemma A-1, let g_1 denote the (unique) tradeoff function defining c_1 and similarly, let g_2 be the tradeoff function defining c_2. We consider the slopes of these tradeoff functions at certain subdomain borders of c_1 and c_2. Without loss of generality, assume that $g_1'(\underline{\tau}_1) \le g_2'(\underline{\tau}_2)$ holds (the other case is symmetric). This leaves us with only three possible cases, which are presented below.

1. $g_1'(\underline{\tau}_1) \le g_2'(\underline{\tau}_2) \le g_1'(\bar{\tau}_1) \le g_2'(\bar{\tau}_2)$: Consider the relevant subdomain borders $\underline{\tau}_1 + \underline{\tau}_2$ and $\bar{\tau}_1 + \bar{\tau}_2$ corresponding to the minimum and maximum driving time of the function c obtained after linking c_1 and c_2. By the first, second, and third statement of Lemma A-1, we know that

$$\underline{\tau}_1^* \le \underline{\tau}_1 + \underline{\tau}_2 \le \underline{\tau}_2^* \le \bar{\tau}_1^* \le \bar{\tau}_1 + \bar{\tau}_2 \le \bar{\tau}_2^*.$$

For arbitrary values $x \in [\underline{\tau}_1 + \underline{\tau}_2, \underline{\tau}_2^*)$, we use the fact $x \ge \underline{\tau}_1^*$ and Equation (1) to infer the inequality $\Delta_x^* \ge \underline{\tau}_1$. Similarly, the fact $x < \bar{\tau}_2^*$ and Equation (2) yield $\Delta_x^* > x - \bar{\tau}_2$. Hence, we have $\Delta_x^* \ge \max\{\underline{\tau}_1, x - \bar{\tau}_2\}$ and the first case of Equation (8) does not apply. On the other hand, we know that both $x < \bar{\tau}_1^*$ and $x < \underline{\tau}_2^*$ hold, so by Equation (3) and Equation (4) we get $x - \underline{\tau}_2 < \Delta_x^* < \bar{\tau}_1$. This means that we are in the second case of Equation (8) and therefore, $\Delta_{\mathrm{opt}}(x) = \min\{\bar{\tau}_1, x - \underline{\tau}_2\} = x - \underline{\tau}_2$ is the optimal choice for any $x \in [\underline{\tau}_1 + \underline{\tau}_2, \underline{\tau}_2^*)$. Making similar observations for the cases $x \in [\underline{\tau}_2^*, \bar{\tau}_1^*)$ and $x \in [\bar{\tau}_1^*, \bar{\tau}_1 + \bar{\tau}_2)$, we obtain corresponding values $\Delta_{\mathrm{opt}}(x) = \Delta_x^*$ and $\Delta_{\mathrm{opt}}(x) = \bar{\tau}_1$, respectively. This yields the desired function $c = \mathrm{link}(c_1, c_2)$, which is given by

$$c(x) = \begin{cases} \infty & \text{if } x < \underline{\tau}_1 + \underline{\tau}_2, \\ c_1(x - \underline{\tau}_2) + c_2(\underline{\tau}_2) & \text{if } \underline{\tau}_1 + \underline{\tau}_2 \le x < \underline{\tau}_2^*, \\ c_1(\Delta_x^*) + c_2(x - \Delta_x^*) & \text{if } \underline{\tau}_2^* \le x < \bar{\tau}_1^*, \\ c_1(\bar{\tau}_1) + c_2(x - \bar{\tau}_1) & \text{if } \bar{\tau}_1^* \le x < \bar{\tau}_1 + \bar{\tau}_2, \\ c_1(\bar{\tau}_1) + c_2(\bar{\tau}_2) & \text{otherwise.} \end{cases}$$

2. $g_1'(\underline{\tau}_1) \le g_1'(\bar{\tau}_1) \le g_2'(\underline{\tau}_2) \le g_2'(\bar{\tau}_2)$: In this case, we know by the four statements in Lemma A-1 that the order $\underline{\tau}_1^* \le \bar{\tau}_1^* \le \underline{\tau}_1 + \underline{\tau}_2 \le \bar{\tau}_1 + \bar{\tau}_2 \le \underline{\tau}_2^* \le \bar{\tau}_2^*$ must hold. Equations (1)–(4) yield $\Delta_x^* \ge \max\{\underline{\tau}_1, x - \bar{\tau}_2\}$ and $\Delta_x^* > \min\{\bar{\tau}_1, x - \underline{\tau}_2\}$ in the whole subdomain $[\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2)$. Consequently, we obtain the optimal value $\Delta_{\mathrm{opt}}(x) = \min\{\bar{\tau}_1, x - \underline{\tau}_2\}$ and the function $c = \mathrm{link}(c_1, c_2)$ is defined as

$$c(x) = \begin{cases} \infty & \text{if } x < \underline{\tau}_1 + \underline{\tau}_2, \\ c_1(\bar{\tau}_1) + c_2(x - \bar{\tau}_1) & \text{if } \underline{\tau}_1 + \underline{\tau}_2 \le x < \bar{\tau}_1 + \underline{\tau}_2, \\ c_1(x - \underline{\tau}_2) + c_2(\underline{\tau}_2) & \text{if } \bar{\tau}_1 + \underline{\tau}_2 \le x < \bar{\tau}_1 + \bar{\tau}_2, \\ c_1(\bar{\tau}_1) + c_2(\bar{\tau}_2) & \text{otherwise.} \end{cases}$$

3. $g_1'(\underline{\tau}_1) \le g_2'(\underline{\tau}_2) \le g_2'(\bar{\tau}_2) \le g_1'(\bar{\tau}_1)$: Along the lines of the first case, Lemma A-1 and Equations (1)–(4) yield that the function $c = \mathrm{link}(c_1, c_2)$ evaluates to

$$c(x) = \begin{cases} \infty & \text{if } x < \underline{\tau}_1 + \underline{\tau}_2, \\ c_1(x - \underline{\tau}_2) + c_2(\underline{\tau}_2) & \text{if } \underline{\tau}_1 + \underline{\tau}_2 \le x < \underline{\tau}_2^*, \\ c_1(\Delta_x^*) + c_2(x - \Delta_x^*) & \text{if } \underline{\tau}_2^* \le x < \bar{\tau}_2^*, \\ c_1(x - \bar{\tau}_2) + c_2(\bar{\tau}_2) & \text{if } \bar{\tau}_2^* \le x < \bar{\tau}_1 + \bar{\tau}_2, \\ c_1(\bar{\tau}_1) + c_2(\bar{\tau}_2) & \text{otherwise.} \end{cases}$$

Due to convexity of both g_1 and g_2, no other cases remain. Hence, the function c constructed above is defined by at most five subfunctions (two of which are constant). In each expression, we can expand the functions c_1 and c_2 to obtain a term that has the general form of a tradeoff function as in Equation (5). In particular, the denominator in the tradeoff functions of both c_1 and c_2 is (strictly) positive in all cases, i. e., we always have $x > \beta$. Moreover, it is easy to verify that c is continuous and decreasing on the interval $[\underline{\tau}_1 + \underline{\tau}_2, \infty)$, by inspecting the corresponding limits at the endpoints of each subdomain of c. In conclusion, the link operation requires constant time in the special case where both input functions are defined by a single tradeoff function. Lemma 1 from Section 3.2 summarizes these insights. It is restated below.

LEMMA 1. *Given two consumption functions c_1 and c_2, each defined by a single tradeoff subfunction, the link operation* $\operatorname{link}(c_1, c_2)$ *requires constant time and its result is a consumption function that is uniquely described by at most three tradeoff subfunctions.*

A.2. Proof of Lemma 2

Lemma 2 from Section 3.2 claims that linking two consumption functions results in a function that is continuous on its interval of admissible driving time. The proof is given below.

LEMMA 2. *Given two consumption functions c_1 and c_2, the function $c := \operatorname{link}(c_1, c_2)$ is continuous on the interval $[\underline{\tau}, \infty)$, where $\underline{\tau} \in \mathbb{R}_{>0}$ denotes the minimum driving time of c.*

Proof. Assume for contradiction that c has a discontinuity at some $x > \underline{\tau}$. By construction of c, a discontinuity in the interval $[\underline{\tau}, \infty)$ corresponds to a discontinuity of some candidate function c^* induced by a tradeoff subfunction of c_1 or c_2 (see Section 3.2). Without loss of generality, let c^* be a subfunction of c_1. We know that c^* has exactly one discontinuity at its minimum driving time $\underline{\tau}^* \in \mathbb{R}_{>0}$. Thus, we have $c(x) = c_1(\underline{\tau}^*) + c_2(x - \underline{\tau}^*)$. We distinguish two cases.

Case 1: If $\underline{\tau}^*$ is not the minimum driving time of c_1, we know that c_1 is continuous in the neighborhood of $\underline{\tau}^*$. Since c is decreasing and has a discontinuity at x, there exists an $\varepsilon > 0$ such that $c_1(\underline{\tau}^* - \varepsilon) + c_2(x - \underline{\tau}^*) < c(x - \varepsilon)$, contradicting the fact that c minimizes this term.

Case 2: If $\underline{\tau}^*$ is in fact the minimum driving time of c_1, we know that the corresponding driving time $x - \underline{\tau}^*$ of c_2 must exceed its minimum driving time $\underline{\tau}_2$, since $x > \underline{\tau} = \underline{\tau}^* + \underline{\tau}_2$. Hence, c_2 is continuous in the neighborhood of $x - \underline{\tau}^*$ and along the lines of the first case we obtain a contradiction, because now $c_1(\underline{\tau}^*) + c_2(x - \underline{\tau}^* - \varepsilon) < c(x - \varepsilon)$ must hold for some $\varepsilon > 0$. Q.E.D.

A.3. Linking General Consumption Functions in Linear Time

In Section 3.2, we described a naïve link operation, which has quadratic running time in the number of subfunctions of c_1 and c_2. In what follows, we show how the complexity of the link operation for general consumption functions can be reduced to linear time. We say that a consumption function c with minimum driving time $\underline{\tau} \in \mathbb{R}_{>0}$ is *convex* if it is convex on the interval $[\underline{\tau}, \infty)$. Note that this holds true for consumption functions of single arcs; see Section 2. Our more sophisticated link operation exploits the fact that consumption functions are always convex, which we now prove formally. Consider the Δ-function Δ_{opt} of $c = \operatorname{link}(c_1, c_2)$, defined as the optimal choice of Δ in Equation (6). See Figure 1 for an example. (Note that we did not

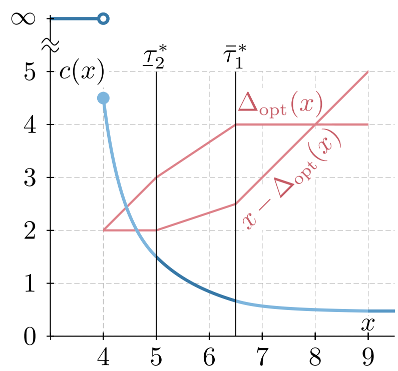

Figure 1 **The functions Δ_{opt} and $x - \Delta_{\mathrm{opt}}$ (red), indicating how the available time is shared between the functions c_1 and c_2 from Figure 5, in order to obtain the function c.**

formally prove that the value Δ is distinct in the general case, but we may as well pick the *minimum* value Δ that fulfills Equation (6) to ensure that Δ_{opt} is well-defined.) Presuming that c_1 and c_2 are convex, the following Lemma A-2 shows that both Δ_{opt} and the value $x - \Delta_{\mathrm{opt}}(x)$ increase wrt. $x \in [\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2]$.

LEMMA A-2. *Let $\Delta_{\mathrm{opt}} \colon [\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2] \to \mathbb{R}_{\geq 0}$ denote the Δ-function for two convex consumption functions c_1 and c_2 with corresponding minimum and maximum driving times $\underline{\tau}_1$, $\bar{\tau}_1$, $\underline{\tau}_2$, and $\bar{\tau}_2$. Moreover, let $\bar{\Delta}_{\mathrm{opt}}(x) := x - \Delta_{\mathrm{opt}}(x)$ be defined on the domain of Δ_{opt}. Then both Δ_{opt} and $\bar{\Delta}_{\mathrm{opt}}$ are continuous and increasing.*

Proof. We begin by showing that Δ_{opt} is increasing. Assume for contradiction that this is not the case, i. e., for some value x in the domain of Δ_{opt}, there are values $\varepsilon > 0$ and $\delta > 0$ such that $\Delta = \Delta_{\mathrm{opt}}(x) > \Delta_{\mathrm{opt}}(x+\varepsilon) = \Delta - \delta$. First of all, note that the inequality $c_1(\Delta) + c_2(x-\Delta) \leq c_1(\Delta-\delta) + c_2(x-\Delta+\delta)$ must hold, since Δ minimizes this term by definition of Δ_{opt}. Further, $\Delta = \Delta_{\mathrm{opt}}(x)$ is the *smallest* among all values that minimize the term by definition, so plugging in $\Delta - \delta < \Delta$ actually yields a *strictly* greater result. Analogously, we have $c_1(\Delta-\delta) + c_2(x+\varepsilon-\Delta+\delta) \leq c_1(\Delta) + c_2(x+\varepsilon-\Delta)$, as this term is minimized by $\Delta - \delta$. Therefore, we obtain

$$\begin{aligned} c_1(\Delta-\delta) - c_1(\Delta) &\leq c_2(x+\varepsilon-\Delta) - c_2(x+\varepsilon-\Delta+\delta) \\ &\leq c_2(x-\Delta) - c_2(x-\Delta+\delta) \\ &< c_1(\Delta-\delta) - c_1(\Delta), \end{aligned}$$

a contradiction. Here, we exploit the fact that c_2 is convex and decreasing and hence, $c_2(x-\Delta) - c_2(x-\Delta+\delta)$ must be decreasing wrt. x for fixed values Δ and δ (the gap between two function values with constant difference on the x-axis must decrease if x increases).

Regarding $\bar{\Delta}_{\mathrm{opt}}$, monotonicity follows from a very similar argument. As before, assume for contradiction that $\bar{\Delta}_{\mathrm{opt}}(x) > \bar{\Delta}_{\mathrm{opt}}(x+\varepsilon)$ for some $\varepsilon > 0$, so the inequality $x - \Delta_{\mathrm{opt}}(x) > x + \varepsilon - \Delta_{\mathrm{opt}}(x+\varepsilon)$ holds. We plug in $\Delta = \Delta_{\mathrm{opt}}(x)$ and $\Delta + \delta = \Delta_{\mathrm{opt}}(x+\varepsilon)$ to obtain $\delta > \varepsilon > 0$. As in the first case, we get $c_1(\Delta) + c_2(x-\Delta) \leq c_1(\Delta+\delta) + c_2(x-\Delta-\delta)$ and $c_1(\Delta+\delta) + c_2(x+\varepsilon-\Delta-\delta) < c_1(\Delta) + c_2(x+\varepsilon-\Delta)$ by the definition of Δ_{opt}. Along the lines of the first case, this yields a contradiction.

Finally, we show that Δ_{opt} and $\bar{\Delta}_{\text{opt}}$ are continuous. Since the sum of $\Delta_{\text{opt}}(x)$ and $x - \Delta_{\text{opt}}(x)$ always equals x, their sum increases by exactly ε for any $x + \varepsilon$ with $\varepsilon > 0$ in the neighborhood of x. Hence, a discontinuity in either function would imply that at least one of the two terms must decrease, but we showed before that both $\Delta_{\text{opt}}(x)$ and $x - \Delta_{\text{opt}}(x)$ increase wrt. x. Q.E.D.

Note that in order to show that a consumption function c with minimum driving time $\underline{\tau}$ and maximum driving time $\bar{\tau}$ is convex, it is sufficient to consider the interval $[\underline{\tau}, \bar{\tau})$, since we already know that c is decreasing and that for all $x \in [\bar{\tau}, \infty)$, the value $c(x) = c(\bar{\tau})$ is constant. The following Lemma A-3 proves this property and hence, implies that linking two convex consumption functions indeed yields a decreasing and convex function. Consequently, Lemma A-2 applies to all consumption functions of the general form.

Lemma A-3. *Given two consumption functions c_1 and c_2 that are convex on their subdomains $[\underline{\tau}_1, \bar{\tau}_1]$ and $[\underline{\tau}_2, \bar{\tau}_2]$ of admissible driving times, the function $c := \text{link}(c_1, c_2)$ is convex on the interval $[\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2)$.*

Proof. Assume for the sake of contradiction that $c = \text{link}(c_1, c_2)$ is not convex on the indicated interval. We use the previous Lemma A-2, which implies that the Δ-function wrt. c_1 and c_2 is increasing. Moreover, both (right) derivatives c_1' and c_2' are increasing on their respective subdomains $[\underline{\tau}_1, \bar{\tau}_1)$ and $[\underline{\tau}_2, \bar{\tau}_2)$ by assumption, as c_1 and c_2 are decreasing and convex. Given that c is not convex, its (right) derivative c' must be decreasing on some subinterval of $[\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2)$. Thus, there exist values $x \in [\underline{\tau}_1 + \underline{\tau}_2, \bar{\tau}_1 + \bar{\tau}_2)$ and $\varepsilon > 0$ such that $x + \varepsilon < \bar{\tau}_1 + \bar{\tau}_2$ and we get

$$\begin{aligned} c'(x) &> c'(x+\varepsilon) \\ &= c_1'(\Delta_{\text{opt}}(x+\varepsilon)) + c_2'(x+\varepsilon-\Delta_{\text{opt}}(x+\varepsilon)) \\ &\geq c_1'(\Delta_{\text{opt}}(x)) + c_2'(x-\Delta_{\text{opt}}(x)) \\ &= c'(x), \end{aligned}$$

a contradiction. This completes the proof. Q.E.D.

A Linear-Time Algorithm for Linking Consumption Functions. Given that the functions Δ_{opt} and $\bar{\Delta}_{\text{opt}}$ (as defined in Lemma A-2) of an arbitrary pair of consumption functions are continuous and increasing, we are able to perform the link operation in a single coordinated linear scan, in which we keep track of Δ_{opt} and $\bar{\Delta}_{\text{opt}}$. For two piecewise functions c_1 and c_2, let $g_1^1, \dots, g_1^k$ and $g_2^1, \dots, g_2^\ell$ denote their defining tradeoff functions, given in increasing order of their subdomains. For some subfunction g_1^j with $j \in \{1, \dots, k\}$ or g_2^j with $j \in \{1, \dots, \ell\}$, we denote by $[\underline{\tau}_i^j, \bar{\tau}_i^j)$ its subdomain and by c_i^j its induced consumption function. The linear-time link operation proceeds as follows. First, it links the consumption functions c_1^1 and c_2^1 induced by the two tradeoff functions g_1^1 and g_2^1 with least admissible driving times. This results in a new convex consumption function $\text{link}(c_1^1, c_2^1)$, which is defined by at most three tradeoff functions. Let Δ_{opt} and $\bar{\Delta}_{\text{opt}}$ be the Δ-functions associated with this link operation. We determine the next pair of consumption functions that are linked. To this end, we consider the points $x_1^1 := \Delta_{\text{opt}}^{-1}(\bar{\tau}_1^1)$ and $x_2^1 := \bar{\Delta}_{\text{opt}}^{-1}(\bar{\tau}_2^1)$ at which the induced consumption functions c_1^1 and c_2^1 reach their maximum driving time in the linked function. If $x_1^1 < x_2^1$, we continue with $\text{link}(c_1^2, c_2^1)$, so Δ_{opt} can take values greater than $\bar{\tau}_1^1$. Similarly, if $x_1^1 > x_2^1$ holds, we compute $\text{link}(c_1^1, c_2^2)$ next, so that $\bar{\Delta}_{\text{opt}}$ may exceed $\bar{\tau}_2^1$. Finally, in the special case $x_1^1 = x_2^1$ we proceed with $\text{link}(c_1^2, c_2^2)$.

We continue this procedure until we reach the maximum driving time and link the consumption functions induced by g_1^k and g_2^ℓ. The lower envelope of the (linear number of) computed consumption functions yields the desired result $\operatorname{link}(c_1, c_2)$. Obviously, the running time of this procedure is in $\mathcal{O}(k+\ell)$.

A.4. Omitted Details from Section 3.3

The unique extreme point of the difference $g_1^i - g_2^j$ of the subfunctions g_1^i and g_2^j of two consumption functions (c. f. Section 3.3) is given by

$$x = \frac{\beta_2 - \beta_1 \sqrt[3]{\frac{\alpha_2}{\alpha_1}}}{1 - \sqrt[3]{\frac{\alpha_2}{\alpha_1}}}.$$

Appendix B: Omitted Details and Proofs from Section 4

Below, Appendix B.1 mentions technical details from Section 4.2, whereas Appendix B.2 proves Lemma 3 from Section 4.3.

B.1. Omitted Details from Section 4.2

As mentioned in Section 4.2, we store a function $\bar{\Delta}_{\text{opt}}$ with each label c to enable path and speed retrieval after the search has terminated. The function $\bar{\Delta}_{\text{opt}}$ is given as $\bar{\Delta}_{\text{opt}}(x) = x - \Delta_{\text{opt}}(x)$, wrt. the link operation that resulted in the label c, and it is a piecewise function with subfunctions that have the general form

$$\bar{\Delta}_{\text{opt}}(x) = x - \frac{x - \lambda}{\mu},$$

with nonnegative coefficients $\lambda \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathbb{R}_{>0}$; c. f. Equation (8) in Section 3.2.

B.2. Proof of Lemma 3

Lemma 3 proves that the number of settled labels per vertex of the heuristic approach described in Section 4.3 is upper bounded by $\lceil 1/\varepsilon \rceil + 1$, which implies that the algorithm itself runs in polynomial time.

LEMMA 3. *The number of settled labels contained in the set $L_{set}(v)$ of each vertex $v \in V$ is at most $\lceil 1/\varepsilon \rceil + 1$ when running tradeoff function propagating (TFP) with exact improved dominance checks.*

Proof. Let $k := \lceil 1/\varepsilon \rceil + 1$ and assume for contradiction that after running TFP, the label set $L_{\text{set}}(v)$ of some vertex $v \in V$ contains at least $k+1$ consumption functions. We denote these functions by $c_1, \ldots, c_{k+1}$ and without loss of generality, we assume they are given in increasing order of minimum driving time (and hence, were inserted into the label set in this order). Since exact dominance checks are applied, we know that c_2 yields an improvement over c_1 by at least εM at its minimum driving time $\underline{\tau}_2 \in \mathbb{R}_{>0}$. Consequently, we have $c_2(\underline{\tau}_2) \leq c_1(\underline{\tau}_2) - \varepsilon M$. We can apply the same argument to any function c_i with $i \in \{2, \ldots, k+1\}$ and it follows that $c_i(\underline{\tau}_i) \leq c_{i-1}(\underline{\tau}_i) - \varepsilon M$. Thus, we obtain

$$\begin{aligned} c_{k+1}(\underline{\tau}_{k+1}) &\leq c_k(\underline{\tau}_{k+1}) - \varepsilon M \\ &\leq c_1(\underline{\tau}_2) - k\varepsilon M \\ &\leq M - \left(\left\lceil \frac{1}{\varepsilon} \right\rceil + 1 \right) \varepsilon M \\ &< 0. \end{aligned}$$

This contradicts the fact that c_{k+1} must have nonnegative consumption for all admissible driving times, since the algorithm ensures that battery constraints are not violated for a consumption function before running any dominance checks with it. Q.E.D.

Algorithm 1: Pseudocode of the function-propagating potential search for TFP

```
   // initialize labels
1  foreach v ∈ V do
2      φ_v ⟵ ∅
3  φ_t ⟵ [(0,0)]
4  Q.insert(t,0)

   // run main loop
5  while Q.isNotEmpty() do
6      u ⟵ Q.deleteMin()

       // scan outgoing arcs
7      foreach (u,v) ∈ Ā do
8          φ_(u,v) ⟵ convert(c_(u,v))
9          φ ⟵ link(φ_u, φ_(u,v))
10         if ∃x ∈ ℝ: φ(x) < φ_v(x) then
11             φ_v ⟵ merge(φ_v, φ)
12             Q.update(v, key(v, φ_v))
```

Appendix C: Omitted Details and Proofs from Section 5

This appendix contains omitted details about the potential function based on piecewise linear functions, introduced in Section 5.2. We first provide pseudocode and a brief description of the backward search to compute the potential function π_φ (Appendix C.1). Afterwards, we formally prove that the resulting potential function is consistent (Appendix C.2).

C.1. Pseudocode of the Label-Correcting Backward Search

Algorithm 1 shows pseudocode of the search that is executed from the target $t \in V$ in order to compute the functions representing the vertex potential $\pi_\varphi \colon V \times [0, M] \cup \{-\infty\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Recall that each vertex stores a *single* label consisting of its (tentative) lower bound function. The search is initialized with a function φ_t at the target $t \in V$ that evaluates to 0 for arbitrary nonnegative state of charge (SoC), represented by the single breakpoint $(0,0)$; see lines 1–4 of Algorithm 1. In the main loop, scanning an outgoing arc $(u,v) \in A$ of some vertex $u \in V$ corresponds to *linking* the two lower bounds representing the label at u and the arc (u,v), respectively. We first *convert* the consumption function $c_{(u,v)}$ mapping driving time to energy consumption to a piecewise linear function $\varphi_{(u,v)}$ mapping SoC to a lower bound on driving time (line 8 of Algorithm 1). Afterwards, the algorithm *merges* the result with the existing label at v to obtain a new convex function (line 11).

C.2. Potential Function Consistency

Below, Lemma C-1 formally proves that the potential function π_φ is consistent when setting $\pi_\varphi(v,b) := \varphi_v(b)$ for all $v \in V$ and $b \in [0, M]$, using the piecewise linear functions computed as described above.

LEMMA C-1. *The potential function π_φ is consistent.*

Proof. For some arc $(u,v) \in A$, let φ_u and φ_v be the piecewise linear functions at u and v, respectively, after the backward search has terminated. Let $\varphi_{(u,v)}$ denote the lower bound function of (u,v). We know that φ_u is upper bounded by the function computed by $\text{link}(\varphi_v, \varphi_{(u,v)})$, since this function was merged into φ_u at some point during the backward search. For an arbitrary driving time $x \in \mathbb{R}_{\geq 0}$, let $b^* := c_{(u,v)}(x)$ denote the corresponding energy consumption on the arc (u,v). Then, we obtain for all $b \in [0,M]$ that

$$\begin{aligned} x + \pi_\varphi(v, f_{(u,v)}(x,b)) &= x + \varphi_v(f_{(u,v)}(x,b)) \\ &\geq x + \varphi_v(b - c_{(u,v)}(x)) \\ &= c^{-1}_{(u,v)}(b^*) + \varphi_v(b - b^*) \\ &\geq \varphi_{(u,v)}(b^*) + \varphi_v(b - b^*) \\ &\geq \min_{b^* \in \mathbb{R}} \varphi_{(u,v)}(b^*) + \varphi_u(b - b^*) \\ &\geq \pi_\varphi(u,b), \end{aligned}$$

which implies that the term $x - \pi_\varphi(u,b) + \pi_\varphi(v, f_{(u,v)}(x,b))$ is nonnegative. This completes the proof. Q.E.D.

Appendix D: Omitted Details and Proofs from Section 6

The following Appendix D.1 provides proofs from Section 6 that were omitted from the main manuscript. Afterwards, Appendix D.2 describes how upper bound functions represented by a single tradeoff subfunction are derived for the witness search presented in Section 6.4.

D.1. Omitted Proofs from Section 6.1 and Section 6.3

Lemma 4 formally proves that the bivariate SoC function of a path consisting solely of nonnegative consumption functions can be represented by a single univariate (consumption) function.

LEMMA 4. *Let $P = [v_1, \ldots, v_k]$ be a path in G and let c_i denote the consumption function of the arc (v_i, v_{i+1}) for $i \in \{1, \ldots, k-1\}$. If all consumption functions are nonnegative, i. e., $c_i(x) \geq 0$ holds for all $x \in \mathbb{R}_{\geq 0}$ and $i \in \{1, \ldots, k-1\}$, the SoC function of P evaluates to*

$$f(x,b) = \begin{cases} -\infty & \text{if } b < c(x), \\ b - c(x) & \text{otherwise,} \end{cases}$$

where c denotes the function obtained after iteratively linking the functions $c_1, \ldots, c_{k-1}$.

Proof. More formally, let $c := \text{link}(\ldots\text{link}(\ldots\text{link}(c_1, c_2), \ldots), c_{k-1})$. First, consider the SoC function f_i of the consumption function c_i of an individual arc (v_i, v_{i+1}) with $i \in \{1, \ldots, k-1\}$. It equals

$$f_i(x,b) = \begin{cases} -\infty & \text{if } b < c_i(x), \\ b - c_i(x) & \text{otherwise,} \end{cases}$$

since c_i has only nonnegative values by assumption, so the SoC at v_{i+1} never exceeds M. We prove the lemma by induction. Assume that for some $i \in \{1, \ldots, k-2\}$, we are given the result $c_{1,\ldots,i} := \text{link}(\ldots\text{link}(\ldots\text{link}(c_1, c_2), \ldots), c_i)$ of linking all arcs in $[v_1, \ldots, v_{i+1}]$ and the corresponding SoC function is

$$f_{1,\ldots,i}(x,b) = \begin{cases} -\infty & \text{if } b < c_{1,\ldots,i}(x), \\ b - c_{1,\ldots,i}(x) & \text{otherwise.} \end{cases}$$

We construct the SoC function $f_{1,\dots,i+1}$ using $c_{1,\dots,i+1} := \text{link}(c_{1,\dots,i}, c_{i+1})$. Since $c_{i+1}(x)$ is nonnegative for all $x \in \mathbb{R}_{\geq 0}$, we obtain $c_{1,\dots,i}(x) \leq c_{1,\dots,i+1}(x)$ for arbitrary driving times $x \in \mathbb{R}_{\geq 0}$. Hence, the path is infeasible for a pair of driving time $x \in \mathbb{R}_{\geq 0}$ and initial SoC $b \in [0, M]$ if and only if $b < \max\{c_{1,\dots,i}(x), c_{1,\dots,i+1}(x)\} = c_{1,\dots,i+1}(x)$. Otherwise, the function $c_{1,\dots,i+1}$ minimizes consumption on the path by definition of the link operation (still, no recuperation is possible). We obtain the function

$$f_{1,\dots,i+1}(x,b) = \begin{cases} -\infty & \text{if } b < c_{1,\dots,i+1}(x), \\ b - c_{1,\dots,i+1}(x) & \text{otherwise,} \end{cases}$$

which completes the proof. Q.E.D.

Lemma 5, restated below, is used in Section 6.3 to derive a simple method for comparing bivariate SoC function when inserting new shortcuts. It is proven formally here.

Lemma 5. *Given a nonpositive or discharging SoC function f_1 and a discharging SoC function f_2, such that their respective consumption functions are c_1^+, c_1^-, c_2^+, and c_2^-, let the value $\varepsilon \geq 0$ be defined as described above. If $c_1^+(x^+) + \varepsilon \leq c_2^+(x^+)$ holds for some $x^+ \in \mathbb{R}_{\geq 0}$, any solution where x^+ is the (optimal) amount of time spent for c_2^+ is dominated by f_1, i. e., we obtain either $c_2^+(x^+) = \infty$ or $f_1(x^+ + x^-, b) \geq f_2(x^+ + x^-, b) = b - (c_2^+(x^+) + c_2^-(x^-))$ for all $x^- \in \mathbb{R}_{\geq 0}$ and $b \in [0, M]$.*

Proof. Assume for the sake of contradiction that there exists some value $x^+ \in \mathbb{R}_{\geq 0}$ such that $c_1^+(x^+) + \varepsilon \leq c_2^+(x^+)$ and for some time $x^- \in \mathbb{R}_{\geq 0}$ and SoC $b \in [0, M]$, the value $b - (c_2^+(x^+) + c_2^-(x^-))$ is a feasible solution that is not dominated by $f_1(x^+ + x^-, b)$. Since $\varepsilon \geq 0$, we know that $c_1^+(x^+) \leq c_2^+(x^+)$ holds. This implies that $c_1^+(x^+) + c_1^-(x^-)$ is a feasible solution for an SoC of b (recall that the minimum driving time of c_1^- is 0). Finally, we know that $c_1^+(x^+) \leq c_2^+(x^+) - \varepsilon$ holds by assumption and $c_1^-(x^-) \leq c_2^-(x^-) + \varepsilon$ holds by the definition of ε. This yields

$$\begin{aligned} f_1(x^+ + x^-, b) &\geq b - \big(c_1^+(x^+) + c_1^-(x^-)\big) \\ &\geq b - \big(c_2^+(x^+) - \varepsilon + c_2^-(x^-) + \varepsilon\big), \end{aligned}$$

which contradicts our assumption and completes the proof. Q.E.D.

D.2. Upper Bound Functions Described by a Constant Number of Coefficients

In Section 6.4, we mentioned that our witness search can utilize upper bounds represented by single tradeoff subfunctions to enable faster operations and better data locality. To derive such bounds, consider a piecewise-defined consumption function c with minimum and maximum driving time $\underline{\tau} \in \mathbb{R}_{>0}$ and $\bar{\tau} \in \mathbb{R}_{>0}$, respectively, that is defined by several tradeoff subfunctions $g_1, \dots, g_k$. We seek a tradeoff function $\bar{g}$ that has the general form $\bar{g}(x) = \alpha/(x-\beta)^2 + \gamma$ as in Equation (5) for all $x \in [\underline{\tau}, \bar{\tau}]$ in the interval of admissible driving times. Further, we demand that $\bar{g}(x) \geq c(x)$ holds for all $x \in [\underline{\tau}, \bar{\tau}]$. To achieve this, we first set

$$\beta := \min_{i \in \{1,\dots,k\}} \beta_i \tag{5}$$

where β_i denotes the coefficient of the tradeoff subfunction g_i of c (see Equation (5)). Then, we can fix the function values $\bar{g}(\underline{\tau}) := c(\underline{\tau})$ and $\bar{g}(\bar{\tau}) := c(\bar{\tau})$ at its domain borders to uniquely define the two remaining coefficients α and γ from Equation (5). In particular, we obtain

$$\gamma = \frac{c(\bar{\tau})(\beta - \bar{\tau})^2 - c(\underline{\tau})(\beta - \underline{\tau})^2}{(\beta - \bar{\tau})^2 - (\beta - \underline{\tau})^2}, \tag{6}$$

$$\alpha = (c(\underline{\tau}) - \gamma)(\underline{\tau} - \beta)^2. \tag{7}$$

Lemma D-1 shows that the resulting function is in fact an upper bound on c. Upper bounds are also *robust* towards incremental linking in the sense that the error does not increase if we recompute the upper bound whenever linking several bound functions results in a bound consisting of multiple subfunctions. This is due to the fact that the bounds are uniquely defined by their (minimum) coefficient β and the domain borders of the original functions, which remain unchanged in the upper bound.

During witness search, whenever linking two bound functions results in a function defined by more than one tradeoff subfunction, we compute and store the upper bound instead. Note that we do not even have to link functions explicitly, but simply compute the new coefficient β in a linear scan that simulates the link operation.

LEMMA D-1. *The function $\bar{g}$ defined by Equations (5)–(7) is an upper bound on the original consumption function c within the interval $[\underline{\tau}, \bar{\tau}]$, i. e., $\bar{g}(x) \geq c(x)$ holds for all $x \in [\underline{\tau}, \bar{\tau}]$.*

Proof. Let $g_1, \dots, g_k$ denote the tradeoff subfunctions defining c and without loss of generality, assume that these subfunctions are given in increasing order of their admissible driving times. First, we argue that it is sufficient to prove the lemma for the case $k = 2$. To show this, we define an operation $\mathrm{bound}\colon \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ that takes as input two consumption functions, each defined by a *single* tradeoff subfunction, and computes an upper bound as described above. For $i \in \{1, \dots, k-1\}$, consider two consumption functions c_i and c_{i+1} induced by two consecutive tradeoff functions g_i and g_{i+1} (see Section 3.2 for the definition of induced consumption functions). Let their corresponding minimum and maximum driving times be $\underline{\tau}_i$, $\bar{\tau}_i = \underline{\tau}_{i+1}$, and $\bar{\tau}_{i+1}$. The bound operation computes the consumption function $c_{i,i+1} := \mathrm{bound}(c_i, c_{i+1})$ with minimum driving time $\underline{\tau}_i$, maximum driving time $\bar{\tau}_{i+1}$, and a single tradeoff subfunction $\bar{g}_{i,i+1}$. According to Equations (5)–(7), the coefficients of this tradeoff function depend only on the values $\beta = \min\{\beta_i, \beta_{i+1}\}$, the driving times $\underline{\tau}_i$ and $\bar{\tau}_{i+1}$, as well as the consumption values $c_i(\underline{\tau}_i) = \bar{g}_{i,i+1}(\underline{\tau}_i)$ and $c_{i+1}(\bar{\tau}_{i+1}) = \bar{g}_{i,i+1}(\bar{\tau}_{i+1})$ at the domain borders of $\bar{g}_{i,i+1}$. Linking the consumption function $c_{i,i+1}$ with another consecutive (induced) consumption function yields a new function defined by the same corresponding values. Consequently, the result $\mathrm{bound}(\dots\mathrm{bound}(\dots\mathrm{bound}(c_1, c_2), \dots), c_k)$ of iteratively applying the bound operation to the k induced consumption functions of c is the function that is defined by the coefficient $\beta = \min_{i \in \{0,\dots,k\}} \beta_i$ (see Equation (5)), the minimum driving time $\underline{\tau} = \underline{\tau}_1$, the maximum driving time $\bar{\tau} = \bar{\tau}_k$, the maximum consumption $c(\underline{\tau}) = c_1(\underline{\tau}_1)$, and the minimum consumption $c(\bar{\tau}) = c_k(\bar{\tau}_k)$. This is exactly the function $\bar{g}$ defined above. Thus, we can construct $\bar{g}$ by iteratively applying the bound operation to consumption functions induced by the tradeoff subfunctions of c. To prove the lemma, we now show that each function constructed by the operation bound is in fact an upper bound on its *two* input functions. Observe that this implies that $\bar{g}$ is an upper bound on c within the interval $[\underline{\tau}, \bar{\tau}]$.

In the remainder of the proof, let c be a consumption function defined by two tradeoff subfunctions g_1 and g_2, which induce two consumption functions c_1 and c_2. We prove that the function $\bar{g}\colon \mathbb{R}_{>0} \to \mathbb{R}$ computed by $\mathrm{bound}(c_1, c_2)$ yields an upper bound on c on the interval $[\underline{\tau}, \bar{\tau}]$. Let the subdomains of g_1 and g_2 be $[\underline{\tau}, \tau)$ and $[\tau, \bar{\tau})$, respectively. By continuity of c on the interval $[\underline{\tau}, \bar{\tau}]$ and by continuity of both g_1 and g_2 on $\mathbb{R}_{>0}$, we know that $g_1(\tau) = g_2(\tau)$. To prove the lemma, we use the following three claims.

1. The inequality $\bar{g}(\tau) \geq g_1(\tau) = g_2(\tau)$ holds.

2. The slopes (i. e., the derivatives) of $\bar{g}$ and g_1 are equal at $\underline{\tau}$ if and only if $\bar{g} \equiv g_1 \equiv g_2$. Otherwise, the slope of $\bar{g}$ is *greater* at this point.
3. The slopes of $\bar{g}$ and g_2 are equal at $\bar{\tau}$ if and only if $\bar{g} \equiv g_1 \equiv g_2$. Otherwise, the slope of $\bar{g}$ is *smaller* at this point.

Then, $\bar{g}(\tau) \geq g_1(\tau)$ holds by our first claim, $\bar{g}$ and g_1 intersect at $\underline{\tau}$ by construction, and $\bar{g}(\underline{\tau}+\varepsilon) \geq g_1(\underline{\tau}+\varepsilon)$ holds for $\varepsilon > 0$ in the neighborhood of $\underline{\tau}$ by our second claim. This implies that $\bar{g}$ must be an upper bound on g_1 on the interval $[\underline{\tau}, \tau]$, because the functions $\bar{g}$ and g_1 can intersect at most twice in this interval unless $\bar{g} \equiv g_1$. This is easy to verify by determining the number of zeros of $\bar{g} - g_1$ within the considered interval $[\underline{\tau}, \tau]$. A similar argument holds for $\bar{g}$ and g_2 on the interval $[\tau, \bar{\tau}]$. Hence, the lemma follows after proving the three claims made above. We detail the rather technical proofs of these claims below.

Assume that the functions g_1 and g_2 are given as $g_i(x) = \alpha_i/(x-\beta_i)^2 + \gamma_i$ for all $x \in \mathbb{R}_{>0}$ and for $i \in \{1,2\}$. For the sake of simplicity and without loss of generality, we presume that $\min\{\beta_1, \beta_2\} = 0$. Note that we can always enforce this property by *shifting* both functions (and their subdomains) along the x-axis. Afterwards, we obtain the same function $\bar{g}$ on the shifted subdomains. By a similar argument, we presume that $\gamma_1 = 0$ holds. Below, we consider the case $\beta_1 = 0$ and $\beta_2 \geq 0$. The case $\beta_1 \geq 0$ and $\beta_2 = 0$ is analogous. Since $\gamma_2 \in \mathbb{R}$ is allowed to become negative, no further case distinction is necessary.

To prove the first claim, we have to show that $\bar{g}(\tau) \geq g_1(\tau) = g_2(\tau)$ holds. For the sake of contradiction, assume $\bar{g}(\tau) < g_1(\tau)$. As mentioned before, continuity of the consumption function c on the interval $[\underline{\tau}, \bar{\tau}]$ implies that $g_1(\tau) = g_2(\tau)$, i. e.,

$$\frac{\alpha_1}{\tau^2} = \frac{\alpha_2}{(\tau-\beta_2)^2} + \gamma_2. \tag{8}$$

Furthermore, we know that c is a *convex* function (see Lemma A-3), so when evaluating the derivatives of g_1 and g_2 at τ, we get the inequality

$$-\frac{2\alpha_1}{\tau^3} \leq -\frac{2\alpha_2}{(\tau-\beta_2)^3}. \tag{9}$$

Finally, we know that the inequalities $0 < \underline{\tau} < \tau < \bar{\tau}$, $\beta_2 \geq 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, and $\tau > \beta_2$ hold by definition for consumption functions composed of multiple tradeoff subfunctions. We now show that altogether, these inequalities yield a contradiction. First, we plug the values of $\alpha \in \mathbb{R}_{\geq 0}$ from Equation (7) and $\gamma \in \mathbb{R}$ from Equation (6) into the term $\bar{g}(\tau) = \alpha/\tau^2 + \gamma$. Afterwards, we replace $\gamma_2 = \alpha_1/\tau^2 - \alpha_2/(\tau-\beta_2)^2$ according to Equation (8) and exploit that the inequality $\alpha_1 \geq \alpha_2\tau^3/(\tau-\beta_2)^3 > 0$ holds by Equation (9) to obtain

$$\begin{aligned}
& & \bar{g}(\tau) = \frac{g_1(\underline{\tau})\underline{\tau}^2(\bar{\tau}^2-\tau^2) + g_2(\bar{\tau})\bar{\tau}^2(\tau^2-\underline{\tau}^2)}{\tau^2(\bar{\tau}^2-\underline{\tau}^2)} &< g_1(\tau) \\
&\Leftrightarrow & g_1(\underline{\tau})\underline{\tau}^2(\bar{\tau}^2-\tau^2) - g_1(\tau)\tau^2(\bar{\tau}^2-\underline{\tau}^2) + g_2(\bar{\tau})\bar{\tau}^2(\tau^2-\underline{\tau}^2) &< 0 \\
&\Leftrightarrow & (\tau^2-\underline{\tau}^2)\Big(\alpha_1(\bar{\tau}^2-\tau^2)(\bar{\tau}-\beta_2)^2(\tau-\beta_2)^2 + \alpha_2\tau^2\bar{\tau}^2\big((\tau-\beta_2)^2-(\bar{\tau}-\beta_2)^2\big)\Big) &< 0 \\
&\Rightarrow & \frac{\alpha_2\tau^2}{\tau-\beta_2}\Big(\tau(\bar{\tau}^2-\tau^2)(\bar{\tau}-\beta_2)^2 + \bar{\tau}^2(\tau-\beta_2)\big((\tau-\beta_2)^2-(\bar{\tau}-\beta_2)^2\big)\Big) &< 0 \\
&\Leftrightarrow & \beta_2(\bar{\tau}^2-\tau^2)(2\bar{\tau}\tau-2\bar{\tau}\beta_2+\bar{\tau}^2-\tau\beta_2) &< 0 \\
&\Leftrightarrow & 2\bar{\tau}\tau-2\bar{\tau}\beta_2+\bar{\tau}^2-\tau\beta_2 &< 0.
\end{aligned}$$

This yields a contradiction, because we know that $0 \leq \beta_2 < \tau < \bar{\tau}$ holds. Thus, both $2\bar{\tau}\tau - 2\bar{\tau}\beta_2$ and $\bar{\tau}^2 - \tau\beta_2$ are positive terms and their sum cannot be negative.

For the second claim, we examine the slopes of g_1 and $\bar{g}$ at the domain border $\underline{\tau}$. Let the parameter $\alpha \in \mathbb{R}_{\geq 0}$ of $\bar{g}$ be defined as in Equation (7). Plugging in the coefficient $\beta = 0$ and the value of $\gamma \in \mathbb{R}$ according to Equation (6), we obtain

$$\alpha = \frac{\big(g_1(\underline{\tau}) - g_2(\bar{\tau})\big)\underline{\tau}^2\bar{\tau}^2}{\bar{\tau}^2 - \underline{\tau}^2}.$$

As before, we use $\gamma_2 = \alpha_1/\tau^2 - \alpha_2/(\tau - \beta_2)^2$ and the inequality $\alpha_1 \geq \alpha_2\tau^3/(\tau - \beta_2)^3 > 0$. For the difference between the derivatives $\bar{g}'$ and g_1' at $\underline{\tau}$, this yields

$$\begin{aligned}
\bar{g}'(\underline{\tau}) - g_1'(\underline{\tau}) &= \frac{2\alpha_1}{\underline{\tau}^3} - \frac{2\alpha}{\underline{\tau}^3} \\
&= \frac{2\left(g_2(\bar{\tau})\bar{\tau}^2\underline{\tau}^2 - \alpha_1\underline{\tau}^2\right)}{\underline{\tau}^3\left(\bar{\tau}^2 - \underline{\tau}^2\right)} \\
&= \frac{2\underline{\tau}^2}{\underline{\tau}^3\left(\bar{\tau}^2 - \underline{\tau}^2\right)}\left(\alpha_2\bar{\tau}^2\frac{(\tau - \beta_2)^2 - (\bar{\tau} - \beta_2)^2}{(\bar{\tau} - \beta_2)^2(\tau - \beta_2)^2} + \alpha_1\left(\frac{\bar{\tau}^2}{\tau^2} - 1\right)\right) \\
&\geq \frac{2\alpha_2\underline{\tau}^2}{\underline{\tau}^3(\bar{\tau}^2 - \underline{\tau}^2)}\left(\bar{\tau}^2\frac{(\tau - \beta_2)^2 - (\bar{\tau} - \beta_2)^2}{(\bar{\tau} - \beta_2)^2(\tau - \beta_2)^2} + \frac{\tau^3}{(\tau - \beta_2)^3}\left(\frac{\bar{\tau}^2}{\tau^2} - 1\right)\right) \\
&= \frac{2\alpha_2\beta_2\underline{\tau}^2(\bar{\tau} - \tau)^2(2\bar{\tau}\tau - 2\bar{\tau}\beta_2 + \bar{\tau}^2 - \tau\beta_2)}{\bar{\tau}^3(\bar{\tau}^2 - \underline{\tau}^2)(\bar{\tau} - \beta_2)^2(\tau - \beta_2)^3}.
\end{aligned}$$

As in the proof of the first claim, we observe that each term in the product of the numerator is nonnegative, while each term in the product of the denominator is positive. Moreover, the numerator is equal to 0 if and only if $\beta_2 = 0$ holds. Using Equation (8), it is easy to verify that this implies $\alpha_1 = \alpha_2$ and $\gamma_2 = 0$, which corresponds to the case where the three functions $\bar{g}$, g_1, and g_2 are equivalent.

Finally, we deal with the slopes of g_2 and $\bar{g}$ at $\bar{\tau}$ to prove the third claim. Below, we first replace the values α, α_2, and γ_2 as in our proof of the second claim. Afterwards, we exploit the fact that $(\tau^3 - x)(\bar{\tau} - \beta_2)^3 - (\bar{\tau}^3 - x)(\tau - \beta_2)^3$ decreases with increasing $x \in \mathbb{R}_{\geq 0}$, since its derivative wrt. x is $(\tau - \beta_2)^3 - (\bar{\tau} - \beta_2)^3 < 0$. After some further rearrangements, we obtain

$$\begin{aligned}
g_2'(\bar{\tau}) - \bar{g}'(\bar{\tau}) &= \frac{2\alpha}{\bar{\tau}^3} - \frac{2\alpha_2}{(\bar{\tau} - \beta_2)^3} \\
&\geq \frac{2\alpha_2\big((\tau^3 - \underline{\tau}^2\beta_2)(\bar{\tau} - \beta_2)^3 - (\bar{\tau}^3 - \underline{\tau}^2\beta_2)(\tau - \beta_2)^3\big)}{\bar{\tau}(\bar{\tau}^2 - \underline{\tau}^2)(\bar{\tau} - \beta_2)^3(\tau - \beta_2)^3} \\
&\geq \frac{2\alpha_2\big((\tau^3 - \tau^2\beta_2)(\bar{\tau} - \beta_2)^3 - (\bar{\tau}^3 - \tau^2\beta_2)(\tau - \beta_2)^3\big)}{\bar{\tau}(\bar{\tau}^2 - \underline{\tau}^2)(\bar{\tau} - \beta_2)^3(\tau - \beta_2)^3} \\
&= \frac{2\alpha_2\beta_2(\bar{\tau} - \tau)^2(\bar{\tau}\tau - \bar{\tau}\beta_2 + \bar{\tau}\tau - \tau\beta_2 + \tau^2 - \tau\beta_2)}{\bar{\tau}(\bar{\tau}^2 - \underline{\tau}^2)(\bar{\tau} - \beta_2)^3(\tau - \beta_2)^2}.
\end{aligned}$$

Again, we end up with products for which all factors are nonnegative (and strictly positive in case of the denominator). As before, the numerator equals 0 if and only if $\bar{g} \equiv g_1 \equiv g_2$. Hence, all three claims hold and the proof is complete. Q.E.D.

Appendix E: Implementation Details

Our implementation stores graphs as adjacency arrays (Cormen et al. 2009), following the dynamic data structures of Delling (2009) for efficient insertion and deletion of shortcuts during the preprocessing routine of Contraction Hierarchies (CH). All algorithms use k-heaps (Cormen et al. 2009, Johnson 1975) as priority queue, where $k=4$ except for unsettled label sets, which use $k=2$. Compared to Fibonacci heaps (Fredman and Tarjan 1987), these heaps have a higher worst-case complexity, but are faster on sparse graphs (such as road networks) in practice (Cherkassky, Goldberg, and Radzik 1996). Implementation details of our speedup techniques are given below.

A Search.* When computing the potential function π_φ (see Section 5.2), the number of breakpoints of lower bounds can become quite large. Therefore, we reduce it as follows (while slightly deteriorating the quality of the bounds). Before applying Graham's scan, we replace consecutive pairs of breakpoints in the piecewise linear function by a single one if they are close to each other, i. e., their difference wrt. driving time or SoC is below a certain threshold $\Delta_x \in \mathbb{R}_{\geq 0}$ or $\Delta_b \in \mathbb{R}_{\geq 0}$, respectively. Two such points $p=(b_p,x_p)$ and $q=(b_q,x_q)$ are replaced by $r:=(\min\{b_p,b_q\},\min\{x_p,x_q\})$. Furthermore, if two consecutive segments pq and qr with $p=(b_p,x_p)$, $q=(b_q,x_q)$, and $r=(b_r,x_r)$ have similar slopes $s_{pq}\approx s_{qr}$ (i. e., the difference $|s_{pq}-s_{qr}|$ is below some threshold $\Delta_s \in \mathbb{R}_{\geq 0}$), we replace them by a single segment from (b_p,x_p) to (b^*,x_r) with slope $\min\{s_{pq},s_{qr}\}$, which uniquely defines the value $b^* \in \mathbb{R}$. Clearly, the modified function remains a lower bound. Moreover, consistency of the potential is maintained, as function values can only decrease and changes in the function are propagated by the search. Thus, all steps in the proof of Lemma C-1 still apply.

Graham's scan and the breakpoint reduction step are performed on-the-fly during the merge operation. Moreover, we convert consumption functions c_a of all arcs $a \in A$ to their corresponding lower bounds φ_a during preprocessing for faster query times. The thresholds ε to determine lower bound errors, Δ_x and Δ_b for close points, and Δ_s for similar slopes are tuning parameters. Smaller thresholds increase accuracy of bounds, but also slow down the backward search. Therefore, we set above thresholds to $2^{\delta-\lfloor \log M \rfloor}$ in our experiments, where $\delta \in \mathbb{N}$ is a constant and M is the battery capacity (assumed to be given in kWh). Hence, bounds are more accurate for higher capacities (where the forward search becomes more expensive). The value of δ is again a tuning parameter. In our experiments, we use $\delta=10$ for Δ_x (the resulting threshold is measured in seconds), $\delta=17$ for Δ_b, and $\delta=15$ for ε (both measured in Wh). For example, a battery capacity of 16 kWh yields $\Delta_x=64$ (seconds), $\Delta_b=2^{13}$ (Wh), and $\varepsilon=2^{11}$ (Wh). The value $\Delta_s=2^{-4}$ is constant and chosen independently of M (all parameters were determined in preliminary experiments).

Contraction Hierarchies. During preprocessing, we determine the next vertex to be contracted using the measures Edge Difference (ED) and Cost of Queries (CQ) according to Geisberger et al. (2012). To reflect the complexity of SoC functions, we add another term *Shortcut Complexity (SC)*, which is defined as $|c^+|+k|c^-|$ for the SoC function of a given shortcut candidate, where $|c^+|$ and $|c^-|$ denote the number of tradeoff subfunctions that define the positive and negative part of a shortcut, respectively, and $k \in \mathbb{N}$ is a tuning parameter. Using penalized weights for negative parts, we favor earlier contraction of SoC functions without a negative part (we use $k=4$ in our experiments). The priority of a vertex (higher priority means higher importance) is then set to $64\,\mathrm{ED}+\mathrm{CQ}+\mathrm{SC}$. We set the priority of all inactive vertices to ∞.

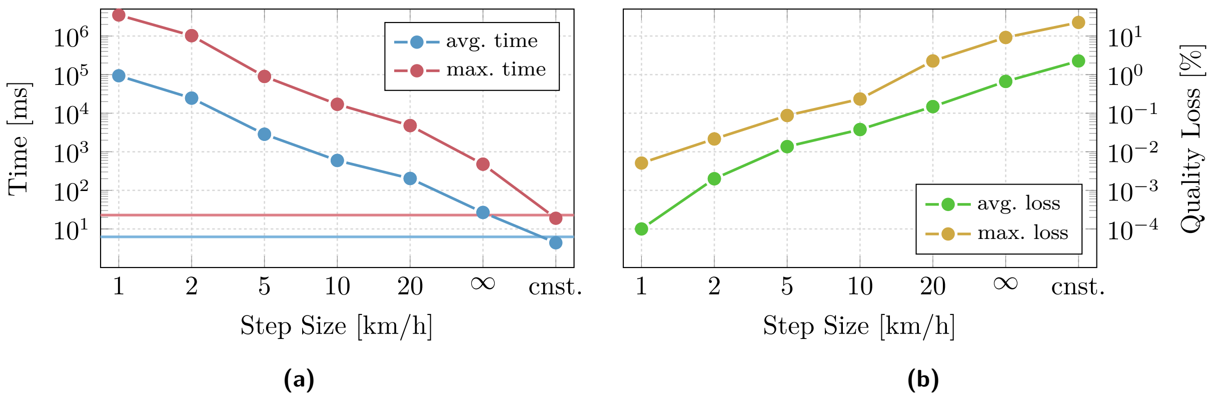

Figure 2 **Trading accuracy for running time when using bicriteria shortest path (BSP) (Ger-PG, 2 kWh). Both plots show query times and result quality for the same set of queries as in Table 3. (a) Average and maximum query times for different speed steps. Horizontal lines correspond to the average and maximum query times when running TFP. (b) Average and maximum loss in quality compared to the result of TFP, i. e., continuous tradeoffs.**

Further, we employ a *settled node limit* (Geisberger et al. 2012) of 128, which limits the maximum number of queue extractions per witness search for better performance. If multiple shortcut candidates with the same tail vertex $u \in V$ are constructed during contraction of a vertex, we save time by running only a *single* multi-target witness search from u. Finally, to improve performance of the backward searches during a query (breadth-first search (BFS) and potential computation), we explicitly construct and store their more lightweight search graphs from the input graph (enriched with shortcuts, but storing less complex cost functions) during preprocessing.

Appendix F: Omitted Plots from Section 8

Figure 2 plots query times against solution qualities for different speed step sizes (visualizing the same figures as in Table 3). Figure 3 shows average running times for the same queries as in Figure 12, comparing the scalability of different approaches. Recall that a run was aborted if at least one of the 100 queries exceeded an hour of computation time.

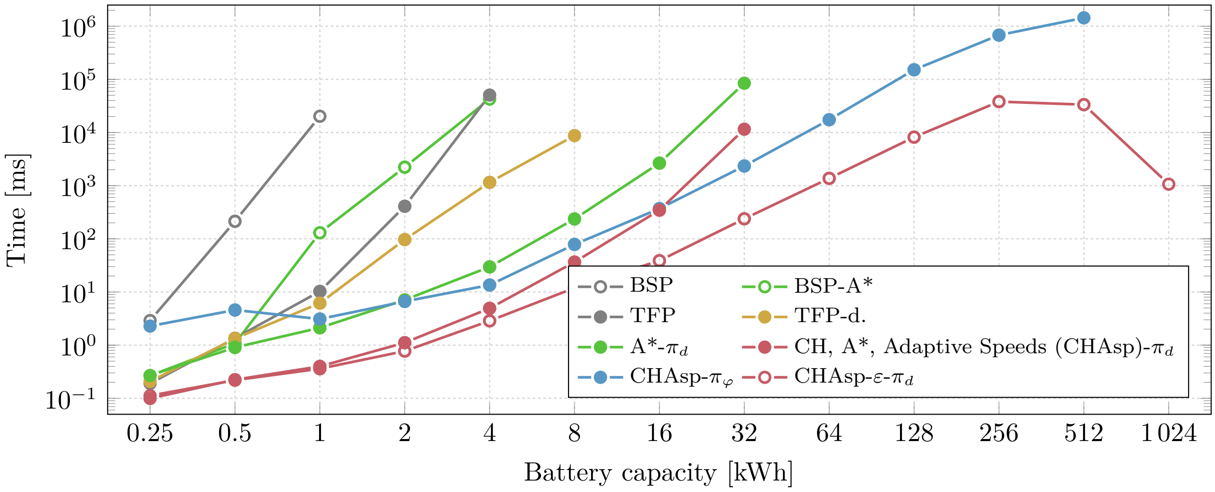

Figure 3 **Average running times for different battery capacities. For the same set of queries as in Figure 12, this plot shows the corresponding average running time of 100 random in-range queries.**

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